(

determined. For instance, if only  $c_1$  and  $b_k$  have been found in Step 6, then from (4) one obtains

$$M_{k-1} = c_1 b_{k-1} + c_2 b_k + c_{n-k+2} b_n a_{n-k-2} \cdots a_{n-1}.$$
(9)

Formulate a set of  $q \times p$  equations using entries in unknown  $b_{k-1}$ and  $c_2$  as variables and the entries in unknown  $c_{n-k+2}b_na_{n-k+2}\cdots a_{n-1}$ are moved to the other side of the equations to be combined with the numerical entries in  $M_{k-1}$ . Upon Gauss-Jordan reduction the consistency of the system of equations will give constraints for the solution of the unknown  $c_{n-k+2}$  and  $b_n$  and the unknown a's. Once again, the solutions are not unique, and additional constraints may be considered. Substituting the values into (9) one can find  $c_{5}$  and  $b_{k-1}$ . It will be seen from the example that computation is simple and computers may be used.

### III. EXAMPLE

The example used in [6], [9] is considered here for illustration.

$$G(s) = \begin{bmatrix} s^3 - s^2 + 1 & 1 & -s^3 + s^2 - 2 \\ 1.5s + 1 & s + 1 & -1.5s - 2 \\ s^3 - 9s^2 - s + 1 & -s^2 + 1 & s^3 - s - 2 \end{bmatrix} / s^4.$$

1) The least common denominator of all minors of G(s) is  $s^8$ . Degree  $\{G(s)\} = n = 8$ .

2) Upon partial-fraction expansion one obtains

$$G(s) = \sum_{j=1}^{k} M_j / (s - \lambda)^j, \quad k = 4$$
$$M_1 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad M_2 = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ -9 & -1 & 0 \end{bmatrix},$$
$$M_3 = \begin{bmatrix} 0 & 0 & 0 \\ 1.5 & 1 & -1.5 \\ -1 & 0 & -1 \end{bmatrix}, \quad M_4 = \begin{bmatrix} 1 & 1 & -2 \\ 1 & 1 & -2 \\ 1 & 1 & -2 \\ 1 & 1 & -2 \end{bmatrix}$$

3) and 4) Construct an  $8 \times 8$  generalized-Jordan-form matrix A and formulate  $(sI - A)^{-1}$ . Setting  $a_4 = 0$  and  $a_1 = a_2 = a_3 = 1$ , one obtains,  $M_4 = c_1b_4 + a_3a_6a_7c_3b_8$ ,  $M_3 = c_1b_3 + c_2b_4 + a_3a_6c_3b_7 + a_4a_6a_7c_3b_8$  $a_{6}a_{7}c_{6}b_{8}, M_{2} = c_{1}b_{2} + c_{2}b_{3} + c_{3}b_{4} + a_{5}c_{5}b_{6} + a_{6}c_{6}b_{7} + a_{7}c_{7}b_{8}, M_{1} =$  $c_1b_1 + c_2b_2 + c_3b_3 + c_4b_4 + c_5b_5 + c_6b_6 + c_7b_7 + c_5b_8.$ 

5) and 6) Rank  $M_4 = 1$ , assume  $a_5$  to be zero.  $M_4 = c_1 b_4$ . One obtains  $c_1 = (1 \ 1 \ 1)^T$ , and  $b_4 = (1 \ 1 \ -2)$ . Since  $1 \le n_2 \le \min(3,3)$ -1 = 2,  $a_6$  and  $a_7$  cannot be zero, and are set to be one each.

