# Hierarchical Aggregation of Linear Systems with Multiple Time Scales 

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#### Abstract

In this paper we carry out a detailed analysis of the multiple time scale behavior of singularly perturbed linear systems of the form $$
\dot{x}^{\epsilon}(t)=A(\epsilon) x^{\epsilon}(t)
$$ where $A(\epsilon)$ is analytic in the small parameter $\epsilon$. Our basic result is a uniform asymptotic approximation to $\exp A(\epsilon) t$ that we obtain under a certain multiple semistability condition. This asymptotic approximation gives a complete multiple time scale decomposition of the above system and specifies a set of reduced order models valid at each time scale.


Our contribution is threefold.

1) We do not require that the state variables be chosen so as to display the time scale structure of the system.
2) Our formulation can handle systems with multiple ( $>2$ ) time scales and we obtain uniform asymptotic expansions for their behavior on $[0, \infty]$.
3) We give an aggregation method to produce increasingly simplified models valid at progressively slower time scales.

## I. Introduction

NOTIONS of time-scale separation are commonly used in heuristic model reduction techniques. It is well known that these notions can be formalized using techniques of singular perturbation theory, e.g., [1]. In this paper we carry out a detailed analysis of the multiple time scale behavior of singularly perturbed (defined in Section III) linear systems of the form

$$
\begin{equation*}
\dot{x}^{\epsilon}(t)=A(\epsilon) x^{\epsilon}(t), \quad x^{\epsilon}(0)=x_{0} \tag{1.1}
\end{equation*}
$$

where $A(\epsilon)$ is analytic in the small parameter $\epsilon \in\left[0, \epsilon_{0}\right]$. Our analysis gives a complete picture of the relationship between weak couplings, singular perturbations, multiple time scale behavior, and reduced order modeling ${ }^{1}$ for these systems. Specifically, we give necessary and sufficient conditions under which (1.1) exhibits well-defined nontrivial behavior at several fundamental time scales. We determine these time scales and we associate a

[^0]reduced order model of (1.1) with each of its fundamental time scales. We then show that these reduced order models can be combined to produce an asymptotic approximation to $x^{\epsilon}(t)$ uniformly valid on $[0, \infty]$.

In previous work it has generally been assumed that the system under consideration has "fast" and "slow" dynamics, and that by a combination of experience and physical insight a choice of state variables is avaitable which displays the two time scale structure of the system. Thus, typically, the starting point for research has been a system of the form

$$
\left[\begin{array}{c}
\dot{x}_{1}^{\epsilon}(t)  \tag{1.2}\\
\epsilon \dot{x}_{2}^{\epsilon}(t)
\end{array}\right]=\left[\begin{array}{ll}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{array}\right]\left[\begin{array}{l}
x_{1}^{\epsilon}(t) \\
x_{2}^{\epsilon}(t)
\end{array}\right] .
$$

While this system is not explicitly of the form (1.1), it can be converted into one of the form of (1.1) by rescaling time $\tau=t / \epsilon$. The rescaling is inconsequential in our development since we recover all the time scales associated with (1.1). We use the form (1.1) throughout our development with the understanding that time has been scaled so that the fastest time scale associated with the system is the order 1 time scale.

Further, in almost all the available literature, the behavior of the system is studied as $\epsilon \downarrow 0$ on intervals of the form $[0, T / \epsilon]$. The existence of nontrivial behavior for times of order $1 / \epsilon^{k}$ or, more generally, on the infinite time interval $[0, \infty]$ is either excluded by assumptions imposed on the matrices $A_{i j}$, or not considered at all. An example of the former is [2] where in the context of (1.2) it is proved that if $A_{22}$ and $A_{11}-A_{12} A_{22}^{-1} A_{21}$ are stable then (1.2) exhibits only two time scales. Two time scale systems are the only ones studied so far in the context of control and estimation problems (see [3] for a bibliography).

Our main contribution, we feel, is threefold.

1) We relax the requirement that state variables be chosen so as to display the time scale structure of the system.
2) Our formulation can handle systems with multiple ( $>2$ ) time scales and we obtain uniform asymptotic expansions for their behavior on $[0, \infty]$.
3) We give a method of aggregation to produce increasingly simplified models valid at progressively slower time scales. (We have applied this method to hierarchically aggregate finite state Markov processes with rare events. A brief description of this application of our methods is given in Section V; the details appear elsewhere [4].)

Systems with more than two time scales have been studied by other authors in different settings. In [5] the authors considered the asymptotic behavior of the quasi-linear system

$$
\begin{equation*}
\epsilon^{2} \dot{x}^{\epsilon}(t)=A(t) x^{\epsilon}(t)+\epsilon f\left(x^{\epsilon}(t), t, \epsilon\right) \tag{1.3}
\end{equation*}
$$

on the time interval $[0, T]$ and found that the asymptotic expansion of $x^{\epsilon}(t)$ requires three series: in $t, t / \epsilon$, and $t / \epsilon^{2}$, respectively. They did not, however, study the behavior of (1.3) on
$[0, \infty]$ and so left open the possibility of additional time scales. More recently, and with a formulation similar to ours. Campbell and Rose [6]-[8], [23] have studied the asymptotic behavior of

$$
\begin{equation*}
\dot{x}^{\epsilon}(t)=\left(\sum_{n=0}^{N} \epsilon^{\prime \prime} A_{n}\right) x^{\epsilon}(t) . \tag{1.4}
\end{equation*}
$$

For the case $N=1$, they showed that a necessary and sufficient condition for

$$
\lim _{\epsilon \downarrow 0} x^{\epsilon}(t / \epsilon)
$$

to exist pointwise (i.e., for fixed $t$-not uniformly on $t$ ) is the semistability of $\boldsymbol{A}_{0}$. For the more general case $N>1$, they give necessary and sufficient conditions so that

$$
\lim _{\epsilon \downarrow 0} x^{\epsilon}\left(t / \epsilon^{N}\right)
$$

exists pointwise, and they also give an expression for the limit. They do not address, however, the question of uniform asymptotic approximations to $x^{e}(t)$ or, equivalently, the question of how to determine the number and the time scales exhibited by (1.4) and how to combine the different pointwise limits to construct a uniform approximation (if possible). Furthermore, it does not seem to be widely appreciated that the system (1.3) may have nontrivial behavior at time scales $t / \epsilon^{2}, t / \epsilon^{3}, \cdots$. In the context of Markov processes with rare events, several authors [9]-[12] have used aggregated models to describe the evolution of these processes. As in the work mentioned before, however, the connection between a hierarchy of increasingly consolidated models and uniform approximation is absent. In this paper we address the foregoing questions within a framework that unifies the partial results cited above. For a more detailed account, the reader is referred to [4] and [20].
Finally, in a setting similar to ours, Hoppensteadt [21] studies uniform asymptotic approximations for the dynamics of a system of the form of (1.1). However he assumes that $A(\epsilon)$ has been decomposed in a form which explicitly displays the time scale structure. Specifically, he assumes that $A(\epsilon)$ is given in the form

$$
\begin{equation*}
A(\epsilon)=\bigoplus_{i=1}^{M} \epsilon^{r_{1}} A_{i}(\epsilon)+\tilde{A}(\epsilon) \tag{1.5}
\end{equation*}
$$

and then shows that the dynamics of (1.1) can be uniformly approximated under certain stability conditions by the dynamics given by the $A_{i}(\epsilon)$ at time scales of order $t / \epsilon^{r_{1}}, i=1, \cdots, M$. As we show in this paper, the transition from (1.1) to (1.4) is neither obvious nor always possible. In fact, from this perspective a major contribution of this paper is in providing an explicit algorithm for determining if a general $A(\epsilon)$ can be put in this form and if a uniform asymptotic approximation exists. This algorithm is constructive, and thus, if the answers it provides are in the affirmative, it will produce the uniform asymptotic approximation and, in effect, a transformation which explicitly displays the time scale structure as in (1.4).

The outline of the paper is as follows. In Section II we present the basic mathematical machinery for our approach: perturbation theory for linear operators. The fundamental results on perturbation of the resolvent, the eigenvalues, and the eigenprojections are stated without proof and are due to Kato [13]. In Section III we define regular and singular perturbations, and indicate the difficulties associated with uniform asymptotic approximations. In Section IV we apply the theory of Section II to obtain, under a certain multiple semistability condition, a uniform asymptotic approximation to $\exp \{A(\epsilon) t\}$ that gives a complete multiple time scale decomposition of the system (1.1), and specifies a set of reduced order models valid at each time scale. We then show that our results are tight, in that when the multiple semistability
condition is not satisfied, the system does not have well-defined behavior at some time scale. A partial time scale decomposition is sometimes possible in this instance, and it is carried out in Section IV-E. In Section V we summarize our results and explain briefly how they may be applied to the hierarchical aggregation of finite state Markov processes with rare transitions.

## II. Mathematical Preliminaries-Perturbation Theory for Linear Operators

We survey here the notation and some results on the perturbation of the eigenvalues, resolvent and eigenprojections of a linear operator $T: \mathbb{C}^{n} \rightarrow \mathbb{C}^{n}$ (for details see [13], [22]). These are the major mathematical tools for our development.

## A. The Resolvent

The set of all eigenvalues of $T$, denoted $\sigma(T)$, is called the spectrum of $T$. The function $R(\xi, T): \mathbb{C}-\sigma(T) \rightarrow \mathbb{C}^{n \times n}$ defined by

$$
\begin{equation*}
R(\xi, T):=(T-\xi I)^{-1} \tag{2.1}
\end{equation*}
$$

is called the resolvent of $T$. The resolvent of $T$ is an analytic function with singularities at $\lambda_{k} \in \sigma(T), k=0,1, \cdots, s$. The Laurent series of $R(\xi, T)$ at $\lambda_{k}$ has the form

$$
\begin{align*}
& R(\xi, T)=-\left(\xi-\lambda_{k}\right)^{-1} P_{k}-\sum_{i=1}^{m_{k}-1}\left(\xi-\lambda_{k}\right)^{-i-1} D_{k}^{l} \\
&+\sum_{i=0}^{\infty}\left(\xi-\lambda_{k}\right)^{i} S_{k}^{i+1} \tag{2.2}
\end{align*}
$$

where

$$
\begin{equation*}
P_{k}:=\frac{-1}{2 \pi i} \int_{\Gamma_{k}} R(\xi, T) d \xi \in \mathbb{C}^{n \times n} \tag{2.3}
\end{equation*}
$$

(with $\Gamma_{k}$ a positively oriented contour enclosing $\lambda_{k}$ but no other eigenvalue of $T$ ) is a projection (i.e., $P_{k}^{2}=P_{k}$ ) called the eigenprojection of the eigenvalue $\lambda_{k}$; and

$$
\begin{equation*}
m_{k}:=\operatorname{dim} \mathscr{R}\left(P_{k}\right) \tag{2.4}
\end{equation*}
$$

is the algebraic multiplicity of $\lambda_{k}$.

$$
\begin{equation*}
D_{k}:=-\frac{1}{2 \pi i} \int_{\Gamma_{k}}\left(\xi-\lambda_{k}\right) R(\xi, T) d \xi \tag{2.5}
\end{equation*}
$$

is the eigennilpotent (i.e., $D_{k}^{m_{k}}=0$ ) for the eigenvalue $\lambda_{k}$; and

$$
\begin{equation*}
S_{k}=\frac{1}{2 \pi i} \int_{\Gamma_{k}}\left(\xi-\lambda_{h}\right)^{-1} R(\xi, T) d \xi \tag{2.6}
\end{equation*}
$$

The following relations between $P_{k}, S_{k}$, and $D_{k}$ hold:

$$
\begin{align*}
P_{k} S_{k} & =S_{k} P_{k}=0  \tag{2.7}\\
P_{k} D_{k} & =D_{k} P_{h}=D_{k}  \tag{2.8}\\
P_{k} T & =T P_{k}  \tag{2.9}\\
\left(T-\lambda_{k} I\right) S_{k} & =I-P_{k}  \tag{2.10}\\
\left(T-\lambda_{k} I\right) P_{k} & =D_{k}  \tag{2.11}\\
P_{k} P_{l} & =\delta_{k \prime} P_{k}  \tag{2.12}\\
\sum_{k=1}^{s} P_{k} & =I . \tag{2.13}
\end{align*}
$$

From (2.12) and (2.13), it follows that

$$
\mathbb{C}^{n}=\mathscr{R}\left(P_{1}\right) \oplus \cdots \oplus \mathscr{R}\left(P_{s}\right)
$$

The $\mathscr{R}\left(P_{k}\right)$ is the algebraic eigenspace (or generalized eigenspace) for the eigenvalue $\lambda_{k}$. From (2.8) and (2.11), it follows that

$$
T P_{k}=P_{k} T=P_{k} T P_{k}=\lambda_{k} P_{k}+D_{k} .
$$

This, together with (2.13), yields the spectral representation of $T$ :

$$
T=\sum_{k=0}^{s}\left(\lambda_{k} P_{k}+D_{k}\right)=-\frac{1}{2 \pi i} \int_{\Gamma} \xi R(\xi, T) d \xi
$$

An eigenvalue $\lambda_{k}$ is said to be semisimple if the associated eigennilpotent $D_{k}$ is zero and simple if, in addition, $m_{k}=1$.

