A DETERMINISTIC AND STOCHASTIC THEORY
FOR
TWO-POINT BOUNDARY-VALUE DESCRIPTOR SYSTEMS

by

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ABSTRACT

A deterministic system theory is developed for two-point boundary-value descriptor systems (TPBVDS's). In particular, detailed characterizations of the properties of reachability, observability and minimaltiy are obtained. In addition, extendibility, i.e. the concept of considering a TPBVDS as being defined on a sequence of intervals of increasing length, is defined and studied. These system-theoretic properties are derived for general TPBVDS's and then specialized to the case of stationary systems for which the input-output map (weighting pattern) is shift-invariant.

Next, the deterministic realization problem of constructing a minimal extendible, stationary TPBVDS that realizes a given weighting pattern is considered and solved using an original transform technique. This transform technique is then used in studying the stochastic realization problem, which consists of constructing a TPBVDS of minimal dimension from its output covariance.

Finally, the optimal estimation (smoothing) problem for TPBVDS's is considered. Two solutions to this problem are proposed. The first solution is based on the fact that the smoothed estimates satisfy a TPBVDS structure and thus the smoother can be implemented by a two-filter method. This implementation requires block diagonalization of the smoother dynamics, i.e. the Hamiltonian-diagonalization problem. This problem is studied and shown to be related to generalized Riccati equations. These Riccati equations are also studied. The second solution proposed consists of a generalization of the Rauch-Tung-Striebel formulation of the smoother for causal systems. For this approach, a generalization of the Kalman filter is proposed. This filter also provides probabilistic interpretations for the solutions of the generalized Riccati equations.

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CHAPTER I:

INTRODUCTION

In discrete-time, Luenberger [4,5] has shown that descriptor systems exhibit noncausal behaviour and thus are useful for modeling noncausal phenomena usually corresponding to spatial processes, i.e. when the independent variable is space rather than time. Lewis has shown in [19] that the dynamics of discrete-time, descriptor systems are completely symmetric with respect to forward and backward time directions. This has led to the observation [1,2,46] that boundary conditions, specifically two-point boundary value conditions, are better adapted than initial conditions, to descriptor dynamics not only because they preserve the symmetry but also because they can guarantee well-posedness. Descriptor systems with such boundary conditions are called two-point boundary-value descriptor systems.

A two-point boundary-value descriptor system (TPBVDS) can be represented as follows

$$Ex(k+1) = Ax(k) + Bu(k) \qquad (1.1)$$

$$V_i x(0) + V_f x(N) = v \qquad (1.2)$$

$$y(k) = Cx(k), \qquad (1.3)$$

where E, A, V_i and V_f are possibly singular, square matrices. TPBVDS's are generalizations of classical causal systems (by letting $E=V_i=I$, $V_f=0$, and $N=\infty$, (1.1)-(1.3) reduces to a causal system), descriptor systems and periodic systems ($V_i=-V_f=I$, $v=0$). These systems are particularly useful for modeling processes defined over finite intervals such as the heat distribution on a rod or deflection of a beam, since, in such situations, a complete symmetry must exist with respect to the two ends of the interval as far as the boundary specifications are concerned. This has been one of the motivations

for developing system theory for these systems..

Another motivation behind studying TPBVDS has been the fact that finite interval optimal smoothers for standard causal systems have TPBVDS structure. Thus, in trying to better underestand the standard causal, finite-interval smoothing problem, some system-properties of TPBVDS's must first be investigated. We will show that in fact the optimal smoother of a larger class of systems, namely TPBVDS's, have TPBVDS structure as well, and so it makes sense to consider the smoothing problem for this larger class of systems. This has first been pointed out by Adams, et al. [46] and studied further in [3]. This investigation is carried even further in this thesis.

TPBVDS's also arise in the study of 2D systems. In particular, the solution and linear estimation of 2D discrete processes satisfying local recursions, such as the nearest-neighbor models or Roesser's model, have been studied in [38], and a general solution technique has been proposed based on converting the 2D system into a 1D TPBVDS of large dimension.

Another motivation for studying system-theoretical properties of TPBVDS has been the hope to extend the results to the multi-dimensional case since the natural description of multi-dimensional systems involves boundary conditions, and moreover, one of the well known properties of these systems is that they are inherently noncausal. We have not yet considered the multi-dimensional case in our work; however, it is our hope that our work will provide the basis for such an investigation. We have made a few comments regarding this problem in Chapter V.

A system theory for TPBVDS's has been developed in [1,2,16]. These works have been largely influenced by the works of Krener [6,7], and Gohberg and

Kaashoek [11,12,13] on the system-theoretic properties of standard (non-descriptor) continuous-time boundary-value linear systems.

In our opinion, the two most important contributions of Krener, to the study of boundary-value linear systems, have been: (1) the introduction of the notions of inward and outward processes which correspond to a natural, causal-anticausal decomposition of the boundary value process; and, (2) the introduction and characterization of the property of stationarity, i.e. shift-invariance of the weighting pattern or impulse response of time-invariant (autonomous) systems. Based on the inward and outward processes, Krener has been able to develop two distinct notions of both reachability and observability, and relate them to the problem of minimality. Also, as Krener points out, unlike the causal case, shift-invariance of the impulse reponse is <u>not</u> implied by the time-invariance of the model. Gohberg and Kaashoek have also studied boundary-value linear systems and, in particular, they have considered the time-invariant case which is the focus of our work as well.

The work described in this thesis builds on our research described in [1,2]. In these previous studies, we extended many of the continuous-time, boundary-value linear system results to the case of TPBVDS and also developed notions of and results on stability that represent original contributions even in the case of continuous-time boundary-value linear systems. The work in [1,2] can also be viewed as contributing to the theory of descriptor systems. In particular, much of the literature on descriptor systems [21-24,47-49] has focused on the continuous-time case which has fundamental differences with the discrete-time case. The results in the literature on the discrete-time case [4,5,17-20], for the most part, do not deal with boundary

conditions (and hence system noncausality) explicitly and also do not make use of the standard form, developed in [1,2], which makes many results and concepts more transparent.

The results presented in this thesis provide several contributions. In particular, we have completed the study begun in [1,2] to extend results on time-invariant, continuous-time boundary-value linear systems to the case of TPBVDS's. In particular, we have completely characterized the concepts of inward and outward processes leading to a complete study of properties of reachability and observability, and minimality. We have also introduced and studied extendibility, i.e. the concept of considering a TPBVDS as being defined on a sequence of intervals of increasing length. The major new aspects of these results are (1), the correction of the minimality result of [2] for the class of stationary TPBVDS's (see Chapter II) and (2), the extension of all of our results and concepts to the complete class of autonomous TPBVDS's.

Using the concept of extendibility, we have been able to pose and solve a realization problem for TPBVDS's. In particular, we have considered the problem of realizing a minimal, stationary, extendible TPBVDS from its input-output weighting pattern. In solving this problem, we have introduced an original transform technique, the (s,t)-transform, which is better adapted to descriptor dynamic than the standard z-transform technique because of the symmetrical way that the zero and infinite modes of the system are treated. Using this transform, we have developed a realization method that involves the solution of a generalized factorization problem and a generalization of the McMillan degree of a rational matrix. As we will there are some important

differences with the causal case. This represents one of the most important results in this thesis.

The concept of extendiblity has also allowed us to consider the stochastic version of the realization problem for TPBVDS's driven by a white Gaussian sequences. In particular, we have used this concept to define stochastic extendibility and pose the stochastic realization problem of constructing a minimal TPBVDS from its output covariance. It turns out that the class of covariances that can be realized by stochastically extendible TPBVDS's is not larger than the class of covariances realizable by causal systems. However, we can now characterize a larger class of (possibly acausal) minimal realizations. This problem is also solved using the (s,t)-transform technique emphasizing the importance of this transform.

The last part of the thesis, which is devoted to a study of optimal estimation for TPBVDS's includes several results which we feel are of significance. In particular, we have shown that the optimal smoother for a TPBVDS has a TPBVDS structure as well, with twice the dimension of the original system. In implementing this smoother using the two-filter solution method [46], the smoother dynamics must be decoupled into forward and backward stable recursions (Hamiltonian diagonalization). We have shown how this can be done by the use of transformation matrices involving the solutions of a new type of generalized Riccati equations. A theory paralleling the existing theory for standard Riccati equations is developed for these generalized Riccati equations. In particular, we have shown that the solutions of these equations can be interpreted as error variances of a generalization of the Kalman filter which we have introduced and studied. Finally, this generalized Kalman filter has been used to construct a

generalization, to the case of TPBVDS's, of the Rauch-Tung-Striebel smoothing algorithm for causal systems.

The outline of the thesis is as follows.

In Chapter II, we review and extend the system-theoretic properties of TPBVDS's introduced in [1,2]. In particular, we obtain a detailed characterization of concepts of reachability and observability, and minimality. We also define and study stationarity and displacement which are two notions of shift-invariance for TPBVDS's. The concept of extendibility is also introduced in this chapter. In the last section of Chapter II, a modal analysis of the TPBVDS is done. This allows us to consider reachability and observability of individual eigenmodes as it is done in the causal case.

In Chapter III, realization theory for extendible stationary TPBVDS's is considered. In particular, this chapter consists of two main sections. In the first section we consider the problem of deterministic realization, i.e. the problem of constructing a minimal, extendible, stationary TPBVDS realization of an input-output weighting pattern. As indicated previously, the most significant part of this study is the introduction of an original transform technique called the (s,t)-transform. By studying the properties of this transform, we obtain results characterizing the dimension of the minimal realization and an algorithm for its construction. In contrast with the causal case, in general here we must perform (generalized) factorizations of more than one rational matrix to obtain the minimal realization.

In the second section of Chapter III, we consider the problem of stochastic realization. We start by obtaining conditions under which a TPBVDS, driven by a white Gaussian sequence, has a stochastically stationary

and extendible output. The final result of this section is that the stochastic realization problem is related to a spectral factorization problem just as in the causal case, although we now have more freedom in its solution since we are not tied to causal realizations.

In Chapter IV, we consider the problem of optimal estimation for TPBVDS's driven by white Gaussian sequences and with independent white Gaussian observation noises. The results of the first section and the first half of the second section of this chapter can also be found in [3]. There, it is shown that the optimal smoother for a TPBVDS has also a TPBVDS structure which can be solved, for example, by the two-filter solution technique which is described in the Appendix. Applying this technique requires the block-diagonalization of the smoother, for which, positive-definite solutions to a new class of generalized Riccati equations must be found.

In Section 4.2, we study these generalized Riccati equations and develop a theory for them paralleling the existing theory for standard Riccati equations. In Section 4.3, we define a generalization of the Kalman filter which leads to both a probabilistic interpretation of the generalized Riccati equations of Section 4.2 and to a generalization of the Rauch-Tung-Striebel smoothing method for causal systems. The actual implementation of this for a general TPBVDS involves inward and outward recursive computations.

Finally, in Chapter V, we conclude with a list of contributions and a list of open questions.

Chapter II:

REACHABILITY, OBSERVABILITY AND MINIMALITY FOR

TWO-POINT BOUNDARY-VALUE DESCRIPTOR SYSTEMS

2.1-Introduction

In this chapter we study the system-theoretic properties of two-point boundary-value descriptor systems (TPBVDS's) and two related classes of shift-invariant two-point boundary-value descriptor systems namely displacement systems for which the Green's function is shift-invariant, and stationary systems for which the input-output map is stationary. We present detailed characterizations of the properties of strong and weak reachability and observability introduced in [1] and of minimality as well. Another property that is studied in this chapter is that of extendibility, i.e. the concept of considering a TPBVDS as being defined on a sequence of intervals of increasing length.

Some of the results in this chapter, such as results concerning the concepts of well-posedness, standard-form, generalized Cayley-Hamilton theorem, inward and outward processes, and strong reachability and observability have already been discussed in [1,2] and so we shall simply review them here. Other results, such as those concerning extendibility, weak reachability and observability, and minimality have only been considered in a much more restricted setting (essentially in the displacement case). The attempt to generalize these concepts have generally failed because no closed-form expressions for the inward process could be found. Here we shall obtain the necessary closed-form expressions and completely resolve the

problems of extendibility, weak reachability and observability, and minimality in the most general case.

In the next section we introduce TPBVDS's, and define and characterize two notions of shift-invariant systems, namely displacement systems and stationary systems.

In Section 2.3 we review the notions of inward and outward processes introduced for TPBVDS's in [1,2] and characterize these processes. We also introduce the concept of extendibility and characterize this property. Section 2.4 discusses the properties of reachability and observability for TPBVDS's, while in Section 2.5 we present minimality results. Some extensions are presented in Section 2.6, and we conclude with a brief discussion in Section 2.7.

2.2-Two-Point Boundary-Value Descriptor Systems

A TPBVDS is described by the following dynamic equation

$$Ex(k+1) = Ax(k) + Bu(k), \quad 0 \leq k \leq N-1 \qquad (2.2.1)$$

with boundary condition

$$V_i x(0) + V_f x(N) = v \qquad (2.2.2)$$

and output

$$y(k) = Cx(k), \quad k=0,1,\ldots,N. \qquad (2.2.3)$$

Here x and v are n-dimensional, u is m-dimensional, y is p-dimensional, and E, A, B, V_i, V_f, and C are constant matrices. In [1] it is shown that if (2.2.1)-(2.2.2) is well-posed (i.e. it yields a well-defined map from {u,v} to x), we can assume, without loss of generality that (2.2.1)-(2.2.2) is in normalized form, i.e. that there exist scalars α and β such that

$$\alpha E + \beta A = I \qquad (2.2.4)$$

(this is referred to as the standard form for the pencil {E,A}) and in addition

$$V_i E^N + V_f A^N = I . \qquad (2.2.5)$$

Note that (2.2.4) implies that E and A commute, that E, A and the system have a common set of eigenvectors[1], and also that $\{E^k, A^k\}$ is a regular pencil for all $k \geq 0$ (see[1]). But most importantly (2.2.4) implies that the space of matrices $A^K E^L$, $K, L \geq 0$, is spanned by the n matrices $\{A^k E^{n-1-k} | k=0,\ldots,n-1\}$; this property has been introduced in [1] as the generalized Cayley-Hamilton

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[1] v is an eigenvector of the system if $v \neq 0$ and for some σ, $(\sigma E - A)v = 0$. σ is called an eigenmode of the system; for descriptor systems σ can be ∞ as well.

theorem. We assume throughout this paper that (2.2.4) and (2.2.5) hold. We also assume that the interval of definition of our system is sufficiently large to excite and observe all system modes. Specifically, we assume that $N \geq 2n$, unless explicitly stated otherwise.

As derived in [1], the map from $\{u,v\}$ to x has the following form:

$$x(k) = A^k E^{N-k} v + \sum_{j=0}^{N-1} G(k,j) Bu(j), \qquad (2.2.6)$$

where the Green's function $G(k,j)$ is given by

$$G(k,j) = \begin{bmatrix} A^k (A - E^{N-k}(V_i A + \omega V_f E) E^k) E^{j-k} A^{N-j-1} \Gamma^{-1} & j \geq k \\ E^{N-k}(\omega E - A^k(V_i A + \omega V_f E) A^{N-k}) E^j A^{k-j-1} \Gamma^{-1} & j < k \end{bmatrix} \qquad (2.2.7)$$

and where ω is any number such that

$$\Gamma \triangleq \omega E^{N+1} - A^{N+1} \qquad (2.2.8)$$

is invertible.

In marked contrast to the case for causal systems ($E=I, V_f=0$), $G(k,j)$ does <u>not</u>, in general, depend on the difference of its arguments. Borrowing some terminology from [11-13], we have the following definition of our first notion of shift-invariance:

Definition 2.2.1

The TPBVDS (2.2.1)-(2.2.2) is a <u>displacement system</u> if (with the usual abuse of notation) for $0 \leq k \leq N$, $0 \leq j \leq N-1$

$$G(k,j) = G(k-j). \qquad (2.2.9)$$

A second notion of shift-invariance is the one associated with the input-output map. Specifically, with v=0 in (2.2.2), we have that (2.2.1)-(2.2.3) defines a linear map of the form

$$y(k) = \sum_{j=0}^{N-1} W(k,j)Bu(j), \qquad (2.2.10)$$

where, obviously

$$W(k,j) = CG(k,j)B. \qquad (2.2.11)$$

Definition 2.2.2

The TPBVDS (2.2.1)-(2.2.3) is <u>stationary</u> if (again with the usual abuse of notation)

$$W(k,j) = W(k-j) \qquad (2.2.12)$$

for $0 \leq k \leq N$, $0 \leq j \leq N-1$.

The following results characterize the conditions under which a TPBVDS is displacement and stationary.

Theorem 2.2.1

The TPBVDS (2.2.1)-(2.2.3) is stationary if and only if

$$O_s[V_i, E]R_s = O_s[V_i, A]R_s = 0 \qquad (2.2.13a)$$

$$O_s[V_f, E]R_s = O_s[V_f, A]R_s = 0, \qquad (2.2.13b)$$

where [X,Y] denotes the commutator product of X and Y

$$[X,Y] = XY - YX \qquad (2.2.14)$$

and

$$R_s = [A^{n-1}B \vdots EA^{n-2}B \vdots \ldots \vdots E^{n-1}B] \qquad (2.2.15)$$

$$O_s = \begin{bmatrix} CA^{n-1} \\ CEA^{n-2} \\ \vdots \\ CE^{n-1} \end{bmatrix}. \qquad (2.2.16)$$

Before proving this result, let us also state a corollary (which will also require proof) and make several comments:

Corollary

The TPBVDS (2.2.1)-(2.2.2) is a displacement system if and only if

$$[V_i,E] = [V_i,A] = 0 \qquad (2.2.17a)$$

$$[V_f,E] = [V_f,A] = 0. \qquad (2.2.17b)$$

The matrices R_s and O_s in (2.2.15), (2.2.16) are, respectively, the strong reachability and strong observability matrices of the TPBVDS as discussed in [1] (see also Section 2.4). Thus (2.2.13) states that V_i and V_f must commute with E and A except for parts that are either in the left nullspace of R_s or the right nullspace of O_s. For example, if R_s and O_s are of full rank - i.e. if the TPBVDS is strongly reachable and strongly observable - V_i and V_f must commute with E and A. Turning to the corollary, we see that this is precisely

the condition for a TPBVDS to be displacement. Thus as expected from (2.2.11), a displacement system is always stationary. Furthermore, the only way in which a TPBVDS can be stationary without being a displacement system is if the system is not strongly reachable or strongly observable.

The results of causal system theory might then suggest that this distinction between displacement and stationary is a trivial artifact caused by the use of possible non-minimal realizations. However, as in the case of continuous-time boundary-value systems [7], we will see that the story is different for TPBVDS. Specifically, as will be shown in Section 2.5, a TPBVDS can be minimal without being strongly reachable or strongly observable.

Proof of the Corollary

Assume that Theorem 2.1 holds. Then, from (2.2.11) we see that the concepts of stationarity and displacement are the same if $C=B=I$. Thus from Theorem 2.1, a TPBVDS is displacement if and only if (2.2.13) holds with R_s and O_s defined with $C=B=I$. However, thanks to the generalized Cayley-Hamilton theorem for pencils in standard form [1], the matrices $\{A^k E^{n-k-1} \mid k=0,\ldots n-1\}$ span the same set as $\{E^k A^j \mid k,j \geq 0\}$. Thus R_s and O_s are of full rank, so that (2.2.13) is equivalent to (2.2.17).

Proof of Theorem 2.2.1

What we must show is that (2.2.13) is equivalent to

$$W(k+1,j+1) = W(k,j) \qquad (2.2.18)$$

for $0 \leq k \leq N-1$, $0 \leq j \leq N-2$. Then, using (2.2.7), the commutativity of E and A, and

performing some algebra we find that (2.2.18) is equivalent to

$$CA^{k+1}E^{N-k-1}[V_iA+\omega V_fE]A^{N-j-2}E^{j+1}\Gamma^{-1}B =$$

$$CA^kE^{N-k}[V_iA+\omega V_fE]A^{N-j-1}E^j\Gamma^{-1}B. \qquad (2.2.19)$$

Now thanks to the generalized Cayley-Hamilton theorem and to (2.2.4), we have that the set of matrices

$$\{A^kE^{M-k} \mid k=0,1,\ldots,M\}$$

spans the same space as

$$\{A^kE^{n-k-1} \mid k=0,1,\ldots,n-1\}$$

as long as $M \geq n-1$. Thus, since $N \geq 2n$, (2.2.19) yields

$$O_sA[V_iA+\omega V_fE]E\Gamma^{-1}R_s = O_sE[V_iA+\omega V_fE]A\Gamma^{-1}R_s. \qquad (2.2.20)$$

The range of the matrix $\Gamma^{-1}R_s$ is independent of ω. To see this, define the strong reachability subspace

$$\mathcal{R}_s = Im(R_s). \qquad (2.2.21)$$

Then the generalized Cayley-Hamilton theorem implies that for all $M \geq 0$

$$A^M\mathcal{R}_s \subset \mathcal{R}_s \quad , \quad E^M\mathcal{R}_s \subset \mathcal{R}_s \qquad (2.2.22)$$

which implies

$$(\omega E^{N+1} - A^{N+1})\mathcal{R}_s = \Gamma\mathcal{R}_s \subset \mathcal{R}_s. \qquad (2.2.23)$$

Thus $\mathcal{R}_s$ is Γ-invariant which implies that $\mathcal{R}_s$ is Γ^{-1}-invariant. In fact, for all ω such that Γ^{-1} exists

$$\Gamma^{-1}\mathcal{R}_s = \Gamma\mathcal{R}_s = \mathcal{R}_s. \qquad (2.2.24)$$

Since the range of $\Gamma^{-1}R_s$ does not depend on ω and (2.2.20) must hold for almost all values of ω (i.e. all values for which Γ is invertible), we can deduce that (2.2.20) is equivalent to the following pair of equalities

$$O_s[AV_iE-EV_iA]A\Gamma^{-1}R_s = 0 \qquad (2.2.25)$$

$$O_s[AV_fE-EV_fA]E\Gamma^{-1}R_s = 0. \qquad (2.2.26)$$

Now, note that (2.2.25) is equivalent to the pair of equalities

$$O_s[AV_iE-EV_iA]A\Gamma^{-1}A^NR_s = 0 \tag{2.2.27}$$

$$O_s[AV_iE-EV_iA]A\Gamma^{-1}E^NR_s = 0. \tag{2.2.28}$$

To see this observe that since $\{E^N, A^N\}$ is regular

$$\mathfrak{R}_s = \text{Im}([A^NR_s \vdots E^NR_s]). \tag{2.2.29}$$

In a similar fashion we have that (2.2.26) is equivalent to the pair of equalities

$$O_s[AV_fE-EV_fA]E\Gamma^{-1}A^NR_s = 0 \tag{2.2.30}$$

$$O_s[AV_fE-EV_fA]E\Gamma^{-1}E^NR_s = 0. \tag{2.2.31}$$

Using the commutativity of E and A together with (2.2.5), we see that (2.2.30) is equivalent to

$$O_s[-AV_iE+EV_iA]E^{N+1}\Gamma^{-1}R_s = 0. \tag{2.2.32}$$

Using the definition of Γ, we see that (2.2.27) and (2.2.32) imply that

$$O_s[AV_iE-EV_iA]R_s = 0. \tag{2.2.33}$$

In a similar fashion (2.2.28) and (2.2.31) can be shown to imply

$$O_s[AV_fE-EV_fA]R_s = 0. \tag{2.2.34}$$

Note also that

$$E\Gamma^{-1}\mathfrak{R}_s \subset \mathfrak{R}_s \quad , \quad A\Gamma^{-1}\mathfrak{R}_s \subset \mathfrak{R}_s \tag{2.2.35}$$

so that (2.2.33), (2.2.34) imply and thus are equivalent to (2.2.25), (2.2.26).

Finally, note that thanks to the commutativity of E and A, (2.2.13a) implies (2.2.33) and (2.2.13b) implies (2.2.34). To see that the reverse of these implications holds, suppose that $\alpha \neq 0$ in (2.2.4) (if $\alpha = 0$, reverse the role of E and A in what follows). Then

$$E = \gamma I + \delta A \quad , \quad \gamma \neq 0. \tag{2.2.36}$$

Substituting this into (2.2.33) yields

$$O_s[V_i,A]R_s = 0, \qquad (2.2.37)$$

and (2.2.36) implies

$$O_s[V_i,E]R_s = 0. \qquad (2.2.38)$$

Similarly (2.2.34) implies (2.2.13b), and the theorem is proved.

As we shall see, the characterization of the displacement property in (2.2.13) simplifies many of the computations associated with TPBVDS's. In particular, it is not difficult to check that the Green's function of a displacement system is given by

$$G(k) = \begin{bmatrix} V_i A^{k-1}E^{N-k} & k>0 \\ -V_f E^{-k}A^{N+k-1} & k\leq0 \end{bmatrix} \qquad (2.2.39)$$

Similarly, the weighting pattern of a stationary TPBVDS is given by

$$W(k) = \begin{bmatrix} CV_i A^{k-1}E^{N-k} B & k>0 \\ -CV_f E^{-k}A^{N+k-1}B & k\leq0 \end{bmatrix} . \qquad (2.2.40)$$

Before closing this section we consider another problem, namely that of the degree of freedom in the choice of boundary matrices V_i and V_f.

Theorem 2.2.2

Consider two TPBVDS's with the same matrices C, E, A, and B and identical weighting patterns. Then, if one has boundary matrices V_i and V_f, and the other $\hat{V}_i$ and $\hat{V}_f$, we must have

$$O_s V_i R_s = O_s \hat{V}_i R_s \qquad (2.2.41a)$$

$$O_s V_f R_s = O_s \hat{V}_f R_s. \qquad (2.2.41b)$$

Conversly if (2.2.41) holds for two TPBVDS's with identical C, E, A, and B

system matrices, then their weighting patterns must be identical.

Proof

By setting

$$W(k,j) = \hat{W}(k,j) \tag{2.2.42}$$

we get that

$$0_s V_i A \Gamma^{-1} R_s = 0_s \hat{V}_i A \Gamma^{-1} R_s \tag{2.2.43a}$$

$$0_s V_f E \Gamma^{-1} R_s = 0_s \hat{V}_f E \Gamma^{-1} R_s \tag{2.2.43b}$$

which in turn implies that

$$0_s V_i A^{N+1} \Gamma^{-1} R_s = 0_s \hat{V}_i A^{N+1} \Gamma^{-1} R_s \tag{2.2.44a}$$

$$0_s V_f A^N E \Gamma^{-1} R_s = 0_s \hat{V}_f A^N E \Gamma^{-1} R_s. \tag{2.2.44b}$$

Now using the fact that both systems are in normalized form, we can rewrite

(2.2.44b) as follows:

$$0_s V_i E^{N+1} \Gamma^{-1} R_s = 0_s \hat{V}_i E^{N+1} \Gamma^{-1} R_s \tag{2.2.45}$$

which in conjunction with (2.2.44a) implies (2.2.41a). Equation (2.2.41b) can

be verified similarly.

The converse is easy to see to be true by noting that in the expression

for W, V_i and V_f appear only premultiplied by 0_s and postmultiplied by R_s.

2.3- Inward Processes, Outward Processes, and Extendibility

As discussed in [1] (with motivation from [7]) the process x in a TPBVDS can be recovered from two processes that each have interpretations as state processes. The outward process, which expands outward toward the boundaries, summarizes all that one needs to know about the input inside any interval in order to determine x outside the interval. The inward process uses input values near the boundary to propagate the boundary condition inward.

The outward process has a simple definition and characterization [1]. In Krener's context the outward process represents the "jump" corresponding to the difference between x at one end of any interval and the value predicted for x at that point given x at the other end of the interval and assuming zero input inside the interval. In our context, we can't predict in either direction (due to the possible singularity of E and A) and thus we use a somewhat different definition

$$z_0(k,j) = E^{j-k}x(j) - A^{j-k}x(k) \qquad k<j. \qquad (2.3.1)$$

This agrees with Krener's definition if E=I but in general in our case $z_0(k,j)$ can only be propagated in an outward direction. Also, it is possible to write a closed-form expression in terms of the intervening inputs:

$$z_0(k,j) = \sum_{r=k}^{j-1} E^{r-k}A^{j-r-1}Bu(r) \qquad (2.3.2)$$

and to write outward recursions

$$z_0(k-1,j) = Ez_0(k,j) + A^{j-k}Bu(k-1) \qquad (2.3.3a)$$

$$z_0(k,j+1) = Az_0(k,j) + E^{j-k}Bu(j) . \qquad (2.3.3b)$$

Note that $z_o(k,j)$ does not involve the boundary matrices V_i and V_f – i.e. it only involves (2.2.1) – and thus none of the expressions (2.3.1)-(2.3.3b) are affected if the TPBVDS has the displacement property.