7) Formulate  $c_1b_3 + c_2b_4 = M_3 - c_6b_8$ , a set of nine equations of six variables, namely, b31, b32, b33, c12, c22, and c32. Upon Gauss-Jordan reduction one obtains the following constraints for the system of nine equations to be consistent:

$$\begin{aligned} (c_{26} - c_{16})(b_{51} - b_{52}) &= 0.5, (c_{16} - c_{36})(b_{51} - b_{52}) = 1, \\ (c_{26} - c_{16})(2b_{51} + b_{53}) &= 1.5, (c_{16} - c_{36})(b_{53} + 2b_{51}) = -3 \end{aligned}$$

There are four equations of six unknown variables. Again, additional constraints may be considered. One finds  $c_6 = (-1, -1, 5, 0)^T$ , and  $b_{\rm s} = (-1 \ 0 \ -1)$ . From the original nine equations one obtains  $c_2 = (0 \ 1 \ 0)^T$ , and  $b_3 = (-1 \ 0 \ -1)$ . Repeat the procedures for the two sets of equations successively,  $c_1b_2 + c_3b_4 + c_6b_7 + c_7b_8 = M_2 - M_2$  $c_2b_3$ , and  $c_1b_1 + c_4b_4 + c_6b_6 + c_5b_8 = M_1 - c_2b_2 - c_3b_3 - c_5b_5 - c_7b_7$ , and solve for b2, b7, c3, c7, and b2, c3, b1, b6, c4, c8, respectively. They are:  $b_2 = (-9 - 1 \ 0), b_7 = (-20/3 \ -2/3 \ -2/3), c_3 = (1/3 \ 0 \ 0)^T, c_7 =$  $(-1 \ 0 \ 0)^T$ , and  $b_5 = (0 \ 0 \ 6)$ ,  $c_5 = (1 \ 0 \ 0)^T$ ,  $c_5 = (31 \ 3 \ 0 \ 0)^T$ ,  $c_4 =$  $(-4/3\ 0\ 0)^T$ ,  $b_1 = (1\ 0\ 1)$ ,  $b_6 = (-16/3\ -2/3\ 2\ 3)$ .

## IV. CONCLUSION

The procedure can be applied to the synthesis of networks to reduce the number of physical elements needed. Only simple techniques are required in the computation. Digital computers can be used to further simplify the computation. The procedure needs to determine the least common denominator of all the minors of G(s)and to know the factors of the denominator.

#### References

- E. G. Gilbert, "Controllability and observability in multivariable control systems," SIAM J. Contr., vol. 1, pp. 128-151, 1963.
   R. E. Kalman, "Mathematical description of linear dynamical systems," SIAM J. Contr., vol. 1, pp. 152-192, 1963.
   B. L. Ho, "Effective construction of linear state variable models from in-put output functions," PhD. dissertation, Stanford University, Stanford, Calif., 1966.
   D. Q. Mayne, "Computational procedures for the minimal realization of transfer-function matrices," Proc. Inst. Elec. Eng., vol. 115, pp. 1363-1368, 1968.
- [5]
- [6] [7] L
- [8]
- [9]
- b. S. P. Panda and C. T. Chen, "Irreducible Jordan form realization of a rational matrix," *IEEE Trans. on Automat. Contr.* (Short Papers), vol. AC-14, pp. 66-69, Feb. 1969.
  C. T. Chen, Introduction to Linear System Theory. N. Y.: Holt, Rinehart and Winston, 1970.
  L. M. Silverman, "Realization of linear dynamical systems," *IEEE Trans. Automat. Contr.*, vol. AC-16, pp. 554-567, Dec. 1971.
  C. T. Chen and D. P. Mitral, "A simplified irreducible realization algorithm," *IEEE Trans. Automat. Contr.*, (Short Papers), A.C.-17, pp. 535-537, Aug. 1972.
  Y. L. Kuo, "On the irreducible Jordan form realization and the degree of a rational matrix," *IEEE Trans. Circuit Theory*, vol. AC-17, pp. 322-332, 1970.
- rational matrix," *IEEE Trans. Circuit Theory*, vol. AC-17, pp. 322-332, 1970.
  [10] H. L. Huang, "Use of complex quadruple (*F,G,H,J*) in system minimal realization," *Proc. 16th Midwest Symp. Circuit Theory*, Univ. of Waterloo, Waterloo, Canada, vol. II, Apr. 1973.
  [11] H. L. Huang, "State space synthesis of minimal active RC microelectronic p-port transferunction admittance matrix," in 1973 Int. Symp. Circuit Theory Digest, Toronto, Canada, pp. 252-255.