Using the resolvent $R(\xi, T)$ and a contour enclosing all the eigenvalues of $T$ in its interior we may define

$$
\exp \{T t\}=-\frac{1}{2 \pi i} \int_{\Gamma} \exp (\xi t) R(\xi, T) d \xi
$$

## B. Semisimple and Semistable Operators

An operator $T$ is said to have semisimple null structure (SSNS) if zero is a semisimple eigenvalue of $T$. The following lemma establishes some properties of operators with SSNS.

Lemma 2.1: The following statements are equivalent:

1) $T$ has SSNS,
2) $\mathbb{C}^{n}=\mathscr{R}(T)+\mathscr{N}(T)$,
3) $\mathscr{R}(T)=\mathscr{R}\left(T^{2}\right)$,
4) $\operatorname{rank} T=\operatorname{rank} T^{2}$,
5) $\mathscr{N}(T)=\mathscr{N}\left(T^{2}\right)$.

Proof: See [14].
Comment: When $T$ has SSNS, $P_{0}$, the eigenprojection for the zero eigenvalue, is the projection onto $\mathscr{N}(T)$ along $\mathscr{R}(T)$. Further, it follows [13] that if $T$ has SSNS, $T+P_{0}$ is nonsingular. Now, if $T^{\#}$ is defined to be $\left(T+P_{0}\right)^{-1}-P_{0}$, then it may be verified that

$$
\begin{equation*}
T T^{\#} T=T, \quad T^{\#} T T^{\#}=T^{\#} \tag{2.14}
\end{equation*}
$$

and

$$
\begin{equation*}
T^{\ddagger} T=T T^{\#} \tag{2.15}
\end{equation*}
$$

$T^{\#}$ is thus the group generalized inverse of $T$ (see [14]). Further, if $T$ has SSNS, then $P_{0}$ and $T^{*}$ determine the Laurent expansion of $\mathscr{R}(\lambda, T)$ at zero.

Lemma 2.2: If $T$ has SSNS, then for $\left\{\lambda:|\lambda| \leqslant\left|T^{*}\right|^{-1}\right\}$

$$
\begin{equation*}
\mathscr{R}(\lambda, T)=-\frac{P_{0}}{\lambda}+\sum_{k=0}^{\infty} \lambda^{k}\left(T^{\#}\right)^{k+1} \tag{2.16}
\end{equation*}
$$

Proof: Using (2.14) and (2.15)

$$
\begin{aligned}
& (T-\lambda I)\left(-\frac{P_{0}}{\lambda}+\sum_{k=0}^{\infty} \lambda^{k}\left(T^{\#}\right)^{k-1}\right) \\
& \quad=\left(I-P_{0}\right) \sum_{k=0}^{\infty} \lambda^{k}\left(T^{\mp}\right)^{k}+P_{0}-\sum_{k=1}^{\infty} \lambda^{k}\left(T^{\#}\right)^{k} \\
& \quad=I .
\end{aligned}
$$

Similarly

$$
\left(-\frac{P_{0}}{\lambda}+\sum_{k=0}^{\infty} \lambda^{k}\left(T^{\#}\right)^{k+1}\right)(T-\lambda I)=I .
$$

Also of interest in the sequel are semistable operators: $T$ is said to be semistable if $T$ has SSNS and all the eigenvalues of $T$ except the zero eigenvalue lie in $\dot{\mathbb{C}}_{-}$(the open left half plane).

## C. Perturbation of Eigenvalues

Before we discuss perturbation of the resolvent of an operator $T$, we discuss perturbation of its eigenvalues, when $T$ is of the form

$$
\begin{equation*}
T(\epsilon)=T+\sum_{n=1}^{\infty} \epsilon^{n} T^{(n)} \quad \epsilon \in\left[0, \epsilon_{0}\right] . \tag{2.17}
\end{equation*}
$$

Here (2.17) is assumed to be an absolutely convergent power series expansion. The eigenvalues of $T(\epsilon)$ satisfy

$$
\begin{equation*}
\operatorname{det}(T(\epsilon)-\xi I)=0 \tag{2.18}
\end{equation*}
$$

This is an algebraic equation in $\xi$ whose coefficients are $\epsilon$-analytic. From elementary analytic function theory, e.g., [15], the roots of (2.18) are branches of analytic functions of $\epsilon$ with only algebraic singularities. Hence, the number of (distinct) eigenvalues of $T(\epsilon)$ is a constant $s$, independent of $\epsilon$, except at some isolated values of $\epsilon$. Without loss of generality, let $\epsilon=0$ be such an exceptional point and further let it be the only such point in [ $0, \epsilon_{0}$ ]. In a neighborhood of the exceptional point, the eigenvalues of $T(\epsilon)$ can be expressed by $s$ distinct, analytic functions $\lambda_{1}(\epsilon), \cdots, \lambda_{s}(\epsilon)$. These may be grouped as

$$
\begin{equation*}
\left\{\lambda_{1}(\epsilon), \cdots, \lambda_{p}(\epsilon)\right\},\left\{\lambda_{p+1}(\epsilon), \cdots, \lambda_{p+1}(\epsilon)\right\}, \cdots \tag{2.19}
\end{equation*}
$$

so that each group has a Puiseux series of the form (written below for the first group)

$$
\lambda_{h}(\epsilon)=\lambda+\alpha_{1} \omega^{h} \epsilon^{1 / p}+\alpha_{2} \omega^{2 h} \epsilon^{2 / p}+\cdots \quad h=0,1, \cdots, p-1
$$

where $\lambda$ is an eigenvalue of the unperturbed operator $T$ and $\omega=\exp \{i 2 \pi / p\}$. Each group is called a cycle and the number of elements its period. $\lambda$ is called the center of the cycle and the group of eigenvalues having $\lambda$ as center is called the $\lambda$-group splitting at $\epsilon=0$ (the exceptional point).

## D. Perturbation of the Resolvent

The resolvent of $T(\epsilon)$ is defined on $\rho(T)=\mathbb{C}-\sigma(T(\epsilon))$

$$
R(\xi, T(\epsilon))=(T(\epsilon)-\xi I)^{-1}
$$

Lemma 2.3: If $\xi \in \rho(T)$, then for $\epsilon$ small enough, say $\epsilon \in$ $\left[0, \epsilon_{0}\right], \xi \in \rho(T(\epsilon))$, and

$$
\begin{equation*}
R(\xi, T(\epsilon))=R(\xi, T)+\sum_{n=1}^{\infty} \epsilon^{n} R^{(n)}(\xi) \tag{2.20}
\end{equation*}
$$

where

$$
\begin{align*}
& R^{(n)}(\xi)=\sum_{\substack{\nu_{1}+\cdots+\nu_{p}=n \\
\nu_{i} \geqslant 1}}(-1)^{p} R(\xi, T) T^{\left(\nu_{1}\right)} \\
& \cdot R(\xi, T) T^{\left(\nu_{2}\right)} \cdots T^{\left(\nu_{p}\right)} R(\xi, T), \tag{2.21}
\end{align*}
$$

the sum being taken over all integers $p$ and $\nu_{1}, \cdots, \nu_{p} \geqslant 1$ satisfying $\nu_{1}+\cdots+\nu_{p}=n$.

The series (2.20) is uniformly convergent on compact subsets of $\rho(T)$.

Proof: See [13].

## E. Perturbation of the Eigenprojections <br> We require first a preliminary lemma.

Lemma 2.4 (Taken Verbatim from [13, p. 34]): Let $P(t)$ be a projection matrix depending continuously on a parameter $t$ varying in a connected subset of $\mathbb{C}$. Then the ranges $\mathscr{R}(P(t))$ for different $t$ are isomorphic, i.e., the dimension of $\mathscr{R}(P(t))$ is constant.

Let $\lambda$ be an eigenvalue of $T=T(0)$ with (algebraic) multiplicity $m$. Let $\Gamma$ be a closed contour (positively oriented) in $\rho(T)$ enclosing $\Gamma$ but no other eigenvalues of $T$. From Lemma 2.3, it follows that for $\epsilon$ small enough, $R(\xi, T(\epsilon))$ exists for $\xi \in \Gamma$. and hence, there are no eigenvalues of $T(\epsilon)$ on $\Gamma$. Further, the matrix

$$
\begin{equation*}
P(\epsilon)=-\frac{1}{2 \pi i} \int_{\Gamma} R(\xi, T(\epsilon) d \xi \tag{2.22}
\end{equation*}
$$

is a projection which is equal to the sum of the eigenprojections for all the eigenvalues of $T(\epsilon)$ lying inside $\Gamma$. Using (2.21) and integrating term by term (recall uniform convergence from Lemma 2.3) we have

$$
\begin{equation*}
P(\epsilon)=P+\sum_{n=1}^{\infty} \epsilon^{n} P^{(n)} \quad \epsilon \in\left[0, \epsilon_{0}\right] \tag{2.23}
\end{equation*}
$$

where

$$
\begin{equation*}
P=-\frac{1}{2 \pi i} \int_{\Gamma} R(\xi, T) d \xi \tag{2.24}
\end{equation*}
$$

and

$$
\begin{equation*}
P^{(n)}=-\frac{1}{2 \pi i} \int_{\Gamma} R^{(n)}(\xi) d \xi \tag{2.25}
\end{equation*}
$$

Note that $P$ is the eigenprojection for the eigenvalue $\lambda$. Further, note that $P(\epsilon)$ is continuous in $\epsilon \in\left[0, \epsilon_{0}\right]$. By Lemma 2.4,

$$
\begin{equation*}
\operatorname{dim} \mathscr{R}(P(\epsilon))=\operatorname{dim} \mathscr{R}(P)=m \quad(\text { say }) . \tag{2.26}
\end{equation*}
$$

From (2.26), it follows that the eigenvalues of $T(\epsilon)$ lying inside $\Gamma$ form the $\lambda$ group. Hence, $P(\epsilon)$ is called the total projection and $\mathscr{R}(P(\epsilon))$ the total eigenspace for the $\lambda$-group. The following is a central proposition.

Proposition 2.5: Let $\lambda$ be an eigenvalue of $T=T(0)$ of (algebraic) multiplicity $m$ and $P(\epsilon)$ be the total projection for the $\lambda$-group of $T$. Then,

$$
\begin{align*}
\frac{(T(\epsilon)-\lambda I) P(\epsilon)}{\epsilon} & =-\frac{1}{\epsilon 2 \pi i} \int_{\Gamma}(\xi-\lambda) R(\xi, T(\epsilon)) d \xi \\
& =\frac{D}{\epsilon}+\sum_{n=0}^{\infty} \epsilon^{n} \tilde{T}^{(n)} \quad \text { for } \epsilon \in\left[0, \epsilon_{0}\right] \tag{2.26}
\end{align*}
$$

where $\Gamma$ is a closed positive contour enclosing $\lambda$ and no other eigenvalues of $T, D$ is the eigennilpotent for $\lambda$, and $\tilde{T}^{(n)}$ is given by

$$
\begin{align*}
& \tilde{T}^{(m)}=-\sum_{p=1}^{n+1} \sum_{\substack{p_{1}+\cdots+\nu_{p}=n+1 \\
k_{1}+\cdots+k_{n+1}=p-1 \\
\nu_{1} \geqslant 1 k_{j} \geqslant-m+1}} S^{\left(k_{1}\right)} T^{\left(\nu_{1}\right)} S^{\left(k_{2}\right)} \ldots \\
& S^{\left(k_{p}\right)} T^{\left(p_{p}\right)} S^{\left(k_{p+1}\right)} \tag{2.27}
\end{align*}
$$

with $S^{(0)}=-P, S^{(k)}=D^{-k}$ for $k<0$, and

$$
S^{(k)}=\left[\frac{1}{2 \pi i} \int_{\Gamma}(\xi-\lambda)^{-1} R(\xi, T) d \xi\right]^{k} \quad \text { for } k>0 .
$$

[^1]of the form (3.1) are said to be singularly perturbed if $A(\epsilon)$ has a Laurent series about $\epsilon=0$,
\[

$$
\begin{equation*}
A(\epsilon)=\sum_{p=-r}^{\infty} \epsilon^{p} A_{p} \tag{3.5}
\end{equation*}
$$

\]

with $r>0$, and regularly perturbed if $r=0$. We find this characterization deficient on two counts.