The situation is considerably different, however, for the inward process. As developed in [1], for any $K \leq L$ the inward process $z_i(K,L)$ is a function of the boundary value v and the inputs $\{u(0), \ldots, u(K-1)\}$ and $\{u(L), \ldots, u(N-1)\}$ so that the TPBVDS

$$Ex(k+1) = Ax(k) + Bu(k) \qquad (2.3.4a)$$

$$V_i(K,L)x(K) + V_f(K,L)x(L) = z_i(K,L) \qquad (2.3.4b)$$

yields the same solution as (2.2.1), (2.2.2) for $K \leq k \leq L$. Here $V_i(K,L)$ and $V_f(K,L)$ are assumed to be such that (2.3.4) is in normalized form, i.e.

$$V_i(K,L)E^{L-K} + V_f(K,L)A^{L-K} = I. \qquad (2.3.5)$$

Theorem 2.3.1

The inwardly-propagated boundary matrices and the inward process can be expressed as follows:

$$V_i(K,L) = E^{N-L}(\omega E - A^K(\omega V_f E + V_i A)A^{N-K})\Gamma^{-1}E^K \qquad (2.3.6a)$$

$$V_f(K,L) = -A^K(A - E^{N-L}(\omega V_f E + V_i A)E^L)\Gamma^{-1}A^{N-L} \qquad (2.3.6b)$$

and

$$z_i(K,L) = E^{N-L}A^K v + E^{N-L}(\omega E - A^K(\omega V_f E + V_i A)A^{N-K})\Gamma^{-1}z_o(0,K)$$
$$+ A^K(A - E^{N-L}(\omega V_f E + V_i A)E^L)\Gamma^{-1}z_o(L,N). \qquad (2.3.7)$$

Note in particular the starting values

$$z_i(0,N)=v, \quad V_i(0,N)=V_i, \quad V_f(0,N)=V_f \tag{2.3.8}$$

and the "final values"

$$z_i(k,k) = x(k) \qquad \text{for all } k. \tag{2.3.9}$$

<u>Proof</u>

Let S_h be the following $h \times (h+1)$ block matrix,

$$S_h = \begin{bmatrix} -A & E & 0 & .. & 0 \\ 0 & -A & E & .. & 0 \\ : & & & & : \\ 0 & 0 & & -A & E \end{bmatrix} \tag{2.3.10}$$

then (2.2.1)-(2.2.2) can be expressed as

$$\begin{bmatrix} & S_N & \\ V_i & 0 \ldots 0 & V_f \end{bmatrix} \begin{bmatrix} x(0) \\ : \\ x(N) \end{bmatrix} = \begin{bmatrix} Bu(0) \\ : \\ Bu(N-1) \\ v \end{bmatrix}. \tag{2.3.11}$$

Also it is easy to see that

$$\begin{bmatrix} -A^K & E^K & 0 & \ldots\ldots & 0 \\ 0 & & & & : \\ : & : & S_{L-K} & : & 0 \\ & & & & \\ 0 & & \ldots\ldots & 0 & -A^{N-L} & E^{N-L} \\ V_i & 0 & \ldots\ldots\ldots & 0 & V_f \end{bmatrix} \begin{bmatrix} x(0) \\ x(K) \\ x(K+1) \\ : \\ x(L) \\ x(N) \end{bmatrix} = \begin{bmatrix} z_0(0,K) \\ Bu(K) \\ : \\ Bu(L-1) \\ z_0(L,N) \\ v \end{bmatrix}. \tag{2.3.12}$$

Now to find $V_i(K,L)$ and $V_f(K,L)$, we need to first construct a full-rank matrix

$$[T_i(K,L) \quad T_f(K,L) \quad P(K,L)]$$

so that

$$[T_i(K,L) \quad T_f(K,L) \quad P(K,L)] \begin{bmatrix} -A^K & 0 \\ 0 & E^{N-L} \\ V_i & V_f \end{bmatrix} = 0. \tag{2.3.13}$$

If we now premultiply both sides of (2.3.12) by

$$\Omega(K,L) = \begin{bmatrix} 0 & I & 0 & 0 \\ T_i(K,L) & 0 & T_f(K,L) & P(K,L) \end{bmatrix} \tag{2.3.14}$$

we obtain

$$\begin{bmatrix} & S_{L-K} & \\ T_i(K,L)E^K & -T_f(K,L)A^{N-L} \end{bmatrix} \begin{bmatrix} x(K) \\ \vdots \\ x(L) \end{bmatrix} = \begin{bmatrix} & Bu(K) & \\ & \vdots & \\ & Bu(L-1) & \\ T_i(K,L)z_0(0,K)+T_f(K,L)z_0(L,N)+P(K,L)v \end{bmatrix}$$

(2.3.15)

then clearly,

$$V_i(K,L) = T_i(K,L)E^K \qquad (2.3.16a)$$

$$V_f(K,L) = -T_f(K,L)A^{N-L} \qquad (2.3.16b)$$

and

$$z_i(K,L) = T_i(K,L)z_0(0,K)+T_f(K,L)z_0(L,N)+P(K,L)v. \qquad (2.3.17)$$

It is straightforward to check that

$$T_i(K,L) = E^{N-L}(\omega E-A^K(\omega V_f E+V_i A)A^{N-K})\Gamma^{-1} \qquad (2.3.18a)$$

$$T_f(K,L) = A^K(A-E^{N-L}(\omega V_f E+V_i A)E^L)\Gamma^{-1} \qquad (2.3.18b)$$

and

$$P(K,L) = E^{N-L}A^K \qquad (2.3.18c)$$

satisfy our requirements. Clearly then (2.3.6) and (2.3.7) can be obtained from (2.3.18).

Theorem 2.3.1 can be slightly generalized to give us a relationship between all inwardly-propagated boundary matrices:

$$V_i(K,L) = E^{J-L}(\omega E-A^{K-I}(\omega V_f(I,J)E+V_i(I,J)A)A^{J-K})\Gamma_{J-I}^{-1}E^{K-I} \qquad (2.3.19a)$$

$$V_f(K,L) = -A^{K-I}(A-E^{J-L}(\omega V_f(I,J)E+V_i(I,J)A)E^{L-I})\Gamma_{J-I}^{-1}A^{J-L} \qquad (2.3.19b)$$

when [K,L] is contained in [I,J] and where

$$\Gamma_M = \omega E^{M+1} - A^{M+1}.$$

Corollary

Suppose that (2.2.1)-(2.2.2) is a displacement system. Then,

$$V_i(K,L) = V_i E^{N-L+K} \qquad (2.3.20a)$$

$$V_f(K,L) = V_f A^{N-L+K} \qquad (2.3.20b)$$

and

$$z_i(K,L) = E^{N-L} A^K v + V_i E^{N-L} z_0(0,K) - V_f A^K z_0(L,N). \qquad (2.3.21)$$

Proof

Equations (2.3.20) and (2.3.21) are easily derived from (2.3.6) and (2.3.7) using the fact that E and A must commute with V_i and V_f.

An important interpretation of the inward process, or more specifically the inwardly-propagated boundary matrices (2.3.6) is that the Green's function for the system (2.3.4) on the smaller interval [K,L] is the restriction of the Green's function of the original system (2.2.1)-(2.2.2) defined on [0,N]. A logical question then is whether we can also move the boundary conditions outward so that the Green's function for the resulting system, when restricted to [0,N] yields the original Green's function. This is roughly the property of extendibility. In particular, it makes a good deal of sense to consider extendibility when one considers shift-invariance, as the intuitive notion of shift-invariance includes the idea that there is no real time origin, while the TPBVDS (2.2.1)-(2.2.2) is defined on an interval [0,N] of fixed length.

We now make the following precise definitions:

Definition 2.3.1

The TPBVDS (2.2.1)-(2.2.3) is <u>left</u> (<u>right</u>) <u>input-output extendible</u> if given any interval $[K,N]$ $([0,L])$ containing $[0,N]$, there exists a TPBVDS over this larger interval with the same dynamics as in (2.2.1) but with new boundary matrices $V_i(K,N)$, $V_f(K,N)$ $(V_i(0,L)$, $V_f(0,L))$ such that the weighting pattern $W(k,j)$ of the original system is the restriction of the weighting pattern $W_e(k,j)$ of the new extended system, i.e.

$$W(k,j) = W_e(k,j), \qquad 0 \leq k \leq N, \ 0 \leq j \leq N-1. \qquad (2.3.22)$$

The TPBVDS (2.2.1)-(2.2.3) is <u>input-output extendible</u> if it is both left and right input-output extendible.

Definition 2.3.2

The TPBVDS (2.2.1)-(2.2.2) is <u>left</u> (<u>right</u>) <u>extendible</u> if given any interval $[K,N]$ $([0,L])$ containing $[0,N]$, there exists a TPBVDS over this larger interval with the same dynamics as in (2.2.1) but with new boundary matrices $V_i(K,N)$, $V_f(K,N)$ $(V_i(0,L)$, $V_f(0,L))$ such that the Green's function $G(k,j)$ of the original system is the restriction of the Green's function $G_e(k,j)$ of the new extended system, i.e.

$$G(k,j) = G_e(k,j), \qquad 0 \leq k \leq N, \ 0 \leq j \leq N-1. \qquad (2.3.23)$$

The TPBVDS (2.2.1)-(2.2.2) is <u>extendible</u> if it is both left and right extendible.

In order to characterize the conditions under which each of these types of extendibility hold, let us first define two matrices that will appear on several occasions. Specifically, with any matrix F we associate the _Drazin Inverse_ F^D and its _invertible modification_ $\tilde{F}$. To define these, let T be an invertible matrix such that

$$F = T \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix} T^{-1} \qquad (2.3.24a)$$

where M is invertible and N is nilpotent (e.g. the real Jordan form has this structure). Then

$$F^D = T \begin{bmatrix} M^{-1} & 0 \\ 0 & 0 \end{bmatrix} T^{-1}. \qquad (2.3.24b)$$

$$\tilde{F} = T \begin{bmatrix} M & 0 \\ 0 & N+I \end{bmatrix} T^{-1}. \qquad (2.3.24c)$$

These matrices have a number of important properties:

(i) F^D and $\tilde{F}$ commute with each other and with F.

(ii) If F is invertible, $F^D = F^{-1}$ and $\tilde{F} = F$.

(iii)
$$F^D F = F^D \tilde{F} \qquad (2.3.25)$$

and if μ is the degree of nilpotency of N, i.e. $N^{\mu-1} \neq 0$, $N^\mu = 0$, then for $k \geq \mu$

$$F^{k+1} F^D = F^k , \quad F^k \tilde{F} = F^{k+1}. \qquad (2.3.26)$$

(iv) Let G be any matrix, then the conditon

$$\text{Ker}(F^\mu) \subset \text{Ker}(G) \qquad (2.3.27)$$

is equivalent to

$$G F^D F = G. \qquad (2.3.28)$$

(v) If $\mathcal{X}$ is an F-invariant subspace, then $F^D F \mathcal{X} \subset \mathcal{X}$ and is also F-invariant.

(vi) Let {E,A} be a regular pencil in standard-form, then

$$EE^D + AA^D - AA^D EE^D = I. \qquad (2.3.29)$$

Properties (i)-(v) can be easily checked. To see why property (vi) is true we need to first pre and post multiply (2.3.29) by T and T^{-1} chosen such that

$$TET^{-1} = \begin{bmatrix} E_1 & & \\ & E_2 & \\ & & N_e \end{bmatrix}, \quad TAT^{-1} = \begin{bmatrix} N_a & & \\ & A_2 & \\ & & A_3 \end{bmatrix} \qquad (2.3.30)$$

where E_1, E_2, A_2 and A_3 are invertible (see Section 2.6), in which case

$$TE^DT^{-1} = \begin{bmatrix} E_1^{-1} & & \\ & E_2^{-1} & \\ & & 0 \end{bmatrix}, \quad TA^DT^{-1} = \begin{bmatrix} 0 & & \\ & A_2^{-1} & \\ & & A_3^{-1} \end{bmatrix}. \qquad (2.3.31)$$

Then clearly

$$TAA^DT^{-1} = \begin{bmatrix} 0 & & \\ & I & \\ & & I \end{bmatrix}, \quad TEE^DT^{-1} = \begin{bmatrix} I & & \\ & I & \\ & & 0 \end{bmatrix} \qquad (2.3.32)$$

and

$$TAA^DEE^DT^{-1} = \begin{bmatrix} 0 & & \\ & I & \\ & & 0 \end{bmatrix} \qquad (2.3.33)$$

which imply the desired result.

Note that without loss of generality, it can always be assumed that the E and A matrices of a TPBVDS in normalized form are in the block form (2.3.30). This can always be achieved by a coordinate transformation. The corresponding boundary matrices, in this coordinate system, must have the following form

$$V_i = \begin{bmatrix} E_1^{-N} & v_{12}^i & v_{13}^i \\ 0 & v_{22}^i & v_{23}^i \\ 0 & v_{32}^i & v_{33}^i \end{bmatrix}, \quad V_f = \begin{bmatrix} v_{11}^f & v_{12}^f & 0 \\ v_{21}^f & v_{22}^f & 0 \\ v_{31}^f & v_{32}^f & A_3^{-N} \end{bmatrix}. \qquad (2.3.34)$$

This is because the TPBVDS is supposed to be in normalized form which means that V_i and V_f must satisfy (2.2.5) and E^N and A^N have the following block

structures respectively,

$$\begin{bmatrix} E_1^N & & \\ & E_2^N & \\ & & 0 \end{bmatrix}, \begin{bmatrix} 0 & & \\ & A_2^N & \\ & & A_3^N \end{bmatrix}.$$

Theorem 2.3.2

A TPBVDS is left extendible if and only if

$$V_i - V_i E^D E = 0 \qquad\qquad (2.3.35a)$$

$$V_f - A^D A V_f = 0. \qquad\qquad (2.3.35b)$$

It is right extendible if and only if

$$V_i - E^D E V_i = 0 \qquad\qquad (2.3.36a)$$

$$V_f - V_f A^D A = 0. \qquad\qquad (2.3.36b)$$

It is extendible if and only if

$$V_i - E^D E V_i E^D E = 0 \qquad\qquad (2.3.37a)$$

$$V_f - A^D A V_f A^D A = 0. \qquad\qquad (2.3.37b)$$

Corollary

For a displacement TPBVDS the following statements are equivalent

(i) The TPBVDS is right extendible.

(ii) It is left extendible.

(iii) It is extendible.

(iv) The following equations hold

$$V_i - V_i E^D E = 0 \qquad\qquad (2.3.38a)$$

$$V_f - V_f A^D A = 0. \qquad\qquad (2.3.38b)$$

The corollary follows from the theorem because in the displacement case E, E^D, A and A^D commute with V_i and V_f.

Proof of Theorem 2.3.2

First we show necessity. Let the TPBVDS be left extendible then it must be obtained by moving in the left boundary of another TPBVDS. Then from (2.3.6) it can be seen that

$$Ker(V_i) \subset Ker(E^k) \tag{2.3.39a}$$

$$Ker(V_f^{\cdot}) \subset Ker[(A^k)^{\cdot}] \tag{2.3.39b}$$

where k is the number of steps that the boundary has moved. If k is larger than the maximum of the nilpotency degrees of E and A, then equations (2.3.39) and (2.3.35) are of course equivalent. If the system is right extendible then (2.3.36) can be shown to be true similarly. And of course (2.3.35) and (2.3.36) imply (2.3.37).

To show the sufficiency of (2.3.35) we need to construct matrices $V_i(K,N)$ and $V_f(K,N)$, for each K<0 so that when we move in these boundary matrices to [0,N] we recover V_i and V_f. Assume then that (2.3.35) holds and let

$$V_i(K,N) = [I-(A^D)^{-K}V_f A^{N-K}](E^D)^{N-K} \tag{2.3.40a}$$

$$V_f(K,N) = (A^D)^{-K}V_f. \tag{2.3.40b}$$

First we need to make sure that the extended system is in normalized form, i.e.

$$V_i(K,N)E^{N-K}+V_f(K,N)A^{N-K} = I. \tag{2.3.41}$$

From (2.3.40) and using the fact that V_i and V_f are in normalized form, we get that

$$V_i(K,N)E^{N-K}+V_f(K,N)A^{N-K} = (I-AA^D)EE^D+AA^D \qquad (2.3.42)$$

which is equal to the identity matrix (see property (vi)).

Now we have to make sure that by moving in $V_i(K,N)$ and $V_f(K,N)$ to $V_i(0,N)$ and $V_f(0,N)$ we recover V_i and V_f. This can be verified by substituting the matrices in (2.3.40) into (2.3.19) with K=0, L=J=N and I=K.

The necessity (2.3.36) for right extendibility can be proven similarly. Specifically we construct right extended matrices as follows

$$V_i(0,L) = (E^D)^{L-N}V_i \qquad (2.3.43a)$$

$$V_f(0,L) = [I-(E^D)^{L-N}V_iE^L](A^D)^L. \qquad (2.3.43b)$$

To see the necessity of (2.3.37) for extendibility simply note that (2.3.37) clearly implies (2.3.35) and (2.3.36).

Theorem 2.3.3

Let a TPBVDS be left (right) input-output extendible. Then we can find an equivalent TPBVDS using the freedom in its boundary matrices such that this new TPBVDS is left (right) extendible.

Conversely, every left (right) extendible TPBVDS is left (right) input-output extendible.

Proof

Let a TPBVDS defined over [0,N] be left input-output extendible then there exist a TPBVDS defined over [-n,N] such that when we move in its boundaries to [0,N] we get a TPBVDS with weighting pattern identical to the

weighting pattern of our original TPBVDS, possibly with different boundary matrices. This new representation of our TPBVDS is clearly left extendible because it has been obtained by moving in the left boundary of another system n steps. A similar argument can be used for the case of right extendibility.

The converse of the theorem is trivial.

Theorem 2.3.4

A TPBVDS is left input-output extendible if and only if

$$O_s(V_i - V_iE^DE)R_s = 0 \tag{2.3.44a}$$

$$O_s(V_f - A^DAV_f)R_s = 0. \tag{2.3.44b}$$

It is right input-output extendible if and only if

$$O_s(V_i - E^DEV_i)R_s = 0 \tag{2.3.45a}$$

$$O_s(V_f - V_fA^DA)R_s = 0. \tag{2.3.45b}$$

It is input-output extendible if and only if

$$O_s(V_i - E^DEV_iE^DE)R_s = 0 \tag{2.3.46a}$$

$$O_s(V_f - A^DAV_fA^DA)R_s = 0. \tag{2.3.46b}$$

Corollary

For a stationary TPBVDS the following statements are equivalent

(i) The TPBVDS is right input-output extendible.

(ii) It is left input-output extendible.

(iii) It is input-output extendible.

(iv) The following equations hold

$$O_s(V_i - V_iE^DE)R_s = 0 \tag{2.3.47a}$$

$$O_s(V_f - V_fA^DA)R_s = 0 \tag{2.3.47b}$$

Proof of Theorem 2.3.4

Suppose that the TPBVDS is left input-output extendible, then from
Theorem 2.3.3 there exists a TPBVDS with the same weighting pattern which is
left extendible, i.e. there exist matrices V_i^* and V_f^* such that

$$O_s V_i^* R_s = O_s V_i R_s \qquad (2.3.48a)$$

$$O_s V_f^* R_s = O_s V_f R_s \qquad (2.3.48b)$$

and such that they satisfy (2.3.35).

Notice that (2.3.48) implies that

$$O_s V_i^* E^D E R_s = O_s V_i E^D E R_s \qquad (2.3.49a)$$

$$O_s A^D A V_f^* R_s = O_s A^D A V_f R_s \qquad (2.3.49b)$$

because of the invariance properties of the strong reachability and
observability matrices (see Section 2.4). Premultiplying and postmultiplying
(2.3.35) (with V replaced by V^*) by O_s and R_s respectively and using (2.3.48)
and (2.3.49) we obtain (2.3.44).

Now suppose that (2.3.44) holds. Let

$$V_i^* = (I-AA^D)(E^D)^N + AA^D V_i E^D E \qquad (2.3.50a)$$

$$V_f^* = AA^D V_f \qquad (2.3.50b)$$

then we have to show that the new system obtained by replacing V_i and V_f with
these boundary matrices is just another representation of the original
system. First we need to make sure that this new system is in normalized
form:

$$V_i^* E^N + V_f^* A^N = [(I-AA^D)(E^D)^N + AA^D V_i E^D E]E^N + AA^D V_f A^N \qquad (2.3.51a)$$

$$= (I-AA^D)(EE^D)^N + AA^D = I. \qquad (2.3.51b)$$

Equation (2.3.51b) can be checked easily by using the block form (2.3.30). What remains to be shown is that (2.3.48) holds for these matrices. Clearly (2.3.48b) holds from (2.3.50) and (2.3.44b). Showing (2.3.48a) is more complicated and again we suppose that the system is in the block form (2.3.30), (2.3.34). The matrix V_i^* in (2.3.50a), is given by

$$V_i^* = \begin{bmatrix} I & & \\ & 0 & \\ & & 0 \end{bmatrix} \begin{bmatrix} E_1^{-N} & & \\ & E_2^{-N} & \\ & & 0 \end{bmatrix} + \begin{bmatrix} 0 & & \\ & I & \\ & & I \end{bmatrix} \begin{bmatrix} E_1^{-N} & V_{12}^i & V_{13}^i \\ 0 & V_{22}^i & V_{23}^i \\ 0 & V_{32}^i & V_{33}^i \end{bmatrix} \begin{bmatrix} I & & \\ & I & \\ & & 0 \end{bmatrix} =$$

$$\begin{bmatrix} E_1^{-N} & 0 & 0 \\ 0 & V_{22}^i & 0 \\ 0 & V_{32}^i & 0 \end{bmatrix} \tag{2.3.52}$$

where V_{22}^i and V_{32}^i are (2,2) and (3,2) blocks of V_i.

The strong reachability and observability matrices have a block structure as well , i.e.

$$O_s = W \cdot \begin{bmatrix} O_s^1 & & \\ & O_s^2 & \\ & & O_s^3 \end{bmatrix}, \quad R_s = \begin{bmatrix} R_s^1 & & \\ & R_s^2 & \\ & & R_s^3 \end{bmatrix} \cdot Z \tag{2.3.53}$$

for some invertible matrices Z and W (this is due to the fact that the three blocks of the system have distinct eigenvalues, see Section 2.6). Also observe that

$$O_s(V_i E^N + V_f A^N) R_s = O_s V_i E^N R_s + O_s A A^D V_f A^N R_s = O_s R_s. \tag{2.3.54}$$

By pre and post multiplying (2.3.54) by W^{-1} and Z^{-1}, respectively, and inspecting the (1,2) block we get that

$$O_s^1 V_{12}^i E_2^N R_s^2 = 0 \tag{2.3.55a}$$

which, since R_s^2 is E_2-invariant (again see Section 2.6), and E_2 is invertible, implies that

$$O_s^1 V_{12}^i R_s^2 = 0. \tag{2.3.55b}$$

Also note that (2.3.44a) implies that

$$O_s^k V_{k3}^i R_s^3 = 0, \quad k=1,2,3. \tag{2.3.56}$$

Now by noting the expression for V_i^* in (2.3.52) and equations (2.3.55b) and

(2.3.56) it becomes clear that we must have that

$$O_s V_i R_s = O_s V_i^* R_s. \tag{2.3.57}$$

which is the desired result. The other cases can be argued similarly.

The corollary is immediate by noting that E and A and their Drazin

inverses commute with V_i and V_f when premultiplied by O_s and postmultiplied

by R_s.

We shall emphasize the fact that the notions of left and right

extendibility are indeed distinct notions and we could very well have a

system that is right extendible but not left extendible and vice versa. The

following example demonstrates this fact:

Example 2.3.1

Consider the following TPBVDS

$$x(k+1) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x(k) + u(k) \tag{2.3.58a}$$

$$x(0) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(N) = v. \tag{2.3.58b}$$

This TPBVDS is well-posed and in normalized form. It is easy to check that

for this system (2.3.36b) is violated but (2.3.35a) and (2.3.35b) hold. Thus

this sytem is left extendible but not right extendible.

The input-output extendibility feature is a property of the weighting pattern of the system and not of any specific representation so that it is possible to refer to this property as extendibility of the weighting pattern. The following theorem justifies this:

Theorem 2.3.5

Let 2 TPBVDS's (of possibly different dimension) defined over [0,N] have identical weighting patterns. Then if one is input-output extendible so is the other.

We shall prove this result in Section 2.5.

The extendibility property is a very important property because it allows us to associate to each system a sequence of systems defined over any desired interval. We present a way of constructing this sequence. But first we give the following characterization of extendible systems:

Theorem 2.3.6

Let a TPBVDS be extendible and in block form (2.3.30), (2.3.34). Then the boundary matrices must have the following structure

$$V_i = \begin{bmatrix} E_1^{-N} & 0 & 0 \\ 0 & V_{22}^i & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad V_f = \begin{bmatrix} 0 & 0 & 0 \\ 0 & V_{22}^f & 0 \\ 0 & 0 & A_3^{-N} \end{bmatrix} \tag{2.3.59}$$

which means that the TPBVDS is separated into a purely causal part and a
purely anticausal part, each having nilpotent dynamics, and a non-descriptor
acausal part.

Note that if a system is input-output extendible then it has a
representation of the form (2.3.59).

<u>Proof</u>

The structure in equation (2.3.59) can be easily derived from the
extendibility condition and the fact that the system is in normalized form.

Theorem 2.3.6 allows us to simplify the expression for the Green's
function solution of an extendible system. By replacing the V_i and V_f in the
general Green's function solution by V_i and V_f in (2.3.59), we obtain the
following expression for the Green's function of an extendible system:

$$G(k,j) = \begin{cases} -\tilde{A}^k\tilde{E}^{N-k}V_f EE^D\tilde{E}^{j-N}\tilde{A}^{N-j-1} - (I-EE^D)E^{j-k}(A^D)^{j-k+1} & j \geq k \\ \tilde{A}^k\tilde{E}^{N-k}V_i AA^D\tilde{A}^{-j-1}\tilde{E}^j + (I-AA^D)(E^D)^{k-j}A^{k-j-1} & j < k \end{cases}$$

$$= \begin{cases} -\tilde{A}^k\tilde{E}^{-k}[I-(E^N V_i)]EE^D\tilde{A}^{-j-1}\tilde{E}^j - (I-EE^D)E^{j-k}(A^D)^{j-k+1} & j \geq k \\ \tilde{A}^k\tilde{E}^{-k}(E^N V_i)AA^D\tilde{A}^{-j-1}\tilde{E}^j + (I-AA^D)(E^D)^{k-j}A^{k-j-1} & j < k \end{cases}$$

(2.3.60)

And of course the weighting pattern of an input-output extendible system can
be expressed as:

$$W(k,j) = \begin{cases} -C\{\tilde{A}^k\tilde{E}^{-k}[I-(E^N V_i)]EE^D\tilde{A}^{-j-1}\tilde{E}^j - (I-EE^D)E^{j-k}(A^D)^{j-k+1}\}B & j \geq k \\ C\{\tilde{A}^k\tilde{E}^{-k}(E^N V_i)AA^D\tilde{A}^{-j-1}\tilde{E}^j + (I-AA^D)(E^D)^{k-j}A^{k-j-1}\}B & j < k \end{cases}$$

(2.3.61a)

Note that (2.3.60) expresses the Green's function of an extendible TPBVDS and all of its extension. Similarly, (2.3.61a) expresses the weighting pattern of an input–output extendible TPBVDS and all of its extensions. This observation deserves further comment. Specifically, what we have done is the following. We begin with a specific extendible TPBVDS defined on [0,N], with boundary matrices V_i, V_f so that the system is in standard form over this specific interval. Equations (2.3.60) and (2.3.61a) then provide us with the Green's function and weighting pattern for <u>all</u> extensions of the TPBVDS. Thus we use the parameters associated with any one of the family of extensions to obtain G and W for the whole family. These expressions must of course, not depend on the particular member of the family used in the computation. In particular (2.3.60) and (2.3.61a) do not depend on N. Rather $E^N V_i$ is, in a sense, an invariant of the entire family (remember that V_i also depends on N, as it is chosen so that the system is in standard form over [0,N]). In the simpler stationary case this point can be made much more explicit.