On the Invertibility of Linear Systems

# ALAN S. WILLSKY

Abstract-Sain and Massey [1] have obtained necessary and sufficient conditions for the invertibility of continuous and discrete time linear systems and have also found bounds for what they termed the "inherent integration" (continuous time) or the "inherent delay" (discrete time) of an invertible linear system. In this note, we tighten these bounds by modifying an argument used to prove the Sain-Massey result.

# I. INTRODUCTION

In recent years there have been a number of papers [1]-[5] written that deal with the question of the invertibility of linear time-invariant (LTI) systems-both discrete time and continuous time and over the real field or over an arbitrary field. These studies have led to a number of different algebraic criteria. Massey and Sain [2] have devised a test for invertibility involving the ranks of a set of input sequence - output sequence maps, while Sain and Massey [1] and Brockett and Mesarovic [3] have developed a necessary and sufficient test based upon a rank condition for a single large matrix. Recently, Wang and Davison [5] have come up with an invertibility test that involves the matrices associated with a particular realization, as opposed to the preceding tests which depend only on the system weighting pattern.

One notion that is intimately tied to the concept of invertibility is that of inverse delay or integration. In general, there exists no inverse for a LTI system such that the input to the original system is reproduced as the output of the inverse without delay or integration. The number of units of delay for the discrete time case is called the inverse delay, while the number of integrations in the continuous time case is called the inverse integration. The techniques of Sain and Massey [1],[2] use these notions rather strongly. We note here only that the amount of computation involved in checking to see if there is an inverse with an L-delay (or L-integration) increases with  $L_{i}$  and one of the most important results in [1] is that Sain and Massey are able to bound the largest value of L that we need check.

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In this note, we tighten the Sain-Massey bound by modifying a linear-algebraic argument in [1]. As a direct result of the tightened bound, we also obtain a sharper version of the single matrix result reported in [1] and [3].

## II. NOTATION AND PROBLEM STATEMENT

We consider the LTI continuous and discrete time linear systems over a field K

$$\dot{x}(t) = A x(t) + B u(t)$$
 (1)

$$y(t) = C x(t) + D u(t)$$
 (2)

$$x(k+1) = A x(k) + B u(k)$$
(3)

$$y(k) = C x(k) + D u(k).$$
 (4)

Here  $x \in K^n$ ,  $u \in K^m$ ,  $y \in K^{\rho}$ , and A,B,C,D are matrices of appropriate dimensions (in the continuous time case we assume that  $K = \mathbf{R}$ ). The transfer functions of these systems are

$$G(s) = C(I s - A)^{-1}B + D$$
(5)

$$G(z) = C(I z - A)^{-1}B + D.$$
 (6)

We now recall two definitions from [1].

Definition 1: The continuous time LTI system (1),(2) is L-integral invertible if there exists a linear system

$$\dot{z}(t) = \hat{A} z(t) + \hat{B} v(t)$$
 (7)

$$w(t) = \hat{C} z(t) + \hat{D} v(t)$$
 (8)

with transfer function  $\hat{G}(s)$ , such that

$$\hat{G}(s)G(s) = \frac{1}{s^L} I.$$
(9)

The system (7),(8) is called an *L*-integral inverse. If (1),(2) is *L*-integral invertible for some *L*, the system is called *invertible*, and the smallest *L* such that (1),(2) is *L*-invertible is called the *inherent* integration  $L_0$  of an invertible system.