1) Using this definition, a system is regularly or singularly perturbed depending on the time scale used to write its dynamics.
2) The Laurent series formulation singles out from the very start a certain time scale of interest neglecting the system's evolution at slower and faster time scales.
With a simple normalization of the time variable, a system of the form (3.5) can be rewritten as having a system matrix with a convergent power series as in (3.1). By studying the evolution of the system on the infinite time interval $[0, \infty]$ as in (3.4), we can characterize the perturbation as regular or singular in a more fundamental way which will depend now on the structure of the system matrix. Further, such a study will give equal importance to all time scales present in the system.

In what follows, we focus on the singularly perturbed case, since failure of (3.4) is symptomatic of distinct behavior at several time scales. Formally, the following holds.

Definition 3.1 (Time Scale Behavior): Consider (3.1) and let $\alpha(\epsilon)$ be an order function $\left(\alpha:\left[0, \epsilon_{0}\right] \rightarrow \mathbb{R}_{+} ; \alpha(0)=0\right.$, and $\alpha(\cdot)$ continuous and monotone increasing), $x^{\epsilon}(t)$ is said to have well-defined behavior at time scale $t / \alpha(\epsilon)$ if there exists a continuous matrix $Y(t)$ such that, for any $\delta>0, T<\infty$,

$$
\lim _{\epsilon \downarrow 0} \sup _{t \in[\delta, T]}\|\exp \{A(\epsilon) t / \alpha(\epsilon)\}-Y(t)\|=0
$$

The following proposition shows that regularly perturbed (unlike singularly perturbed) systems have extremely simple time scale behavior.

Proposition 3.2: Let (3.1) be a regularly perturbed version of (3.3). Then, for any order function $\alpha(\epsilon), \delta>0, T<\infty$

$$
\begin{equation*}
\lim _{\epsilon \downarrow 0} \sup _{t \in[\delta, T]}\left\|\exp \{A(\epsilon) t / \alpha(\epsilon)\}-P_{0}\right\|=0 \tag{3.6}
\end{equation*}
$$

where $P_{0}$ is the eigenprojection for the zero eigenvalue of $A_{0}$. Proof:

$$
\begin{align*}
& \left\|\exp \{A(\epsilon) t / \alpha(\epsilon)\}-P_{0}\right\| \\
& \leqslant\left\|\exp \{A(\epsilon) t / \alpha(\epsilon)\}-\exp \left\{A_{0} t / \alpha(\epsilon)\right\}\right\| \\
& \quad+\left\|\exp \left\{A_{0} t / \alpha(\epsilon)\right\}-P_{0}\right\| \tag{3.7}
\end{align*}
$$

By the definition of regular perturbation, the first term of the RHS of (3.7) converges to 0 as $\epsilon \downarrow 0$ uniformly in $t$. For the second term, we write

$$
\begin{equation*}
\exp \left\{A_{0} t\right\}=P_{0}-\frac{1}{2 \pi i} \int_{\Gamma_{0}} e^{\lambda t} R\left(\lambda, A_{0}\right) d \lambda \tag{3.8}
\end{equation*}
$$

where $\Gamma_{0}$ is a contour enclosing all nonzero eigenvalues of $A_{0}$. By the assumption of semistability of $A_{0}$, we may choose $\Gamma_{0}$ to be in the left half plane bounded away from the $j \omega$-axis, say by the line $\{\lambda: \operatorname{Re} \lambda=-\beta\}$. Using (3.8), we then have

$$
\begin{equation*}
\left\|\exp \left\{A_{0} t / \alpha(\epsilon)\right\}-P_{0}\right\| \leqslant K e^{-\beta \delta / \alpha(\epsilon)} \quad \text { for } t \in[\delta, \infty] . \tag{3.9}
\end{equation*}
$$

Taking limits on both sides of (3.7) using (3.9) proves (3.6).
To complete our discussion of the distinction between regular and singularly perturbed systems, we give a necessary and suffi-
cient condition for (3.1) to be a singularly perturbed version of (3.3).

Proposition 3.3: The system (3.1) is singularly perturbed if and only if $\operatorname{rank} A_{0}<n \operatorname{rank} A(\epsilon)$.

Proof: Necessity is established by contradiction. Let $n \operatorname{rank} A(\epsilon)=\operatorname{rank} A_{0}$. Since the set of eigenvalues of $A(\epsilon)$ is a continuous function of $\epsilon$, the zero eigenvalue of $A(\epsilon)$ does not split. Hence, for $\epsilon$ small, a contour $\gamma_{0}$ enclosing the origin can be found such that it only encloses the zero eigenvalue of $A(\epsilon)$. Since $A(\epsilon)$ is assumed to be semistable, the only singularity of the resolvent $R\left(\lambda, A(\epsilon)\right.$ ) within $\gamma_{0}$ is a pole at $\lambda=0$ with residue $P_{0}(\epsilon)$, and we obtain

$$
\begin{equation*}
-\frac{1}{2 \pi i} \int_{\gamma_{0}} e^{\lambda t} R(\lambda, A(\epsilon)) d \lambda=P_{0}(\epsilon) . \tag{3.10}
\end{equation*}
$$

From Section II, we have that $P_{0}(\epsilon) \rightarrow P_{0}$ as $\epsilon \downarrow 0$, where $P_{0}$ is the eigenprojection for the zero eigenvalue of $A_{0}$,

$$
\begin{equation*}
P_{0}=-\frac{1}{2 \pi i} \int_{\gamma_{0}} e^{\lambda t} R\left(\lambda, A_{0}\right) d \lambda \tag{3.11}
\end{equation*}
$$

Using (3.10) and (3.11) we have

$$
\begin{aligned}
& \left\|\exp \{A(\epsilon) t\}-\exp \left\{A_{0} t\right\}\right\| \\
& \quad \leqslant \frac{1}{2 \pi} \int_{\Gamma_{0}}\left\|R(\lambda, A(\epsilon))-R\left(\lambda, A_{0}\right)\right\| e^{\operatorname{Re} \lambda t} d \lambda+\left\|P_{0}(\epsilon)-P_{0}\right\|
\end{aligned}
$$

where $\Gamma_{0}$ is a positive contour enclosing all nonzero eigenvalues of $A_{0}(\epsilon)$ for $\epsilon$ small. Since $R\left(\lambda, A_{0}(\epsilon)\right)$ converges uniformly to $R\left(\lambda, A_{0}\right)$ on $\Gamma_{0}$ and $\Gamma_{0}$ can be chosen to lie in $\mathbb{C}_{-}$bounded away from the $j \omega$-axis (by semistability of $A_{0}$ ), we have

$$
\begin{equation*}
\lim _{\epsilon \downarrow 0} \sup _{t \geqslant 0}\left\|\exp \{A(\epsilon) t\}-\exp \left\{A_{0} t\right\}\right\|=0 \tag{3.12}
\end{equation*}
$$

which establishes the contradiction.
Sufficiency is also established by contradiction. If (3.1) is a regularly perturbed version of (3.3), then

$$
\lim _{\epsilon \downarrow 0} P(\epsilon) \triangleq \lim _{\epsilon \downarrow 0} \lim _{t \rightarrow \infty} \exp \{A(\epsilon) t\}=\lim _{t \rightarrow \infty} \exp \left\{A_{0} t\right\}=P_{0}
$$

But $P(\epsilon), P_{0}$ are the eigenprojections for the zero eigenvalue of $A(\epsilon), A_{0}$, respectively; and, by Proposition 2.6, $\operatorname{rank} P(\epsilon)=$ rank $P_{0}$, thus establishing a contradiction because $\operatorname{rank} P(\epsilon)=$ null $A(\epsilon)$ and rank $P_{0}=$ null $A_{0}$.
Remarks:

1) If $A_{0}$ is asymptotically stable, then any perturbation is regular.
2) There is a heuristic connection between the time scale evolution of (3.1) and the eigenvalues of $A(\epsilon)$. In particular, eigenvalues of order $\epsilon^{k}$ are symptomatic of system behavior at time scale $t / \epsilon^{k}$. However, there are several detailed assumptions and delicate analysis to be performed to validate this heuristic reasoning. This is the focus of our attention in the following sections.

## IV. Complete Time Scale Decomposition

## A. Spatial and Temporal Decomposition of $\exp \{A(\epsilon) t\}$-The

 Multiple Semisimple Null Structure ConditionTo facilitate the notation in the development that follows, we choose for the perturbed system (3.1) the notation

$$
\begin{equation*}
\dot{x}^{\epsilon}(t)=A_{0}(\epsilon) x^{\epsilon}(t), \quad x^{\epsilon}(0)=x_{0} \tag{4.1}
\end{equation*}
$$

with

$$
\begin{equation*}
A_{0}(\epsilon)=\sum_{p=0}^{\infty} \epsilon^{p} A_{0 p} \tag{4.2}
\end{equation*}
$$

Of obvious interest here is when (4.1) is singularly perturbed-time scale behavior is trivial when the perturbation is regular as shown by Proposition 3.2. We thus restrict our attention to the case rank $A_{0}<n \operatorname{rank} A_{0}(\epsilon)$. For our development we need to construct a sequence of matrices $A_{k}(\epsilon), k=1, \cdots, m$ obtained recursively from $A_{0}(\epsilon)$ as indicated below.

Recall the notation of Section II. Let $P_{0}(\epsilon)$ denote the total projection for the zero group of eigenvalues of $A_{0}(\epsilon)$. From Corollary 2.8, it follows that if $A_{00}$ has semisimple null structure (SSNS), then the matrix

$$
A_{1}(\epsilon) \triangleq \frac{P_{0}(\epsilon) A_{0}(\epsilon)}{\epsilon}=\frac{A_{0}(\epsilon) P_{0}(\epsilon)}{\epsilon}=\frac{P_{0}(\epsilon) A_{0}(\epsilon) P_{0}(\epsilon)}{\epsilon}
$$

has a series expansion of the form

$$
\begin{equation*}
A_{1}(\epsilon)=\sum_{p=0}^{\infty} \epsilon^{p} A_{1 p} \tag{4.3}
\end{equation*}
$$

If the first term in the series (4.3), namely $A_{10}$, has SSNS, it follows that

$$
A_{2}(\epsilon) \triangleq \frac{P_{1}(\epsilon) A_{1}(\epsilon)}{\epsilon}=\frac{P_{1}(\epsilon) P_{0}(\epsilon) A_{0}(\epsilon)}{\epsilon^{2}}
$$

where $P_{1}(\epsilon)$ is the total projection for the zero group of eigenvalues of $A_{1}(\epsilon)$, has series expansion

$$
\begin{equation*}
A_{2}(\epsilon)=\sum_{p=0}^{\infty} \epsilon^{p} A_{2 p} . \tag{4.4}
\end{equation*}
$$

The recursion ends at step $m$, i.e., at

$$
\begin{align*}
A_{m}(\epsilon) & \triangleq \frac{P_{m-1}(\epsilon) A_{m-1}(\epsilon)}{\epsilon}=\frac{P_{m-1}(\epsilon) P_{m-2}(\epsilon) \cdots P_{0}(\epsilon) A_{0}(\epsilon)}{\epsilon^{m}} \\
& =\sum_{p=0}^{\infty} \epsilon^{p} A_{m p} \tag{4.5}
\end{align*}
$$

if the matrix $A_{m 0}$ does not have SSNS. The following proposition establishes several properties of the matrices $A_{k}(\epsilon), P_{k}(\epsilon)$. Define $Q_{k}(\epsilon)=I-P_{k}(\epsilon)$; note that $Q_{k}(\epsilon)$ is also a projection [onto the eigenspaces of the nonzero groups of eigenvalues of $\left.A_{k}(\epsilon)\right]$.
Proposition 4.1: For $\epsilon$ small enough, including zero, and $k=$ $1, \cdots, m$

1) $P_{i}(\epsilon) P_{j}(\epsilon)=P_{j}(\epsilon) P_{i}(\epsilon) \quad i, j=0,1, \cdots, m$
2) $Q_{i}(\epsilon) Q_{i}(\epsilon)=0 \quad i \neq j, i, j=0,1, \cdots, m$
3) $\mathbb{C}^{n}=\mathscr{R}\left(Q_{0}(\epsilon)\right) \oplus \cdots \oplus \mathscr{R}\left(Q_{k}(\epsilon)\right)$

$$
\oplus \mathscr{R}\left(P_{0}(\epsilon) \cdots P_{k}(\epsilon)\right)
$$

4) $\operatorname{rank} Q_{k}(\epsilon)=\operatorname{rank} A_{k 0}$
and for $\epsilon$ small enough but not zero,

$$
\text { 5) } \quad \begin{aligned}
Q_{k}(\epsilon) A_{0}(\epsilon) & =\epsilon^{k} Q_{k}(\epsilon) A_{k}(\epsilon)=\epsilon^{k} A_{k}(\epsilon) Q_{k}(\epsilon) \\
& =A_{0}(\epsilon) Q_{k}(\epsilon) .
\end{aligned}
$$

The proof of this result is a modification of results in [1]. See [20] for details.
The following proposition establishes that the sequence $A_{k}(\epsilon)$ always terminates at some finite $m$.