Note also that if we are in the basis (2.3.30), then by letting C and B equal $[C_1 \quad C_2 \quad C_3]$, $\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}$ respectively, $W(k,j)$ can be expressed as

$$W(k,j) = \begin{bmatrix} -C_2 A_2^k E_2^{N-k} V_{22}^f E_2^{j-N} A_2^{N-j-1} B_2 - C_3 A_3^{k-j-1} N_e^{j-k} B_3 & j \geq k \\ C_2 A_2^k E_2^{N-k} V_{22}^i E_2^j A_2^{-j-1} B_2 + C_1 E_1^{j-k} N_a^{k-j-1} B_1 & j < k \end{bmatrix}. \quad (2.3.61b)$$

We can construct the sequence of (inward and outward) extensions (in standard form) of our extendible or input–ouput extendible TPBVDS as follows

$$V_i(I,J) = \tilde{E}^{-J} \tilde{A}^I (E^N V_i) A A^D \tilde{E}^I \tilde{A}^I + (I - A A^D) \tilde{E}^{I-J},$$

$$V_f(I,J) = \tilde{E}^{-J} \tilde{A}^I [I - (E^N V_i)] E E^D \tilde{E}^J \tilde{A}^{-J} + (I - E E^D) \tilde{A}^{I-J}. \quad (2.3.62a)$$

In the basis (2.3.30), (2.3.62a) becomes

$$
V_i(I,J) = \begin{bmatrix} E_1^{-(J-I)} & 0 & 0 \\ 0 & E_2^{N-J}A_2^I V_{22}^i E_2^I A_2^{-I} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad V_f(I,J) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_2^I E_2^{N-J} V_{22}^f E_2^{J-N} A_2^{N-J} & 0 \\ 0 & 0 & A_3^{-(J-I)} \end{bmatrix}.
$$

$$(2.3.62b)$$

In the stationary case the situation is even simpler. The weighting pattern of an input–output extendible stationary TPBVDS can be expressed as follows

$$
W(k) = \begin{bmatrix} C(E^N V_i)E^D(AE^D)^{k-1}B & k>0 \\ -C(A^N V_f)A^D(EA^D)^{-k}B & k\leq 0 \end{bmatrix}
$$
$$
= \begin{bmatrix} C(E^N V_i)E^D(AE^D)^{k-1}B & k>0 \\ C[I-(E^N V_i)]A^D(EA^D)^{-k}B & k\leq 0 \end{bmatrix}.
\quad (2.3.63)
$$

Notice that the Green's function G(k,j) of an extendible system and the weighting pattern W(k,j) --W(k) in the stationary case-- of an input–output extendible system are completely determined in terms of matrices C, E, A, B and $P_0 = E^N V_i$. Matrix P_0 contains the contribution of the boundary conditions to the Green's function and to the weighting pattern of the system. When P_0 is used to describe the contribution of the boundary conditions of an extendible TPBVDS to the Green's function, P_0 is unique because in this case V_i is unique. However, when P_0 is used to describe the contribution of the boundary conditions of an input–output extendible TPBVDS to the weighting pattern, P_0 is in general not unique because V_i in this case is not unique (see Theorem 2.2.2). We shall study in detail this degree of freedom in the choice of P_0 in the stationary case because it is needed for the realization

theory of Chapter III. A similar study can be done for non-stationary systems, although the calculations are significantly more complicated and unenlightening.

Let us now consider an input-output extendible system. And let us define the projection matrix P as follows.

Definition 2.3.3

Let P be a matrix such that

$$W(k) = \begin{bmatrix} CPE^D(AE^D)^{k-1}B & k > 0 \\ -C(I-P)A^D(EA^D)^{-k}B & k \leq 0 \end{bmatrix} \qquad (2.3.64)$$

where W(k) is the weighting pattern of the input-output extendible stationary TPBVDS (2.2.1)-(2.2.2) given by (2.3.63), and such that P satisfies

$$O_s(PA-AP)R_s = O_s(PE-EP)R_s = 0 \qquad (2.3.65a)$$

$$O_s(P-PEE^D)R_s = O_s((I-P)AA^D-(I-P))R_s = 0. \qquad (2.3.65b)$$

Then, P is called the projection matrix of (2.2.1)-(2.2.2).

Every input-output extendible stationary TPBVDS has a projection matrix P in particular

$$P = P_0 = E^N V_i. \qquad (2.3.66a)$$

The above choice for the projection matrix is not unique in general. It is not difficult to see that if P is a projection matrix then so is P+Q where Q is any matrix such that $O_s Q R_s$ equals zero. In fact, we show later that P is a projection matrix if and only if it satisfies

$$O_s P R_s = O_s(E^N V_i)R_s. \qquad (2.3.66b)$$

The weighting pattern of an input-output extendible stationary TPBVDS (C,V_i,V_f,E,A,B,N) and the weighting patterns of all of its extensions are completely described by (2.3.64). Thus in specifying this system, the boundary condition (i.e. the contribution of the boundary condition to the weighting pattern) is completely specified in terms of the projection matrix P, and since P is independent of N, this TPBVDS and all of its extensions are completely characterized by the 5-tuple (C,P,E,A,B). The 5-tuple (C,P,E,A,B) characterizes a family of stationary extendible systems (i.e. one for each interval length N) but since these systems all have identical weighting patterns (i.e. $W(k)$ as in (2.3.64) restricted to their domain of definitions), we shall simply refer to this family of systems as an input-output extendible stationary TPBVDS (C,P,E,A,B) having $W(k)$, as defined in (2.3.64), for weighting pattern.

The following results justify this representation of input-output extendible stationary TPBVDS's.

Theorem 2.3.7

Consider the 5-tuple (C,P,E,A,B) such that {E,A} is in standard form and such that (2.3.65) is verified. Then for any interval length N, there exist matrices V_i and V_f such that the TPBVDS (C,V_i,V_f,E,A,B,N) is normalized, input-output extendible and stationary and its weighting pattern (2.3.63) is equal to $W(k)$ in (2.3.64).

Proof

Let

$$V_i = P(E^D)^N + \sigma X(\sigma E^N + A^N)^{-1} \qquad (2.3.67)$$

$$V_f = (I-P)(A^D)^N + X(\sigma E^N + A^N)^{-1} \qquad (2.3.68)$$

where

$$X = I - PEE^D - (I-P)AA^D = (I-P)EE^D + PAA^D - EE^D AA^D \qquad (2.3.69)$$

and σ is any scalar such that

$$\sigma E^N + A^N$$

is invertible. Then with V_i and V_f defined as in (2.3.67)–(2.3.68), we have

that (2.3.63) and (2.3.64) are equal thanks to the following

$$O_s X R_s = 0. \qquad (2.3.70)$$

Equations (2.3.65) imply (2.2.13) and (2.3.47). Also by direct calculation we

can see that V_i and V_f are normalized. Thus the TPBVDS $(C, V_i, V_f, E, A, B, N)$

satisfies the conditions of the theorem.

Theorem 2.3.8

Let $(C, V_i, V_f, E, A, B, N)$ be an input–output extendible, stationary TPBVDS.

Then matrix P is a projection matrix of this TPBVDS if and only if

$$O_s V_i E^N R_s = O_s P R_s \qquad (2.3.71a)$$

$$O_s V_f A^N R_s = O_s(I-P)R_s. \qquad (2.3.71b)$$

Proof

Suppose that P satisfies (2.3.71). Since the system is supposed to be

stationary, (2.2.13) must hold. From (2.2.13), we get

$$O_s V_i E^{N+1} R_s = O_s E V_i E^N R_s. \qquad (2.3.72)$$

Also thanks to the E-invariance of $\text{Ker}(O_s)$, there exists a matrix Z_1 such that

$$O_s E = Z_1 O_s. \qquad (2.3.73)$$

By combining (2.3.72) and (2.3.73), we get

$$O_s V_i E^{N+1} R_s = O_s E V_i E^N R_s = Z_1 O_s V_i E^N R_s$$
$$= Z_1 O_s P R_s = O_s E P R_s. \qquad (2.3.74)$$

Similarly, Thanks to the E-invariance of $\text{Im}(R_s)$, there exists a matrix Z_2 such that

$$E R_s = R_s Z_2 \qquad (2.3.75)$$

and

$$O_s V_i E^{N+1} R_s = O_s V_i E^N R_s Z_2 = O_s P R_s Z_2 = O_s P E R_s. \qquad (2.3.76)$$

By combining (2.3.74) and (2.3.76), we get

$$O_s V_i E^{N+1} R_s = O_s E P R_s = O_s P E R_s. \qquad (2.3.77)$$

Similarly, by looking at $O_s V_f A^{N+1} R_s$, we find

$$O_s V_f A^{N+1} R_s = O_s A(I-P) R_s = O_s (I-P) A R_s. \qquad (2.3.78)$$

Expressions (2.3.77) and (2.3.78) clearly imply (2.3.65a). Similarly, (2.3.65b) follows from (2.3.47). Finally, (2.3.71) clearly implies (2.3.64) and the if part of the theorem is proven.

To show the only if part note that by setting (2.3.63) equal to (2.3.64) and using (2.3.65a) we obtain

$$O_s P E^D R_s = O_s V_i E^N E^D R_s \qquad (2.3.79a)$$
$$O_s (I-P) A^D R_s = O_s V_f A^N A^D R_s \qquad (2.3.79b)$$

which thanks to E- and A-invariance of $\text{Im}(R_s)$ imply that

$$O_s PE^D ER_s = O_s V_i E^N EE^D R_s \qquad (2.3.80a)$$

$$O_s (I-P)A^D AR_s = O_s V_f A^N AA^D R_s. \qquad (2.3.80b)$$

But (2.3.80), using (2.3.65b) and (2.3.47), implies (2.3.71) which is the desired result.

Thus we have shown that there is complete equivalence between the boundary representation in terms of the boundary matrices V_i and V_f and the representation in terms of the projection matrix P. This latter representation is used in the next chapter which deals with the realization theory for input-output extendible, stationary TPBVDS's.

Example 2.3.2

Consider the TPBVDS

$$x(k+1) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x(k) + u(k) \qquad (2.3.81a)$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x(0) + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x(N) = v \qquad (2.3.81b)$$

$$y(k) = x(k). \qquad (2.3.81c)$$

This TPBVDS is in standard-form, stationary and input-output extendible (in fact it is extendible). The projection matrix for this system is

$$P = 0 \qquad (2.3.82)$$

(in this case P is unique because (2.3.81) is strongly reachable and observable). The weighting pattern W(k) of (2.3.81) and its extensions is given by

$$W(k) = \begin{bmatrix} 0 & k>0 \\ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} & k\leq 0 \end{bmatrix} . \qquad (2.3.83)$$

Note that (2.3.81) is an anti-causal system. This in fact can be seen
directly from the projection matrix. In general, if P=0, the system is
anti-causal, if P=I, it is causal. If P≠0 and P≠I, it is not necessarily true
that we have an acausal system. For example if C=0, for any P, W=0 which is
clearly not an acausal weighting pattern. However, for minimal systems (which
are the subjects of Section 2.5), we can deduce acausality if P≠0 and P≠I
(also see Section 3.2).

2.4-Reachability and Observability

As discussed in [1], there are two notions for both reachability and observability for TPBVDS's. In this section we provide brief reviews of these definitions and present additional results.

Definition 2.4.1

The system (2.2.1)-(2.2.2) is <u>strongly reachable</u> on [K,L] if the map

$$\{u(k)\colon k\epsilon[K,L]\} \to z_o(K,L) \tag{2.4.1}$$

is onto. The system is <u>strongly reachable</u> if it is strongly reachable on some interval.

From (2.3.2) we can write

$$z_o(K,L) = R_s(L-K)\begin{bmatrix} u(K) \\ \vdots \\ u(L-1) \end{bmatrix} \tag{2.4.2}$$

where

$$R_s(j) = [A^{j-1}B \vdots EA^{j-2}B \vdots \ldots \vdots E^{j-1}B]. \tag{2.4.3}$$

Note that $R_s = R_s(n)$. Furthermore a TPBVDS is strongly reachable if and only if R_s has full rank (this is a consequence of the generalized Cayley-Hamilton Theorem). Furthermore, the strongly reachable spaces have the usual nesting property, i.e.

$$\mathcal{R}_s(k) = Im[R_s(k)] \subset Im[R_s(k+1)] = \mathcal{R}_s(k+1). \tag{2.4.4}$$

We refer the reader to [1] for proofs of these and other results related to strong reachability. For future reference we define the strongly reachable subspace

$$\mathcal{R}_s = Im[R_s]. \tag{2.4.5}$$

Also in [1] it is shown that the system (2.1)-(2.2) is strongly reachable if and only if

$$[sE-tA \vdots B]$$

has full row-rank for all $(s,t) \neq 0$. Or equivalently, if no left eigenvector of the system is orthogonal to the columns of B. Note that this test of strong reachability can be applied even if the system is not in normalized-form. To see this, let us suppose that (2.2.1)-(2.2.2) is well-posed, but not in normalized-form. Now remember that any well-posed system can be put into normalized-form by premultiplying (2.2.1) and (2.2.2) by some invertible matrices T and S, respectively. Premultiplying (2.2.1) by an invertible matrix T means replacing E, A, and B with TE, TA and TB, respectively. Since the new system is in normalized form, we can test whether or not it is strongly reachable by testing the full rankedness of $[sTE-tTA \vdots TB]=T[sE-tA \vdots B]$. By noting that T is invertible, we obtain the desired result.

Defintion 2.4.2

The system (2.2.1)-(2.2.3) is <u>strongly</u> <u>observable</u> on [K,L] if the map

$$z_i(K,L) \rightarrow \{y(k): k \in [K,L]\} \tag{2.4.6}$$

defined by (2.3.5), (2.3.6), and (2.2.3) with $u \equiv 0$ on [K,L] is one to one. The system is <u>strongly</u> <u>observable</u> if it is strongly observable on some [K,L].

With $u \equiv 0$, we have

$$\begin{bmatrix} y(K) \\ \vdots \\ y(L) \end{bmatrix} = O_s(L-K)z_i(K,L) \tag{2.4.7}$$

where

$$O_s(j) = \begin{bmatrix} CE^j \\ CAE^{j-1} \\ \vdots \\ CA^j \end{bmatrix} . \tag{2.4.8}$$

Note that $O_s = O_s(n-1)$. Furthermore, a TPBVDS is strongly observable if and only if O_s has full rank. In addition, the strong unobservability subspaces have the usual nesting property

$$\mathcal{O}_s(k+1) = \text{Ker}(O_s(k+1)) \subset \mathcal{O}_s(k) = \text{Ker}(O_s(k)). \tag{2.4.9}$$

Again for future reference we define the strongly unobservable subspace

$$\mathcal{O}_s = \text{Ker}(O_s). \tag{2.4.10}$$

Again form [1] we have that the TPBVDS (2.1)-(2.2) is strongly observable if and only if

$$\begin{bmatrix} sE-tA \\ C \end{bmatrix}$$

has full column-rank for all $(s,t) \neq (0,0)$, or equivalently, if no right eigenvector of the system is orthogonal to the rows of C. Again, as for the case of strong reachability, this test of strong observability is valid even if the system is not in normalized-form.

Note that the properties of strong reachability and observability involve only the matrices C, E, A, and B. As we shall see, the other weaker set of notions of reachability and observability involve the boundary matrices as well.

Definition 2.4.3

The system (2.2.1)-(2.2.2) is weakly reachable off [K,L] if the map

$$\{u(k): k\epsilon[0,K-1]\cup[L,N-1]\} \to z_i(K,L) \qquad (2.4.11)$$

with v=0 is onto. The weakly reachable subspace $\mathcal{R}_w(K,L)$ is the range of this map. The system is called weakly reachable if

$$\mathcal{R}_w \triangleq \bigcup_{K,L} \mathcal{R}_w(K,L) = \mathbb{R}^n. \qquad (2.4.12)$$

The space $\mathcal{R}_w$ is called the weak reachability space.

While it is shown in [1] that for K and L far from the boundaries the dimension of $\mathcal{R}_w(K,L)$ is constant, it is not generally true that this space is fixed or that any nesting of weak reachability spaces occurs as K and L move inward from the boundaries. That is why we may very well have a system which is weakly reachable, but where $\mathcal{R}_w(K,L)$ is not the whole space for any K and L. In [1] we defined weak reachability differently, specifically we called a system weakly reachable if $\mathcal{R}_w(K,L)$ equaled $\mathbb{R}^n$ for K and L far from the boundaries. We shall see later that Definition 2.4.3 is more appropriate.

Theorem 2.4.1

The weak reachability space $\mathcal{R}_w$ can be expressed as

$$\mathcal{R}_w = \bigcup_{0 \leq k < n} A^k E^{n-1-k} Im[V_i R_s \quad V_f R_s]. \qquad (2.4.13)$$

Corollary

For an extendible system, the weak reachability space $\mathcal{R}_w$ can be expressed as

$$\mathcal{R}_w = \bigcup_{0 \leq k < n} A^k E^{n-1-k} V_i \mathcal{R}_s + \mathcal{R}_s. \qquad (2.4.14)$$

Proof

First we prove the following Lemma which justifies the use of the terms "strong" and "weak".

Lemma 2.4.1

For any TPBVDS

$$\mathcal{R}_s \subset \mathcal{R}_w. \qquad (2.4.15)$$

Proof

We will show a stronger result that

$$\mathcal{R}_s \subset \mathcal{R}_w(K,L) \quad \text{for } K, L \in [n, N-n]. \qquad (2.4.16)$$

From expression (2.3.7) for $z_i(K,L)$ with $v=0$, and the fact that the space reached by $z_0(0,K)$ and $z_0(L,N)$ is exactly $\mathcal{R}_s$, we can easily deduce that

$$\mathcal{R}_w(K,L) = E^{N-L}(\omega E - A^K(\omega V_f E + V_i A)A^{N-K})\Gamma^{-1}\mathcal{R}_s$$
$$+ A^K(A - E^{N-L}(\omega V_f E + V_i A)E^L)\Gamma^{-1}\mathcal{R}_s \qquad (2.4.17)$$

By noting that $A^{N-K}\mathcal{R}_s \subset \mathcal{R}_s$ and $E^L\mathcal{R}_s \subset \mathcal{R}_s$, (2.4.17) implies that

$$\mathcal{R}_w(K,L) \supset E^{N-L}(\omega E - A^K(\omega V_f E + V_i A)A^{N-K})E^L\Gamma^{-1}\mathcal{R}_s + A^K(A - E^{N-L}(\omega V_f E + V_i A)E^L)A^{N-K}\Gamma^{-1}\mathcal{R}_s$$
$$= \mathcal{R}_s + E^{N-L}A^K(\omega V_f E + V_i A)A^{N-K}E^L\Gamma^{-1}\mathcal{R}_s \qquad (2.4.18)$$

which in turn implies (2.4.16). Clearly then (2.4.16) implies (2.4.15).

To prove the Theorem, observe that

$$\mathcal{R}_w \supset \mathcal{R}_s + \bigcup_K \mathcal{R}_w(K,K). \tag{2.4.19}$$

Using (2.2.7) and the fact that $\Gamma^{-1}\mathcal{R}_s = \mathcal{R}_s$, this implies that

$$\mathcal{R}_w \supset \bigcup_K \{A^K E^{N-K}[\omega V_f E + V_i A]\mathcal{R}_s\} + \mathcal{R}_s, \text{ for all } \omega. \tag{2.4.20}$$

But thanks to (2.2.5) and the E- and A-invariance of $\mathcal{R}_s$,

$$\mathcal{R}_s + [\omega V_f E + V_i A]\mathcal{R}_s = V_i \mathcal{R}_s + V_f \mathcal{R}_s \tag{2.4.21}$$

which along with (2.4.19) and Cayley-Hamilton proves that

$$\mathcal{R}_w \supset \bigcup_{0 \leq k < n} A^k E^{n-1-k} \text{Im}[V_i R_s \quad V_f R_s]. \tag{2.4.22}$$

The other inclusion is trivial since in expression (2.3.7) for $z_i(K,L)$, the range of the map $u \to z_i$ is essentially the range of matrices $A^r E^s V_i A^t E^u$ and $A^r E^s V_f A^t E^u$.

To prove the corollary simply note that we can decompose the system into 3 subsystems as in (3.2.30), in which case V_i and V_f are expressed as in (2.3.59). Now using the fact that for an extendible system $V_f = (I - V_i E^N)(A^D)^N$, we can show that

$$\text{Im}[V_i R_s \vdots R_s] = \text{Im}[V_i R_s \vdots V_f R_s] \tag{2.4.23}$$

which yields the desired result.

In the case of displacement systems, expression (2.4.13) simplifies and $\mathcal{R}_w$ can be expressed as follows

$$\mathcal{R}_w = \text{Im}[V_i R_s \quad V_f R_s]. \tag{2.4.24}$$

If in addition the system is extendible then $\mathcal{R}_w = \mathcal{R}_w(K,L)$ for K and L far from the boundaries (see [1]).

In analogy with the strong reachability result, we state without proof the following characterization of weak reachability, which is proved in [2]:

Theorem 2.4.2

A displacement system is weakly reachable if and only if the matrix

$$[sE-tA \vdots V_iB \vdots V_fB]$$

has full rank for all $(s,t) \neq (0,0)$. If in addition it is extendible, then it is weakly reachable if and only if

$$[sE-tA \vdots V_iB \vdots B]$$

has full rank for all $(s,t) \neq (0,0)$.

As one would expect, there is a dual set of concepts and results for weak observability:

Definition 2.4.4

The system (2.2.1)-(2.2.3) is weakly observable off [K,L] if the map

$$z_o(K,L) \rightarrow \{y(k) : k \in [0,K] \cup [L,N]\} \qquad (2.4.25)$$

with $v=0$ and $u(j)=0$, $j \in [0,K-1] \cup [L,N-1]$ is one to one. The weakly unobservable subspace $O_w(K,L)$ is the null space of this map. The system is called weakly observable if

$$O_w \overset{\Delta}{=} \underset{K,L}{\cap} O_w(K,L) = \{0\}. \qquad (2.4.26)$$

The space O_w is the weak unobservability space.

By analogy with the weak reachability case we simply present the dual set of results concerning weak observability.

Theorem 2.4.3

The weakly unobservable space can be expressed as follows

$$O_w = \bigcap_{0 \leq k < n} \text{Ker} \begin{bmatrix} O_s V_i \\ O_s V_f \end{bmatrix} E^{n-1-k} A^k. \tag{2.4.27}$$

Corollary

For an extendible system the weakly unobservable space can be expressed as follows

$$O_w = O_s \cap \{ \bigcap_{0 \leq k < n} \text{Ker}(O_s V_i E^{n-1-k} A^k) \}. \tag{2.4.28}$$

Lemma 2.4.2

For any TPBVDS

$$O_w \subset O_s. \tag{2.4.29}$$

This Lemma shows that weak observability is a weaker condition that strong observability.

If the TPBVDS is displacement, (2.4.27) simplifies as follows

$$O_w = \text{Ker} \begin{bmatrix} O_s V_i \\ O_s V_f \end{bmatrix} \tag{2.4.30}$$

and if in addition it is extendible $O_w = O_w(K,L)$ for all K and L far from the boundaries.

Theorem 2.4.4

A displacement system is weakly observable if and only if the matrix

$$\begin{bmatrix} sE-tA \\ CV_i \\ CV_f \end{bmatrix}$$

has full rank for all $(s,t) \neq (0,0)$. If in addition, it is extendible, then it is extendible if and only if

$$\begin{bmatrix} sE-tA \\ CV_i \\ C \end{bmatrix}$$

has full rank for all $(s,t) \neq (0,0)$.

In this section we have introduced two distinct notions of reachability and observability. In some cases the two notions coincide, for example for causal systems where $V_i = E = I$ and $V_f = 0$. In this case $\mathscr{R}_s = \mathscr{R}_w$ and $\mathcal{O}_s = \mathcal{O}_w$. In general, however, that is not the case. We shall see in the next section that both of these notions are indeed needed to study minimality.

The following example illustrates the difference between the concept of strong and that of weak reachability:

Example 2.4.1

Consider the following displacement TPBVDS

$$x(k+1) = x(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k) \qquad (2.4.31)$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} x(0) + \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} x(N) = 0 \qquad (2.4.32)$$

where

$$x(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}.$$

(2.4.33)

This system is well-posed and in normalized form. The strong reachability space for this system is just

$$Im \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

so that the system is not strongly reachable. In fact, we can easily see that only x_1 is strongly reachable and x_2 is not. However, using (2.4.24), we can check that the system is weakly reachable. In fact, we can check that this system is weakly reachable off any interval $[K,L]$, $0 < K, L < N$. To intuitively illustrate this fact, note that boundary condition (2.4.32) can be rewritten as follows

$$x_1(0) = 0$$

(2.4.34)

$$x_2(0) = x_1(N).$$

(2.4.35)

It is clear that $x_1(k)$ can be made arbitrary by proper choice of inputs $u(j)$, $j < k$. On the other hand, $x_1(N)$, and thus $x_2(0)$, can also be independently made arbitrary by proper choice of $u(j)$, $k \leq j < N$. But (2.4.31) implies that $x_2(k)$ is constant for all k, so that it must equal $x_2(0)$ and $x_1(N)$. The result is that $x_1(k)$ and $x_2(k)$, which form $x(k)$, can be made arbitrary by proper choice of the input u. This explains why this system is weakly reachable.

2.5-Minimality

In this section we present minimality results for TPBVDS's. We also specifically consider the stationary and extendible stationary cases. These results are analogous to those in [7] and [12], with differences due to possible singularity of E and A.

Definition 2.5.1

A TPBVDS is minimal if x has the lowest dimension among all TPBVDS's having the same weighting pattern.

Theorem 2.5.1

A TPBVDS with $N \geq 4n$ is minimal if and only if

(a) $\mathcal{R}_w = \mathbb{R}^n$ (2.5.1)

(b) $\mathcal{O}_w = \{0\}$ (2.5.2)

(c) $\mathcal{O}_s \subset \mathcal{R}_s$ (2.5.3)

(i.e. if it is weakly reachable and observable, and any strongly unobserved mode is strongly reached).

As in the causal case, the proof of this result involves the introduction of Hankel matrices and the description of a method for reducing the dimension of systems violating any of the conditions (a)-(c). As we will

see, in the present context we actually have 3 different Hankel matrices and also, as in [7] we may have a certain level of nonuniquness in minimal realizations that is not present in the causal case.

The length of the interval here is assumed to be larger than 4 times the dimension of the system so that all the modes on both sides of a state in the middle of the interval can be reached and observed (see the proof for details on where this assumption is needed). If N is not large enough, the conditions of Theorem 2.5.1 become necessary but not sufficient.

<u>Proof</u>

We begin with the description of reduction procedures if any of the conditions (2.5.1)-(2.5.3) are not satisfied. Consider first the case in which $\mathfrak{R}_w \neq \mathbb{R}^n$. Let $\mathfrak{R}_2$ be any subspace such that

$$\mathfrak{R}_w \oplus \mathfrak{R}_2 = \mathbb{R}^n. \qquad (2.5.4)$$

Then, by performing a similarity transformation on x to represent it in a basis compatible with (2.5.4) we arrive at a system of the form (2.2.1)-(2.2.3) with

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad x_1 \in \mathfrak{R}_w, \; x_2 \in \mathfrak{R}_2, \qquad (2.5.5a)$$

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad E = \begin{bmatrix} E_{11} & E_{12} \\ 0 & E_{22} \end{bmatrix}, \quad V_i = \begin{bmatrix} V_{11}^i & V_{12}^i \\ V_{21}^i & V_{22}^i \end{bmatrix}, \quad (2.5.5b)$$

$$V_f = \begin{bmatrix} V_{11}^f & V_{12}^f \\ V_{21}^f & V_{22}^f \end{bmatrix}, \quad C = [C_1 \mathrel{\vdots} C_2], \quad B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}. \qquad (2.5.5c)$$

The 0-blocks in A and E follow from the A- and E-invariance of $\mathfrak{R}_w$. The 0-block in B is due to the fact that $\mathrm{Im}[B] \subset \mathfrak{R}_s$ (Cayley-Hamilton) and $\mathfrak{R}_s \subset \mathfrak{R}_w$.

In addition, since

$$\mathcal{R}_w \supset V_i \mathcal{R}_s + V_f \mathcal{R}_s \tag{2.5.6}$$

we must have

$$V_{21}^i A_{11}^k E_{11}^j B_1 = V_{21}^f A_{11}^k E_{11}^j B_1 = 0. \tag{2.5.7}$$

From the form (2.2.7) for the weighting pattern of a TPBVDS we can then conclude that the weighting pattern of our system is given by

$$W(k,j) = \begin{bmatrix} C_1 A_{11}^k (A_{11} - E_{11}^{N-k}(V_{11}^i A_{11} + \omega V_{11}^f E_{11}) E_{11}^k) E_{11}^{j-k} A_{11}^{N-j-1} \Gamma_1^{-1} B_1 & j \ge k \\ C_1 E_{11}^{N-k} (\omega E_{11} - A_{11}^k (V_{11}^i A_{11} + \omega V_{11}^f E_{11}) A_{11}^{N-k}) E_{11}^j A_{11}^{k-j-1} \Gamma_1^{-1} B_1 & j < k \end{bmatrix} \tag{2.5.8}$$

where $\Gamma_1 = \omega E_{11}^{N+1} - A_{11}^{N+1}$, so that we have apparently reduced our system to

$$E_{11} x_1(k+1) = A_{11} x_1(k) + B_1 u(k)$$

$$V_{11}^i x_1(0) + V_{11}^f x_1(N) = v_1 \tag{2.5.9}$$

$$y(k) = C_1 x_1(k).$$

Note since E and A are in standard form, so are E_{11} and A_{11}. However, the boundary matrices V_{11}^i and V_{11}^f need not be normalized and indeed there is no guarantee that (2.5.9) is well-posed. However, note that V_{11}^i and V_{11}^f in (2.5.8) are pre- and post-multiplied by $C_1 A_{11}^k E_{11}^{N-k}$ and $\Gamma_1^{-1} A_{11}^j E_{11}^{N-j} B_1$ so that there may be a degree of freedom in the choices for V_{11}^i and V_{11}^f that verify (2.5.8). In fact, if O_s^1 and R_s^1 denote the strong observability and reachability matrices of (2.5.9), since the null space of $C_1 A_{11}^k E_{11}^{N-k}$ includes that of O_s^1 and the range of $\Gamma_1^{-1} A_{11}^j E_{11}^{N-j} B_1$ is included in R_s^1, we can modify V_{11}^i and V_{11}^f as long as $O_s^1 V_{11}^i R_s^1$ and $O_s^1 V_{11}^f R_s^1$ remain unchanged and preserve the weighting pattern (2.5.8). Thanks to the following result, we can modify the boundary matrices in order to make (2.5.9) well-posed while leaving the weighting pattern unchanged.