Definition 2: The discrete time LTI system (3),(4) is L-delay invertible if there exists a linear system

$$z(k+1) = \hat{A} z(k) + \hat{B} v(k)$$
(10)

$$w(k) = \hat{C} z(k) + \hat{D} v(k)$$
(11)

with transfer function  $\hat{G}(z)$ , such that

$$\hat{G}(z)G(z) = \frac{1}{z^L} I.$$
 (12)

The system (10), (11) is called an *L*-delay inverse. If (3), (4) is *L*-delay invertible for some *L*, the system is called invertible and the smallest *L* such that (3), (4) is *L*-delay invertible is called the *inherent delay L*<sub>0</sub>.

Thus, it is clear that we can determine L-integral invertibility and the inherent integration of a system (1),(2) by looking at the question of L-delay invertibility and the inherent delay of the discrete system (3),(4) formed by using the same A,B,C, and D matrices. Therefore, we restrict our attention to the discrete time system (3),(4). To do this, we first define some notation and an equivalent definition of L-delay invertibility [1],[2]. Let  $U_k \in \mathbf{R}^{(k+1)m}$  and  $Y_k \in \mathbf{R}^{(k+1)p}$  be the vectors of the first k + 1 inputs and outputs, respectively,

$$U_{k} = \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ \vdots \\ u(k) \end{bmatrix}, \quad Y_{k} = \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ \vdots \\ y(k) \end{bmatrix}.$$
(13)

Assuming that x(0) = 0 (or at least that it is known so that we can subtract out its effect), we have

where

$$Y_k = M_k U_k \tag{14}$$

$$M_{k} = \begin{bmatrix} T_{0} & 0 & \cdots & 0 \\ T_{1} & T_{0} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ T_{k} & T_{k-1} & \cdots & T_{0} \end{bmatrix},$$
(15)

The following is an equivalent definition of L-delay invertibility.

Definition 3: The system (3),(4) is L-delay invertible if  $U_k$  is uniquely determined by the response segment  $Y_{k+L}$  for  $k = 0, 1, 2, \cdots$ . Note that it is immediate from Definition 3 that if (3),(4) is L-delay invertible, it is k-delay invertible  $\forall k \geq L$ . Also, it is clear that if the system (3),(4) is invertible, we must have  $\rho,n \geq m$ . In the next section, we assume that this holds.

## III. NECESSARY AND SUFFICIENT CONDITIONS FOR THE INVERTIBILITY OF LINEAR SYSTEMS

We first recall a characterization of invertibility proven in [1],[2] (here rank  $M_{-1}$  is defined to be zero).

Theorem 1: For any nonnegative integer L,

$$\operatorname{rank}(M_L) - \operatorname{rank}(M_{L-1}) \le m \tag{17}$$

with equality if and only if (3),(4) has an *L*-delay inverse. We now strengthen the two corollaries in [1] to Theorem 1 and

also the single matrix invertibility condition stated in [1] and [3]. Corollary 1: Let q = dimension of the nullspace of D. The system

(3),(4) is invertible if and only if

rank 
$$(M_{n-q+1})$$
 - rank  $(M_{n-q}) = m$ , (18)

i.e., if (3),(4) is invertible, its inherent delay  $L_0$  cannot exceed n-q+1.

*Proof:* Note that

 $T_{0}$ 

$$\operatorname{rank} M_0 = \operatorname{rank} D = m - q. \tag{19}$$

Thus if q = 0, the system is 0-delay invertible, and (18) is satisfied, since the system is k-delay invertible for all  $k \ge 0$ . Suppose  $q \ge 1$  and that the system is not L-delay invertible. Then it is not k-delay invertible for any  $k \le L$ . Thus, from Theorem 1 and (19), we have

$$\operatorname{rank} M_L \le L(m-1) + m - q.$$
 (20)

Thus, the  $(L + 1)\rho \times (L + 1)m$  matrix has column nullity

$$(L+1)m - \operatorname{rank}(M_L) \ge L + q. \tag{21}$$

Let  $\mathfrak{N}_L$  be the subspace of  $\mathbf{R}^{(L+1)m}$  that is annihilated by  $M_L$ .