Proposition 4.2: Let $A_{k}(\epsilon), k=0,1, \cdots$, be the sequence of matrices defined recursively by (4.5). At least one of the following two conditions (possibly both) are satisfied at some $m<\infty$

1) $A_{m o}$ does not have SSNS
2) $A_{m+1}(\epsilon)=0$ or, equivalently,

2') $\sum_{k=0}^{m} \operatorname{rank} A_{k 0}=d$.
Proof: It only needs to be shown that 2) occurs for $m<\infty$ if (1) does not. From Proposition 4.1, for all $j \geqslant 0$

$$
\begin{equation*}
\mathbb{C}^{n}=\mathscr{R}\left(Q_{0}(\epsilon)\right) \oplus \cdots \oplus \mathscr{R}\left(Q_{,}(\epsilon)\right) \oplus \mathscr{R}\left(P_{0}(\epsilon) \cdots P_{j}(\epsilon)\right) . \tag{4.6}
\end{equation*}
$$

Since $\operatorname{rank} Q_{k}(\epsilon)=\operatorname{rank} A_{k 0}$, only a finite number of $A_{k 0}$ 's can be nonzero. Let $m$ be such that $A_{m 0} \neq 0$ and $A_{k 0}=0$ for $k>m$. If $A_{k 0}=0, P_{k}(\epsilon)=I$. Hence, $A_{k 0}=0$ for $k>m$ implies that $A_{m+1}(\epsilon)=0$.

To show the equivalence of 2 ) and $2^{\prime}$ ), note that

$$
A_{m+1}(\epsilon)=\frac{A_{0}(\epsilon) P_{0}(\epsilon) \cdots P_{m}(\epsilon)}{\epsilon^{m}}
$$

Hence, $A_{m+1}(\epsilon)=0$ implies that $\mathscr{R}\left(P_{0}(\epsilon) \cdots P_{m}(\epsilon)\right) \subset$ $\mathscr{N}\left(A_{0}(\epsilon)\right)$. On the other hand, if $x \in \mathscr{N}\left(A_{0}(\epsilon)\right)$, then $x \in$ $\mathscr{N}\left(A_{k}(\epsilon)\right)$ and therefore $P_{k}(\epsilon) x=x$. Thus, $\mathscr{N}\left(A_{0}(\epsilon)\right)=$ $\mathscr{R}\left(P_{0}(\epsilon) \cdots P_{m}(\epsilon)\right)$. Using this in (4.6) yields that 2$\left.) \Rightarrow 2^{\prime}\right)$. The proof of the converse is similar.

Definition 4.3: An analytic matrix function $A_{0}(\epsilon)$ of $\epsilon$ satisfies the multiple semisimple null structure (MSSNS) condition if the sequence of matrices $A_{k}(\epsilon)$ can be constructed until the stopping condition 2') of Proposition 4.2 has been met with all the matrices

$$
A_{k 0}=\lim _{\epsilon \downarrow 0} \frac{P_{k-1}(\epsilon) \cdots P_{0}(\epsilon) A_{0}(\epsilon)}{\epsilon^{i}} \quad k=0,1, \cdots, m
$$

having semisimple null structure (SSNS).
Proposition 4.4: If $A_{0}(\epsilon)$ satisfies the MSSNS condition, then for some $\epsilon_{1}>0$

1) $A_{k}(\epsilon)$ has SSNS for $\epsilon \in\left[0, \epsilon_{1}\right], \quad k=0, \cdots, m$.
2) For $\left.\epsilon \in] 0, \epsilon_{1}\right]$

$$
\mathscr{R}\left(A_{k}(\epsilon)\right)=\mathscr{R}\left(Q_{k}(\epsilon)\right) \oplus \cdots \oplus \mathscr{R}\left(Q_{m}(\epsilon)\right) \quad k=0, \cdots, m
$$

$$
\begin{array}{r}
\mathscr{N}\left(A_{k}(\epsilon)\right)=\mathscr{R}\left(Q_{0}(\epsilon)\right) \oplus \cdots \oplus \mathscr{R}\left(Q_{k-1}(\epsilon)\right) \oplus \mathscr{N}\left(A_{0}(\epsilon)\right)  \tag{4.7}\\
k=1, \cdots, m \quad 4
\end{array}
$$

$\mathscr{N}\left(A_{0}(\epsilon)\right)=\mathscr{R}\left(P_{0}(\epsilon) \cdots P_{m}(\epsilon)\right)$.
3) If $\lambda(\epsilon)$ is an eigenvalue of $A_{k}(\epsilon)$ not belonging to its zero group, then $\epsilon^{h} \lambda(\epsilon)$ is an eigenvalue of $A_{0}(\epsilon)$ in $\mathscr{R}\left(Q_{k}(\epsilon)\right)$. Conversely, if $\mu(\epsilon)$ is an eigenvalue of $A_{0}(\epsilon)$ in $\mathscr{R}\left(Q_{k}(\epsilon)\right)$, then $\epsilon^{-k} \mu(\epsilon)$ is an eigenvalue of $A_{k}(\epsilon)$ not belonging to its zero group.

Proof: Equation (4.9) has been established in the proof of Proposition 4.2. Further, if $y \in \mathscr{R}\left(A_{0}(\epsilon)\right)$ then $y=A_{0}(\epsilon) x$ for some $x$. Now, using 3) of Proposition 4.1, and (4.9) above,

$$
y=\sum_{k=0}^{m} A_{0}(\epsilon) Q_{k}(\epsilon) x=\sum_{k=0}^{m} Q_{k}(\epsilon) A_{0}(\epsilon) x
$$

so that $\mathscr{R}\left(A_{0}(\epsilon)\right) \subset \mathscr{R}\left(Q_{0}(\epsilon)\right) \oplus \cdots \oplus \mathscr{R}\left(Q_{m}(\epsilon)\right)$. Equality of the subspaces follows from counting dimensions. Check that this finishes the proof of 1$)-3$ ) for $k=0$.

Consider $\mathscr{N}\left(A_{k}(\epsilon)\right)$. By definition of $A_{k}(\epsilon)$ we have, for $\epsilon$ small enough but nonzero.

$$
\begin{equation*}
\mathscr{N}\left(A_{k}(\epsilon)\right) \supset \mathscr{N}\left(A_{0}(\epsilon)\right) \oplus \mathscr{R}\left(Q_{0}(\epsilon)\right) \oplus \cdots \oplus \mathscr{R}\left(Q_{k-1}(\epsilon)\right) \tag{4.10}
\end{equation*}
$$

Establish inclusion in the other direction by contradiction. Let $x \in \mathscr{N}\left(A_{k}(\epsilon)\right)$ but not the right-hand side of (4.10). From 3) of Proposition 4.1,

$$
\mathbb{C}^{n}=\mathscr{R}\left(Q_{0}(\epsilon)\right) \oplus \cdots \oplus \mathscr{R}\left(Q_{m}(\epsilon)\right) \oplus \mathscr{N}\left(A_{0}(\epsilon)\right) .
$$

Hence, write $x=x_{1}+x_{2}$ with $x_{1} \in \mathscr{R}\left(Q_{0}(\epsilon)\right) \oplus \cdots \oplus$ $\mathscr{R}\left(Q_{k-1}(\epsilon)\right) \oplus \mathscr{N}\left(A_{0}(\epsilon)\right)$ and $0 \neq x_{2} \in \mathscr{R}\left(Q_{k}(\epsilon)\right) \oplus \cdots \oplus$ $\mathscr{R}\left(Q_{m}(\epsilon)\right)$ with $P_{l}(\epsilon) x=x$ if $l<k$. Now $x \in \mathscr{N}\left(A_{k}(\epsilon)\right)$ implies that

$$
\begin{aligned}
0 & =A_{k}(\epsilon) x=\frac{A_{0}(\epsilon) P_{0}(\epsilon) \cdots P_{k-1}(\epsilon)}{\epsilon^{k}} x \\
& =\frac{A_{0}(\epsilon) P_{0}(\epsilon) \cdots P_{k-1}(\epsilon)}{\epsilon^{k}} x_{2} \\
& =\frac{A_{0}(\epsilon)}{\epsilon^{k}} x_{2}
\end{aligned}
$$

i.e., $x_{2} \in \mathscr{N}\left(A_{0}(\epsilon)\right)$, thereby yielding a contradiction. This establishes (4.8). To prove (4.7), note that by definition of $A_{k}(\epsilon)$

$$
\mathscr{R}\left(A_{k}(\epsilon)\right) \subset \mathscr{R}\left(P_{0}(\epsilon) \cdots P_{k-1}(\epsilon)\right) \cap \mathscr{R}\left(A_{0}(\epsilon)\right)
$$

and it follows from Proposition 4.1 and the SSNS of $A_{0}(\epsilon)$ that

$$
\begin{aligned}
& \mathscr{R}\left(P_{0}(\epsilon) \cdots P_{k-1}(\epsilon)\right) \cap \mathscr{R}\left(A_{0}(\epsilon)\right) \\
& \quad=\mathscr{R}\left(Q_{k}(\epsilon)\right) \oplus \cdots \oplus \mathscr{R}\left(Q_{m}(\epsilon)\right) .
\end{aligned}
$$

Equality (4.7) follows now from counting dimensions. To prove 3 ), notice that if $A_{k}(\epsilon) u=\lambda(\epsilon) u$ and $\lambda(\epsilon)$ does not belong to the zero group of eigenvalues of $A_{k}(\epsilon)$ then $Q_{k}(\epsilon) u=u$ and, therefore, it follows from 5) of Proposition 4.1 that

$$
A_{0}(\epsilon) u=A_{0}(\epsilon) Q_{k}(\epsilon) u=\epsilon^{k} A_{k}(\epsilon) u=\epsilon^{k} \lambda(\epsilon) u
$$

Conversely, let $A_{0}(\epsilon) u=\mu(\epsilon) u$ with $u \in \mathscr{R}\left(Q_{k}(\epsilon)\right)$. Then

$$
\epsilon^{-k} \mu(\epsilon) u=\epsilon^{-k} A_{0}(\epsilon) Q_{0}(\epsilon) u=A_{k}(\epsilon) u
$$

Proposition 4.4 establishes that if $A_{0}(\epsilon)$ has MSSNS, then it may be decomposed as

$$
\begin{equation*}
A_{0}(\epsilon)=\sum_{k=0}^{m} \epsilon^{k} A_{k}(\epsilon) Q_{k}(\epsilon) \tag{4.10}
\end{equation*}
$$

and that the eigenvalues of $A_{0}(\epsilon)$ may be divided into ( $m+1$ ) groups corresponding to eigenvalues of order $\epsilon^{j}, j=0, \cdots, m$, in the invariant subspaces $\mathscr{R}\left(Q_{j}(\epsilon)\right)$. Further, the eigenvalues of order $\epsilon^{k}$ coincide with $\epsilon^{k}$ times the order one eigenvalues of $A_{k}(\epsilon)$. The ranges and nullspaces of $A_{k}(\epsilon)$ are shown in Fig. 1; in addition to $\mathscr{N}\left(A_{0}(\epsilon)\right), \mathscr{N}\left(A_{k}(\epsilon)\right)$ includes the eigenspaces of $A_{0}(\epsilon)$ corresponding to eigenvalues of order $1, \epsilon, \cdots, \epsilon^{k-1}$; $\mathscr{R}\left(A_{k}(\epsilon)\right)$, on the other hand, includes the eigenspaces of $A_{0}(\epsilon)$ corresponding to all eigenvalues of order $\epsilon^{k}$.

The following theorem (one of two central results) illustrates the consequences of MSSNS for the time scale behavior of $\exp \left\{A_{0}(\epsilon) t\right\}$.