Lemma 2.5.1

Consider a (possibly not well-posed) TPBVDS (2.2.1)-(2.2.2), with E and A in standard form and for which the following holds:

$$0_s(V_iE^N + V_fA^N)R_s = 0_sR_s \qquad (2.5.10)$$

Then we can find $\tilde{V}_i$, $\tilde{V}_f$ so that

$$\tilde{V}_iE^N + \tilde{V}_fA^N = I \qquad (2.5.11)$$

and

$$0_sV_iR_s = 0_s\tilde{V}_iR_s \quad , \quad 0_sV_fR_s = 0_s\tilde{V}_fR_s. \qquad (2.5.12)$$

Proof

Let

$$X = I - [V_iE^N + V_fA^N] \qquad (2.5.13)$$

so that

$$0_sXR_s = 0. \qquad (2.5.14)$$

Let a and b be any scalars such that (aE^N+bA^N) is invertible, and then take

$$\tilde{V}_i = V_i + aX(aE^N + bA^N)^{-1} \qquad (2.5.15a)$$

$$\tilde{V}_f = V_f + bX(aE^N + bA^N)^{-1} \qquad (2.5.15b)$$

From (2.5.14) and the A- and E-invariance of $\mathfrak{R}_s$ and 0_s we have that

$$0_sA^kE^jXA^rE^sR_s = 0 \qquad k,j,r,s \geq 0 \qquad (2.5.16)$$

from which we can easily check that (2.5.12) holds. Finally (2.5.11) can also be checked by direct calculation.

To apply this lemma to (2.5.9) we must show that (2.5.10) holds for this

system. Expressions (2.5.5) and (2.5.7) imply that:

$$CA^kE^jB = C_1A_{11}^kE_{11}^jB_1 \tag{2.5.17a}$$

$$CA^rE^sV_iA^kE^jB = C_1A_{11}^rE_{11}^sV_{11}^iA_{11}^kE_{11}^jB_1 \tag{2.5.17b}$$

$$CA^rE^sV_fA^kE^jB = C_1A_{11}^rE_{11}^sV_{11}^fA_{11}^kE_{11}^jB_1. \tag{2.5.17c}$$

Therefore, since our original system was assumed to be in normalized form

$$C_1A^rE^s[V_{11}^iE_{11}^N+V_{11}^fA_{11}^N]A_{11}^kE_{11}^jB_1 = CA^rE^s[V_iE^N+V_fA^N]A^kE^jB$$

$$= CA^{k+r}E^{j+s}B = C_1A_{11}^{k+r}E_{11}^{j+s}B_1 \tag{2.5.18}$$

from which we conclude that

$$0_s^1[V_{11}^iE_{11}^N+V_{11}^fA_{11}^N]R_s^1 = 0_s^1R_s^1 \tag{2.5.19}$$

where 0_s^1 and R_s^1 are the strong observability and reachability matrices for (2.5.9).

To continue with the proof of the theorem, note that the problem of reducing the dimension of the realization if (2.5.2) is violated is merely the dual of the problem that we have just considered. Consequently, we omit the details. We turn then to the case in which condition (2.5.3) is not satisfied. In this case, there is a subspace $\mathcal{Z} \neq \{0\}$ such that

$$\mathcal{R}_s \oplus \mathcal{Z} = \mathcal{R}_s + \mathcal{O}_s. \tag{2.5.20}$$

Let $\mathcal{W}$ be any subspace such that $\mathcal{W} \oplus \mathcal{Z} = \mathbb{R}^n$ and perform a similarity transformation of the TPBVDS to represent it in a basis compatible with (2.5.4). This yields a model as in (2.5.5b), (2.5.5c) with the additional fact that $C_2=0$. To put the reduced system in normalized form we once again apply Lemma 2.5.1.

What remains to show is that two TPBVDS's with the same weighting pattern and both satisfying (2.5.1)-(2.5.3) must have the same dimension and consequently are minimal. To proceed with the proof we need the following Lemma:

Lemma 2.5.2

Let $\{E_i, A_i\}$, $i=1,2$, be two regular pencils so that $\alpha E_i + \beta A_i = I$, $i=1,2$, where $\dim(E_i) = \dim(A_i) = n_i$. Suppose that $N \geq 2\max(n_1, n_2)$. Also suppose that for some matrices $\{M_i, N_i\}$, $i=1,2$,

$$M_1 A_1^k E_1^{N-1-k} N_1 = M_2 A_2^k E_2^{N-1-k} N_2 \quad , \quad 0 \leq k \leq N-1. \tag{2.5.21}$$

Then for all K, L,

$$M_1 A_1^K E_1^L N_1 = M_2 A_2^K E_2^L N_2. \tag{2.5.22}$$

Proof

Note first that for $K+L \leq N-1$ we can write

$$E_i^K A_i^L = E_i^K A_i^L (\alpha E_i + \beta A_i)^{N-1-K-L} \tag{2.5.23}$$

and in this case (2.5.22) follows directly from (2.5.21). For $K+L \geq N$, let us suppose for simplicity that $\alpha \neq 0$. From what we have first shown for $K+L \leq N-1$, we know that

$$M_1 A_1^k N_1 = M_2 A_2^k N_2 \quad , \quad 0 \leq k \leq N-1. \tag{2.5.24}$$

From results on the causal partial realization problem [15] and the fact that $N \geq 2n_i$, we can conclude that

$$M_1 A_1^k N_1 = M_2 A_2^k N_2 \quad , \quad k \geq 0. \tag{2.5.25}$$

Equation (2.5.22) then follows since we can write E_i as $(I - \beta A_i)/\alpha$.

We note that as discussed in [15] the condition on the size of N is important here, although slightly smaller bounds on the interval size can be obtained --(essentially the sum of observability and reachability indices).

Proceeding with the proof, consider two systems $(C_j, E_j, A_j, V_j^i, V_j^f, B_j)$, $j=1,2$, satisfying minimality conditions (2.5.1)-(2.5.3), and without loss of generality assume that both are in normalized form with the same α and β. What we know is that

$$C_1 A_1^k (A_1 - E_1^{N-k}(V_1^i A_1 + \omega V_1^f E_1)E_1^k)E_1^{j-k}A_1^{N-j-1}\Gamma_1^{-1}B_1$$
$$= C_2 A_2^k (A_2 - E_2^{N-k}(V_2^i A_2 + \omega V_2^f E_2)E_2^k)E_2^{j-k}A_2^{N-j-1}\Gamma_2^{-1}B_2 \quad , \quad j \geq k \quad (2.5.26a)$$

$$C_1 E_1^{N-k}(\omega E_1 - A_1^k(V_1^i A_1 + \omega V_1^f E_1)A_1^{N-k})E_1^j A_1^{k-j-1}\Gamma_1^{-1}B_1$$
$$= C_2 E_2^{N-k}(\omega E_2 - A_2^k(V_2^i A_2 + \omega V_2^f E_2)A_2^{N-k})E_2^j A_2^{k-j-1}\Gamma_2^{-1}B_2 \quad , \quad j < k. \quad (2.5.26b)$$

Let $k \in [2n, N-2n]$ (remember that $N \geq 4n$), then we can apply Lemma 2.3.2 to get

$$C_1 A_1^k (A_1 - E_1^{N-k}(V_1^i A_1 + \omega V_1^f E_1)E_1^k)E_1^K A_1^L \Gamma_1^{-1}B_1$$
$$= C_2 A_2^k (A_2 - E_2^{N-k}(V_2^i A_2 + \omega V_2^f E_2)E_2^k)E_2^K A_2^L \Gamma_2^{-1}B_2 \quad , \text{ for all } K,L \quad (2.5.27a)$$

$$C_1 E_1^{N-k}(\omega E_1 - A_1^k(V_1^i A_1 + \omega V_1^f E_1)A_1^{N-k})E_1^K A_1^L \Gamma_1^{-1}B_1$$
$$= C_2 E_2^{N-k}(\omega E_2 - A_2^k(V_2^i A_2 + \omega V_2^f E_2)A_2^{N-k})E_2^K A_2^L \Gamma_2^{-1}B_2 \quad , \text{ for all } K,L. \quad (2.5.27b)$$

By taking $K=r$, $L=N-k+s$ in (2.5.27a) and $K=k+r$ and $L=s$ in (2.5.27b) and subtracting the two sides of (2.5.27a) from (2.5.27b) we obtain

$$C_1 E_1^r A_1^s B_1 = C_2 E_2^r A_2^s B_2 \qquad (2.5.28)$$

and this for all $r,s \geq 0$.

Using (2.5.26) and (2.5.28) we can show that

$$C_1 A_1^k E_1^{N-k}(V_1^i A_1 + \omega V_1^f E_1)E_1^j A_1^{N-j-1}\Gamma_1^{-1}B_1 = C_2 A_2^k E_2^{N-k}(V_2^i A_2 + \omega V_2^f E_2)E_2^j A_2^{N-j-1}\Gamma_2^{-1}B_2.$$
$$(2.5.29)$$

and taking into account Lemma 2.5.2, this implies that

$$C_1 A_1^r E_1^s (V_1^i A_1 + \omega V_1^f E_1)E_1^t A_1^u \Gamma_1^{-1}B_1 = C_2 A_2^r E_2^s (V_2^i A_2 + \omega V_2^f E_2)E_2^t A_2^u \Gamma_2^{-1}B_2$$
$$(2.5.30)$$

for all $r,s,t,u \geq 0$. Then using the fact that both systems are in normalized

form we obtain

$$C_1 A_1^r E_1^s V_1^i E_1^t A_1^u B_1 = C_2 A_2^r E_2^s V_2^i E_2^t A_2^u B_2 \qquad (2.5.31a)$$

$$C_1 A_1^r E_1^s V_1^f E_1^t A_1^u B_1 = C_2 A_2^r E_2^s V_2^f E_2^t A_2^u B_2 \qquad (2.5.31b)$$

for all $r, s, t, u \geq 0$.

As in the case of causal systems, Hankel matrices are extremely useful in proving our minimality result. In the present context, however, there are three different Hankel matrices.

$$H_{in} = 0_s^1 R_w^1 = 0_s^2 R_w^2 \qquad (2.5.32)$$

$$H_{out} = 0_w^1 R_s^1 = 0_w^2 R_s^2 \qquad (2.5.33)$$

$$H_s = 0_s^1 R_s^1 = 0_s^2 R_s^2 \qquad (2.5.34)$$

where R_s^j and 0_s^j are the strong reachability and observability matrices of system j, respectively, and where

$$R_w^j = [A^{n-1}(V_i R_s^j \vdots V_f R_s^j) \quad EA^{n-2}(V_i R_s^j \vdots V_f R_s^j) \quad \cdots \quad E^{n-1}(V_i R_s^j \vdots V_f R_s^j)],$$

$$0_w^j = \begin{bmatrix} \begin{bmatrix} 0_s^j V_i \\ 0_s^j V_f \end{bmatrix} A^{n-1} \\ \vdots \\ \begin{bmatrix} 0_s^j V_i \\ 0_s^j V_f \end{bmatrix} E^{n-1} \end{bmatrix}, \quad j=1,2.$$

R_w^j and 0_w^j are the __weak reachability__ and __weak observability matrices__ of system j, respectively. Clearly

$$\mathscr{R}_w^j = Im(R_w^j) \qquad (2.5.35a)$$

$$\mathcal{O}_w^j = Ker(0_w^j) \qquad (2.5.35b)$$

for $j=1,2$. Equations (2.5.32)-(2.5.34) are direct consequences of (2.5.28) and (2.5.31). From (2.5.33), we get that

$$R_s^2 = UR_s^1, \qquad (2.5.36)$$

where

$$U = \{O_w^2 {}^{\cdot} O_w^2\}^{-1} O_w^2 {}^{\cdot} O_w^1, \qquad (2.5.37)$$

and where O_w^2 has full rank because of the weak observability assumption. Similarly we can obtain an analogous expression for R_s^1 in terms of R_s^2. These allow us to conclude that

$$\text{rank}(R_s^1) = \text{rank}(R_s^2) = \rho, \qquad (2.5.38)$$

and in an analogous way we can show that

$$\text{rank}(O_s^1) = \text{rank}(O_s^2) = \omega. \qquad (2.5.39)$$

Finally, condition (2.5.3) together with (2.5.34) imply that

$$\rho - (n_1 - \omega) = \text{rank } H_s = \rho - (n_2 - \omega) \qquad (2.5.40)$$

from which we see that

$$n_1 = n_2, \qquad (2.5.41)$$

completing the proof of the Theorem.

Corollary 2.5.1a

Let $(C_j, V_j^i, V_j^f, E_j, A_j, B_j, N)$, $j=1,2$, be two minimal realizations of the same weighting pattern, where $\{E_j, A_j\}$, $j=1,2$, are in standard form for the same α and β. Then there exists an invertible matrix T so that

$$B_2 = TB_1 \qquad (2.5.42a)$$

$$C_2 = C_1 T^{-1} \qquad (2.5.42b)$$

$$O_s^1(V_1^i - T^{-1}V_2^i T)R_s^1 = 0 \qquad (2.5.43a)$$

$$O_s^1(V_1^f - T^{-1}V_2^f T)R_s^1 = 0 \qquad (2.5.43b)$$

and

$$(A_1 - T^{-1}A_2 T)R_s^1 = 0 \qquad (2.5.44a)$$

$$(E_1 - T^{-1}E_2 T)R_s^1 = 0 \qquad (2.5.44b)$$

$$O_s^1(A_1 - T^{-1}A_2 T) = 0 \qquad (2.5.44c)$$

$$O_s^1(E_1 - T^{-1}E_2 T) = 0 \qquad (2.5.44d)$$

where R_s^1 and O_s^1 are the strong reachability and observability matrices for system 1.

Proof

From (2.5.36) we have that

$$R_s^2 = U R_s^1 \qquad (2.5.45)$$

with U defined as in (2.5.37). While this choice for U is not necessarily invertible, we can always find an invertible T so that

$$R_s^2 = T R_s^1 \qquad (2.5.46)$$

since $\mathscr{R}_s^1$ and $\mathscr{R}_s^2$ have the same dimension. In a similar way we can always find an invertible matrix W so that

$$O_s^2 W = O_s^1. \qquad (2.5.47)$$

From (2.5.34) we can then conclude that

$$O_s^2[W - T]R_s^1 = 0. \qquad (2.5.48)$$

The question, then is whether we can choose W=T. To see that this can be done, assume that we have chosen a basis for each of the two systems compatible with the following direct sum decomposition:

$$\mathcal{O}_s \oplus [\mathcal{O}_s^\perp \cap \mathscr{R}_s] \oplus [\mathcal{O}_s^\perp \cap \mathscr{R}_s].$$

The requirement (2.5.46) implies that T must have the form

$$T = \begin{bmatrix} T_1 & T_2 & * \\ T_3 & T_4 & * \\ 0 & 0 & * \end{bmatrix} \qquad (2.5.49)$$

where T_1, T_2, T_3 and T_4 are fixed and $*$ are arbitrary. Similarly, (2.5.47)

implies that W must have the form

$$W = \begin{bmatrix} * & * & * \\ 0 & W_1 & W_2 \\ 0 & W_3 & W_4 \end{bmatrix}. \tag{2.5.50}$$

Finally, by direct computation we can check that (2.5.48) implies

$$W_1 = T_4 , \qquad T_3 = W_3 = 0 \tag{2.5.51}$$

so that with the indicated degrees of freedom we can take

$$W = T = \begin{bmatrix} T_1 & T_2 & * \\ 0 & T_4 & W_2 \\ 0 & 0 & W_4 \end{bmatrix}. \tag{2.5.52}$$

Proceeding with the proof, note that (2.5.42a), (2.5.42b) follow from

(2.5.47), (2.5.48) plus the fact that $\{E_j, A_j\}$, $j=1,2$, are in standard form

for the same α and β. Also, the equality of the weighting patterns of the two

systems is equivalent to

$$O^1_s V^i_1 R^1_s = O^2_s V^i_2 R^2_s \tag{2.5.53a}$$

$$O^1_s V^f_1 R^1_s = O^2_s V^f_2 R^2_s \tag{2.5.53b}$$

from which (2.5.43a) and (2.5.43b) follow. Finally, recall that R_s is A- and

E-invariant. Thus, thanks to Cayley-Hamilton we can conclude that

$$A_2 R^2_s = T A_1 R^1_s , \qquad E_2 R^2_s = T E_2 R^2_s \tag{2.5.54}$$

from which (2.5.44a) and (2.5.44b) follow. Equation (2.5.44c) and (2.5.44d)

are verified in a similar fashion.

Corollary 2.5.1b

(a) Every left (right) input-output extendible TPBVDS has a minimal

realization that is also left (right) input-output extendible.

(b) Every left (right) extension of a minimal left (right) input-output

extendible TPBVDS is minimal.

Proof

Part (a) follows Theorem 2.3.5 which we prove here.

Proof of Theorem 2.3.5

Suppose that we have two realizations $(C_j, E_j, A_j, V_j^i, V_j^f, B_j)$, $j=1,2$, of the same weighting pattern. Then we would like to show that if one of these is left (right) input-output extendible, so is the other. This result can be seen to be true as follows. First, it is not difficult to see that the following genralization of Lemma 2.5.2 holds. Specifically, if (2.5.21) holds, then for all $P, Q, K, L \geq 0$,

$$M_1(A_1^D)^P(E_1^D)^Q A_1 E_1 N_1 = M_2(A_2^D)^P(E_2^D)^Q A_2 E_2 N_2. \tag{2.5.55}$$

Then not only do we have that

$$O_s^1 V_1^i R_s^1 = O_s^2 V_2^i R_s^2 \tag{2.5.56a}$$

$$O_s^1 V_1^f R_s^1 = O_s^2 V_2^f R_s^2 \tag{2.5.56b}$$

(since both systems have the same weighting pattern) but also

$$O_s^1 V_1^i E_1^D E_1 R_s^1 = O_s^2 V_2^i E_2^D E_2 R_s^2 \tag{2.5.57a}$$

$$O_s^1 A_1^D A_1 V_1^f R_s^1 = O_s^2 A_2^D A_2 V_2^f R_s^2. \tag{2.5.57b}$$

Suppose that system 1 is left input-output extendible and thus satisfies (2.3.44). Then (2.5.56) and (2.5.57) imply that system 2 also satisfies (2.3.44) which means that it is left input-output extendible. Right extendibility can be proven similarly.

To show part (b), suppose that an extension of a minimal system defined on the interval $[0,N]$ is not minimal and thus can be reduced. Reduce the extension and move in its boundaries to the interval $[0,N]$. The system obtained has clearly the same weighting pattern as the original system defined on $[0,N]$ but has lower dimension, which is a contradiction.

Theorem 2.5.2

A stationary TPBVDS, with $N \geq 2n$, is minimal if and only if

(a) $\operatorname{Im}[V_i R_s \,\vdots\, V_f R_s] = \mathbb{R}^n$ (2.5.58)

(b) $\operatorname{Ker} \begin{bmatrix} O_s V_i \\ \\ O_s V_f \end{bmatrix} = \{0\}$ (2.5.59)

(c) $\mathcal{O}_s \subset \mathcal{R}_s$ (2.5.60)

Proof

First, note that the minimality conditions of Theorem 2.5.1 are necessary and sufficient for this case as well, even though we have a weaker condition on the length of the interval. This is because the only place that the assumption $N \geq 4n$ was used in the proof of Theorem 2.5.1, was in the derivation of (2.5.28) and (2.5.31). But in the stationary case, as long as $N \geq 2n$, (2.5.31) immediately follows from Lemma 2.5.2 and the assumption that the weighting patterns of the two systems must be identical. In addtion, (2.5.28) follows from (2.5.31) and the assumption that the two systems are in normalized form. So all we need to show is that conditions (2.5.1)-(2.5.3) and (2.5.58)-(2.5.60) are equivalent in the stationary case.

Note that since

$$\operatorname{Im}[V_i R_s \,\vdots\, V_f R_s] \subset \mathcal{R}_w \qquad (2.5.61)$$

$$\operatorname{Ker} \begin{bmatrix} O_s V_i \\ \\ O_s V_f \end{bmatrix} \supset \mathcal{O}_s \qquad (2.5.62)$$

condition (2.5.58)-(2.5.60) are sufficient for minimality. To show necessity let us assume that (2.5.1)-(2.5.3) hold. Suppose also that (2.5.58) fails in which case there exists a vector $q \neq 0$ such that

$$q'[V_iR_s | V_fR_s] = 0, \qquad (2.5.63)$$

which implies that

$$q'R_s = 0. \qquad (2.5.64)$$

By noting that condition (2.5.60) is equivalent to

$$\text{Left-Ker}(R_s) = \mathcal{R}_s^{\perp} \subset \mathcal{O}_s^{\perp} = \text{Row-Im}(O_s) \qquad (2.5.65)$$

(2.5.64) implies that

$$q' \in \text{Row-Im}(O_s) \qquad (2.5.66)$$

which thanks to the stationarity conditions (2.2.13a) and (2.2.13b) implies that

$$q'[V_iE^rA^s - E^rA^sV_i]R_s = 0 \qquad (2.5.67a)$$

$$q'[V_fE^rA^s - E^rA^sV_f]R_s = 0 \qquad (2.5.67b)$$

for all r and s. Thanks to E and A-invariance of R_s, there exists a matrix D such that

$$E^{n-1-k}A^kR_s = R_sD. \qquad (2.5.68)$$

Then (2.5.67) implies

$$q'E^{n-1-k}A^k[V_iR_s | V_fR_s] = q'[V_iR_sD | V_fR_sD] = 0. \qquad (2.5.69)$$

Since (2.5.69) holds for all $k \in [0, n-1]$ we obtain

$$q'R_w = 0, \qquad (2.5.70)$$

which violates (2.5.1). Similarly we can show that if (2.5.59) fails, then (2.5.2) is violated.

We have shown above that conditions (2.5.1)-(2.5.3) are equivalent to conditions (2.5.58)-(2.5.60) for stationary systems. However, note that this does not imply that (2.5.1) is equivalent to (2.5.58), and (2.5.2) to (2.5.59). As can be seen from the proof of Theorem 2.5.2, condition (2.5.60) must be true to have (2.5.1) be equivalent to (2.5.58) and for (2.5.2) to be equivalent to (2.5.59). The following example illustrates this point:

Example 2.5.1

Consider the following stationary TPBVDS in normalized form defined over an interval of length N

$$C = [0 \ 0 \ 1], \ V_i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \ V_f = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & N & -N^2/2 \end{bmatrix},$$

$$E = I, \ A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \ B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \tag{2.5.71}$$

For this system, the strong reachability space R_s is

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and so $[V_i R_s \vdots V_f R_s]$ is equal to

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & -1 \end{bmatrix}$$

and thus does not have full rank. This implies that condition (2.5.58) is not satisfied. On the other hand, we have that

$$\mathcal{R}_w = \bigcup_{k=0,\ldots,n-1} \{ \mathrm{Im}(E^{n-k-1} A^k [V_i R_s \vdots V_f R_s]) \}$$

$$= \bigcup_{k=0,1,2} \{ \mathrm{Im}(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^k \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & -1 \end{bmatrix}) \} = \mathbb{R}^3 \tag{2.5.72}$$

which means that condition (2.5.1) is satisfied. This example illustrates

that, if (2.5.3) does not hold, (2.5.1) and (2.5.58) are not equivalent. In this example, (2.5.3) does not hold since the strong observability matrix O_s is equal to $[0\ 0\ 1]$ which implies that O_s equals

$$\text{Im}(\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix})$$

which is clearly not included in the strongly reachable space

$$\text{Im}(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}).$$

In Section 2.3 we examined the family of input-output extendible stationary TPBVDS's having all identical weighting patterns restricted to their domains of definition and identical system matrices C, E, A and B. We showed that this family of systems is completely characterized by its projection matrix P. We refered to this family of systems as the input-output extendible stationary TPBVDS (C,P,E,A,B).

Thanks to Corollary 2.5.1a, we can see that if one of the members of the family of systems (C,P,E,A,B) defined over an interval of length N is minimal then all the members defined over intervals longer than N are also minimal. Also it can be verified that by moving in the boundary conditions of a minimal system (i.e. satisfying (2.5.58)-(2.5.60)) we obtain minimal systems as long as $N \geq 2n$. Thus, either all the members of the family defined over intervals of at least length 2n are minimal or no member is minimal. This property allows us to talk about minimality of the input-output extendible stationary TPBVDS (C,P,E,A,B).

Theorem 2.5.3

The input-output extendible stationary TPBVDS (C,P,E,A,B) is minimal if
and only if

(a) $[sE-tA \vdots PB \vdots B]$ has full row rank for all $(s,t) \neq (0,0)$ (2.5.73a)

(b) $\begin{bmatrix} sE-tA \\ CP \\ C \end{bmatrix}$ has full column rank for all $(s,t) \neq (0,0)$ (2.5.73b)

(c) $\mathcal{O}_s \subset \mathcal{R}_s$. (2.5.73c)

Proof

The proof here is similar in structure to the proof of Theorem 2.5.2,
although a bit more involved. We have to show that conditions (2.5.1)-(2.5.3)
applied to any paticular member (C,V_i,V_f,E,A,B,N), with $N \geq 2n$, of the family
of input-output extendible stationary TPBVDS's (C,P,E,A,B), are equivalent to
conditions (2.5.73). Condition (2.5.73a) is equivalent to

$$\underset{0 \leq k < n}{U} \ Im\{E^{n-1-k}A^k[PB \vdots B]\} = \mathbb{R}^n.$$ (2.5.74)

We would first like to show that (2.5.73a) (or equivalently (2.5.74)) and
(2.5.73c) imply (2.5.1). Let v be a vector such that

$$v'R_w = 0$$ (2.5.75)

where R_w is the weak reachability matrix of any member $(C,V_i,V_f E,A,B,N)$ of
(C,P,E,A,B). Equation (2.5.75) implies that

$$v'R_s = 0.$$ (2.5.76)

As argued in the proof of the Theorem 2.5.2, we have that

$$v' \in Row\text{-}Im(\mathcal{O}_s).$$ (2.5.77)

Using (2.5.77) and (2.3.73a) we get

$$v'[\bigcup_{0 \le k < n} \mathrm{Im}\{E^{n-1-k}A^k[PB \vdots B]\}] =$$

$$v'[\bigcup_{0 \le k < n} \mathrm{Im}\{E^{n-1-k}A^k[V_iE^N B \vdots V_f A^N B]\}] = 0, \qquad (2.5.78)$$

which is the desired result. Similarly, we can show that (2.5.73b) and

(2.5.73c) imply (2.5.2). Thus we have shown that conditions (2.5.73) imply

(2.5.1)-(2.5.3) applied to any member of the family (C,P,E,A,B).

Conversely, let v be a vector such that

$$v'[\bigcup_{0 \le k < n} \mathrm{Im}\{E^{n-1-k}A^k[PB \vdots B]\}] = 0 \qquad (2.5.79)$$

which together with (2.5.3) implies that

$$v'R_s = 0 \qquad (2.5.80)$$

which implies (2.5.77). But then (2.3.73a) yields

$$v'[\bigcup_{0 \le k < n} \mathrm{Im}\{E^{n-1-k}A^k[V_iE^N B \vdots V_f A^N B]\}] = 0 \qquad (2.5.81)$$

where (C,V_i,V_f,E,A,B,N) is any member of the family (C,P,E,A,B). Equation

(2.5.81) using (2.2.13), (2.3.47) and the generalized Cayley-Hamilton theorem

gives us the following

$$v'(E^r A^s V_i E^t A^u B) = v'(E^r A^s V_f E^t A^u B) = 0 \qquad (2.5.82)$$

for all (r,s,u,t), which clearly contradicts (2.5.1). Thus we have shown that

(2.5.1) and (2.5.3) imply (2.5.73a). Similarly we can show that (2.5.73b) is

implied by (2.5.2)-(2.5.3) and thus proving the theorem.

Corollary

Let $(C_j, P_j, E_j, A_j, B_j)$, j=1,2, represent 2 minimal input-output extendible

stationary TPBVDS's realizing the same weighting pattern, where $\{E_j, A_j\}$,

j=1,2, are in standard form for the same α and β. Then there exists an

invertible matrix T such that (2.5.42) and (2.5.44) hold and such that

$$O_s^1 (P_1 - T^{-1} P_2 T) R_s^1 = 0 \qquad (2.5.83)$$

where R_s^1 and O_s^1 denote the strong reachability and observability matrices of

the system 1.

Proof

Let $(C_j, V_j^i, V_j^f, E_j, A_j, B_j, N)$, $N \geq 2n$, be a member of $(C_j, P_j, E_j, A_j, B_j)$, $j=1,2$,

then equations (2.5.42) and (2.5.44) must hold thanks to Corollary 5.1a. Also

(2.5.43) implies that

$$O_s^1 V_1^i R_s^1 = O_s^2 V_2^i R_s^2 \qquad (2.5.84)$$

which thanks to Lemma 2.5.2 implies that

$$O_s^1 V_1^i E_1^N R_s^1 = O_s^2 V_2^i E_2^N R_s^2 \qquad (2.5.85)$$

which in turn thanks to Theorem 2.3.8 implies that

$$O_s^1 P_1 R_s^1 = O_s^2 P_2 R_s^2 \qquad (2.5.86)$$

which clearly implies (2.5.83).