$$\dim \mathfrak{N}_{\mathrm{L}} \ge L + q. \tag{22}$$

Consider the map that sends  $u(0), u(1), \dots, u(L)$  into the state at time L + 1, and restrict this map to the control sequences in  $\mathfrak{N}_L$ . If dim  $\mathfrak{N}_L \ge n + 1$ , this restricted map has a nontrivial kernel—i.e., there exists an input sequence  $u(0), \dots, u(L)$ , not identically zero, such that the corresponding output segment is

$$y(0) = y(1) = \cdots = y(L) = 0$$
 (23)

and

$$x(L+1) = 0. (24)$$

Thus, the output sequence corresponding to the input sequence  $u(0), \dots, u(L)$ , 0, 0,  $\dots$  is identically zero, and the system is not invertible.

Noting from (22) that dim  $\mathfrak{N}_L \ge n+1$  if L = n-q+1, we have that the system is invertible if and only if it is (n-q+1) - delay invertible.

We remark that for q = 0 the invertibility question is trivial, for q = 1 our result is the same as the Sain-Massey result, and our result is stronger for q > 1. In particular, if D = 0, we have

Corollary 2: The system

$$x(k+1) = A x(k) + B u(k)$$
(25)

$$y(k) = C x(k) \tag{26}$$

is invertible if and only if it is (n - m + 1) – delay invertible.

We also have the following strengthened corollary, the proof of which involves a trivial modification of the analogous result in [1] if we keep the above proof of Corollary 1 in mind.

Corollary 3: The system (3),(4) is invertible if and only if there is no input segment  $U_{n-q+1} \neq 0$  followed by all zeroes, which produces the all zero output sequence in (3),(4) when  $x_0 = 0$ .

Similarly, we obtain a strengthened version of the single matrix result in [1] and [3].

Theorem 2: The system (3),(4) is invertible if and only if

$$rank(N) = (n - q + 2)m$$
 (27)

where N is the  $(2n - q + 2)\rho \times (n - q + 2)m$  matrix

$$N = \begin{bmatrix} D & 0 & \cdots & 0 \\ C & B & D & \cdots & 0 \\ \vdots & & & & \\ C & A^{n-q}B & C & A^{n-q-1}B & \cdots & D \\ C & A^{n-q+1}B & C & A^{n-q}B & \cdots & C & B \\ \vdots & & \vdots & & \vdots \\ C & A^{2n-q}B & C & A^{2n-q-1}B & \cdots & C & A^{n-1}B \end{bmatrix}.$$
(28)

Corollary 4: The rank condition

$$\operatorname{rank}(N) = (n - q + 2)m$$
 (29)

holds if and only if

$$\operatorname{rank}(M_{n-q+1}) - \operatorname{rank}(M_{n-q}) = m.$$
 (30)

We also note that in a similar manner one can obtain strengthened versions of the necessary and sufficient conditions, presented in [1] and [3], for the dual concept of functional controllability.

## IV. CONCLUSIONS

In this note we have obtained a strengthened version of the necessary and sufficient conditions, derived in [1]-[3], for linear system invertibility. These results reduce the question of invertibility to a set of rank tests for certain matrices, and our strengthening of these results depends on a careful counting argument.

The question of system invertibility is important in such applications as the design of encoding-decoding systems, and has received a great deal of attention in the literature. We refer the reader to more general invertibility results in [6]-[9]. In particular the finite group system results in [6]-[8] are quite similar in flavor to the results in [1]-[3] and in this note.