Theorem 4.5: If $A_{0}(\epsilon)$ satisfies the MSSNS condition, then
$\exp \left\{A_{0}(\epsilon) t\right\}=\sum_{k=0}^{m} Q_{k}(\epsilon) \exp \left\{A_{k}(\epsilon) \epsilon^{k} t\right\}+P_{0}(\epsilon) \cdots P_{m}(\epsilon)$
$\mathbb{R}^{n}=R\left(Q_{m}(\varepsilon)\right) \oplus \cdots \cdots \oplus \in \mathcal{R}\left(Q_{1}(\varepsilon)\right) \oplus R\left(Q_{0}(\varepsilon)\right) \oplus \mathcal{N}\left(A_{0}(\varepsilon)\right)$


Fig. 1. Geometric content of Proposition 4.4.

$$
\begin{align*}
& =\sum_{k=0}^{m} \exp \left\{Q_{k}(\epsilon) A_{k}(\epsilon) \epsilon^{k} t\right\}-m I  \tag{4.12}\\
& =\prod_{k=0}^{m} \exp \left\{Q_{k}(\epsilon) A_{k}(\epsilon) \epsilon^{k} t\right\} \tag{4.13}
\end{align*}
$$

Proof: Write

$$
\begin{aligned}
& \exp \left\{A_{0}(\epsilon) t\right\} \\
& \quad=P_{0}(\epsilon) \exp \left\{A_{0}(\epsilon) t\right\}+Q_{0}(\epsilon) \exp \left\{A_{0}(\epsilon) t\right\} \\
& \quad=\exp \left\{A_{1}(\epsilon) \epsilon t\right\}-Q_{0}(\epsilon)+Q_{0}(\epsilon) \exp \left\{A_{0}(\epsilon) t\right\}
\end{aligned}
$$

Repeating this manipulation for $\exp \left\{A_{1}(\epsilon) \in t\right\}$, we have

$$
\begin{aligned}
\exp \left\{A_{0}(\epsilon) t\right\}= & \exp \left\{A_{2}(\epsilon) \epsilon^{2} t\right\}+Q_{1}(\epsilon) \exp \left\{A_{1}(\epsilon) \epsilon t\right\} \\
& +Q_{0}(\epsilon) \exp \left\{A_{0}(\epsilon) t\right\}-Q_{1}(\epsilon)-Q_{0}(\epsilon)
\end{aligned}
$$

Repeating this procedure $m$ times yields

$$
\begin{align*}
\exp \left\{A_{0}(\epsilon) t\right\}= & \exp \left\{A_{m+1}(\epsilon) \epsilon^{m+1} t\right\}-\sum_{k=0}^{m} Q_{k}(\epsilon) \\
& +\sum_{k=0}^{m} Q_{k}(\epsilon) \exp \left\{A_{k}(\epsilon) \epsilon^{k} t\right\} \tag{4.14}
\end{align*}
$$

But $A_{m+1}(\epsilon)=0$ and $I-\sum_{k=0}^{m} Q_{k}(\epsilon)=P_{0}(\epsilon) \cdots P_{m}(\epsilon)$ so that (4.14) yields (4.11). Use the identity

$$
Q_{k}(\epsilon) \exp \left\{A_{k}(\epsilon) \epsilon^{k} t\right\}=\exp \left\{Q_{k}(\epsilon) A_{k}(\epsilon) \epsilon^{k} t\right\}-I+Q_{k}(\epsilon)
$$

in (4.16) to obtain (4.17). Equation (4.18) follows directly from (4.10), the property 5) of Proposition 4.1, and the fact that

$$
Q_{k}(\epsilon) A_{k}(\epsilon) \cdot Q_{j}(\epsilon) A_{j}(\epsilon)=0 \quad j \neq k .
$$

Remark: Under the MSSNS condition, (4.11) of Theorem 4.5 gives a spatial and temporal decomposition of $\exp \left\{A_{0}(\epsilon) t\right\}$, e.g., $Q_{k}(\epsilon) \exp \left\{A_{k}(\epsilon) \epsilon^{k} t\right\}$ does not change significantly in time until $t$ is of order $1 / \epsilon^{k}$. This decomposition is of crucial importance in studying multiple time scale behavior, uniform asymptotic approximations, and reduced order models for the system (4.1).

## B. Uniform Asymptotic Approximation of $\exp \left\{A_{0}(\epsilon) t\right\}$ :

## The Multiple Semistability Assumption

As stated in the previous section, $\exp \left\{A_{0}(0) t\right\}$ is a uniform approximation to $\exp \left\{A_{0}(\epsilon) t\right\}$ on any compact time interval $[0, T]$. To capture all the multiple time scale behavior, however, it is necessary to have a uniform asymptotic approximation on $[0, \infty]$. For this we need the following condition.

Definition 4.6: $A_{0}(\epsilon)$ satisfies the multiple semistability (MSST) condition if

1) $\boldsymbol{A}_{0}(\epsilon)$ satisfies the MSSNS condition, and
2) the matrices

$$
A_{k 0}=\lim _{\epsilon \downarrow 0} \frac{P_{k-1}(\epsilon) \cdots P_{0}(\epsilon) A_{0}(\epsilon)}{\epsilon^{k}}
$$

for $k=0, \cdots, m$ are semistable.
The following is a central result in uniform approximation.
Theorem 4.7: Let $A_{0}(\epsilon)$ satisfy the MSST condition. Then,

$$
\begin{equation*}
\lim _{\epsilon \downarrow 0} \sup _{t \geqslant 0}\left\|\exp \left\{A_{0}(\epsilon) t\right\}-\phi(t, \epsilon)\right\|=0 \tag{4.15}
\end{equation*}
$$

where $\phi(t, \epsilon)$ is any of the following expressions:

$$
\begin{align*}
\phi(t, \epsilon) & =\sum_{k=0}^{m} Q_{k} \exp A_{k 0} \epsilon^{k} t+P_{0} \cdots P_{m}  \tag{4.16}\\
& =\sum_{k=0}^{m} \exp \left\{A_{k 0} \epsilon^{k} t\right\}-m I  \tag{4.17}\\
& =\prod_{k=0}^{m} \exp \left\{A_{k 0} \epsilon^{k} t\right\}  \tag{4.18}\\
& =\exp \left\{\sum_{k=0}^{m} A_{k 0} \epsilon^{k} t\right\}
\end{align*}
$$

where $A_{k 0}=\lim _{\epsilon 10} A_{k}(\epsilon), \quad P_{k}=\lim _{\epsilon \downarrow 0} P_{k}(\epsilon)$, and $Q_{k}=$ $\lim _{\epsilon \downarrow 0} Q_{k}(\epsilon)$. Furthermore,

$$
\begin{equation*}
\mathbb{C}^{n}=\mathscr{R}\left(A_{00}\right) \oplus \cdots \oplus \mathscr{R}\left(A_{m 0}\right) \oplus\left(\bigcap_{k=0}^{m} \mathscr{N}\left(A_{k 0}\right)\right) . \tag{4.19}
\end{equation*}
$$

Proof: We first establish (4.15) with $\phi(t, \epsilon)$ as in (4.16). Using (4.11) from Theorem 4.5 for $\exp \left\{A_{0}(\epsilon) t\right\}$

$$
\begin{aligned}
\exp \{ & \left.A_{0}(\epsilon) t\right\}-\dot{\phi}(t, \epsilon) \\
= & \left(P_{0}(\epsilon) \cdots P_{m}(\epsilon)-P_{0} \cdots P_{m}\right) \\
& +\sum_{k=0}^{m}\left(Q_{k}(\epsilon) \exp \left\{A_{k}(\epsilon) \epsilon^{k} t\right\}-Q_{k} \exp \left\{A_{k 0} \epsilon^{k} t\right\}\right) .
\end{aligned}
$$

The first term in the above equation tends to zero as $\epsilon \downarrow 0$ independently of $t$. For the second term, write

$$
\begin{aligned}
\psi_{k}(t, \epsilon) & \triangleq Q_{k}(\epsilon) \exp \left\{A_{k}(\epsilon) \epsilon^{k} t\right\}-Q_{k} \exp \left\{A_{k 0} \epsilon^{\epsilon^{t} t}\right\} \\
& =-\frac{1}{2 \pi i} \int_{\Gamma_{k}} e^{\lambda \epsilon^{k} t}\left(R\left(\lambda, A_{k}(\epsilon)\right)-R\left(\lambda, A_{k 0}\right)\right) d \lambda
\end{aligned}
$$

where $\Gamma_{k}$ is a contour enclosing all nonzero eigenvalues of $A_{k 0}$. By semistability of $A_{k 0}, \Gamma_{k}$ can be chosen to lie in the left half plane bounded away from the $j \omega$-axis. Hence, we have for some $\alpha<0$

$$
\begin{aligned}
\left\|\psi_{k}(t, \epsilon)\right\| & \leqslant \frac{1}{2 \pi} e^{\alpha \epsilon^{k} t} \int_{\Gamma_{k}}\left\|R\left(\lambda, A_{k}(\epsilon)\right)-R\left(\lambda, A_{k 0}\right)\right\| d \lambda \\
& \leqslant \frac{1}{2 \pi} \int_{\Gamma_{k}}\left\|R\left(\lambda, A_{k}(\epsilon)\right)-R\left(\lambda, A_{k 0}\right)\right\| d \lambda
\end{aligned}
$$

Since $R\left(\lambda, A_{k}(\epsilon)\right)$ converges uniformly to $R\left(\lambda, A_{k 0}\right)$ on compact subsets of $\mathbb{C}$ (by Lemma 2.5), we have that $\left\|\psi_{k}(t, \epsilon)\right\|$ tends to zero as $\in \downarrow 0$ (uniformly in $t$ ). Equality between the different expressions of $\phi(t, \epsilon)$ is established as in Theorem 4.5.

To establish (4.19) we have from 3) of Proposition 4.1 that

$$
\mathbb{C}^{n}=\mathscr{R}\left(Q_{0}(\epsilon)\right) \oplus \cdots \oplus \mathscr{R}\left(Q_{m}(\epsilon)\right) \oplus \mathscr{R}\left(P_{0}(\epsilon) \cdots P_{m}(\epsilon)\right)
$$

and by continuity of the projections $Q_{h}(\epsilon), P_{k}(\epsilon)$,

$$
\mathbb{C}^{n}=\mathscr{R}\left(Q_{0}\right) \oplus \cdots \oplus \mathscr{R}\left(Q_{m}\right) \oplus \mathscr{R}\left(P_{0} \cdots P_{m}\right)
$$

The direct sum decomposition (4.19) follows now from the fact that, by construction, $Q_{k}$ is the projection on $\mathscr{R}\left(A_{h 0}\right)$ along $\mathscr{N}\left(A_{k 0}\right)$.

In the next section we use the result of Theorem 4.7 to determine the complete multiple time scale behavior for (4.1) and obtain a set of reduced order models.

## C. Multiple Time Scale Behavior and Reduced Order Models

Multiple time scale behavior is explicated by the following corollary to Theorem 4.7.

Corollary 4.8: Let $A_{0}(\epsilon)$ satisfy the MSST condition. Then, 1)

$$
\begin{align*}
& \lim _{\epsilon \downarrow 0} \sup _{\delta \leqslant t \leqslant T}\left\|\exp \left\{A_{0}(\epsilon) t / \epsilon^{k}\right\}-\Phi_{k}(t)\right\|=0 \\
& \forall \delta>0, T<\infty ; k=0,1, \cdots, m-1 . \tag{4.20}
\end{align*}
$$

2) 

$$
\begin{equation*}
\lim _{\epsilon \downarrow 0} \sup _{\delta \leqslant t<\infty}\left\|\exp \left\{A_{0}(\epsilon) t / \epsilon^{m}\right\}-\Phi_{m}(t)\right\|=0 \quad \forall \delta>0 \tag{4.21}
\end{equation*}
$$

where $\Phi_{h}(t)$ is given by

$$
\begin{align*}
\Phi_{k}(t) & =Q_{k} \exp \left\{A_{k 0} t\right\}+P_{0} \cdots P_{k}  \tag{4.22}\\
& =P_{0} \cdots P_{k-1} \exp \left\{A_{k 0} t\right\} \quad k=0,1, \cdots, m \tag{4.23}
\end{align*}
$$

Proof: From Theorem 4.7 we have that

$$
\begin{align*}
& \exp \left\{A_{0}(\epsilon) t / \epsilon^{k}\right\} \\
&= \sum_{l=0}^{k-1} Q_{l} \exp \left\{A_{l 0} t / \epsilon^{k-l}\right\}+Q_{k} \exp \left\{A_{k 0} t\right\} \\
&+\sum_{l=k+1}^{m} Q_{l} \exp \left\{A_{10} t \epsilon^{l-k}\right\}+P_{0} \cdots P_{m}+o(1) \tag{4.24}
\end{align*}
$$

uniformly for $t \in[0, \infty]$. Now, by the semistability of $A_{l 0}$

$$
Q_{1} \exp \left\{A_{10} t\right\}=-\frac{1}{2 \pi i} \int_{\Gamma_{1}} e^{\lambda t} R\left(\lambda, A_{10}\right) d \lambda
$$

for some $\Gamma_{l}$ in $\mathbb{C}_{-}$bounded away from the $j \omega$-axis. By the boundedness of $R\left(\lambda, A_{k 0}\right)$ on $\Gamma_{I}$

$$
\left\|Q_{1} \exp \left\{A_{10} t\right\}\right\| \leqslant M_{l} e^{-\alpha_{l} t} \quad \text { with } \alpha_{l}>0
$$

This yields

$$
\begin{equation*}
\lim _{\epsilon \downarrow 0} \sup _{\delta \leqslant t \leqslant \infty} \sum_{l=0}^{k-1} Q_{l} \exp \left\{A_{l 0} t / \epsilon^{k-l}\right\}=0 \tag{4.25}
\end{equation*}
$$

On the other hand, it is clear that

$$
\lim _{\epsilon \downarrow 0} \sup _{0 \leqslant t \leqslant T}\left\|\exp \left\{A_{10} \epsilon^{l-k} t\right\}-I\right\|=0 \quad \forall l>k, \quad T<\infty .
$$

Using (4.25) and (4.26) in (4.24) yields (4.20) and (4.21) with

$$
\begin{equation*}
\Phi_{k}(t)=Q_{k} \exp \left\{A_{k 0} t\right\}+\sum_{t=k+1}^{m} Q_{l}+P_{0} \cdots P_{m} \tag{4.27}
\end{equation*}
$$

Equality of expressions (4.22) and (4.23) follows from (4.27).