Note that what Theorem 2.5.2 says is that one can consider minimality

within the smaller class of stationary systems. One might ask the same

question about the class of displacement systems. This question, however,

remains open. Specifically, if we start with a displacement system and follow

the reduction procedure described in the proof of Theorem 2.5.1, we do not

necessarily end up with a displacement system. Note, however that from

Corollary 2.5.1a we see that there may be a certain level of nonuniqueness in

minimal realizations -- both in the state space isomorphism T and, more

importantly, in the boundary and system matrices. A conjecture that remains open is that one can use this freedom to choose a minimal realization that is also a displacement system.

2.6-Block Standard and Normalized Forms

A well-known result for causal systems is the following. Suppose that the A-matrix is block diagonalized with no common eigenvalues among the blocks. Then reachability and observability of the entire system is equivalent to the reachability and observability of all of the individual subsystems defined by the block structure of A. The same type of result is easily shown to hold as well for stationary TPBVDS's once we define generalized notions of standard and normalized form.

Definition 2.6.1

The regular pencil $\{E,A\}$ is in block standard form if

(i)   for some invertible matrix T, we have

$$TET^{-1} = \text{diag}(E_1, E_2, \ldots, E_M) \qquad (2.6.1)$$

$$TAT^{-1} = \text{diag}(A_1, A_2, \ldots, A_M) \qquad (2.6.2)$$

where

(ii) each $\{E_i, A_i\}$ pair is in standard form, i.e. there exist α_i, β_i such that

$$\alpha_i E_i + \beta_i A_i = I \quad , \quad i=1,\ldots,M \qquad (2.6.3)$$

and furthermore $\{E_i, A_i\}$ and $\{E_j, A_j\}$, $i \neq j$, have no eigenmode in common. That is, for any pair $(s,t) \neq (0,0)$, $|sE_i - tA_i| = 0$ for at most one value of i.

If a system is in block standard form, we can always, by a change of the coordinate system, transform it such that E and A are block diagonal as in (2.6.1) and (2.6.2). Thus for simplicity we can assume that if the system is in block standard form, E and A are block diagonal.

Note that E and A in (2.6.1), (2.6.2) commute, and from the proof of the well-posedness result in [1], we can readily check that well-posedness of (2.2.1)-(2.2.3) when E and A commute is equivalent to the invertibility of $V_iE^N+V_fA^N$. Consequently, if this is true we can premultiply (2.2.2) by the inverse of this matrix to obtain a generalization of normalized form:

Definition 2.6.2

The TPBVDS (2.2.1)-(2.2.2) is in block normalized form (BNF) if {E,A} is in block standard form and (2.2.5) holds.

In general, there is no reason for V_i and V_f to be block-diagonal for a system in BNF. However, in the stationary case we have the following result:

Theorem 2.6.1

A TPBVDS in BNF is stationary if and only if it has a representation where V_i and V_f are in the same block diagonal form as E and A, i.e.

$$TV_iT^{-1} = diag(V_1^i,\ldots,V_M^i) \qquad (2.6.4)$$
$$TV_fT^{-1} = diag(V_1^f,\ldots,V_M^f) \qquad (2.6.5)$$

and moreover, each of the subsystems $(C_k,V_k^i,V_k^f,E_k,A_k,B_k,N)$ is stationary.

As before, we have an immediate corollary:

Corollary

A TPBVDS in BNF is displacement if and only if V_i and V_f are in the same block-diagonal form (2.6.4), (2.6.5) as E and A, and moreover, each of the subsystems $(C_k,V_k^i,V_k^f,E_k,A_k,B_k,N)$ is displacement.

Proof of Theorem 2.6.1

Consider a TPBVDS in BNF and assume without loss of generality that E and A are in block-form (2.6.1), (2.6.2), respectively. We first prove the following:

Lemma 2.6.1

The strong reachability and observability matrices of the overall system have the following form

$$R_s = \text{diag}(R_s^1, \ldots R_s^M).W \tag{2.6.6}$$

$$O_s = Z.\text{diag}(O_s^1, \ldots O_s^M) \tag{2.6.7}$$

where W and Z are invertible matrices and R_s^k and O_s^k are strong reachability and strong observability matrices of the kth block of the system.

Proof

We begin by putting the pencil in standard form by premultiplying E and A by $(\alpha E + \beta A)^{-1}$ for some α and β. Note that $(\alpha E + \beta A)^{-1}$ is block-diagonal, as are the new E and A matrices. Indeed all we have done is to modify the system so that (2.6.3) is satisfied with all α_i equal to α and all β_i equal to β. Suppose $\alpha \neq 0$ (otherwise reverse the roles of E and A). It is not difficult to check in this case that the condition that no two blocks of E and A have the same eigenmode now implies that no two blocks of A have the same eigenvalue. Also in this case

$$\mathcal{R}_s = \text{Im}[B \vdots AB \vdots \ldots \vdots A^{n-1}B] \tag{2.6.8}$$

(replace E by $(I-\beta A)/\alpha$ in R_s and use the usual Cayley-Hamilton theorem).

Equation (2.6.6) then follows from the usual causal system result. Equation (2.6.7) can be verified similarly.

Note that Lemma 2.6.1 demonstrates the equivalence of strong reachability/observability of the overall system and of all of the subsystems. Also, since every block is in standard form we can see that the strong reachability and observability spaces are E- and A-invariant, as when E and A are in standard form.

An examination of the proof of Theorem 2.2.1 shows that if we simply assume that E and A commute and that $\mathcal{R}_s$ and $\mathcal{O}_s$ are E- and A-invariant, the necessary and sufficient conditions for stationarity are

$$O_s[EV_iA-AV_iE]R_s = 0 \qquad (2.6.9)$$

$$O_s[EV_fA-AV_fE]R_s = 0. \qquad (2.6.10)$$

Consider next the following modification of our TPBVDS. Specifically, we keep C, E, A, B the same and simply null out the off-diagonal blocks of V_i and V_f. That is, let

$$V_i = \begin{bmatrix} V_{11}^i & \cdots & V_{1m}^i \\ \vdots & & \vdots \\ V_{M1}^i & \cdots & V_{MM}^i \end{bmatrix} \qquad (2.6.11)$$

with the blocks of V_f defined similarly. Then let

$$\tilde{V}_i = \text{diag}(V_{11}^i, V_{22}^i, \ldots, V_{MM}^i) \qquad (2.6.12)$$

$$\tilde{V}_f = \text{diag}(V_{11}^f, V_{22}^f, \ldots, V_{MM}^f). \qquad (2.6.13)$$

What we wish to show is that $(C, \tilde{V}_i, \tilde{V}_f, E, A, B, N)$ is in BNF and has the same weighting pattern as the original system.

The fact that it is in BNF follows immediatly since we have not changed E and A and

$$\tilde{V}_i E^N + \tilde{V}_f A^N = V_i E^N + V_f A^N = I. \tag{2.6.14}$$

Thus what we need to show is that

$$O_s \tilde{V}_i R_s = O_s V_i R_s \tag{2.6.15}$$

$$O_s \tilde{V}_f R_s = O_s V_f R_s \tag{2.6.16}$$

or thanks to (2.6.6) and (2.6.7) that

$$O_s^k V_{kj}^i R_s^j = 0 \qquad j{\neq}k \tag{2.6.17}$$

$$O_s^k V_{kj}^f R_s^j = 0 \qquad j{\neq}k. \tag{2.6.18}$$

We focus on (2.6.17), as (2.6.18) follows similarly.

From (2.6.9) we immediately find that for $j{\neq}k$

$$O_s^k [E_k V_{kj}^i A_j] R_s^j = O_s^k [A_k V_{kj}^i E_j] R_s^j. \tag{2.6.19}$$

Recall that $\{E_j, A_j\}$ and $\{E_k, A_k\}$ are in standard form, and indeed by a block-diagonal transformation we can assume that $\alpha E_j + \beta A_j = \alpha E_k + \beta A_k = I$ for a single, given pair α and β. Without any loss of generality we can assume that this is true. Furthermore, assume that $\alpha{\neq}0$ (otherwise reverse the roles of E and A), so that

$$E_j = \gamma I + \delta A_j \quad , \quad E_k = \gamma I + \delta A_k. \tag{2.6.20}$$

Using (2.6.20) in (2.6.19) implies that

$$O_s^k [V_{kj}^i A_j] R_s^j = O_s^k [A_k V_{kj}^i] R_s^j. \tag{2.6.21}$$

Since $\mathscr{R}_s^j$ is A_j-invariant and $\mathcal{O}_s^k$ is A_k-invariant, we have that (2.6.21) implies that

$$O_s^k [V_{kj}^i p(A_j)] R_s^j = O_s^k [p(A_k) V_{kj}^i] R_s^j \tag{2.6.22}$$

for any polynomial p. Take any generalized eigenvector v of A_j in $\mathscr{R}_s^j$ corresponding to the eigenvalue λ_j of A_j. then there is an integer m so that

$$(\lambda_j I - A_j)^m v = 0. \tag{2.6.23}$$

Let $p(x)=(\lambda_j-x)^m$. Also, let w be any generalized left-eigenvector of A_k in $(\mathcal{O}_s^k)^\perp$ corresponding to the eigenvalue μ_k of A_k. Then, from (2.6.22) we have that

$$0 = w'V_{kj}^i p(A_j)v = w'p(A_k)V_{kj}^i v = (\lambda_j-\mu_k)^m w'V_{kj}^i v. \qquad (2.6.24)$$

Thanks to (2.6.20) and the fact that $\{E_j,A_j\}$ and $\{E_k,A_k\}$ have no eigenmodes in common, $(\lambda_j-\mu_k)^m\neq0$, so we can conclude that

$$w'V_{kj}^i v = 0. \qquad (2.6.25)$$

But, since $\mathcal{R}_s^j$ is A_j-invariant and $\mathcal{O}_s^k$ is A_k-invariant, the columns of R_s^j and rows of O_s^k are spanned by such v's and w's, respectively, yielding (2.6.17).

Note that if the overall system (and therefore at least one of the subsystems) is not both strongly reachable and observable, there is some freedom in the choices of V_i and V_f. What the theorem says is that we can always choose these to be block-diagonal. If, however, all of the subsystems are strongly reachable and observable then the only possibility is for V_i and V_f to be block-diagonal. This is what happens in the Corollary (which, as before, corresponds to the case B=C=I). Note also, that since we can always take the boundary matrices to be block-diagonal, we have, as in the causal case, the fact that minimality of the overall system is equivalent to minimality of all of the subsystems.

There are several other important consequences of this theorem. First, note that for a stationary TPBVDS in BNF with V_i and V_f as in (2.6.4), (2.6.5), Theorem 2.6.1 and Theorem 2.2.1, applied to each subsystem, allow us to deduce that in fact not only does (2.6.9) hold, but so does (2.2.13). This in turn allows us to obtain the simple form for the weighting pattern given in (2.2.40).

Lemma 2.6.1 allows us to study strong reachability and observability of individual eigenmodes. To see this, consider a TPBVDS transformed into the following normalized or block normalized form[1] where

$$E = \text{diag}(E_1, \ldots, E_M) \qquad (2.6.26a)$$

$$A = \text{diag}(A_1, \ldots, A_M) \qquad (2.6.26b)$$

where $\{E_i, A_i\}$ has a unique eigenmode σ_i with $\sigma_i \neq \sigma_j$ for $i \neq j$. Then we say that the eigenmode σ_j is strongly reachable if (E_j, A_j, B_j) is strongly reachable (i.e. R_s^j has full rank). It can easily be verified that σ_j is strongly reachable if and only if

$$[\sigma_j E - A \vdots B]$$

has full row rank ($\sigma_j = \infty$ is strongly reachable if and only if $[E \vdots B]$ has full row rank). Similarly, we say that an eigenmode σ_j is strongly observable if (C_j, E_j, A_j) is strongly observable (i.e. O_s^j has full rank). Eigenmode σ_j is strongly observable if and only if

$$\begin{bmatrix} \sigma_j E - A \\ C \end{bmatrix}$$

has full column rank ($\sigma_j = \infty$ is strongly observable if and only if $\begin{bmatrix} E \\ C \end{bmatrix}$ has full column rank). In the displacement case, the boundary matrices are also in block diagonal form:

$$V_i = \text{diag}(V_1^i, \ldots, V_M^i) \qquad (2.6.27a)$$

$$V_f = \text{diag}(V_1^f, \ldots, V_M^f). \qquad (2.6.27b)$$

---

[1]We can always transform any regular $\{E, A\}$ into the block form (2.6.26). Assume $\{E, A\}$ is in standard-form, then we find T such that TAT^{-1} and TET^{-1} are in real Jordan form (thanks to the standard-form, E and A can be put into Jordan form simultaneously). Then (2.6.26) can be obtained by reordering the Jordan blocks of TAT^{-1} and TET^{-1}.

The BNF (2.6.26)-(2.6.27) allows us to consider weak reachability and observability of individual eigenmodes. An eigenmode σ_j is called weakly reachable (observable) if subsystem j is weakly reachable (observable). Also σ_j is weakly reachable if and only if

$$[\sigma_j E-A \mid V_i B \mid V_f B]$$

has full row rank; it is weakly observable if and only if

$$\begin{bmatrix} \sigma_j E-A \\ CV_i \\ CV_f^i \end{bmatrix}$$

has full column rank.

We can also use Theorem 2.6.1 to obtain the following result:

Theorem 2.6.2

Consider a minimal, stationary TPBVDS, then any eigenmode of the strongly unreachable (unobservable) part of the system is also an eigenmode of the strongly reachable (observable) part of the system.

Proof

Suppose that σ_k is an eigenmode of the strongly unreachable part of the system but not of the strongly reachable part. Theorem 2.6.1 allows us to break-down the system into subsystems each one of which has a distinct eigenmode. In particular, let $\Sigma_k = (C_k, V_k^i, V_k^f, E_k, A_k, B_k, N)$ denote the subsystem associated to eigenmode σ_k. Then, since Σ_k is minimal, it has a strongly reachable part (otherwise, B_k must be zero, the subsystem has weighting pattern 0 and the minimal realization has dimension 0). Thus, σ_k is an

eigenmode of the strongly reachable part of Σ_k and of the original system. This, of course, is a contradiction.

Before closing this section, we should mention that the motivation behind introducing the concepts of block standard form and BNF has been the usefulness of the following block standard form

$$E = \text{diag}(I, I, A_b) \qquad (2.6.28a)$$

$$A = \text{diag}(A_f, U, I) \qquad (2.6.28b)$$

where the eigenvalues of A_f and A_b are all inside the unit circle and the eigenvalues of U on the unit circle. This particular block standard form has been used in [2,3,16] and shall be used in the next chapter for studying the stability and the stochastic realization problem for TPBVDS's. When $\{E,A\}$ has no eigenmode on the unit circle, the block standard form (6.2.28) is called the forward-backward stable form.

2.7-Conclusions

In this chapter we have developed some of the system-theoretic properties of two-point boundary-value descriptor systems. We have derived detailed characterizations of reachability, observability, and minimality with particular attention paid to the shift-invariant case. As had already been noted for continuous-time, non-descriptor boundary-value systems, minimality for TPBVDS's is a bit more complicated than for causal systems. Indeed there is a certain degree of nonuniqueness in minimal realizations. One open problem that we have noted concerns whether one can use this freedom to guarantee that a displacement system always has a minimal realization that is also displacement.

Another concept that we have introduced and studied in this paper is extendibility, i.e. the idea of thinking of a TPBVDS as being defined on a sequence of intervals of increasing length. Once one introduces such a notion, it becomes possible to talk about the realization (as opposed to partial realization) problem and asymptotic properties such as stability (see [16]). These subjects are studied in the next chapter.

Chapter III:

REALIZATION THEORY FOR INPUT-OUTPUT EXTENDIBLE, STATIONARY,

TWO-POINT BOUNDARY-VALUE DESCRIPTOR SYSTEMS

3.1-Introduction

In this chapter, we consider both the deterministic and the stochastic realization problem for input-output extendible, stationary TPBVDS's. In the previous chapter we saw that to a stationary TPBVDS (C,V_i,V_f,E,A,B,N) which is input-output extendible we can associate a family of input-output extendible stationary TPBVDS's denoted by (C,P,E,A,B) such that all members of this family have identical C, E, A and B matrices and identical weighting patterns restricted to their proper domains of definition. We simply refer to this family of systems as an input-output extendible stationary TPBVDS (C,P,E,A,B). The matrix P is called the projection matrix and contains all the information about the boundary conditions that is reflected in the weighting pattern. The weighting pattern associated with this family of systems, which naturally is defined from $-\infty$ to $+\infty$, is completely determined in terms of the projection matrix P and system matrices C, E, A and B. The fact that this weighting pattern is defined everywhere allows us to develop a realization theory (as opposed to a partial realization theory which could also be very interesting to consider) for input-output extendible stationary TPBVDS's.

This chapter consists of two more sections. In the first section, we consider the problem of deterministic realization and in the second the problem of stochastic realization.

3.2-Deterministic Realization Theory

The deterministic realization theory for input-output extendible stationary TPBVDS's consists of constructing an input-output extendible stationary TPBVDS (C,P,E,A,B) from its weighting pattern $W(k)$. In the causal case this problem is usually studied using the z-transform method. In the case considered here, however, the stationary weighting pattern W is acausal, i.e. $W(k)$ is in general non-zero for $k \leq 0$ as well as $k > 0$, and no stability constraint is imposed on W. For these reasons, the classical one-sided or two-sided z-transform techniques may not be applied to this problem. We shall use a variant of the z-transform technique to handle this difficulty. Using this transform technique we are able to obtain the degree of the minimal realization and construct the minimal realization directly.

In Section 2.3 we saw that the weighting pattern of an input-output extendible stationary TPBVDS (C,P,E,A,B) can be expressed as follows:

$$W(k) = \begin{bmatrix} CPE^D(AE^D)^{k-1}B & k > 0 \\ -C(I-P)A^D(EA^D)^{-k}B & k \leq 0 \end{bmatrix} \qquad (3.2.1)$$

where P, the projection matrix, must satisfy the following equations:

$$O_s(PA-AP)R_s = O_s(PE-EP)R_s = 0 \qquad (3.2.2a)$$

$$O_s(P-PEE^D)R_s = O_s((I-P)-(I-P)AA^D)R_s = 0. \qquad (3.2.2b)$$

We also obtained minimality conditions for this system (see Section 2.5). Specifically, we showed that an input-output extendible stationary TPBVDS is minimal if and only if

a) $[sE-tA \vdots PB \vdots B]$ has full row rank for all $(s,t) \neq (0,0)$ (3.2.3)

b) $\begin{bmatrix} sE-tA \\ CP \\ C \end{bmatrix}$ has full column rank for all $(s,t) \neq (0,0)$ (3.2.4)

c) $Ker(O_s) \subset Im(R_s)$.   (3.2.5)

The deterministic realization problem considered in this section can be formulated as follows: given an infinite sequence of matrices $W(k)$, find matrices C, E, A, B and P such that (3.2.1) and (3.2.2) hold. We are particularly interested in realizations (C,P,E,A,B) of lowest dimension i.e. those satisfying (3.2.3)-(3.2.5). The first problem we consider is under what conditions the sequence $W(k)$ admits a finite dimensional realization.

3.2.1-Realizability Conditions

In this section we study the conditions under which a given sequence $W(k)$ is realizable as the weighting pattern of a finite dimensional input-output extendible, stationary TPBVDS. At the same time we will propose a method for constructing such a TPBVDS.

Theorem 3.2.1

A sequence of matrices $W(k)$ is the weighting pattern of an input-output extendible stationary TPBVDS if and only if for some scalars α_i, β_i, n_1 and n_2.

$$W(n_1+j) = \sum_{i=1}^{n_1} \alpha_i W(n_1-i+j) \qquad \text{for all } j>0. \qquad (3.2.6)$$

$$W(-n_2+j) = \sum_{i=1}^{n_2} \beta_i W(-n_2+i+j) \qquad \text{for all } j\leq 0. \qquad (3.2.7)$$

Proof

The only if part is deduced easily from (3.2.1) and the usual Cayley-Hamilton result. To show the if part note that we can decompose $W(k)$ as follows

$$W_f(k) = u(k-1)W(k) \qquad (3.2.8)$$

$$W_b(k) = u(-k)W(k) \qquad (3.2.9)$$

where $u(k)=1$ for $k \geq 0$ and $u(k)=0$ otherwise. Clearly then

$$W(k) = W_f(k) + W_b(k). \qquad (3.2.10)$$

Thanks to (3.2.6) and (3.2.7), $W_f(k)$ and $W_b(k)$ can be realized by finite dimensional causal and anticausal systems, respectively. Let (C_f, A_f, B_f) and (C_b, A_b, B_b) be such realizations, i.e.

$$W_f(k) = C_f A_f^{k-1} B_f \qquad \text{for } k>0 \qquad (3.2.11a)$$

$$W_b(k) = C_b A_b^{-k} B_b \qquad \text{for } k \leq 0. \qquad (3.2.11b)$$

Then it is clear that input-output extendible stationary TPBVDS

$$(C,P,E,A,B) = \left([C_f \quad -C_b], \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} I & 0 \\ 0 & A_b \end{bmatrix}, \begin{bmatrix} A_f & 0 \\ 0 & I \end{bmatrix}, \begin{bmatrix} B_f \\ B_b \end{bmatrix} \right) \qquad (3.2.12)$$

realizes $W(k)$. This completes the proof of the theorem.

An input-output extendible stationary TPBVDS having a representation of the form (3.2.12) is called separable. The TPBVDS (3.2.12) can be realized over any desired interval as follows

$$\begin{bmatrix} I & 0 \\ 0 & A_b \end{bmatrix} x(k+1) = \begin{bmatrix} A_f & 0 \\ 0 & I \end{bmatrix} x(k) + \begin{bmatrix} B_f \\ B_b \end{bmatrix} u(k) \qquad (3.2.13)$$

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} x(0) + \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} x(N) = \begin{bmatrix} \xi_f \\ \xi_b \end{bmatrix} \qquad (3.2.14)$$

$$y(k) = [C_f \quad -C_b]x(k). \qquad (3.2.15)$$

The extendible, stationary and separable TPBVDS (3.2.13)-(3.2.15) realizes W(k), restricted to the interval [-N+1,N], for any N.

The method described in the proof of Theorem 3.2.1 can be used to realize any realizable sequence. The realization obtained, however, is not in general minimal and must be reduced (the minimal TPBVDS is not in general separable). In the next section, we will propose another method based on a transform theory and a factorization problem which yields directly a minimal realization.

3.2.2-(s,t)-Transform

One problem with using the z-transform in cases where the dynamic of the system is singular is that the infinite frequencies cannot be handled in the same way as other frequencies, even though in such systems (at least in the discrete case) there should be total symmetry between zero and infinite frequencies. For this reason, we propose the following transform

$$H(s,t) = \sum_{k=-\infty}^{+\infty} H(k) t^{k-1}/s^k. \qquad (3.2.16)$$

Clearly if H(s,t) exists, then it is strictly proper in (s,t) but not necessarily in s and t separately. Strictly proper in (s,t) means that for all s and t for which H(s,t) is defined

$$\lim_{\gamma \to \infty} H(\gamma s, \gamma t) = 0. \qquad (3.2.17)$$

This can be easily seen by noting that

$$H(\gamma s, \gamma t) = (1/\gamma) H(s,t). \qquad (3.2.18)$$

In the case in which we are interested, $H(s,t)$ has rational entries in s and t, and strictly proper in this case implies that the denominators of these entries have higher degrees[1] than their corresponding numerators. Note that the z-transform can be obtained from the (s,t)-transform simply by replacing (s,t) with $(z,1)$, and that the (s,t)-transform is obtained from the z-transform by replacing z with s/t and dividing the result by t, so that all rational matrices in z, proper or not, translate into proper rational matrices in (s,t). Thus the (s,t)-transform is proper for all the cases in which we are interested.

In the causal case, the z-transform has an important role in the realization problem. Specifically, the realization problem is reduced to the following factorization of the z-transform of the weighting pattern (impulse response) $H(z)$ of the system

$$H(z) = K(zI-F)^{-1}G \qquad (3.2.19a)$$

for some matrices K, F and G. Hence the realization problem in the causal case reduces to the factorization of a proper rational matrix. For the boundary value systems that we are considering, the situation is more complex: even though we do need to consider the following factorization of rational matrices (in s and t this time)

$$H(s,t) = K(sD-tF)^{-1}G, \qquad (3.2.19b)$$

the realization problem and the factorization problems are not identical.

Let W_f and W_b represent the causal and anticausal parts of $W(k)$ respectively (as defined in $(3.2.8)-(3.2.10)$), and let $W(k)$ be realized as in $(3.2.1)$. Then, thanks to $(3.2.2b)$, we can compute the transforms of $W_f(k)$ and

---

[1] By degree of a polynomial $p(s,t)$ we mean the degree in s and t, e.g. st^2 has degree 3. Clearly, the degree of $p(s,t)$ is just the usual degree of $p(t,t)$.

$W_b(k)$ as follows

$$W_f(s,t) = \sum_{k=1}^{\infty} (t^{k-1}/s^k)CPE^D(AE^D)^{k-1}B$$

$$= CPE^D(sI-tAE^D)^{-1}B = CP(sE-tA)^{-1}B \qquad (3.2.20a)$$

$$W_b(s,t) = \sum_{k=-\infty}^{0} -(t^{k-1}/s^k)C(I-P)A^D(EA^D)^kB$$

$$= C(I-P)A^D(sEA^D-tI)^{-1}B = C(I-P)(sE-tA)^{-1}B. \qquad (3.2.20b)$$

Note that in general $W_f(s,t)$ and $W_b(s,t)$ do not have the same regions of convergence[2]. However, we will consider their analytical extensions instead (while using the same notation). In that case

$$W_f(s,t) + W_b(s,t) = C(sE-tA)^{-1}B. \qquad (3.2.21)$$

Also observe that

$$[W_f(s,t) \quad W_b(s,t)] = C(sE-tA)^{-1}[PB \quad (I-P)B], \qquad (3.2.22)$$

$$\begin{bmatrix} W_f(s,t) \\ W_b(s,t) \end{bmatrix} = \begin{bmatrix} CP \\ C(I-P) \end{bmatrix}(sE-tA)^{-1}B. \qquad (3.2.23)$$

We shall see that factorizations (3.2.21), (3.2.22) and (3.2.23) are directly tied to the 3 Hankel matrices: $O_s R_s$, $O_s R_w$ and $O_w R_s$, respectively (see Theorem 3.2.2).

Note that given the sequence $W(k)$ we can compute $W_f(s,t)$ and $W_b(s,t)$, so that it appears (thanks to (3.2.21)) that, as in the causal case, the realization problem has been reduced to a factorization problem. This is only partly true, however, because the minimal TPBVDS is not necessarily strongly reachable or observable and thus the situation is more complex than in the

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[2] $W_f(s,t)$ and $W_b(s,t)$ have a common region of convergence only when $W(k)$ is summable (see Section 3.2.5) or when it is left- or right-sided (i.e. there exists σ such that $W(k)=0$ for $k>\sigma$ or $k<\sigma$). In that case, $W_f(s,t)+W_b(s,t)$ is just the (s,t)-transform of $W(k)$.

causal case. In fact we shall see that in general, to construct the realization, we first need to perform the 2 factorizations (3.2.22) and (3.2.23).

The factorization problem in the case where E is invertible is simple. Many ways of constructing the minimal factorization exist (see [17]). The dimension of the minimal factorization has also been studied and it is shown (e.g. [18]) that this dimension is equal to the McMillan degree of the rational transfer matrix. We shall see in the next section that similar results can be obtained for the case where E is not necessarily invertible.

3.2.3-Factorization of Rational Matrices in s and t

From causal realization theory, we know how to construct a minimal factorization of a strictly proper rational matrix H(z), i.e. finding matrices K, F and G with F having smallest possible dimension such that

$$H(z) = K(zI-F)^{-1}G. \qquad (3.2.24)$$

In that case, F, K and G are unique (except for similarity transformations).

The singular factorization problem is more complex: we want to find K, D, F, and G of lowest possible dimension such that a given rational matrix H(s,t) can be expressed as

$$H(s,t) = K(sD-tF)^{-1}G. \qquad (3.2.25)$$

Clearly, even with the assumption that (D,F) is in standard form i.e. for some α and β, $\alpha D+\beta F=I$, D and F are not unique. To insure uniqueness we must also choose α and β a priori. In essence, in the causal case we have done that by forcing D to be equal to I which corresponds to $\alpha=1$ and $\beta=0$. Any pair (α,β) is acceptable as long as $H(\alpha,-\beta)$ is defined.

Theorem 3.2.2

a) Let $H(s,t)$ be a rational matrix in s and t, then $H(s,t)$ is factorizable if and only if (3.2.18) holds for all $\gamma \neq 0$ and for all s and t such that $H(s,t)$ is defined.

b) Let $H(s,t)$ be factorizable, and let (α,β) be a pair of scalars such that $H(\alpha,-\beta)$ exists. Then there exists a unique minimal factorization of $H(s,t)$ (except for similarity transformations) such that

$$\alpha D + \beta F = I \tag{3.2.26}$$

$$H(s,t) = K(sD-tF)^{-1}G. \tag{3.2.27}$$

Moreover, the dimension μ of this minimal factorization is given by

$$\mu[H(s,t)] = \upsilon(H(\alpha z, 1-\beta z)) \tag{3.2.28}$$

where $\upsilon(.)$ denotes the usual McMillan degree, and where $H(\alpha z, 1-\beta z)$ is a strictly proper rational matrix in z.