#### References

- M. K. Sain and J. L. Massey, "Invertibility of linear time-invariant dynamical systems," *IEEE Trans. on Automat. Contr.*, vol. AC-14, pp. 141-149, Apr. 1969.
   J. L. Massey and M. K. Sain, "Inverses of linear sequential circuits," *IEEE Trans. Comput.*, vol. C-17, pp. 330-337, Apr. 1968.
   R. W. Brockett and M. D. Mesarovic, "The reproducibility of multivariable systems," *J. Math. Anal. Appl.*, vol. 11, pp. 548-563, July 1965.
   G. D. Forney, Jr., "Convolutional codes 1: Algebraic structure," *IEEE Trans. Inform. Theory*, vol. 17-16, pp. 720-738, Nov. 1970.
   S. H. Wang and E. J. Davison, "A new invertibility criterion for linear multivariable systems," *IEEE Trans. Automat. Contr.* (Tech. Notes and Corresp.), vol. AC-18, pp. 538-539, Oct. 1973.
   A. Willsky, "Dynamical systems defined on groups: Structural properties and estimation," Ph.D. dissertation, Dept. Aeronautics and Astronautics, Mass. Inst. Technol., Cambridge, Mass., June 1973.

- A. S. Willsky, "Invertibility conditions for a class of finite state systems evolving on groups," Proc. 11th Allerton Conf., Univ. of Illinois, Oct. 3-5, 1973.
   A. S. Willsky, "Invertibility of finite group homomorphic sequential sys-tems," Inform. Contr., to be published.
   R. R. Olson, "On the invertibility of finite state machines," Dep. Elec. Eng., Univ. of Notre Dame, Tech. Rep. EE-703, July 23, 1970.

# Minimal Order Observers and Certain Singular Problems of Optimal Estimation and Control

# HARRY G. KWATNY

Abstract-It is shown that a Riccati equation of particular structure which arises in a number of singular optimal estimation and control processes can be reduced in order. This fact leads directly to a procedure for the design of a class of minimal order observers, the structure of which can be interpreted as the limiting form of appropriate Kalman estimators with vanishing observation noise.

# I. INTRODUCTION

As might be anticipated, the theory of minimal order observers can be closely allied with certain singular problems of optimal estimation and control. This commonality is particularly striking when it is recognized that minimal order observer design can be accomplished through solution of a matrix Riccati equation which is identical in structure to those arising in singular optimal regulator problems and which admits a reduction in order.

It is known that the problem of minimal order observer design for an nth order, completely observable system with r independent outputs can be conveniently solved by solution of an  $(n - r) \times$ (n - r) dimension matrix Riccati equation [1].

In what follows it is shown that the required Riccati equation can be derived through reduction of a larger  $n \times n$  Riccati equation and that, in appropriate circumstances, observers designed in this way are limiting forms of Kalman estimators for vanishing observation noise in the sense of Friedland [2]. Furthermore, it is observed that certain Riccati equations obtained by Friedland [2] and Moylan and Moore [3] for singular optimal regulator problems are structured identically to that obtained for the observer design problem and can be reduced in order. Certain problems of estimating the state of a linear dynamical system from observation of outputs corrupted by correlated noise are duals of these singular regulator problems and consequently can be solved by identical procedures.

#### II. ORDER REDUCTION OF A CLASS OF RICCATI EQUATIONS

Let C be an  $r \times n$  matrix of full rank and let  $C_0^*$  denote a right inverse of C.

Let P be an  $n \times n$  symmetric matrix which satisfies the relation

$$CP = 0$$
 (1)

as well as the algebraic Riccati equation

$$PA'[I - C_0^*C]' + [I - C_0^*C]AP - PA'J'S^{-1}JAP + M = 0 \quad (2)$$

where S > 0 is a symmetric  $r \times r$  matrix,  $M \ge 0$  is a symmetric  $n \times n$  matrix which has the property

$$CM = 0. (3)$$

J is an  $r \times n$  matrix which will be required to satisfy a controllability condition given below. A method will be given for obtaining P satisfying both (1) and (2) by solving a Riccati equation of dimension  $(n-r) \times (n-r).$ 

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