## Remarks:

1) From (4.22) of Corollary 4.8 and (4.16) of Theorem 4.7 it follows that

$$
\begin{equation*}
\exp \left\{A_{0}(\epsilon) t\right\}=\sum_{k=0}^{m} \Phi_{k}\left(\epsilon^{k} t\right)-\sum_{k=0}^{m-1} P_{0} \cdots P_{k}+o(1) \tag{4.28}
\end{equation*}
$$

uniformly for $t>0$. Thus, only the behavior at time scales $t / \epsilon^{k}$, $k=0, \cdots, m$ is needed to capture the evolution of $\exp \left\{A_{0}(\epsilon) t\right\}$ on $[0, \infty]$. From the proof of Corollary 4.8 , it is clear that

$$
\lim _{\epsilon \downarrow 0} \exp \left\{A_{0}(\epsilon) t / \alpha(\epsilon)\right\}
$$

exists for any order function $\alpha(\epsilon)$. Indeed if $\alpha_{k}(\epsilon)=o\left(\epsilon^{k}\right)$ and $\epsilon^{k+1}=o\left(\alpha_{k}(\epsilon)\right)$ then

$$
\begin{equation*}
\lim _{\epsilon \downarrow 0} \exp \left\{A_{0}(\epsilon) t / \alpha_{k}(\epsilon)\right\}=P_{0} \cdots P_{k} \tag{4.29}
\end{equation*}
$$

and for $\alpha(\epsilon)=o\left(\epsilon^{m}\right)$

$$
\begin{equation*}
\lim _{\epsilon \downarrow 0} \exp \left\{A_{0}(\epsilon) t / \alpha(\epsilon)\right\}=P_{0} \cdots P_{m} \tag{4.30}
\end{equation*}
$$

Thus, the system has well-defined behavior at all time scales, even though only a finite number of them ( $t / \epsilon^{k}, k=0,1, \cdots, m$ ), called the fundamental or natural time scales, are required to capture the system evolution (strictly speaking only those for which $A_{k 0} \neq 0$ ).
2) The behavior of the system as given by (4.23) is canonic in the following sense: at a given time scale, say $t / \epsilon^{k}$, all faster time scales $t / \epsilon^{\prime}$ for $l<k$ have come to their equilibria (respectively, $P_{l}$ ); and all slower time scales $t / \epsilon^{l}$ for $l>k$ have yet to evolve.

To interpret the matrices $A_{k 0}$ as reduced order models of (4.1), notice that the uniform asymptotic approximation

$$
\exp \left\{A_{0}(\epsilon) t\right\}=\sum_{k=0}^{m} Q_{k} \exp \left\{A_{k 0} \epsilon^{k} t\right\}+P_{0} \cdots P_{m}+o(1)
$$

together with (4.19) imply that the subspaces $\mathscr{R}\left(Q_{k}\right), k=0, \cdots, m$, are almost invariant subspaces (or $\epsilon$-invariant subspaces, as defined in [16]) of (4.1). The parts of $x^{\epsilon}(t)$ that evolve in different subspaces do so at different time scales.

Corollary 4.9: Consider the linear systems

$$
\begin{equation*}
\dot{y}_{k}=A_{k 0} y_{k}, \quad y_{k}(0)=Q_{k} x_{0}, \quad k=0,1, \cdots, m \tag{4.31}
\end{equation*}
$$

Then

$$
\sum_{k=0}^{m} \operatorname{dim} \mathscr{R}\left(Q_{k}\right)=n \operatorname{rank} A_{0}(\epsilon)
$$

2) In particular, choosing a basis adapted to the direct sum decomposition (4.19), it is possible to asymptotically decouple (4.1) into a set of lower dimensional systems, each evolving at a different time scale as follows. Let $V$ be the ( $\epsilon$-independent) change of basis mentioned above; then Theorem 4.7 can be written as

$$
\begin{aligned}
\exp \left\{A_{0}(\epsilon) t\right\} & =V^{-1}\left\{\exp \sum_{k=0}^{m} V A_{k 0} V^{-1} \epsilon^{k} t\right\} V+o(1) \\
& =V^{-1} \operatorname{diag}\left\{e^{\hat{\hat{A}_{0} t}}, e^{\hat{A}_{1} \epsilon t}, \cdots, e^{\hat{A}_{m} \epsilon^{m} t}, I\right\} V+o(1)
\end{aligned}
$$

where the matrices $\hat{A}_{k}$ are full rank square matrices with dimension equal to rank $A_{k 0}$ corresponding to the nonzero part of $A_{k 0}$ in the new basis.
In the next section we show that such a complete decomposition and simplification as has been elaborated here is possible only if $A_{0}(\epsilon)$ satisfies the MSST condition.

## D. Necessity of the Multiple Semistability Condition

We have shown in Sections IV-B and IV-C the existence of well-defined behavior at several time scales under the MSST condition. If MSST is not satisfied then, at least for some order function $\alpha(\epsilon)$, the limit

$$
\lim _{\epsilon \downarrow 0} \exp \left\{A_{0}(\epsilon) t / \alpha(\epsilon)\right\}
$$

does not exist. To illustrate this, consider the following examples.
Example $4.9\left(A_{k_{0}}\right.$ not $\left.S S N S\right)$ : Consider the matrix

$$
A_{0}(\epsilon)=\left[\begin{array}{lll}
\epsilon & 0 & -2 \epsilon \\
\epsilon & \epsilon & -2 \epsilon \\
1 & 1 & -2
\end{array}\right]
$$

semistable for $\epsilon \in[0,1]$ with eigenvalues $\lambda_{0}=0, \lambda_{1}=-2+o(1)$ and $\lambda_{2}=-\epsilon^{2}+o\left(\epsilon^{2}\right)$. This matrix does not satisfy the MSSNS condition, as may be verified (for a systematic procedure to do this calculation see Section IV-E) that

$$
A_{10}=\left[\begin{array}{ccc}
0 & -1 & 0 \\
0 & 0 & 0 \\
0 & -1 / 2 & 0
\end{array}\right]
$$

is nilpotent. Also, by direct computation, it is found that

$$
\exp \left\{A_{0}(\epsilon) t\right\}=\frac{1}{\lambda_{2}-\lambda_{1}} \sum_{i=1}^{2}\left[\begin{array}{ccc}
\left(\lambda_{2}-\lambda_{1}\right) & \frac{(-1)^{i}}{\lambda_{i}}\left(e^{\lambda_{i} t}-1\right) & \frac{(-1)^{i}}{\lambda_{i}} \epsilon\left(\lambda_{i}-\epsilon\right)\left(e^{\lambda_{i} t}-1\right) \\
0 & (-1)^{k}\left(\lambda_{i}-\epsilon+2\right) e^{\lambda_{i} t} & (-1)^{i-1} 2 \epsilon e^{\lambda_{i} t} \\
0 & (-1)^{i} e^{\lambda_{i} t} & \left(\lambda_{i}-\epsilon\right) e^{\lambda_{,} t}
\end{array}\right]
$$

$$
Q_{k} x^{t}(t)=y_{k}\left(\epsilon^{k} t\right)+o(1), \quad k=0,1, \cdots, m
$$

and

$$
\begin{equation*}
x^{c}(t)=\sum_{k=0}^{m} y_{k}\left(\epsilon^{k} t\right)+P_{0} \cdots P_{m} x_{0}+o(1) \tag{4.32}
\end{equation*}
$$

Proof: A straightforward modification of Corollary 4.8.
with the following time scale behavior:

$$
\lim _{\epsilon \downarrow 0} \exp \left\{A_{0}(\epsilon) t\right\}=\exp \left\{A_{00} t\right\}=\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & \left(1-e^{-2 t}\right) / 2 & e^{-2 t}
\end{array}\right]
$$

and

## Remarks:

1) The linear systems (4.31), although written as equations on $\mathbb{R}^{n}$, are really reduced order models since each evolves in $\mathscr{R}\left(Q_{k}\right)$, $k=0,1, \cdots, m$ and

$$
\lim _{\epsilon \downarrow 0} \exp \left\{A_{0}(\epsilon) t / \epsilon\right\}=P_{0} \exp \left\{A_{10} t\right\}=\left[\begin{array}{ccc}
1 & t / 2 & 0 \\
0 & 1 & 0 \\
0 & 1 / 2 & 0
\end{array}\right]
$$

To see that the limit

$$
\lim _{\epsilon \downarrow 0} \exp \left\{A_{0}(\epsilon) t / \epsilon^{2}\right\}
$$

does not exist, consider the $(1,2)$ entry of $\exp \left\{A_{0}(\epsilon) t / \epsilon^{2}\right\}$ :

$$
\begin{equation*}
\frac{1}{\lambda_{2}-\lambda_{1}}\left[\frac{\epsilon}{\lambda_{2}}\left(e^{\lambda_{2} t / \epsilon^{2}}-1\right)-\frac{\epsilon}{\lambda_{1}}\left(e^{\lambda_{1} t / \epsilon^{2}}-1\right)\right] . \tag{4.33}
\end{equation*}
$$

Since $\lambda_{2}=-\epsilon^{2}+o\left(\epsilon^{2}\right)$, the first term in (4.33) is of order $1 / \epsilon$ as $\epsilon \downarrow 0$. Thus, the system does not have well-defined behavior at time scale $t / \epsilon^{2}$ even though it has a negative real eigenvalue of order $\epsilon^{2}$.

Example 4.10 ( $A_{k 0}$ not MSST): Consider the matrix

$$
A_{0}(\epsilon)=\left[\begin{array}{ccc}
-2 & 0 & 0 \\
0 & -\epsilon^{2} & \epsilon \\
0 & -\epsilon & -\epsilon^{2}
\end{array}\right] .
$$

This matrix is semistable for $\epsilon \geqslant 0$ and it has the three eigenvalues $\lambda_{0}=-2, \lambda_{1}=-\epsilon^{2}+i \epsilon$, and $\lambda_{2}=-\epsilon^{2}-i \epsilon$. Also

$$
A_{00}=\left[\begin{array}{rrr}
-2 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right]
$$

and

$$
A_{10}=\left[\begin{array}{rrr}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & -1 & 0
\end{array}\right] .
$$

MSST is violated since $A_{10}$ has purely imaginary eigenvalues. Calculation yields

$$
\exp \left\{A_{0}(\epsilon) t\right\}=\left[\begin{array}{ccc}
e^{-2 t} & 0 & 0 \\
0 & e^{-\epsilon^{2} t} \cos \epsilon t & -e^{-\epsilon^{2} t} \sin \epsilon t \\
0 & e^{-\epsilon^{2} t} \sin \epsilon t & e^{-\epsilon^{2} t} \cos \epsilon t
\end{array}\right]
$$

The system has well-defined behavior at time scales $t$ and $t / \epsilon$ but $\exp \left\{A_{0}(\epsilon) t / \epsilon^{2}\right\}$ does not have a limit as $\epsilon \downarrow 0$ because of the presence of terms of the form $e^{-t} \sin t / \epsilon$. (The attenuation is slower than the frequency of oscillation.)

In fact, the MSST condition is a necessary and sufficient condition for the existence of multiple time scale behavior.

Theorem 4.11: Let $A_{0}(\epsilon)$ be semistable for $\epsilon \in\left[0, \epsilon_{0}\right]$ and let $A_{k 0}, k \geqslant 0$ be the sequence of matrices constructed in Section IV-A. If $A_{00}, A_{10}, \cdots, A_{i-1}, 0$ are semistable but $A_{10}$ is not, then the limit as $\epsilon \downarrow 0$ of

$$
\begin{equation*}
\exp \left\{A_{0}(\epsilon) t / \epsilon^{q}\right\} \tag{4.34}
\end{equation*}
$$

does not exist for any $l<q \leqslant l+1$. Further, if $A_{l 0}$ has a pole on the imaginary axis (including zero) which is not semisimple, then

$$
\lim _{\epsilon \downarrow 0} \sup _{t \geqslant 0}\left\|\exp \left\{A_{0}(\epsilon) t\right\}\right\|=\infty
$$

Proof: We construct the proof for $l=0$ by contradiction. Assume that the limit

$$
\lim _{\epsilon \downarrow 0} \exp \left\{A_{0}(\epsilon) t / \epsilon^{q}\right\}
$$

exists for $t>0$ and some $q \in] 0,1]$. If this limit exists, then so does the limit of $\exp \left\{P_{0}(\epsilon) A_{0}(\epsilon) t / \epsilon^{q}\right\}$ as $\epsilon \downarrow 0$ because
$\lim _{\epsilon \downarrow 0} P_{0}(\epsilon) \exp \left\{A_{0}(\epsilon) t / \epsilon^{q}\right\}=\lim _{\epsilon \downarrow 0} \exp \left\{P_{0}(\epsilon) A_{0}(\epsilon) t / \epsilon^{q}\right\}-Q_{0}$.