Corollary

The factorization

$$H(s,t) = K(sD-tF)^{-1}G \tag{3.2.29}$$

is minimal if and only if (D,F,G) is strongly reachable and (K,D,F) is strongly observable. Moreover, the dimension of the minimal factorization is equal to the rank of the Hankel matrix $O_s R_s$ where O_s denotes the strong observability matrix (K,D,F) and R_s the strong reachability matrix (D,F,G).

Proof of Theorem

To show part a), notice that the only if part is clearly implied by (3.2.25). To show the if part, we need to construct a realization. For this let α and β be such that $H(\alpha,-\beta)$ exists. Now consider the rational matrix $H(\alpha z, 1-\beta z)$. This matrix is strictly proper in z because

$$\lim_{z\to\infty} H(\alpha z, 1-\beta z) = \lim_{z\to\infty} (1/z)H(\alpha,-\beta) = 0. \tag{3.2.30}$$

Thus it can be realized as

$$H(\alpha z, 1-\beta z) = K(zI-F)^{-1}G. \tag{3.2.31}$$

Now assume that $\alpha \neq 0$ (otherwise reverse the role of D and F) and let

$$w = \alpha/(\alpha t+\beta s) \tag{3.2.32a}$$

$$z = s/(\alpha t+\beta s). \tag{3.2.32b}$$

In this case

$$s = \alpha z/w \tag{3.2.33a}$$

$$t = (1-\beta z)/w, \tag{3.2.33b}$$

which implies that

$$H(s,t) = wH(\alpha z, 1-\beta z) = wK(zI-F)^{-1}G = K(sD-tF)^{-1}G, \tag{3.2.34}$$

where

$$D = (1/\alpha)I-(\beta/\alpha)F. \tag{3.2.35}$$

This is the desired realization, completing the proof of part a).

For part b), we have already done most of the work. Notice simply that the factorizations (3.2.31) and (3.2.34) with D defined in (3.2.35) are different only by a scalar mutiplication so that we can construct one from the other and thus the dimension and uniqueness property of the two must be identical.

Proof of Corollary

Note that factorization (3.2.31) is minimal if and only if (K,F) is observable and (F,G) is reachable, which since α is assumed to be nonzero, are equivalent to (K,D,F) strongly observable and (D,F,G) strongly reachable, respectively.

Also note that we have shown that, when $\alpha \neq 0$, $\mu(H(s,t))$ is equal to the McMillan degree of $H(\alpha z, 1-\beta z)$ as defined in (3.2.31). From results on causal realization theory (see e.g. [27]) we know that this McMillan degree is equal to the rank of the Hankel matrix

$$\hat{H} = \hat{O}.\hat{R} \tag{3.2.36}$$

where $\hat{O}$ and $\hat{R}$ are the observability matrix (K,F) and the reachability matrix (F,G). But with $\alpha \neq 0$, the nullspace of $\hat{O}$ coincides with that of O_s and the image of $\hat{R}$ with that of R_s. Thus, the rank of $\hat{H}$ must equal the rank of $O_s R_s$. This completes the proof of the theorem.

In the proof of Theorem 3.2.2 we have developed a factorization method for the factorizable matrix $H(s,t)$. Namely, first choose α and β for which $H(\alpha,-\beta)$ is defined. Then form $H(\alpha z, 1-\beta z)$ which is a strictly proper rational matrix in z. Factorize this matrix in the regular form (3.2.31) which gives us K, F and G. Finally compute D from (3.2.35).

The dimension μ of the minimal factorization can also be obtained directly from the matrix $H(s,t)$.

Theorem 3.2.3

The dimension of the minimal factorization of a factorizable $H(s,t)$ is equal to the degree of the least common multiple of the denominators of all the minors of $H(s,t)$.

Proof

First note that all the polynomials that appear in the numerators and the denominators of the entries and thus the minors of $H(s,t)$ are homogeneous, i.e. they have the following form

$$p(s,t) = \sum_{i=0}^{k} \alpha_i s^{k-i} t^i \qquad (3.2.37)$$

where k is the degree of p. This follows from condition (3.2.18). Moreover, thanks again to (3.2.18), the degree of the denominator of each entry of $H(s,t)$ is always one plus the degree of the numerator. Therefore, for the minors of $H(s,t)$ the degree of the denominator is the order of the minor plus the degree of the numerator.

Proceeding with the proof, suppose that $K(sD-tF)^{-1}G$ is a minimal factorization of $H(s,t)$. Without loss of generality we can assume that K, D, F and G have the following form (this can always be achieved by a similarity transformation[3]):

$$K = [K_1 \quad K_2], \quad D = \begin{bmatrix} D_1 & 0 \\ 0 & N \end{bmatrix}, \quad F = \begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix}, \quad G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} \qquad (3.2.38)$$

where N is nilpotent, and D_1 and F_2 are invertible. Now consider the rational

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[3] Since $\{D,F\}$ is in standard form, D and F can be simultaneously be put into real Jordan form. Reordering the eigenstructure of this real Jordan form yields (3.2.38).

matrix

$$H_1(s,t) = K_1(sD_1-tF_1)^{-1}G_1. \qquad (3.2.39a)$$

Note that

$$H_1(s,t) = (1/t)K_1(zD_1-F_1)^{-1}G_1 = (1/t)\tilde{H}_1(z) \qquad (3.2.39b)$$

where $z=s/t$. Since $H_1(s,t)$ can be obtained from $\tilde{H}_1(z)$ and vice versa, the dimension of the minimal factorization of $H_1(s,t)$ and $\tilde{H}_1(z)$ must be equal. But $\tilde{H}_1(z)$ is a strictly proper rational matrix in z and thus the dimension of its minimal factorization is equal to its McMillan degree, i.e. the degree of $a_1(z)$, the least common multiple of the denominators of the minors of $\tilde{H}_1(z)$ (see Chapter 3 of [45]). Also note that since D_1 is invertible

$$H_1(s,0) < \infty \qquad (3.2.40)$$

and thus t is not a factor of the denominator of any of the entries and consequently minors of $H_1(s,t)$. Let $p_1(s,t)$ denote the least common multiple of the denominators of the minors of $H_1(s,t)$, then t is not a factor of $p_1(s,t)$ and consequently the degree of $p_1(s,t)$ is just the degree (in z) of $p_1(z,1)$. But

$$p_1(z,1) = a_1(z) \qquad (3.2.41)$$

so that the degree of $p_1(s,t)$ equals the McMillan degree of $\tilde{H}_1(z)$ thus it corresponds to the dimension of D_1 and F_1 (see e.g. [27]).

For block 2 we proceed similarly: let

$$H_2(s,t) = K_2(sN-tF_2)^{-1}G_2. \qquad (3.2.42)$$

Then

$$H_2(0,t) < \infty \qquad (3.2.43)$$

because A_2 is invertible. So s is not a factor of the least common multiple of the denominators of the minors of H_2 denoted by $p_2(s,t)$. Thus, the degree of $p_2(s,t)$ is just the degree in t of $p_2(1,t)$ which, by analogy with the

previous case, is just the dimension N and F_2. Also note that $H_2(s,t)=\infty$ only at t=0 thanks to nilpotency of N and the fact that N and F_2 are in standard form (which imply that the eigenvalues of $sN-tF_2$ are just $t\lambda_j$ where λ_j is an eigenvalue of F_2). Thus,

$$p_2(s,t) = p_2(1,t) = t^{n_2} \qquad (3.2.44)$$

where n_2 denotes the dimension of N and F_2.

Noting that

$$H(s,t) = H_1(s,t) + H_2(s,t) \qquad (3.2.45)$$

and the fact that $p_1(s,t)$ and $p_2(s,t)$ have no common factors, we can easily deduce that the least common multiple $p(s,t)$ of the denominators of the minors of H satisfies

$$p(s,t) = p_1(s,t) \cdot p_2(s,t), \qquad (3.2.46)$$

which proves the theorem.

Example 3.2.1

Consider the following sequence

$$H(k) = \begin{bmatrix} -1 & k=0 \\ 1 & k=1 \\ 0 & \text{elsewhere} \end{bmatrix}. \qquad (3.2.47)$$

The corresponding (s,t)-transform is

$$H(s,t) = 1/s - 1/t \qquad (3.2.48a)$$

and the z-transform is

$$H(z) = -1 + 1/z. \qquad (3.2.48b)$$

Already we can see the advantage of using the (s,t)-transform; H(s,t) has poles at s=0 and at t=0 which means that the sequence has a zero and an infinite mode whereas H(z) has a pole only at z=0.

Applying Theorem 3.2.3 we can see that the degree of the minimal

factorization must be 2 (= degree of st). To construct the minimal

factorization simply choose $\alpha=\beta=1$ and perform the following factorization

$$H(z,1-z) = 1/z - 1/(1-z) = (1 \quad 1)(zI-\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix})^{-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad (3.2.49)$$

which implies that

$$K = (1 \quad 1), \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad G = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \qquad (3.2.50)$$

3.2.4-Direct Realization Method

In previous sections we have defined the minimal factorization and

minimal realization problems. From (3.2.21)-(3.2.23) we can see that the

minimal realization problem involves factorization – in fact several

factorizations – but, unlike for the causal case, the 2 problems are not

identical. In this section, we make the relationship between these clear as

we use the (s,t)-transform and the factorization method discussed in the

previous sections to obtain the degree and construct a minimal realization of

the realizable weighting pattern W(k).

Theorem 3.2.4

The dimension n of a minimal realization of W(k) is given by

$$n = \mu([W_f(s,t) \quad W_b(s,t)]) + \mu(\begin{bmatrix} W_f(s,t) \\ W_b(s,t) \end{bmatrix}) - \mu(W_f(s,t)+W_b(s,t))$$

$$(3.2.51)$$

where $\mu(.)$ denotes the degree of the minimal factorization.

Proof

Let (C,P,E,A,B) be a minimal realization of $W(k)$ and let ρ, ω and τ be defined as follows

$$\omega = \mu(C(sE-tA)^{-1}[PB \quad (I-P)B]) = \mu([W_f(s,t) \quad W_b(s,t)]) \qquad (3.2.52a)$$

$$\rho = = \mu(\begin{bmatrix} CP \\ C(I-P) \end{bmatrix}(sE-tA)^{-1}B) = \mu(\begin{bmatrix} W_f(s,t) \\ W_b(s,t) \end{bmatrix}) \qquad (3.2.52b)$$

$$\tau = \mu(C(sE-tA)^{-1}B) = \mu([W_f(s,t) + W_b(s,t)]). \qquad (3.2.52c)$$

From the corollary of Theorem 3.2.2, it follows that ρ, ω and τ are just the rank of Hankel matrices $O_s R_w$, $O_w R_s$ and $O_s R_s$ respectively, where

$$R_w = [E^{n-1}(PB \quad (I-P)B)\vdots\dots\vdots A^{n-1}(PB \quad (I-P)B)] \qquad (3.2.53a)$$

$$O_w = \begin{bmatrix} \begin{bmatrix} CP \\ C(I-P) \end{bmatrix}E^{n-1} \\ \vdots \\ \begin{bmatrix} CP \\ C(I-P) \end{bmatrix}A^{n-1} \end{bmatrix}. \qquad (3.2.53b)$$

Then from the minimality conditions (3.2.3)-(3.2.4), R_w and O_w have full rank which means that ρ and ω are the ranks of the strong reachability R_s and the strong observability matrices O_s respectively. Expression (3.2.51) then follows from (2.5.40).

Example 3.2.2

Consdier the following weighting pattern

$$W(k) = \begin{bmatrix} \alpha^k & k \geq 1 \\ \beta\alpha^k & k < 1 \end{bmatrix} \qquad (3.2.54)$$

where α and β are scalar parameters and $\alpha < 1$. Using Theorem 3.2.1, it is straightforward to verify that $W(k)$ is realizable. From Theorem 3.2.4, we can compute the dimension of minimal realizations of $W(k)$:

$$n = \mu([\alpha/(s-\alpha t) \quad \alpha\beta/(\alpha t-s)]) + \mu(\begin{bmatrix} \alpha/(s-\alpha t) \\ \alpha\beta/(\alpha t-s) \end{bmatrix}) - \mu[(1-\beta)\alpha/(s-\alpha t)]$$

$$= \begin{bmatrix} 1 + 1 - 1 = 1 & \text{for } \beta\neq 1 \\ 1 + 1 - 0 = 2 & \text{for } \beta=1 \end{bmatrix} .$$

(3.2.55)

When $\beta\neq 1$, a minimal realization (C,P,E,A,B) of W(k) is

$$(\alpha/(1-\beta),1/(1-\beta),1,\alpha,1).$$

The causal part $W_f(s,t)$ and the anticausal part $W_b(s,t)$ of W have the same pole, namely $s/t=\alpha$, that is why we can realize them both with just one eigenmode. The resulting realization is strongly reachable, strongly observable and non-separable. In general, any time a minimal realization is not separable, the causal and anticausal parts of W must share a common pole. On the other hand, if the causal and anticausal parts of W do not share any common pole, then all cooresponding minimal realizations are separable. In particular, this is the case when W(k) is summable which means that the causal part of W has poles inside the unit circle and the anticausal part of W has poles outside the unit circle. We shall further study this case later in this section.

When $\beta=1$, a minimal realization of W is

$$([\alpha \ -\alpha], \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, I, \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}).$$

This separable realization is not strongly reachable and it is not strongly observable. Notice that in the previous realization (for $\beta\neq 1$), as β approachs 1, the system matrices tend to infinity. Thus, in a sense, $\beta=1$ is a singularity point and we can see indeed that the dimension of minimal realizations of W is 2 only when β is exactly equal to 1. Minimality in this case is not a generic property. Gohberg and Kaashoek in their works [11,13] have studied conditions under which minimality is a generic property for boundary value linear systems. They call systems satisfying these conditions stably minimal.

In the proof of Theorem 3.2.4 we have seen that in computing n, we obtain the dimension of the strong reachability and observability matrices which allows us to determine whether the minimal realization is strongly reachable or strongly observable. Thus to do the actual realization, we need to consider three different cases:

a-The minimal system is strongly reachable

What this implies is the following. If we have a minimal realization (C,P,E,A,B) of $W(k)$, of dimension $n=\rho$, then

$$\begin{bmatrix} CP \\ C(I-P) \end{bmatrix} (sE-tA)^{-1}B = \begin{bmatrix} W_f(s,t) \\ W_b(s,t) \end{bmatrix} \qquad (3.2.56a)$$

is a minimal factorization of $\begin{bmatrix} W_f(s,t) \\ W_b(s,t) \end{bmatrix}$. Thanks to the corollary of Theorem 3.2.2, we can conclude that <u>any</u> minimal factorization of $\begin{bmatrix} W_f(s,t) \\ W_b(s,t) \end{bmatrix}$ yields a minimal <u>realization</u> of $W(k)$. Note also that since $\rho=n$, any such minimal realization is strongly reachable. Thus our construction is as follows: for any fixed α and β such that $W_f(\alpha,-\beta)$ and $W_b(\alpha,-\beta)$ are defined construct a minimal factorization

$$\begin{bmatrix} W_f(s,t) \\ W_b(s,t) \end{bmatrix} = \begin{bmatrix} C_f \\ C_b \end{bmatrix} (sE-tA)^{-1}B \qquad (3.2.56b)$$

such that $\alpha E + \beta A = I$. From (3.2.56a), the matrix C is given by

$$C = C_f + C_b. \qquad (3.2.57a)$$

To find P note from (3.2.56a) that P must satisfy

$$CP = C_f. \qquad (3.2.57b)$$

Also since R_s has full rank, condition (3.2.2a) becomes

$$O_s(PE-EP) = O_s(PA-AP) = 0. \qquad (3.2.58)$$

From (3.2.57b) and (3.2.58) we can see that P can be chosen to be any solution to

$$O_s P = O_s^f, \qquad (3.2.59)$$

where

$$O_s^f = (C_f, E, A) = \begin{bmatrix} C_f A^{n-1} \\ C_f E A^{n-2} \\ \vdots \\ C_f E^{n-1} \end{bmatrix}. \qquad (3.2.60)$$

Of course we are guaranteed that there exists a P satisfying (3.2.59). Furthermore, the part of P which is not determined from (3.2.59) is exactly the degree of freedom that exists in the selection of P (see the corollary of Theorem 2.5.3).

b-The minimal system is strongly observable ($\omega=n$)

By analogy with the previous case we construct the factorization

$$[W_f(s,t) \quad W_b(s,t)] = C(sE-tA)^{-1}[B_f \quad B_b]. \qquad (3.2.61)$$

Then

$$B = B_f + B_b, \qquad (3.2.62)$$

and P is any matrix satisfying

$$PR_s = R_s^f \qquad (3.2.63)$$

where

$$R_s^f = (E, A, B_f) = [A^{n-1}B_f \mid EA^{n-2}B_f \mid \ldots \mid E^{n-1}B_f]. \qquad (3.2.64)$$

c-The minimal system is neither strongly reachable or strongly observable

This case is slightly more complicated because E and A cannot be directly obtained from a minimal factorization; this can be seen by noting that E and A are not even uniquely determined in this case (see Section 2.5). The factorizations that we have discussed, in this case only partially characterize the system matrices. To see this, suppose that (C,P,E,A,B) is a minimal realization of $W(k)$ and let us do a 4-part Kalman decomposition of it. Thanks to Theorem 2.6.2, this realization has no strongly unreachable and unobservable part. Thus, it can be represented as follows

$$A = \begin{bmatrix} A_1 & A_4 & A_6 \\ 0 & A_2 & A_5 \\ 0 & 0 & A_3 \end{bmatrix} \tag{3.2.65a}$$

$$E = \begin{bmatrix} E_1 & E_4 & E_6 \\ 0 & E_2 & E_5 \\ 0 & 0 & E_3 \end{bmatrix} \tag{3.2.65b}$$

$$B = \begin{bmatrix} B_1 \\ B_2 \\ 0 \end{bmatrix} \tag{3.2.65c}$$

$$C = \begin{bmatrix} 0 & C_2 & C_3 \end{bmatrix} \tag{3.2.65d}$$

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix}. \tag{3.2.66}$$

By direct calculation we can show that

$$\begin{bmatrix} W_f(s,t) \\ W_b(s,t) \end{bmatrix} =$$

$$\begin{bmatrix} C_2 P_{21} + C_3 P_{31} & C_2 P_{22} + C_3 P_{32} \\ -C_2 P_{21} - C_3 P_{31} & C_2 (I-P_{22}) - C_3 P_{32} \end{bmatrix} \left(s \begin{bmatrix} E_1 & E_4 \\ 0 & E_2 \end{bmatrix} - t \begin{bmatrix} A_1 & A_4 \\ 0 & A_2 \end{bmatrix} \right)^{-1} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \tag{3.2.67a}$$

$$\begin{bmatrix} W_f(s,t) & W_b(s,t) \end{bmatrix} =$$

$$\begin{bmatrix} C_2 & C_3 \end{bmatrix} \left(s \begin{bmatrix} E_2 & E_5 \\ 0 & E_3 \end{bmatrix} - t \begin{bmatrix} A_2 & A_5 \\ 0 & A_3 \end{bmatrix} \right)^{-1} \begin{bmatrix} P_{21}B_1 + P_{22}B_2 & -P_{21}B_1 + (I-P_{22})B_2 \\ P_{31}B_1 + P_{32}B_2 & -P_{31}B_1 - P_{32}B_2 \end{bmatrix}. \tag{3.2.67b}$$

and

$$W_f(s,t) + W_b(s,t) = C(sE-tA)^{-1}B = C_2(sE_2-tA_2)^{-1}B_2. \qquad (3.2.67c)$$

Factorizations (3.2.67) are minimal and thus if we perform minimal factorizations

$$[W_f(s,t) \quad W_b(s,t)] = \tilde{C}(s\tilde{E}-t\tilde{A})^{-1}[\tilde{B}_f \quad \tilde{B}_b]) \qquad (3.2.68a)$$

$$\begin{bmatrix} W_f(s,t) \\ W_b(s,t) \end{bmatrix} = \begin{bmatrix} \hat{C}_f \\ \hat{C}_b \end{bmatrix}(s\hat{E}-t\hat{A})^{-1}\hat{B} \qquad (3.2.68b)$$

for the same α and β (i.e. $\alpha\hat{E}+\beta\hat{A}=\alpha\tilde{E}+\beta\tilde{A}=I$), thanks to part b) of Theorem 3.2.2, we must have that matrices $(\tilde{C},\tilde{E},\tilde{A},[\tilde{B}_f \quad \tilde{B}_b])$ are related to the matrices

$$([C_2 \quad C_3], \begin{bmatrix} E_2 & E_5 \\ 0 & E_3 \end{bmatrix}, \begin{bmatrix} A_2 & A_5 \\ 0 & A_3 \end{bmatrix}, \begin{bmatrix} P_{21}B_1+P_{22}B_2 & -P_{21}B_1+(I-P_{22})B_2 \\ P_{31}B_1+P_{32}B_2 & -P_{31}B_1-P_{32}B_2 \end{bmatrix})$$

by a similarity transformation. Similarly, matrices $(\begin{bmatrix} \hat{C}_f \\ \hat{C}_b \end{bmatrix},\hat{E},\hat{A},\hat{B})$ are related to

$$(\begin{bmatrix} C_2P_{21}+C_3P_{31} & C_2P_{22}+C_3P_{32} \\ -C_2P_{21}-C_3P_{31} & C_2(I-P_{22})-C_3P_{32} \end{bmatrix}, \begin{bmatrix} E_1 & E_4 \\ 0 & E_2 \end{bmatrix}, \begin{bmatrix} A_1 & A_4 \\ 0 & A_2 \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix})$$

by a similarity transformation. Specifically, there exist invertible matrices V and W such that

$$\tilde{C}V = [C_2 \quad C_3], \quad V^{-1}\tilde{E}V = \begin{bmatrix} E_2 & E_5 \\ 0 & E_3 \end{bmatrix}, \quad V^{-1}\tilde{A}V = \begin{bmatrix} A_2 & A_5 \\ 0 & A_3 \end{bmatrix},$$

$$V^{-1}[\tilde{B}_f \quad \tilde{B}_b] = \begin{bmatrix} P_{21}B_1+P_{22}B_2 & -P_{21}B_1+(I-P_{22})B_2 \\ P_{31}B_1+P_{32}B_2 & -P_{31}B_1-P_{32}B_2 \end{bmatrix}), \qquad (3.2.69a)$$

and

$$\begin{bmatrix} \hat{C}_f \\ \hat{C}_b \end{bmatrix}W = \begin{bmatrix} C_2P_{21}+C_3P_{31} & C_2P_{22}+C_3P_{32} \\ -C_2P_{21}-C_3P_{31} & C_2(I-P_{22})-C_3P_{32} \end{bmatrix},$$

$$W^{-1}\hat{E}W = \begin{bmatrix} E_1 & E_4 \\ 0 & E_2 \end{bmatrix}, \quad W^{-1}\hat{A}W = \begin{bmatrix} A_1 & A_4 \\ 0 & A_2 \end{bmatrix}, \quad W^{-1}\hat{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}). \qquad (3.2.69b)$$

Now let

$$\tilde{B} = \tilde{B}_f + \tilde{B}_b \qquad (3.2.70a)$$

$$\hat{C} = \hat{C}_f + \hat{C}_b \qquad (3.2.70b)$$

then it can be seen that $(\tilde{C}, \tilde{E}, \tilde{A}, \tilde{B})$ and $(\hat{C}, \hat{E}, \hat{A}, \hat{B})$ are related respectively to

$$([C_2 \quad C_3], \begin{bmatrix} E_2 & E_5 \\ 0 & E_3 \end{bmatrix}, \begin{bmatrix} A_2 & A_5 \\ 0 & A_3 \end{bmatrix}, \begin{bmatrix} B_2 \\ 0 \end{bmatrix}) \text{ and } ([0 \quad C_2], \begin{bmatrix} E_1 & E_4 \\ 0 & E_2 \end{bmatrix}, \begin{bmatrix} A_1 & A_4 \\ 0 & A_2 \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix})$$

by similarity transformations V and W as well.

Note that factorizations $(\tilde{C}, \tilde{E}, \tilde{A}, \tilde{B})$ and $(\hat{C}, \hat{E}, \hat{A}, \hat{B})$ are strongly observable and strongly reachable respectively. Thus by performing a 4-part Kalman decompositions of $(\tilde{C}, \tilde{E}, \tilde{A}, \tilde{B})$ and $(\hat{C}, \hat{E}, \hat{A}, \hat{B})$, we obtain:

$$\tilde{C} = (\tilde{C}_1 \quad \tilde{C}_2), \ \tilde{A} = \begin{bmatrix} \tilde{A}_1 & \tilde{A}_2 \\ 0 & \tilde{A}_4 \end{bmatrix}, \ \tilde{E} = \begin{bmatrix} \tilde{E}_1 & \tilde{E}_2 \\ 0 & \tilde{E}_4 \end{bmatrix}, \ \tilde{B} = \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix} \qquad (3.2.71a)$$

$$\hat{C} = (0 \quad \hat{C}_2), \ \hat{A} = \begin{bmatrix} \hat{A}_1 & \hat{A}_2 \\ 0 & \hat{A}_4 \end{bmatrix}, \ \hat{E} = \begin{bmatrix} \hat{E}_1 & \hat{E}_2 \\ 0 & \hat{E}_4 \end{bmatrix}, \ \hat{B} = \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix}. \qquad (3.2.71b)$$

Note that

$$W_f(s,t) + W_b(s,t) = C(sE-tA)^{-1}B = C_2(sE_2-tA_2)^{-1}B_2 =$$
$$\tilde{C}(s\tilde{E}-t\tilde{A})^{-1}\tilde{B} = \hat{C}(s\hat{E}-t\hat{A})^{-1}\hat{B} \qquad (3.2.72)$$

which implies that

$$\tilde{C}_1(s\tilde{E}_1-t\tilde{A}_1)^{-1}\tilde{B}_1 = \hat{C}_2(s\hat{E}_4-t\hat{A}_4)^{-1}\hat{B}_2. \qquad (3.2.73)$$

But $(\tilde{C}_1, \tilde{E}_1, \tilde{A}_1, \tilde{B}_1)$ and $(\hat{C}_2, \hat{E}_4, \hat{A}_4, \hat{B}_2)$ are both strongly reachable and observable which implies that they must be related by a similarity transformation, i.e. for some invertible matrix T,

$$\hat{C}_2 T^{-1} = \tilde{C}_1, \ T\hat{A}_4 T^{-1} = \tilde{A}_1, \ T\hat{E}_4 T^{-1} = \tilde{E}_1, \ T\hat{B}_2 = \tilde{B}_1. \qquad (3.2.74a)$$

The matrix T can be computed as follows

$$T = \tilde{R}_s \hat{R}_s^{\cdot} (\hat{R}_s \hat{R}_s^{\cdot})^{-1} \qquad (3.2.74b)$$

where $\hat{R}_s$ and $\tilde{R}_s$ denote, respectively, the strong reachability matrices of $(\hat{E}_4, \hat{A}_4, \hat{B}_2)$ and $(\tilde{E}_1, \tilde{A}_1, \tilde{B}_1)$.

Thus, the C, E, A and B matrices of the minimal realization are given by

$$C=(0 \ \tilde{C}_1 \ \tilde{C}_2), \quad A=\begin{bmatrix} \hat{A}_1 & \hat{A}_2 T^{-1} & * \\ 0 & \tilde{A}_1 & \tilde{A}_2 \\ 0 & 0 & A_4 \end{bmatrix}, \quad E=\begin{bmatrix} \hat{E}_1 & \hat{E}_2 T^{-1} & * \\ 0 & \tilde{E}_1 & \tilde{E}_2 \\ 0 & 0 & E_4 \end{bmatrix}, \quad B=\begin{bmatrix} \hat{B}_1 \\ \tilde{B}_1 \\ 0 \end{bmatrix} \quad (3.2.75)$$

where * indicates an arbitrary matrix. Finally, to solve for P, let

$(C,V_i,V_f,E,A,B,2n-1)$ be a realization of $W(k)$ over an interval of length

$2n-1$. Then the boundary matrix V_i satifies

$$O_s PR_s = O_s V_i E^{2n-1} R_s. \quad (3.2.76)$$

From (2.2.40) we get that

$$O_s V_i R_s = \begin{bmatrix} (W_{kj}) \end{bmatrix}. \quad (3.2.77)$$

where

$$W_{kj} = W_f(2n-1-|k-j|). \quad (3.2.78)$$

Thus we can first compute a V_i from (3.2.77). Then P is obtained from

(3.2.76). Note that the nonunicity in the choice of P corresponds exactly to

the amount of freedom which is available in choosing P (see (2.5.83)) and so

any P satisfying (3.2.76) is a projection matrix. An alternative to (3.2.76)

for solving for P can be obtained as follows. Note that since $Im(R_s)$ is

E-invariant, there exists a matrix Z such that

$$ER_s = R_s Z \quad (3.2.79)$$

and thus

$$E^{2n-1} R_s = R_s Z^{2n-1} \quad (3.2.80)$$

which along with (3.2.76) and (3.2.77) yields

$$O_s PR_s = \begin{bmatrix} (W_{kj}) \end{bmatrix} Z^{2n-1}. \quad (3.2.81)$$

In summary, to construct a minimal realization in this case we have to

proceed as follows. First, perform factorizations (3.2.68) and use (3.2.70)

to construct $(\tilde{C},\tilde{E},\tilde{A},\tilde{B})$ and $(\hat{C},\hat{E},\hat{A},\hat{B})$. Then perform the decompositions

(3.2.71) and compute T from (3.2.74b). System matrices C, E, A, and B are then given by (3.2.75). Finally, compute P from (3.2.76) or (3.2.81).