Define

$$
F_{0}(\epsilon)=\frac{P_{0}(\epsilon) A_{0}(\epsilon)}{\epsilon^{q}} .
$$

The next step is to prove that $\sigma\left(F_{0}(\epsilon)\right)$ remains bounded as $\epsilon \downarrow 0$. Take $0 \neq \lambda(\epsilon) \in \sigma\left(P_{0}(\epsilon) A_{0}(\epsilon)\right)$ and let $\phi(\epsilon)$ be a corresponding eigenvector with $\|\phi(\epsilon)\|=1$. Then

$$
\begin{aligned}
& \exp \left\{P_{0}(\epsilon) A_{0}(\epsilon) t / \epsilon^{q}\right\} \phi(\epsilon) \\
& =\exp \left\{\operatorname{Re} \lambda(\epsilon) t / \epsilon^{q}\right\} \cdot \exp \left\{i \operatorname{Im} \lambda(\epsilon) t / \epsilon^{q}\right\} \phi(\epsilon)
\end{aligned}
$$

And if $\varepsilon_{m} \downarrow 0$ is a sequence for which $\phi\left(\epsilon_{m}\right)$ converges, then

$$
\begin{equation*}
\exp \left\{\operatorname{Re} \lambda\left(\epsilon_{m}\right) t / \epsilon_{m}^{q}\right\} \cdot \exp \left\{i \operatorname{Im} \lambda\left(\epsilon_{m}\right) t / \epsilon_{m}^{q}\right\} \tag{4.35}
\end{equation*}
$$

must also converge as $m \rightarrow \infty$. Now, since the trace of $A_{0}(\epsilon)$ has a series expansion in integer powers of $\epsilon$ and the eigenvalues in the zero group of $A_{0}(\epsilon), \mu(\epsilon)$, have nonpositive real parts,

$$
\operatorname{Re} \mu(\epsilon) / \epsilon^{k} \rightarrow \mu \quad \text { as } \epsilon \downarrow 0
$$

for some integer $k \geqslant 1$ and some constant $\mu$. We thus conclude that $\operatorname{Re\lambda }\left(\epsilon_{m}\right) / \epsilon_{m 1}^{q}$ must converge as $m \rightarrow \infty$. Further, by the convergence of (4.35), $\operatorname{Im} \lambda\left(\epsilon_{m}\right) / \epsilon_{m}^{q}$ must also converge as $\epsilon \downarrow 0$. Because $\sigma\left(F_{0}(\epsilon)\right)$ remains bounded as $\epsilon_{m} \downarrow 0$ we can choose $t_{1}$ such that

$$
\left|\operatorname{Im} \sigma\left(F_{0}(\epsilon) t_{1}\right)\right|<\pi
$$

for $\epsilon$ small enough. Hence, if $\ln$ denotes the principal branch of the logarithmic function, we obtain

$$
\begin{equation*}
\ln \exp \left\{F_{0}(\epsilon) t_{1}\right\}=F_{0}(\epsilon) t_{1}=\frac{D_{0}}{\epsilon^{q}}+G_{0}(\epsilon) t_{1} \tag{4.36}
\end{equation*}
$$

where the last equality follows from Proposition 2.7 with $D_{0}$ being the eigennilpotent for the zero eigenvalue of $A_{00}$ and $G_{0}(\epsilon)$ a continuous function of $\epsilon$. The limit

$$
\lim _{\epsilon \downarrow 0} \ln \left\{\exp F_{0}(\epsilon) t_{1}\right\} \triangleq B\left(t_{1}\right)
$$

is well defined by the boundedness of $\sigma\left(F_{0}(\epsilon)\right)$ and therefore, by (4.36), $D_{0}=0$, i.e., if the limit of (4.34) exists, then $A_{00}$ must be SSNS.

Suppose now that $A_{00}$ has some purely imaginary eigenvalue $\mu$. Then there exists at least one eigenvalue $\mu(\epsilon)$ of $A_{0}(\epsilon)$ such that $\mu(\epsilon) \rightarrow \mu$ as $\epsilon \downarrow 0$. Let $\phi(\epsilon)$ be a corresponding eigenvector with $\|\phi(\epsilon)\|=1$ and $\epsilon_{m} \downarrow 0$ a sequence for which $\phi\left(\epsilon_{m}\right)$ converges. Then, if (4.34) has a limit so does

$$
\phi\left(\epsilon_{m}\right)^{T} \exp \left\{A_{0}\left(\epsilon_{m}\right) t / \epsilon_{m}^{q}\right\} \phi\left(\epsilon_{m}\right)=e^{\mu\left(\epsilon_{m}\right) t / \epsilon_{m}^{q}}
$$

which is a contradiction. We have thus shown that if $A_{00}$ is not semistable, (4.34) cannot have a limit as $\epsilon \downarrow 0$.

To prove the theorem for an arbitrary $l$, notice that using the same algebraic manipulation as in the proof of Theorem 4.5, we can write

$$
\begin{align*}
\exp \left\{A_{0}(\epsilon) t / \epsilon^{q}\right\}= & \exp \left\{A_{l}(\epsilon) t / \epsilon^{q-l}\right\}-\sum_{k=0}^{1-1} Q_{k}(\epsilon) \\
& +\sum_{k=0}^{l-1} Q_{k}(\epsilon) \exp \left\{A_{k}(\epsilon) t / \epsilon^{q-k}\right\} \tag{4.37}
\end{align*}
$$

By semistability of $A_{00}, \cdots, A_{1-1,0}$, the second and third sums on the right-hand side of (4.37) have well-defined limits as $\epsilon \downarrow 0$.

Thus, assuming that

$$
\exp \left\{A_{0}(\epsilon) t / \epsilon^{q}\right\}
$$

has a limit as $\epsilon \downarrow 0$, so does

$$
\exp \left\{A_{l}(\epsilon) t / \epsilon^{q-l}\right\} \quad l<q \leqslant l+1
$$

implying, as proved previously for $l=0$, that $A_{l 0}$ is semistable, a contradiction.

To prove the second part of the theorem, suppose that $A_{i 0}$ has an eigenvalue on the imaginary axis which is not semisimple. Then $\forall M<\infty$ there exists a $T<\infty$ such that

$$
\left\|\exp \left\{A_{/ 0} T\right\}\right\|>M
$$

and because $\exp \left\{A_{l}(\epsilon) T\right\}$ converges to $\exp \left\{A_{10} T\right\}$ as $\epsilon \downarrow 0$, we conclude that

$$
\lim _{\epsilon \downarrow 0} \sup _{t \geqslant 0}\left\|\exp \left\{A_{l}(\epsilon) t\right\}\right\|=\infty .
$$

The desired result follows now from (4.37).
The above theorem can be interpreted as saying that if a system has well-defined behavior at all time scales then its system matrix must be MSST. As we will discuss in Section V, there are systems for which this condition is always satisfied. In general, however, the sequence of matrices $A_{k 0}$ will have to be computed to check for semistability.

## E. Computation of the Multiple Time Scale Behavior

Theorem 4.7 reveals that the matrices $A_{k 0}$ play a fundamental role in the asymptotic analysis of singularly perturbed systems. These are the leading terms in the series expansions of the matrices

$$
A_{k}(\epsilon)=\frac{P_{k-1}(\epsilon) \cdots P_{0}(\epsilon) A_{0}(\epsilon)}{\epsilon^{k}}
$$

for $k=0,1, \cdots, m$. The matrices in the series expansion of $A_{k}(\epsilon)$ are shown in Fig. 2. The $(i+1)$ th row of Fig. 2 is computed from the $i$ th row using Corollary 2.8, i.e.,

$$
\begin{align*}
A_{i+1, j}= & \sum_{p=1}^{j+1}(-1)^{p} \\
& \cdot \sum_{\substack{\nu_{1}+\cdots+\nu_{p}=j+1 \\
k_{1}+\cdots+k_{p+1}=p-1 \\
\nu_{i} \geqslant 1, k_{i} \geqslant 0}} S_{i}^{\left(k_{1}\right)} A_{i \nu_{1}} S_{i}^{\left(k_{2}\right) \cdots A_{i \nu_{p}} S_{i}^{\left(k_{p+1}\right)}} \\
i= & 0,1, \cdots, m ; j \geqslant 0
\end{align*}
$$

where

$$
\begin{aligned}
S_{i}^{(0)} & =-P_{i} \\
S_{i}^{(p)} & =\left(A_{i 0}^{*}\right)^{p}, \quad p>0 .
\end{aligned}
$$

The formula (4.38) enables the array of matrices $A_{i j}$ to be computed triangularly, so that computation of $A_{k 0}$ requires only the computation of $A_{i j}$ for $i=0, \cdots, k-1$ and $j=0, \cdots, k-i$. Thus, the algorithm contained in (4.38) may be implemented recursively. In the following proposition, we illustrate the complexity of the expressions for the $A_{k 0}$ in terms of the given data $A_{00}, A_{01}, \cdots$ [i.e., $\left.A_{0}(\epsilon)\right]$. We note also from (4.34) that the computation of the $A_{k 0}$ 's, and hence the asymptotic limit of $\exp \left\{A_{0}(\epsilon) t\right\}$, involves only $A_{00}, \cdots, A_{0 m}$ [only finitely many matrices in the asymptotic expansion of $\left.A_{0}(\epsilon)\right]$.


Fig. 2. Triangular array of matrices $A_{i j}$.

Proposition 4.12: The matrices $A_{k 0}$ for $k=0,1,2$, and 3 are given by

$$
\begin{aligned}
& A_{00} \\
& A_{10}= P_{0} A_{01} P_{0} \\
& A_{20}= P_{1} P_{0}\left(A_{02}-A_{01} A_{00}^{\#} A_{01}\right) P_{0} P_{1} \\
& A_{30}= P_{2} P_{1} P_{0}\left(A_{03}-A_{01} A_{00}^{\#} A_{02}-A_{02} A_{00}^{\#} A_{01}\right. \\
&+A_{01} A_{00}^{\#} A_{01} A_{00}^{\#} A_{01}-A_{02}\left(P_{0} A_{01} P_{0}\right)^{\#} A_{02} \\
&+A_{02}\left(P_{0} A_{01} P_{0}\right)^{\#} A_{01} A_{00}^{\#} A_{01} \\
&-A_{01} A_{00}^{\#} A_{01}\left(P_{0} A_{01} P_{0}\right)^{\#} A_{02} \\
&-A_{01} A_{00}^{\#} A_{01}\left(P_{0} A_{01} P_{0}\right)^{\#} \\
&\left.\cdot A_{01} A_{00}^{\#} A_{01}\right) P_{0} P_{1} P_{2}
\end{aligned}
$$

Proof: By somewhat laborious calculation.