Example 3.2.3

Let

$$W(k) = \begin{bmatrix} 2 & k=1 \\ 1 & \text{elsewhere} \end{bmatrix} \qquad (3.2.82)$$

then

$$W_f(s,t) = 1/s - 1/(s-t) = -t/[s(s-t)] \qquad (3.2.83a)$$

$$W_b(s,t) = 1/(s-t). \qquad (3.2.83b)$$

Applying Theorem 3.2.4 gives us the dimension of the minimal realization:

$$n = 2 + 2 - 1 = 3. \qquad (3.2.84)$$

It also tells us that the minimal realization is neither strongly reachable or strongly observable. Thus to obtain a minimal realization we follow procedure c) described above. First we perform the following 2 factorizations

$$[W_f \ W_b] = [-t/[s(s-t)] \ \ 1/(s-t)] = [1 \ 1](sI-t\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix})^{-1}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \qquad (3.2.85a)$$

$$\begin{bmatrix} W_f \\ W_b \end{bmatrix} = \begin{bmatrix} -t/[s(s-t)] \\ 1/(s-t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}(sI-t\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix})^{-1}\begin{bmatrix} -1 \\ 1 \end{bmatrix}. \qquad (3.2.85b)$$

We also find that

$$\tilde{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad (3.2.86a)$$

$$\hat{C} = [1 \ 0]. \qquad (3.2.86b)$$

In this case we can verify that T can be chosen to be just the identity matrix and the minimal realization is

$$C = [0 \ 1 \ 1], \ E = \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ A = \begin{bmatrix} 1 & 0 & * \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ B = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \ P = \begin{bmatrix} * & * & * \\ 0 & 1 & * \\ 1 & 0 & * \end{bmatrix} \qquad (3.2.87)$$

where * indicates entries that can be chosen arbitrarily. However, if we want the system (3.2.87) to be in normalized-form we must pick the * in E equal to zero.

The above approach to the construction of a minimal realization is worthwhile only if the resulting realization is not separable. The reason for this is that in the separable case, we can easily perform the realization as was done in the proof of Theorem 3.2.1. The problem is to find a way of recognizing that the minimal realization is separable before actually constructing this realization. The following result solves this problem:

Theorem 3.2.5

W(k) has a separable minimal realization if and only if

$$n = \mu(W_f(s,t)) + \mu(W_b(s,t)). \qquad (3.2.88)$$

Proof

First assume that W has a separable minimal realization, in which case clearly (3.2.88) holds. On the other hand suppose that (3.2.88) holds and realize W_f and W_b separately. Then putting the realizations for W_f and W_b in parallel clearly realizes W which must be minimal because n is the degree of the minimal realization.

In the next section, we consider another class of weighting patterns for which the realization procedure is simple. Namely, we consider stable systems, i.e. systems whose impluse response W(k) is summable. As will be shown below, these systems admit separable realizations where the forward and backward subsystems are forward and backward stable.

3.2.5-The Class of Stable TPBVDS's

In the case where the sequence $W(k)$ is summable, i.e. when

$$\sum_{k=-\infty}^{\infty} |W(k)| < \infty, \tag{3.2.89}$$

it turns out that the realizability condition, as well as finding the degree of the minimal realization and the realization procedure, are simpler than in the general case.

Theorem 3.2.6

a) A summable sequence $W(k)$ is realizable if and only if the (s,t)-transform of $W(k)$, $W(s,t)$, is rational in s and t.

b) A summable and realizable sequence $W(k)$ has a minimal realization which consists of a separable TPBVDS where the forward and backward subsystems are forward and backward stable respectively. Moreover, this realization is strongly reachable and observable and thus has the displacement property.

Proof

To show part a) suppose that $W(s,t)$ is rational. Note that

$$W(s,t) = \sum_{k=-\infty}^{\infty} W(k) t^{k-1}/s^k \tag{3.2.90}$$

is well-defined (i.e. has a region of convergence) thanks to (3.2.89). Also note that

$$W(s,t) = W_f(s,t) + W_b(s,t). \tag{3.2.91}$$

Since $W(s,t)$, $W_f(s,t)$ and $W_b(s,t)$ have a common region of convergence (which includes $|t/s|=1$). $W_f(s,t)$ and $W_b(s,t)$ must be rational as well. Note that $W_f(s,t)$ is analytic for $|s|\geq|t|$ and $W_b(s,t)$ for $|t|\geq|s|$. thus $W_f(s,t)$ and $W_b(s,t)$ have the following minimal factorizations

$$W_f(s,t) = C_f(sI-tA_f)^{-1}B_f \qquad (3.2.92a)$$

$$W_b(s,t) = C_b(sA_b-tI)^{-1}B_b \qquad (3.2.92b)$$

where A_f and A_b have eigenvalues inside the unit circle. Now consider the TPBVDS (3.2.13)-(3.2.15) with C_f. A_f. B_f. C_b. A_b and B_b as defined in (3.2.92). It is easy to check that the weighting pattern of this system is just $W(k)$ proving that $W(k)$ is realizable. The only if part is trivial.

To show part b) simply note that the realization constructed above is strongly reachable and observable and thus it is minimal.

Next, let us introduce the notion of stability for input-output extendible stationary TPBVDS's.

Definition 3.2.1

The input-ouptut extendible stationary TPBVDS (C,P,E,A,B) is called <u>stable</u> if it has a summable weighting pattern.

Essentially a stable system is a separable system where the forward and backward subsystems are forward and backward stable. A stable system has a number of interesting properties:

a- it has a stable minimal realization which is strongly reachable and observable,

b- let $(C, V_i, V_f, E, A, B, N)$, $N \geq 2n$, be any finite interval minimal realization of a stable minimal TPBVDS (C, P, E, A, B) then $(C, V_i, V_f, E, A, B, N)$ is strongly reachable and observable, extendible and separable. If in addition we assume that $\{E, A\}$ has been put in forward-backward stable form (see Section 2.6),

$$E = \begin{bmatrix} I & 0 \\ 0 & A_b \end{bmatrix}, \ A = \begin{bmatrix} A_f & 0 \\ 0 & I \end{bmatrix} \tag{3.2.93}$$

with A_f and A_b having eigenvalues inside the unit circle ($\{E, A\}$ cannot have any eigenmode on the unit circle because $W(s, t)$ has no poles on the unit circle), then the projection matrix P is given by

$$P = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \tag{3.2.94}$$

and the boundary matrices V_i and V_f are equal to P and I-P, respectively, regardless of the length of the interval N,

c- There exists a realization of the stable TPBVDS (C, P, E, A, B) defined on $[-\infty, +\infty]$. This realization denoted by $(C, V_i, V_f, E, A, B, \infty)$ has W(k) for weighting pattern,

d- the projection matrix P of a stable TPBVDS (C, P, E, A, B) is completely determined in terms of the pencil $\{E, A\}$, in fact,

$$P = E^\infty \tag{3.2.95}$$

if the system is in the forward-backward stable BNF (3.2.93)-(3.2.94).

From property c- we can see that the realization procedure for summable sequences just consists in performing the factorization

$$W(s, t) = C(sE - tA)^{-1} B \tag{3.2.96}$$

and transforming {E,A} into the forward-backward stable block standard form (3.2.93). In Section 3.3 we will see that for the problem of stochastic realization, we need only consider stable TPBVDS's.

3.2.6-Conclusion

In this section we have studied the problem of deterministic realization for input-output extendible, stationary TPBVDS's. We have obtained realizability conditions and proposed a method for realizing any realizable weighting pattern with a separable realization. This method however does not always yield a minimal realization. We then proposed a new transform technique, which is well adapted to handling noncausal weighting patterns. This allowed us to obtain a direct method for computing the degree of a minimal realization and for constructing such a realization. This approach generalizes the classical realization theory for causal systems.

In the next section, we shall consider the stochastic realization problem. As in the causal case, it turns out that there are strong ties between the two problems of deterministic and stochastic realization.

3.3-Stochastic Realization Theory

In this section we consider the problem of stochastic realization for extendible stationary systems driven by white noise that have stochastically stationary outputs. In particular, we first study conditions under which a stationary system is stochastically stationary (i.e. it has a stochastically stationary output) and stochastically extendible (i.e. it has an extendible output covariance). Then we show that the stochastic realization problem reduces to a factorization problem and finally we present minimality conditions for stochastic realizations.

Throughout this section we consider minimal, input-output extendible, stationary TPBVDS's in normalized form or block normalized form (BNF):

$$Ex(k+1) = Ax(k) + Bu(k) \tag{3.3.1a}$$

$$V_i x(0) + V_f x(N) = v \tag{3.3.1b}$$

$$y(k) = Cx(k) \tag{3.3.1c}$$

where $u(k)$ is a white, zero-mean, Gaussian unit-variance sequence and v is a zero mean, Gaussian random vector, independent of $u(k)$ for all k, with variance matrix Q. This stochastic TPBVDS is a generalization of the usual causal stochastic systems. Note however that in this case, unlike the causal case, even though v is assumed to be independent of $u(k)$, $x(k)$ is not a Markov process[1].

In [16] we have studied stochastic stationarity of the "state process"

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[1] The process $x(k)$ satisfying (3.3.1a)-(3.3.1b) is a reciprocal process [44].

x. In particular we have shown that x is a stochastically stationary process if and only if Q, the boundary variance matrix, satisfies the following generalized Lyapunov equation

$$EQE' - AQA' = V_i BB'V_i' - V_f BB'V_f',$$ (3.3.2a)

in which case, the variance matrix of x, P_x, satisfies

$$EP_x E' - AP_x A' = V_i E^N BB'(V_i E^N)' - V_f A^N BB'(V_f A^N)'.$$ (3.3.2b)

We have also shown the following lemma:

Lemma 3.3.1

If the pencil {E,A} has no reciprocal eigenmodes (i.e. if σ is an eigenmode, $1/\sigma$ is not) and no eigenmodes on the unit circle, then (3.3.2a) and (3.3.2b) have unique solutions.

The proof can be found in [2,16] and consists of first showing that if pencil {E,A} is in standard form and satisfies the conditions of the lemma, then $E \oplus E - A \oplus A$, where $\oplus$ denotes the Kronecker product, is invertible. Then, noting that if we form a vector ξ from the entries of Q (or P_x) by lexicographic ordering, the left hand side of (3.3.2a) (or (3.3.2b)) is just $(E \oplus E - A \oplus A)\xi$.

If the system does have reciprocal eigenmodes or eigenmodes on the unit circle, the "state" variance matrix P_x of a stochastically stationary system is not uniquely determined by (3.3.2a) and other methods must be used for the computation of P_x. Note that (3.3.2a) and (3.3.2b) both reduce to the standard Lyapunov equation in the causal case.

The problem that we are considering in this section is slightly different in that we are interested in studying systems that have a stochastically stationary output y and not necessarily a stochastically stationary x. In the causal case, since minimal systems are strongly observable, the two problems are exactly the same problem. In the TPBVDS case, however, as we have seen in Chapter II, minimal systems are not necessarily strongly observable. Thus, stochastic stationarity of the output in this case is a strictly weaker condition than stochastic stationarity of the "state". In the following section, we obtain necessary and sufficient conditions for stochastic stationarity of the output y.

3.3.1-Stochastic Stationarity and Extendibility

The first problem that we shall consider consists of determining conditions under which the output of (3.3.1) is a stochastically stationary process, in which case (3.3.1) is called a stochastically stationary TPBVDS.

Definition 3.3.1

The TPBVDS (3.3.1) is stochastically stationary if (with the usual abuse of notation)

$$\overline{y(k+j)y(k)'} = \Lambda(k+j,k) = \Lambda(j) \quad \text{for } 0 \leq k \leq N, \ 0 \leq j \leq N-k.$$

In the causal case, the stochastic stationarity of a system is tested by examining whether Q, the variance matrix of the initial condition, satisfies

a Lyapunov equation. In the case of system (3.3.1), we shall see that a similar test exists.

Theorem 3.3.1

The TPBVDS (3.3.1) is stochastically stationary if and only if Q satisfies

$$O_s(EQE'-AQA'-V_iBB'V_i'+V_fBB'V_f')O_s' = 0 \qquad (3.3.3)$$

where O_s is the strong observability matrix.

Note that if the system is observable, we can replace O_s in (3.3.3) with I and obtain the generalized Lyapunov equation (3.3.2a). If the system is causal as well, (3.3.3) reduces to the usual Lyapunov equation.

Proof

It is clear that (3.3.1) is stochastically stationary if and only if

$$\Lambda(k+1,i+1) = \Lambda(k,i) \qquad k,i\epsilon[0,N-1]. \qquad (3.3.4)$$

By writing out y explicitly in terms of u and v using (2.2.6) and (2.2.40), we find that for $j \geq m$,

$$\Lambda(j,m) = CA^jE^{N-j}QA'^mE'^{N-m}C' + \sum_{p=0}^{m-1} CV_iA^{j-p-1}E^{N-(j-p)}BB'A'^{m-p-1}E'^{N-(m-p)}V_i'C' -$$

$$\sum_{q=m}^{j-1} CV_iA^{j-q-1}E^{N-(j-q)}BB'A'^{N+(m-q)-1}E'^{-(m-q)}V_f'C' +$$

$$\sum_{r=j}^{N-1} CV_fA^{N+(j-r)-1}E^{-(j-r)}BB'A'^{N+(m-r)-1}E'^{-(m-r)}V_f'C', \qquad (3.3.5)$$

from which we obtain

$$\Lambda(k+1,i+1)-\Lambda(k,i) =$$

$$CA^k E^{N-k-1}(EQE'-AQA'-V_i BB'V_i'+V_f BB'V_f')(A^i E^{N-i-1})'C'. \qquad (3.3.6)$$

Now, taking into account expression (2.2.16) for O_s, we see that thanks to the generalized Cayley-Hamilton theorem, (3.3.6) and (3.3.3) are equivalent, thus completing this proof.

Equation (3.3.3) can in general have many positive semi-definite solutions. This situation arises in the causal case when the system has eigenmodes on the unit-circle. The case that we consider here is more complicated; Q is of course not unique when the system has eigenmodes on the unit circle, but there is also nonuniqueness when the system has reciprocal eigenmodes (i.e. if σ and σ^{-1} are both eigenmodes). Furthermore, the fact that minimal systems are not necessarily strongly observable is another source of nonuniqueness.

Theorem 3.3.2

Any TPBVDS obtained by moving in the boundaries of a stochastically stationary TPBVDS is stochastically stationary and the two systems (the original and the one that has been obtained by "moving-in") have identical output covariance sequences (where both are defined). In addition, if the systems are strongly reachable and observable, then the boundary variance

matrix of the "moved-in" system is uniquely determined in terms of the length of the interval over which it is defined.

Proof

Let TPBVDS (3.3.1) be stochastically stationary and let

$$Ex(k+1) = Ax(k) + Bu(k) \qquad (3.3.7a)$$

$$V_i(K,L)x(K) + V_f(K,L)x(L) = z_i(K,L) \qquad (3.3.7b)$$

$$y(k) = Cx(k) \qquad (3.3.7c)$$

be the system obtained by moving in the boundaries of (3.3.1) to the interval [K,L]. Moving in the boundaries, does not affect the mapping from {u,v} to y inside [K,L] (the contribution of u's outside [K,L] and v to y's inside [K,L] are through $z_i(K,L)$) and thus the output covariance of (3.3.7) over its domain of definition is identical to that of (3.3.1).

To show that when the TPBVDS (3.3.7) is strongly reachable and observable, its boundary variance matrix is uniquely determined in terms of L-K, note in this case, that $V_i(K,L)$ and $V_f(K,L)$ are uniquely determined in terms of L-K (see Section 2.3). Also note that, thanks to the first of part of the theroem and the assumption of strong observability, x(k) is stochastically stationary. Therefore, $\overline{x(K)x(L)}'$ is only a function of K-L and thus (3.3.7b) implies that the variance of $z_i(K,L)$ only depends on K-L. This proves the theorem.

Note that the boundary variance matrix for (3.3.7) is just the variance of $z_i(K,L)$ which can be obtained using the expression (2.3.7) for the inward process z_i.

In analogy with the determinitic extendibility concept studied in Chapter II, a natural question to ask here is under what conditions can we extend outwards the boundaries of a stochastically stationary TPBVDS.

Definition 3.3.2

The TPBVDS (3.3.1) is called stochastically extendible if for any $M>N$, there exists a stochastically stationary TPBVDS Σ_M defined over an interval of length M such that TPBVDS (3.3.1) can be obtained by moving in the boundaries of Σ_M.

Stochastic extendibility is a concept similar to deterministic extendibility for stationary systems: we want to be able to extend the domain of the system without modifying the statistics of its output. This allows us to associate an output covariance sequence $\Lambda(k)$, defined everywhere, to any stochastically extendible TPBVDS. Just as in the deterministic case, where we used $W(k)$ to study the problem of deterministic realization for input-output extendible stationary TPBVDS's, we shall use $\Lambda(k)$ to study the problem of stochastic realization for stochastically extendible TPBVDS's.

Example 3.3.1

Consider the following minimal, input-output extendible stationary
TPBVDS

$$x(k+1) = x(k) + u(k) \qquad (3.3.8a)$$

$$(1/2)x(0) + (1/2)x(N) = v \qquad (3.3.8b)$$

$$y(k) = x(k) \qquad (3.3.8c)$$

where $u(k)$ is a white Gaussian sequence of variance 1, and v is independent
of $u(k)$ and has variance q. It is straightforward to verify that this system
is stochastically stationary. Now let us check whether it is stochastically
extendible. For that, let M be some integer such that M>N. Suppose that there
exists a stochastically stationary TPBVDS defined over an interval of length
M, such that by moving in its boundaries we can recover (3.3.8). This TPBVDS
must have the form

$$x(k+1) = x(k) + bu(k) \qquad (3.3.9a)$$

$$(1/2)x(-J) + (1/2)x(M-J) = v_{JM} \qquad (3.3.9b)$$

$$y(k) = x(k) \qquad (3.3.9c)$$

where $[-J,M-J]$ contains $[0,N]$ (this is because the system matrices of a
stochastic extension are the same as that of a deterministic extension and
since in this case $E=A=1$, V_i and V_f do not change as we move in and out the
boundaries) and we must also have that the inward process for this system
must satisfy

$$\overline{z_i(0,N)z_i(0,N)'} = q. \qquad (3.3.10)$$

Noting that (3.3.9) has the displacement property and using expression
(2.3.21), we can show that

$$z_i(0,N) = v_{JM} + (1/2)(u(-J)+u(-J+1)+\ldots+u(-1))$$

$$- (1/2)(u(N)+u(N+1)+\ldots+u(M-J-1)). \qquad (3.3.11)$$

Using the fact that the $u(k)$'s are independent and have unit variance, we get

$$\overline{z_i(0,N)z_i(0,N)'} = \overline{v_{JM}v'_{JM}} + (1/4)(M-N). \qquad (3.3.12)$$

But from (3.3.10) and (3.3.12) we get

$$\overline{v_{JM}v'_{JM}} = q - (1/4)(M-N) = (q+N/4) - M/4 \qquad (3.3.13)$$

which since the left hand side of (3.3.13) must be non-negative, implies that

$$M \leq 4q + N. \qquad (3.3.14)$$

Thus this system can be statistically extended only up to a point but not indefinitely. This system is not stochastically extendible.

Theorem 3.3.3

The stochastically stationary TPBVDS (3.3.1) is stochastically extendible if and only if

a) (C,P,E,A,B), where P is a projection matrix of (3.3.1), is stable (as defined in Section 3.2.5),

b) its boundary variance matrix Q is invertible and satisfies

$$V_i'Q^{-1}V_f = 0. \qquad (3.3.15)$$

Proof

First we prove the only if part. Assume that the TPBVDS (3.3.1) is stochastically extendible. Then for any $M>N$, there exists an extension of (3.3.1) over an interval of length M having output covariance $\Lambda(j)$ identical to the output covariance of (3.3.1) for $-N+1 \leq j \leq N$. Let $W_M(j)$ denote the weighting pattern of Σ_M and v_M its boundary value vector having covariance

Q_M. Then

$$y(M/2) = CA^{M/2}E^{M/2-1}v_M + \sum_{j=0}^{M-1} W_M[(M/2)-j]u(j) \qquad (3.3.16)$$

which implies that

$$\Lambda(0) = \overline{y(M/2)y(M/2)'}$$

$$= CA^{M/2}E^{M/2-1}Q_M A'^{M/2}E'^{M/2-1}C' + \sum_{k=-(M/2)+1}^{M/2} W_M(k)W_M(k)'. \qquad (3.3.17)$$

But

$$W_M(k) = W(k), \quad \text{for} \quad -M+1 \leq k \leq M \qquad (3.3.18)$$

where $W(k)$ is the weighting pattern of (C,P,E,A,B). Thus using the fact that

$$CA^{M/2}E^{M/2-1}Q_M A'^{M/2}E'^{M/2-1}C' \geq 0, \qquad (3.3.19)$$

and that $\Lambda(0)$ does not depend on M. (3.3.17) implies that

$$\sum_{k=-\infty}^{\infty} W(k)W(k)' < \infty \qquad (3.3.20)$$

which implies the a) part of the theorem.

To show part b), observe first that since (C,P,E,A,B) is stable and minimal, it is strongly reachable, strongly observable and has displacement property (see Section 3.2.5). Thus we can assume that E, A and P have the following block forms (see (3.2.93)-(3.2.94))

$$E = \begin{bmatrix} I & 0 \\ 0 & A_b \end{bmatrix}, \quad A = \begin{bmatrix} A_f & 0 \\ 0 & I \end{bmatrix}, \quad P = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \qquad (3.3.21)$$

where A_b and A_f have eigenvalues inside the unit circle. Moreover, the boundary matrices $V_i(K,L)$ and $V_f(K,L)$ of all the elements of (C,P,E,A,B) are given by

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}.$$

respectively.

Let us now move in Σ_M one step, two ways: once by moving in the left boundary and once by moving in the right boundary. By doing so, thanks to Theorem 3.2.2, we should obtain the same boundary variance matrix for the "moved-in" systems. Using expression (2.3.21) for the inward process of a displacement TPBVDS, we can show that if Q_{M-1} denotes the boundary variance matrix of Σ_{M-1}, then by moving in the right boundary we get

$$Q_{M-1} = EQ_M E' + \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} BB' \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}. \tag{3.3.22}$$

By moving in the left boundary, we get

$$Q_{M-1} = AQ_M A' + \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} BB' \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}. \tag{3.3.23}$$

If we now let

$$B = \begin{bmatrix} B_f \\ B_b \end{bmatrix}, \quad Q_M = \begin{bmatrix} Q_M^{1,1} & Q_M^{1,2} \\ (Q_M^{1,2})' & Q_M^{2,2} \end{bmatrix}, \quad Q_{M-1} = \begin{bmatrix} Q_{M-1}^{1,1} & Q_{M-1}^{1,2} \\ (Q_{M-1}^{1,2})' & Q_{M-1}^{2,2} \end{bmatrix} \tag{3.3.24}$$

expressions (3.3.22) and (3.3.23) imply that

$$Q_M^{1,1} = Q_{M-1}^{1,1} = Q_f \tag{3.3.25a}$$

where Q_f is the solution of the following Lyapunov equation

$$Q_f - A_f Q_f A_f' = B_f B_f', \tag{3.3.25b}$$

and

$$Q_M^{2,2} = Q_{M-1}^{2,2} = Q_b \tag{3.3.26a}$$

where Q_b is the solution of the following Lyapunov equation

$$Q_b - A_b Q_b A_b' = B_b B_b'. \tag{3.3.26b}$$

They also imply that

$$Q_{M-1}^{1,2} = A_f Q_M^{1,2} = Q_M^{1,2} A_b' \tag{3.3.27}$$

which in turn implies that

$$Q^{1,2} = A_f^{M-N} Q_M^{1,2} \tag{3.3.28}$$

where $Q^{1,2}$ is the (1,2) block of Q, the boundary variance matrix of the

original TPBVDS (3.3.1). Note that since A_f is stable, if $Q^{1,2}$ is not zero, $Q_M^{1,2}$ must grow unbounded as M grows. But we have shown that $Q_M^{1,1}$ and $Q_M^{2,2}$ are constant which means that if $Q^{1,2}$ is not zero then Q_M cannot remain positive semi-definite as M grows. Thus $Q^{1,2}$ must be zero which proves part b) (note that if {E,A} has no reciprocal eigenmodes, (3.3.27) directly implies that $Q_M^{1,2}$ is zero for all M).

To show the if part, note that thanks to a) and b), we can suppose that the system is in the form (3.3.21) with

$$Q = \begin{bmatrix} Q_f & 0 \\ 0 & Q_b \end{bmatrix} \qquad (3.3.29)$$

where Q_f and Q_b satisfy (3.3.25b) and (3.3.26b). Then any element of (C,P,E,A,B) with boundary variance matrix given by (3.3.29) is a stochastic extension of (3.3.1). This implies that (3.3.1) is stochastically extendible and the theorem is proved.

Corollary

Let (3.3.1) be stochastically extendible, then there exists a family of stochastically extendible TPBVDS's $(C,V_i(M),V_f(M),E,A,B,M)$ for all M>0 such that

(a) TPBVDS (3.3.1) is a member of this family

(b) If $\Sigma_{M_1} = (C,V_i(M_1),V_f(M_1),E,A,B,M_1)$ and $\Sigma_{M_2} = (C,V_i(M_2),V_f(M_2),E,A,B,M_2)$ are any two members of this family and $M_2 > M_1$ then Σ_{M_1} is obtained by moving in the boundaries of Σ_{M_2}.

Proof

We have already constructed this family of systems. In particular, we have shown in the proof of the theorem that, without loss of generality, we can assume that (3.3.1) is in the form (3.3.21). It is not difficult to see in this case that all the deterministic extensions of (3.3.1), i.e. all the members of (C,P,E,A,B), with Q given by (3.3.29) are stochastically extendible regardless of the length of the interval over which they are defined. Also as we have shown in the proof of the theorem, that the boundary variance matrix Q remains unchanged as we move in the boundaries. Thus, the family of systems formed from the members of (C,P,E,A,B) with Q given by (3.3.29) satisfies the conditions of the corollary.

Corollary

The output covariance $\Lambda(j)$ associated to the stochastically extendible TPBVDS (3.3.1) is given by

$$\Lambda(j) = \sum_{k=-\infty}^{\infty} W(k+j)W(k)'. \tag{3.3.30}$$

Proof

Suppose, without loss of generality, that the system is in the forward-backward stable form (3.3.21). Let $M/2 > |j|$, then if Σ_M denotes the stochastic extension of (3.3.1) to the interval [0,M] with Q_M and W_M representing the boundary variance matrix and the weighting pattern of Σ_M,

respectively, we get

$$\Lambda(j) = \overline{y(j+M/2)y(M/2)'} =$$

$$A^{j+M/2}_E{}^{M/2-j}_{Q_M}A'^{,M/2}_E{}^{,M/2}_{C'} + \sum_{m=0}^{M-1} W_M(j+M/2-m)W_M(M/2-m)' =$$

$$A^{j+M/2}_E{}^{M/2-j}_{QA}{}'^{,M/2}_E{}^{,M/2}_{C'} + \sum_{m=0}^{M-1} W(j+M/2-m)W(M/2-m)' \qquad (3.3.31)$$

where Q is given by (3.3.29) and W(k) is the weighting pattern of (C,P,E,A,B). Then as M grows $A^{j+M/2}_E{}^{M/2-j}_{QA}{}'^{,M/2}_E{}^{,M/2}_{C'}$ tends to zero and thus in the limit as M goes to infinity, by letting k=M/2-m inside (3.3.31), we obtain (3.3.30).

Thus we see that to every stochastically extendible TPBVDS (3.3.1), we can associate a family of input-output extendible stationary TPBVDS's having identical output covariance restricted to their domain of definitions. We refer to this family of systems as the stochastically extendible TPBVDS (C,P,E,A,B) having $\Lambda(j)$ for output covariance. To see that the output covariance $\Lambda(j)$ is completely determined in terms of the matrices C, P, E, A and B, simply note (3.3.30) and (3.2.1).

Before ending this section, we shall demonstrate by the following example that condition b) of Theorem 3.3.3 is indeed needed to guarantee stochastic extendibility.

Example 3.3.2

Consider the minimal, extendible, displacement TPBVDS

$$x(k+1) = \begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix} x(k) + u(k) \qquad (3.3.32a)$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x(0) + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x(N) = v \qquad (3.3.32b)$$

$$y(k) = \begin{bmatrix} 1 & 1 \end{bmatrix} x(k) \qquad (3.3.32c)$$

where $u(k)$ is a white sequence of variance 1, and v is independent of $u(k)$ and has variance Q. It is easy to check that this system is stochastically stationary, if

$$Q = \begin{bmatrix} 1/3 & c \\ c & 4/3 \end{bmatrix} \qquad (3.3.33)$$

where c is any scalar smaller than $4/9$. The nonuniqueness of Q is due to the fact that the system has reciprocal eigenmodes, namely $1/2$ and 2 (see Lemma 3.3.1). By applying Theorem 3.3.3, we can see however that (3.3.32) is stochastically extendible if and only if $c=0$.