## Remarks:

1) If $A(\epsilon)$ is of the form $A+\epsilon B$, then we have

$$
\begin{aligned}
& A_{00}=A \\
& A_{10}=P_{0} B P_{0} \\
& A_{20}=-P_{1} P_{0} B A^{\#} B P_{0} P_{1} \\
& A_{30}=P_{2} P_{1} P_{0}\left(B A^{\#} B A^{\#} B-B A^{\#} B\left(P_{0} B P_{0}\right)^{\#} B A^{\#} B\right) P_{0} P_{1} P_{2} .
\end{aligned}
$$

Thus, a system of the form

$$
\dot{x}^{\epsilon}(t)=(A+\epsilon B) x^{\epsilon}(t)
$$

may exhibit time scale behavior at time scales of order $1 / \epsilon, 1 / \epsilon^{2}, \cdots, 1 / \epsilon^{m}$, a fact that is not widely appreciated in the literature. As examination of the characteristic polynomial of $A+\epsilon B$ will show, no eigenvalue of $A+\epsilon B$ can be of $o\left(\epsilon^{n}\right)$ so that at most $m=n$. Similar reasoning leads us to the conclusion that for

$$
A_{0}(\epsilon)=\sum_{k=0}^{p} \epsilon^{k} A_{k}
$$

$m$ can at most be $n p$.
2) For the classical two time scale formulation of (1.2), normalized to the form (1.1), we have

$$
A=A_{00}=\left[\begin{array}{cc}
0 & 0 \\
A_{21} & A_{22}
\end{array}\right]
$$

and

$$
B=A_{01}=\left[\begin{array}{cc}
A_{11} & A_{12} \\
0 & 0
\end{array}\right] .
$$

If $A_{22}$ is stable, then $A_{1}$ is semistable with

$$
P_{0}=\left[\begin{array}{cc}
I & 0 \\
-A_{22}^{-1} A_{21} & 0
\end{array}\right]
$$

so that

$$
A_{10}=\left[\begin{array}{cc}
A_{11}-A_{12} A_{22}^{-1} A_{21} & 0 \\
-A_{22}^{-1} A_{21}\left(A_{11}-A_{12} A_{22}^{-1} A_{21}\right) & 0
\end{array}\right] .
$$

From this, we see that the model at the fast time scale is

$$
\begin{aligned}
& \dot{x}_{2}=A_{22} x_{2}+A_{21} x_{1} \\
& \dot{x}_{1}=0
\end{aligned}
$$

and the reduced order model for the slower dynamics is

$$
\begin{aligned}
& \dot{x}_{1}=\left(A_{11}-A_{12} A_{22}^{-1} A_{21}\right) x_{1} \\
& x_{2}=-A_{22}^{-1} A_{21} x_{1} .
\end{aligned}
$$

If, as is usually assumed, $A_{11}-A_{12} A_{22}^{-1} A_{21}$ is also stable, then $\operatorname{rank} A_{22}+\operatorname{rank}\left(A_{11}-A_{12} A_{22}^{-1} A_{21}\right)=\operatorname{rank}(A+\epsilon B)$, so that the system (1.2) has only two time scales.
3) If $A_{0}(\epsilon)$ is a rational function of $\epsilon$, with Taylor series about $\epsilon=0$ given by

$$
A_{0}(\epsilon)=\sum_{k=0}^{\infty} \epsilon^{k} A_{0 k},
$$

then it is well known (e.g., [17]-[19]) that $m$ is the order of the Smith-MacMillan zero of $A_{0}(\epsilon)$ at $\epsilon=0$. It may also be established that the matrices $A_{k 0}$ defined above are related to block Toeplitz matrices of the form

$$
\left[\begin{array}{cc}
A_{00} & 0 \\
A_{01} & A_{00}
\end{array}\right]\left[\begin{array}{ccc}
A_{00} & 0 & 0 \\
A_{01} & A_{00} & 0 \\
A_{02} & A_{01} & A_{00}
\end{array}\right], \cdots,\left[\begin{array}{cccc}
A_{00} & 0 & & 0 \\
A_{01} & \ddots & & \\
\vdots & \ddots & \ddots & \\
A_{0 m} & \cdots & A_{01} & A_{00}
\end{array}\right]
$$

The details of this connection will be presented elsewhere, since it is not in the mainstream of our development here.

## F. Partial Time Scale Decomposition

We discuss here the multiple time scale behavior of systems that do not satisfy the MSST condition of Section IV-B. Such systems have well-defined behavior at some but not all time scales, and it may be useful to be able to isolate the time scales at which they have well-defined behavior. Consider, for example, the case when time $\left\{A_{k 0}\right\}_{k=0}^{m}$ have SST, but $A_{10}$ for some $l \leqslant m$ violates the semistability condition, i.e., has at least one nonzero eigenvalue $\lambda$, with $\operatorname{Re} \lambda \geqslant 0$. Then, we have the following.

Proposition 4.13: Let the matrix $A_{0}(\epsilon)$ satisfy the MSSNS condition and let the matrices $A_{k 0}, k=0,1, \cdots, m$ for $k \neq l$ be semistable. Then, $\forall \delta>0, T<\infty$

$$
\lim _{\epsilon \downarrow 0} \sup _{\delta \leqslant t \leqslant T}\left\|\exp \left\{A_{0}(\epsilon) t / \epsilon^{k}\right\}-\Phi_{k}(t)\right\|=0 \quad \text { for } k=1, \cdots, l
$$

where

$$
\begin{align*}
\Phi_{k}(t) & =P_{0} \cdots P_{k-1} \exp \left\{A_{k 0} t\right\} \\
& =Q_{k} \exp \left\{A_{k 0} t\right\}+P_{0} \cdots P_{k} \tag{4.40}
\end{align*}
$$

Proof: Follows readily from the proof of Theorem 4.7. Remarks:

1) Equation (4.39) does not hold for $k>l$. For these values of $k$, however, we have

$$
\begin{equation*}
\lim _{\epsilon \downarrow 0} \sup _{\delta \leqslant t \leqslant T}\left\|P_{I}(\epsilon) \exp \left\{A_{0}(\epsilon) t / \epsilon^{k}\right\}-\Phi_{k}(t)\right\|=0 . \tag{4.41}
\end{equation*}
$$

Note that in (4.41) the projection matrix $P_{/}(\epsilon)$ annihilates behavior at time scale $t / \epsilon^{l}$ involving unstable or oscillatory modes. In general, however, $P_{,}(\epsilon)$ in (4.41) cannot be replaced by $P_{/}(0)$, so that (4.41) is of limited use in obtaining a uniform asymptotic series for $\exp \left\{A_{0}(\epsilon) t\right\}$.
2) Sometimes in applications, $A_{0}(\epsilon)$ satisfies a uniform stability condition, viz.,

$$
\begin{equation*}
\left\|\exp \left\{A_{0}(\epsilon) t\right\}\right\| \leqslant K \quad \forall t \geqslant 0, \quad \epsilon \in\left[0, \epsilon_{0}\right] . \tag{4.42}
\end{equation*}
$$

Although (4.42) guarantees that $A_{0}(\epsilon)$ satisfies the MSSNS condition and that any purely imaginary eigenvalue of the matrices $A_{k 0}$ is semisimple, it is not enough to guarantee MSST. A uniformly stable system may not have well-defined behavior at some time scales because of the presence of slightly attenuated oscillations that when seen at slower time scales represent infinite frequency. For uniformly stable systems, however, Proposition 4.13 can be strengthened.

Proposition 4.14: Let the matrix $A_{0}(\epsilon)$ satisfy (4.42) and let the matrices $A_{k 0}, k=0,1, \cdots, m$ be semistable for $k \neq l$. Then $\forall \delta>0, T<\infty$

$$
\begin{aligned}
& \lim _{\epsilon \downarrow 0} \sup _{\delta \leqslant t \leqslant T}\left\|\exp \left\{A_{0}(\epsilon) t / \epsilon^{k}\right\}-\Phi_{k}(t)\right\|=0 \quad k=1, \cdots, l \\
& \lim _{\epsilon \downarrow 0} \sup _{\delta \leqslant t \leqslant T}\left\|P_{l} \exp \left\{A_{0}(\epsilon) t / \epsilon^{k}\right\}-\Phi_{k}(t)\right\|=0
\end{aligned}
$$

$$
k=I+1, \cdots, m
$$

where $\Phi_{k}(t)$ is as in (4.40) and $T$ can be taken equal to $\infty$ for $k=m$.

Proof: Follows readily from Theorem 4.7 and the properties of uniformly stable systems mentioned above.

## V. Conclusions and Application of Our Results to the Hierarchical Aggregation of Finite State Markov Processes

We have studied the asymptotic behavior of $\exp \left\{A_{0}(\epsilon) t\right\}$ over the time interval $[0, \infty)$. We have formalized the notions of multiple time scales and reduced order models valid at different time scales. The most important conclusion is that a certain multiple stability condition referred to as the MSST condition is necessary and sufficient for $\exp \left\{A_{0}(\epsilon) t\right\}$ to have well-defined multiple time scale behavior. We feel that our results will have important computational consequences for the simulation of large-scale linear systems with weak couplings, but this has yet to be explored.

An application of particular interest to $u$ is the hierarchical aggregation of finite state Markov processes (FSMP) with some rare events. The presence of rare events in an FSMP is modeled by a small parameter $\epsilon$ in its matrix of transition rates, e.g., $A_{0}(\epsilon)=A_{0}+\epsilon B$. The matrix of transition probabilities for the FSMP, $\eta_{\epsilon}(t)$, is then given by

$$
\begin{equation*}
P^{\epsilon}(t)=\exp \left\{A_{0}(\epsilon) t\right\} . \tag{5.1}
\end{equation*}
$$

It is shown by us in [4] that when $A_{0}(\epsilon)$ is a matrix of transition rates then $A_{0}(\epsilon)$ satisfies the MSST condition so that $P^{\epsilon}(t)$
always has well-defined multiple time scale behavior. The reduced order models that describe the evolution of $P^{\epsilon}(t)$ at each of its fundamental time scales are then interpreted as increasingly simplified aggregated models of $\eta^{\epsilon}(t)$ obtained by collapsing several states of $\eta^{\epsilon}(t)$ into single states of the reduced order model. The aggregation is hierarchical so that the model at a time scale, say $t / \epsilon^{\prime}$, can be obtained by coalescing some states of the (already simplified) model valid at time scale $t / \epsilon^{t-1}$.
This problem has also been studied in detail in [12] where a sequence of aggregation models is also obtained. However, the question of a uniform asymptotic approximation to (5.1) was not studied in [12]. In [4], we develop the hierarchy of approximations as a uniform asymptotic expansion to (5.1) and relate our results to those of [12].

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# Multivariable Feedback, Sensitivity, and Decentralized Control 

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#### Abstract

In this paper the problem of sensitivity reduction by feedback is studied and related to a problem of decentralized control. A plant will be represented by an $N \times N$ matrix of frequency responses, which may be unstable or irrational. The object will be to find conditions on $P(s)$ under which a diagonal feedback $F(s)$ can make the sensitivity $\left\|\{I+P(s) F(s))^{-1}\right\|$ arbitrarily small over some specified frequency interval $\left[-j \omega_{0}, j \omega_{0}\right]$, without violating a global sensitivity bound $\|\{I+$ $P(s) F(s)\}^{-1} \| \leqslant M,(M \geqslant$ some const. $>1)$ for $\operatorname{Re}(s) \geqslant 0$. It will be shown that such a diagonal feedback of the "high gain" type can be constructed whenever $P^{-1}(s)$ is analytic in $\operatorname{Re}(s) \geqslant 0, P(s)$ satisfies an attenuation condition near $s=\infty$, and $P(s)$ approaches diagonal dominance at high frequencies. It will also be shown that these conditions on the plant can be interpreted as conditions for the existence of a decentralized wide-band control scheme.


## I. Introduction

IN this paper it will be shown that feedback can reduce sensitivity in a system characterized by a matrix of frequency responses. The result will be applied to deduce conditions under which a scheme of decentralized control is possible.

In an earlier paper [2], it was shown that feedback can reduce certain weighted measures of sensitivity whenever the plant $P(s)$

[^2]has an "approximate inverse," and feedback can make weighted sensitivity arbitrarily small whenever $P^{-1}(s)$ is analytic in $\operatorname{Re}(s)$ $\geqslant 0$ and $P(s)$ approaches zero slowly enough as $s \rightarrow \infty$. The approach in [2] involves an a priori parametrization of feedbacks that maintain closed-loop stability. Here an alternative approach will be developed, not dependent on a priori parametrization, and specialized to diagonal feedbacks.

Diagonal feedbacks are interesting because of their simplicity and their relevance to the problem of decentralized control (see Section I-A). In comparison to [2], we shall achieve diagonal feedback at the cost of the restriction that $P(s)$ approach diagonal dominance as $s \rightarrow \infty$.

Diagonal dominance and weak-coupling conditions were introduced into feedback theory during the 1960 's by Zames (see, e.g., [3]-[5]) in conjunction with the (incremental) small-gain and conic-sector theories for stability (sensitivity to perturbations). Rosenbrock has used Ostrowski's theorem to derive stability conditions for matrices of frequency responses that are diagonally dominant at all frequencies.
The requirement of diagonal dominance at all frequencies is quite restrictive, and satisfied by few multivariable systems in practice. On the other hand, many physical systems have highfrequency attenuation rates that increase with distance. Consequently, at high frequencies the magnitudes of their off-diagonal transmissions decrease with frequency more quickly than the diagonal ones. Such systems become diagonally dominant at high frequencies where, indeed, they approach diagonal.

## A. Sensitivity Reduction and Decentralized Control

An example might illustrate the class of problems that interest us. Imagine a house with many rooms, in each of which a man tries to control room temperature by watching a thermometer


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    ${ }^{1}$ In this paper we use the terms "reduced-order models" and "aggregated models" interchangeably. In many references (such as in the economics literature) in which the latter expression is used, what is typically meant by it is a special type of reduced-order model resulting from a procedure which explicitly combines (e.g., adds) groups of variables of the original system. In [4] the results we develop here are taken as a starting point for constructing such an explicit aggregation procedure for singularly perturbed finite state Markov processes.

[^1]:    ${ }^{2}$ The results of our work go through mutatis mutandis when (3.2) is an asymptotic series, provided this rank condition is satisfied.

[^2]:    Manuscript received January 12, 1982; revised January 13, 1983. Paper recommended by E. W. Kamen, Past Chairman of the Linear Systems Committee. This paper is partly based on Bensoussan [1]. It was originally presented at the 1981 CDC [7] and 1982 ACC [8] Conferences.
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