3.3.2-Characterization of the Output Covariance Sequence of Stochastically Extendible TPBVDS's

In the causal case, the covariance sequence of a finite dimensional stationary Gauss-Markov process is deterministically realizable and in fact, the first step in stochastic realization consists of performing a deterministic realization of the covariance sequence (see for example [40]). In this section we develop similar results for the case of TPBVDS's. In particular, we show that the class of output covariance sequences that are stochastically realizable by stochastically extendible TPBVDS's is the same

as the ones that are realizable by causal systems; however, the class of realizations is clearly larger. Constructing the realizations as in the causal case reduces to a spectral factorization problem.

Theorem 3.3.4

The output covariance $\Lambda(k)$ of the stochastically extendible TPBVDS (3.3.1) of dimension n is deterministically realizable and has a stable, minimal realization of dimension at most 2n.

Proof

Since $W(k)$ is summable, thanks to (3.3.30), $\Lambda(j)$ is summable as well. To see this, note that

$$\sum_{j=-\infty}^{\infty} |\Lambda(j)| \leq \sum_{k,j=-\infty}^{\infty} |W(k+j)| |W(k)'| = (\sum_{k=-\infty}^{\infty} |W(k)|)(\sum_{k=-\infty}^{\infty} |W(k)|)' < \infty .$$

(3.3.34)

Thus we can use Theorem 3.2.6 to test the realizability of $\Lambda(k)$. Let

$$\Lambda(s,t) = \sum_{j=-\infty}^{\infty} \Lambda(j) t^{k-1}/s^k = \sum_{k,j=-\infty}^{\infty} W(k+j)W(k)' t^{k-1}/s^k$$

$$= (\sum_{m=-\infty}^{\infty} W(m)t^{m-1}/s^{m-1})(\sum_{k=-\infty}^{\infty} W(k)' t^{-k}/s^{-k+1}) = sW(s,t)W(t,s)'$$

(3.3.35)

where m=k+j. Expression (3.3.35) clearly indicates that $\Lambda(s,t)$ is rational and thanks then to Theorem 3.2.6, all that remains to be shown is that the dimension of the minimal realization of $\Lambda(j)$ is at most twice that of $W(k)$. To see this, simply note that we can construct a factorization for $\Lambda(k)$ of dimension 2n (see the following Lemma).

Lemma 3.3.2

Let (3.3.1) be stochastically extendible. Then $\Lambda(s,t)$, i.e. the (s,t)-transform of its output covariance, can be expressed as follows

$$\Lambda(s,t) = \mathscr{C}(s\mathscr{E}-t\mathscr{A})^{-1}\mathscr{B},$$

(3.3.36)

where

$$\mathscr{C} = [-C \quad 0], \quad \mathscr{E} = \begin{bmatrix} E & -BB' \\ 0 & A' \end{bmatrix}, \quad \mathscr{A} = \begin{bmatrix} A & 0 \\ 0 & E' \end{bmatrix}, \quad \mathscr{B} = \begin{bmatrix} 0 \\ C' \end{bmatrix}.$$

(3.3.37)

If in addition, the equation

$$E\Pi E' = A\Pi A' + BB'$$

(3.3.38)

has a solution Π. Then, $\Lambda(s,t)$ can be expressed as follows

$$\Lambda(s,t) = \hat{\mathscr{C}}(s\hat{\mathscr{E}}-t\hat{\mathscr{A}})^{-1}\hat{\mathscr{B}}$$

(3.3.39)

where

$$\hat{\mathscr{C}} = [-C \quad -C\Pi E'], \quad \hat{\mathscr{E}} = \begin{bmatrix} E & 0 \\ 0 & A' \end{bmatrix}, \quad \hat{\mathscr{A}} = \begin{bmatrix} A & 0 \\ 0 & E' \end{bmatrix}, \quad \hat{\mathscr{B}} = \begin{bmatrix} -A\Pi C' \\ C' \end{bmatrix}.$$

(3.3.40)

Note that Equation 3.3.38 resembles the generalized Lyapunov equations (3.3.2) introduced previously. Since the Π-dependent part of (3.3.38) is the same as in (3.3.2), the existence and uniqueness conditions, namely Lemma 3.3.1, hold. Specifically, (3.3.38) has a unique solution Π if {E,A} has no reciprocal eigenmodes. However, this condition is not necessary for existence of a solution. For example consider the TPBVDS (3.3.32). This system has a pair of reciprocal eigenmodes namely 2 and 1/2, and expression (3.3.38) for

this system has a solution as well, namely

$$\Pi = \begin{bmatrix} -1/3 & 0 \\ 0 & 4/3 \end{bmatrix}.$$

(3.3.41)

Note that Π here is not positive semi-definite (emphasizing in fact that (3.3.38) is not the equation for the variance of any statistic of the TPBVDS. However, for our purposes, we simply need some solution and not necessarily a positive-definite one.

Factorizations (3.3.36) and (3.3.39) are generalizations of the results in [39] where similar factorizations are obtained in the continuous-time, non-descriptor causal case. By analogy with the causal case, we say that (3.3.36) is a cascade realization of $\Lambda(s,t)$ and (3.3.39) a parallel realization (thanks to the summability of Λ, a realization of $\Lambda(s,t)$ can be obtained from the above factorizations).

Proof

Thanks to (3.3.35), we can express $\Lambda(s,t)$ as follows

$$\Lambda(s,t) = sW(s,t)W(t,s)' = sC(sE-tA)^{-1}BB'(tE'-sA')^{-1}C' \quad (3.3.42)$$

which is exactly what we obtain from (3.3.36).

The form (3.3.39) is simply obtained by replacing $\mathcal{C}$, $\mathcal{E}$, $\mathcal{A}$, $\mathcal{B}$ with $\mathcal{C}T$, $S\mathcal{E}T$, $S\mathcal{A}T$ and $S\mathcal{B}$ where

$$T = \begin{bmatrix} I & \Pi E' \\ 0 & I \end{bmatrix}, \ S = \begin{bmatrix} I & -A\Pi \\ 0 & I \end{bmatrix}$$

(3.3.43)

and by noting that

$$\mathcal{C}(s\mathcal{E}-t\mathcal{A})^{-1}\mathcal{B} = \mathcal{C}T(sS\mathcal{E}T-tS\mathcal{A}T)^{-1}S\mathcal{B}.$$

(3.3.44)

A realization of $\Lambda(k)$, thanks to the summability of $\Lambda(k)$, is given by (3.3.37). However, this realization is not in normalized, or block normalized form. For this reason, and also to construct the projection matrix $\mathscr{P}$, we transform $\mathcal{E}$ and $\mathcal{A}$ into the forward-backward stable block form

$$L\mathcal{E}R = \begin{bmatrix} I & 0 \\ 0 & A_b \end{bmatrix}, \quad L\mathcal{A}R = \begin{bmatrix} A_f & 0 \\ 0 & I \end{bmatrix} \tag{3.3.45}$$

where A_f and A_b have eigenvalues inside the unit circle (notice that $\{\mathcal{E},\mathcal{A}\}$ has no eigenmodes on the unit circle because $\{E,A\}$ has no eigenvalue on the unit circle), in which case

$$\mathscr{P} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}. \tag{3.3.46}$$

If $\{E,A\}$ has no reciprocal eigenmodes, then $\Lambda(s,t)$ can be factorized as in (3.3.39) and the two blocks of $\{\mathcal{E},\mathcal{A}\}$ have no eigenmode in common, so that the realization (3.3.40) is in BNF and the corresponding projection matrix $\mathscr{P}$ is simply

$$\mathscr{P} = \begin{bmatrix} P & 0 \\ 0 & (I-P)' \end{bmatrix} \tag{3.3.47}$$

where P is the projection matrix of (3.3.1). Factorizations (realizations) (3.3.36) and (3.3.39) may not be minimal even if (3.3.1) is minimal. This corresponds to the well known result in the causal case that a deterministically minimal realization is not necessarily a minimal stochastic realization (see for example [41]).

Example 3.3.3

Consider the stochastically extendible TPBVDS (3.3.32). This system is
deterministically minimal (because it is strongly reachable and observable).
The output y(k) of this system is the sum of two statistically independent
forward and backward subsystems which have identical dynamics (of course one
is in the forward direction and the other in the backward direction but that
does not affect the output covariance). The output covariance sequences of
these 2 subsystems are related by a scalar multiplication (specifically, the
output covariance of the backward system is 4 times that of the forward
system). Thus the output covariance sequence of this system equals the sum of
the output covariances of the two subsystems and thus equals the output
covariance of the following system

$$x(k+1) = (1/2)x(k) + u(k) \qquad\qquad (3.3.48)$$

$$x(0) = v \qquad\qquad (3.3.49)$$

$$y(k) = \sqrt{5}/2 \ x(k) \qquad\qquad (3.3.50)$$

where u is a white, unit-variance sequence and v has variance 4/3. This
example illustrates that a deterministically minimal system is not
necessarily stochastically minimal.

3.3.3-Stochastic Realization for Stochastically Extendible TPBVDS's

The stochastic realization problem in this case consists of constructing
stochastically extendible TPBVDS's of minimum dimension from their output
covariance $\Lambda(j)$. In the previous section we showed that if $\Lambda(j)$ is the output
covariance of a stochastically extendible TPBVDS (i.e. $\Lambda(j)$ is stochastically

realizable), then

$$\Lambda(s,t) = sW(s,t)W(t,s)' \qquad (3.3.51)$$

where $\Lambda(s,t)$ is the (s,t)-transform of $\Lambda(j)$ and $W(s,t)$ is the (s,t)-transform of $W(k)$, the weighting pattern of the TPBVDS. From (3.3.51) we can obtain the following result:

Theorem 3.3.5

A sequence $\Lambda(j)$, $-\infty < j < \infty$, is stochastically realizable if and only if

a) it is summable,

b) its (s,t)-transform, $\Lambda(s,t)$ is rational,

c) $$t\Lambda(s,t) = s\Lambda(t,s)' \qquad (3.3.52)$$

d) $$\Lambda(e^{j\omega},1) \geq 0 \quad \text{for all } \omega. \qquad (3.3.53)$$

Corollary

A sequence $\Lambda(j)$, $-\infty < j < \infty$, is stochastically realizable if and only if it is stochastically realizable by a causal system.

Proof

The only if part clearly follows (3.3.51). To show the if part, we have to show that given that conditions a)-d) are satisfied, (3.3.51) is also satisfied for some $W(s,t)$. Note that

$$\Lambda(s,t) = (1/t)\Lambda(s/t,1), \qquad (3.3.54)$$

and $\Lambda(z,1)$ satisfies

$$\Lambda(z,1) = \Lambda(z^{-1},1)' \tag{3.3.55}$$

$$\Lambda(e^{j\omega},1) \geq 0. \tag{3.3.56}$$

Using classical spectral factorization results (see [39,40,41,42]), there exists a $H(z)$ such that

$$\Lambda(z,1) = H(z)H(z^{-1})'. \tag{3.3.57}$$

Thus we get

$$\Lambda(s,t) = (1/t)H(s/t)H(t/s)', \tag{3.3.58}$$

which implies (3.3.51) for

$$W(s,t) = (1/t)H(s/t). \tag{3.3.59}$$

Also note that we can always choose $W(s,t)$ satisfying (3.3.51) which has no poles (s,t) such that $|s|<|t|$. Such a $W(s,t)$ yields a causal system and thus the corollary is proven.

So we see that to every $W(s,t)$ that satisfies (3.3.51) corresponds a stochastic realization of the sequence $\Lambda(j)$. The degree of freedom in choosing $W(s,t)$ is larger than in the causal case; in the causal case we can switch zeros between $H(z)$ and $H(z^{-1})'$, here we can switch zeros and poles between $W(s,t)$ and $W(t,s)'$.

Also note that to every W satisfying (3.3.51) corresponds an H satisfying (3.3.57) (see 3.3.59). Thus the factorization problem of $\Lambda(s,t)$ reduces to the standard factorization problem (3.3.57). Note however that the standard factorization problem usually consists of finding all proper and stable (i.e. with poles inside the unit circle) $H(z)$'s satisfying (3.3.57)

whereas, in our case, _every_ H(z) satisfying (3.3.57) yields a deterministically realizable W(s,t). But of course not all W(s,t)'s satisfying (3.3.51) are of interest because many do not yield minimal stochastic realizations.

Example 3.3.4

Let

$$\Lambda(s,t) = 1/t, \qquad (3.3.60)$$

which can be expressed as

$$\Lambda(s,t) = sW_i(s,t)W_i(t,s)' \qquad (3.3.61)$$

where

$$W_i(s,t) = t^{i-1}/s^i. \qquad (3.3.62)$$

Each $W_i(s,t)$ can be realized and the result is a stochastic realization of (3.3.60). However, the dimension μ_i of the minimal realization of $W_i(s,t)$ which is just the degree of the denominator of $W_i(s,t)$ (see Theorem 3.2.3) is given by

$$\mu_i = \begin{bmatrix} i & \text{for } i \geq 1 \\ 1-i & \text{for } i \leq 0 \end{bmatrix} \qquad (3.3.63)$$

which implies that the dimension of minimal stochastic realizations of Λ is 1. A minimal stochastic realization can be constructed by either (deterministically) realizing $W_1(s,t)$ or $W_0(s,t)$.

Theorem 3.3.6

The dimension of the minimal stochastic realization of $\Lambda(j)$ equals one half the degree, $\mu(\Lambda(s,t)/s)$, of the least common multiple of the denominators of all the minors of $\Lambda(s,t)/s$.

Proof

Let $W(s,t)$ be the weighting pattern of any stochastic realization of $\Lambda(s,t)$. Then thanks to (3.3.51) and the fact that

$$\mu(H_1 H_2) \leq \mu(H_1) + \mu(H_2) \tag{3.3.64}$$

for any H_1 and H_2, we can deduce that $\mu(W(s,t))$ cannot be less than $(1/2)\mu(\Lambda(s,t)/s)$. Now we need to show that we can find a $W(s,t)$ that has a minimal realization with exactly this dimension. Note that we can choose, without loss of generality, a $W(s,t)$ that has no poles and zeros outside the unit circle (this is just the minimum phase, causal stochastic realization). In this case, there can be no pole-zero cancellation in the product

$$W(s,t)W(t,s)' = \Lambda(s,t)/s, \tag{3.3.65}$$

which means that $\mu((\Lambda(s,t)/s))$ equals $2\mu(W(s,t))$. Then, from Theorem 3.2.3, we can deduce that the minimal realization of this W has exactly dimension

$$n = \mu(W(s,t)) = (1/2)\mu(\Lambda(s,t)/s). \tag{3.3.66}$$

Example 3.3.5

Consider the following matrix sequence

$$\Lambda(0) = I, \ \Lambda(1) = \Lambda(-1)' = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ \Lambda(k) = 0 \text{ for } |k| > 1. \tag{3.3.67}$$

Clearly, $\Lambda(j)$ is summable and

$$\Lambda(s,t) = \begin{bmatrix} 1/t & 1/s \\ s/t^2 & 1/t \end{bmatrix} \qquad (3.3.68)$$

satifies (3.3.52) and (3.3.53) and thus $\Lambda(j)$ is stochastically realizable.

The dimension of the minimal stochastic realization of $\Lambda(s,t)$ is equal to one

half the degree of s^2t^2 which is just 2. Matrices

$$W(s,t) = \begin{bmatrix} t/s^2 \\ 1/s \end{bmatrix} \qquad (3.3.69)$$

and

$$W(s,t) = \begin{bmatrix} 1/s \\ 1/t \end{bmatrix} \qquad (3.3.70)$$

satisfy (3.3.51).

Realizing (3.3.69) yields the causal system

$$x(k+1) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) \qquad (3.3.71)$$

$$x(-\infty) = v \qquad (3.3.72)$$

$$y(k) = x(k) \qquad (3.3.73)$$

and (3.3.70) yields

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x(k+1) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k) \qquad (3.3.74)$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x(-\infty) + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x(\infty) = v \qquad (3.3.75)$$

$$y(k) = x(k). \qquad (3.3.76)$$

Note that in the first realization we have a double pole at zero and in the

second realization we have one pole at zero and one at infinity.

In the next chapter, we study the problem of optimal smoothing for TPBVDS's; we extend the results on fixed interval smoothing for causal systems to the case of TPBVDS's. But before ending this chapter, as a preview to the next chapter, we consider the optimal smoothing problem for the class of stochastically stationary TPBVDS's considered in this section and show that in this case, as in the causal stationary case, the optimal smoother can be easily derived using transform methods.

3.3.4—Optimal Smoother for the Infinite-horizon Stable TPBVDS

In this section we consider the optimal smoothing problem for the following deterministically minimal, stable TPBVDS defined over the interval $(-\infty, +\infty)$,

$$Ex(k+1) = Ax(k) + Bu(k) \qquad (3.3.77a)$$

$$y(k) = Cx(k) + r(k) \qquad (3.3.77b)$$

where $u(k)$ is a white Gaussian sequence of variance I and $r(j)$ is a white Gaussian sequence of variance R, with $R > 0$, and, $u(k)$ and $r(j)$ are independent for all k and j. The boundary conditions for (3.3.77) need not be specified explicitly because they are uniquely specified in terms of the E and A matrices (see Section 3.2.5).

The optimal smoothing problem consists of finding the optimal estimate $\hat{x}(k)$ of $x(k)$, given by

$$\hat{x}(k) = \overline{[x(k) \,|\, y(j), -\infty < j < +\infty]}, \qquad (3.3.78)$$

and the corresponding smoothing error

$$P_e = \overline{\tilde{x}(k)\tilde{x}(k)'} = \overline{[x(k)-\hat{x}(k)][x(k)-\hat{x}(k)]'}. \qquad (3.3.79)$$

By letting

$$\hat{x}(k) = \sum_{m=-\infty}^{\infty} K(k-m)y(m), \tag{3.3.80}$$

and using the fact that the estimation error is orthogonal to the observation, i.e.

$$\overline{[x(k)-\hat{x}(k)]y(j)'} = 0, \quad \text{for all } k \text{ and } j \tag{3.3.81}$$

we deduce that

$$\Lambda_{xy}(k) = \sum_{m=-\infty}^{\infty} K(k-m)\Lambda_{yy}(m) \tag{3.3.82}$$

where Λ_{xy} denotes the cross-correlation sequence of x and y, and Λ_{yy} the auto-correlation sequence of y. Taking the (s,t)-transform of both sides of (3.3.82) we obtain

$$\Lambda_{xy}(s,t) = tK(s,t)\Lambda_{yy}(s,t). \tag{3.3.83}$$

Thus the mapping from the output y to the optimal estimate $\hat{x}$ is given by $K(s,t)$ where

$$K(s,t) = (1/t)\Lambda_{xy}(s,t)\Lambda_{yy}^{-1}(s,t). \tag{3.3.84}$$

If we are able to realize this transfer function, the estimation problem is resolved. Note that

$$\Lambda_{yy}(s,t) = s[C(sE-tA)^{-1}BB'(tE'-sA')^{-1}C' + R/st] \tag{3.3.85}$$

$$\Lambda_{xy}(s,t) = s(sE-tA)^{-1}BB'(tE'-sA')^{-1}C' \tag{3.3.86}$$

so that

$$K(s,t) = s(sE-tA)^{-1}BB'(tE'-sA')^{-1}C'[R+stC(sE-tA)^{-1}BB'(tE'-sA')^{-1}C']^{-1} =$$

$$s(sE-tA)^{-1}BB'(tE'-sA')^{-1} \times$$

$$[I-stC'R^{-1}C(sE-tA)^{-1}BB'\{stC'R^{-1}C(sE-tA)^{-1}BB'+tE'-sA'\}^{-1}]C'R^{-1} =$$

$$s(sE-tA)^{-1}BB'[stC'R^{-1}C(sE-tA)^{-1}BB'+tE'-sA']C'R^{-1}. \tag{3.3.87}$$

Thus

$$K(s,t) = \mathscr{C}(s\mathscr{E}-t\mathscr{A})^{-1}\mathscr{B} \qquad (3.3.88)$$

where

$$\mathscr{C} = [I \quad 0], \; \mathscr{E} = \begin{bmatrix} E & -BB' \\ 0 & A' \end{bmatrix}, \; \mathscr{A} = \begin{bmatrix} A & 0 \\ C'R^{-1}C & E' \end{bmatrix}, \; \mathscr{B} = \begin{bmatrix} 0 \\ -C'R^{-1} \end{bmatrix}. \qquad (3.3.89)$$

Thus, we have the desired factorization of $K(s,t)$ given by (3.3.88)-(3.3.89). The realization problem is then almost solved. Note first that the weighting pattern of the optimal smoother, K, is summable[2]. Thus, the smoother is stable, so that all that remains to be done is to transform $\{\mathscr{E},\mathscr{A}\}$ into the forward-backward stable form.

For the smoothing error, we can proceed similarly. Since $x(k)$ and $y(k)$ are elements of the Hilbert space spanned by the $r(k)$ and $u(k)$, for some $L(k,m)$,

$$\tilde{x}(k) = x(k) - \hat{x}(k) = \sum_{m=-\infty}^{\infty} L(k,m) \begin{bmatrix} u(m) \\ r(m) \end{bmatrix}. \qquad (3.3.90)$$

Thanks to the stationarity of the problem (no absolute time origin), $L(k,m)$ is only a function of the difference $k-m$ and by abusing notation we shall refer to it as $L(k-m)$. Now we multiply both sides of (3.3.90) by $r(j)'$ and take expectation. Noting that x and r are independent and that $r(j)$ and $y(m)$ are independent for $m\neq j$, and using (3.3.80) we obtain,

$$K(k-j)R = L(k-j) \begin{bmatrix} 0 \\ R \end{bmatrix} \qquad (3.3.91)$$

---

[2] Note that $\Lambda_{yy}(k-j)=C\Lambda_{xx}(k-j)C'+R\delta_{kj}$ and that for some $\alpha>0$, $R\geq\alpha I$. Thus,

$$\overline{\hat{x}(0)\hat{x}(0)'} = \sum_{k=-\infty}^{+\infty} \sum_{j=-\infty}^{+\infty} K(k)\Lambda_{yy}(k-j)K(j)' \geq \sum_{k=-\infty}^{+\infty} K(k)RK(k)' \geq \alpha \sum_{k=-\infty}^{+\infty} K(k)K(k)'.$$

Since $\overline{\hat{x}(0)\hat{x}(0)'}$ is bounded (by the variance of x), $K(k)$ must be summable.

which since R is assumed to be positive definite, implies that

$$K(s,t) = L(s,t) \begin{bmatrix} 0 \\ I \end{bmatrix}. \qquad (3.3.92)$$

Now multiplying (3.3.90) by $u(j)'$ and taking expectation we get that

$$L(k-j) \begin{bmatrix} I \\ 0 \end{bmatrix} = G(k-j)B - \sum_{m=-\infty}^{\infty} K(k-m)CG(m-j)B \qquad (3.3.93)$$

where G is the Green's function of the system. Noting that

$$G(s,t) = (sE-tA)^{-1} \qquad (3.3.94)$$

and $K(s,t)$ is given by (3.3.88), after some algebra we find

$$L(s,t) \begin{bmatrix} I \\ 0 \end{bmatrix} = \mathscr{C}(s\mathscr{E}-t\mathscr{A})^{-1} \begin{bmatrix} B \\ 0 \end{bmatrix}. \qquad (3.3.95)$$

Combining (3.3.88), (3.3.92) and (3.3.95) yields,

$$L(s,t) = \mathscr{C}(s\mathscr{E}-t\mathscr{A})^{-1}\tilde{\mathscr{B}}, \qquad (3.3.96)$$

where

$$\tilde{\mathscr{B}} = \begin{bmatrix} B & 0 \\ 0 & -C'R^{-1} \end{bmatrix}. \qquad (3.3.97)$$

Noting that the smoothing error is stable, from the factorization (3.3.96) we can construct a realization for it. The smoothing error variance P_e is just the variance matrix of the output of this realization so that it can be obtained from the solution of a generalized Lyapunov equation as follows. Let $(\mathscr{C}_s, \mathscr{D}_s, \mathscr{E}_s, \mathscr{A}_s, \mathscr{B}_s)$ denote a forward-backward stable realization of L. Thus for some invertible $\mathscr{F}$ and $\mathscr{T}$,

$$\mathscr{C}_s = \mathscr{C}\mathscr{T}^{-1}, \quad \mathscr{E}_s = \mathscr{F}\mathscr{E}\mathscr{T}^{-1}, \quad \mathscr{A}_s = \mathscr{F}\mathscr{A}\mathscr{T}^{-1}, \quad \mathscr{B}_s = \mathscr{F}\tilde{\mathscr{B}}. \qquad (3.3.98)$$

As seen in Section 3.2.5, in this case,

$$\mathscr{D}_s = \mathscr{E}^{\infty} \qquad (3.3.99)$$

The variance of the "state" of this realization Π must then satisfy (see (3.3.2) and note that $V_i=V_iE^N=\mathscr{D}_s$ for stable systems in forward-backward form)

$$\mathscr{E}_s\Pi\mathscr{E}_s' - \mathscr{A}_s\Pi\mathscr{A}_s' = \mathscr{D}_s\mathscr{B}_s\mathscr{B}_s'\mathscr{D}_s' - (I-\mathscr{D}_s)\mathscr{B}_s\mathscr{B}_s'(I-\mathscr{D}_s)'. \qquad (3.3.100)$$

Then

$$P_e = \mathscr{C}_s \Pi \mathscr{C}_s'. \tag{3.3.101}$$

Thus we see that we need to transform $\{\mathscr{E}, \mathscr{A}\}$ into the forward-backward stable form. This problem is treated in depth in the next chapter, where dynamics (3.3.88) and (3.3.96) are rederived for the smoother and smoothing error corresponding to a general TPBVDS (the corresponding boundary conditions are of course more general in that case).

3.3.5—Conclusion

In this section, we have defined and characterized the class of stochastically extendible TPBVDS's and have studied the problem of stochastic realization for this class of systems. In particular, we have obtained conditions for a sequence of matrices to be stochastically realizable which in fact are exactly the same conditions that are needed for a sequence of matrices to be realizable by a causal system, so that the class of covariance sequences that can be realized by stochastically extendible TPBVDS's is exactly the same as the class of sequences that are stochastically realizable by causal systems. The difference is that there are more TPBVDS realizations than just causal realizations for any given output covariance. We have also shown that as in the causal case, the stochastic realization problem for stochastically extendible TPBVDS's reduces to a spectral factorization problem and we have derived necessary and sufficient conditions for minimality of the stochastic realization. Finally, as a preview to the next chapter where we consider the smoothing problem for TPBVDS's, we have derived some results concerning the smoother for a stochastically extendible TPBVDS.

CHAPTER IV:

ESTIMATION FOR TWO-POINT BOUNDARY-VALUE

DESCRIPTOR SYSTEMS

In this chapter, we develop an estimation (smoothing) theory for

TPBVDS's. In Section 4.1, we show that the optimal smoother for a TPBVDS can

be obtained by solving a TPBVDS of twice the dimension of the original

system. We then propose solving this TPBVDS by the two-filter solution method

described in the Appendix. The two-filter solution requires that the

descriptor dynamics be transformed into forward-backward stable form and thus

in Section 4.2, we develop a method for such a transformation. In particular,

we show that this transformation is related to solutions of generalized

Riccati equations. These Riccati equations are studied, and a theory

paralleling the existing theory for standard Riccati equations is developed

for these equations. In Section 4.3, a different approach to the smoothing

problem is proposed. This approach is essentially a generalization of the

Rauch-Tung-Striebel formulation [34] for smoothing causal systems. For this

purpose, we propose a generalization of the Kalman filter which turns out to

be directly tied to the generalized Riccati equations of Section 4.2. We end

Section 4.3 with an analysis of the limiting behaviour of the smoother.

4.1-Smoother for TPBVDS's

In this section we consider the smoothing problem for the TPBVDS

$$Ex(k+1) = Ax(k) + Bu(k) \qquad (4.1.1a)$$

$$V_i x(0) + V_f x(N) = v \qquad (4.1.1b)$$

where $u(k)$ is a zero-mean, white, unit-variance, Gaussian sequence defined on

the interval $[0,N-1]$, $x(k)$ is the boundary value process, and v is a

zero-mean Gaussian random vector with variance Q and independent of $u(k)$. We

are given the interior observations

$$y(k) = Cx(k) + r(k) \ , \ k=1,\ldots,N-1 \qquad (4.1.2a)$$

and the boundary observation

$$y_b = W_i x(0) + W_f x(N) + r_b \qquad (4.1.2b)$$

where $r(k)$ is a zero-mean, white, Gaussian sequence of variance $R>0$ defined

on the interval $[1,N-1]$ and independent of $u(k)$ and v. Also r_b is a Gaussian

random vector with variance Π and is independent of $r(k)$, $u(k)$ and v. We

assume that the TPBVDS (4.1.1) is well-posed (but not necessarily in

normalized or block normalized form).

We can rewrite (4.1.1) as a single equation

$$\mathcal{G}x = \mathcal{B}u \qquad (4.1.3)$$

where

$$x' = (x'(0),\ldots x'(N)) \qquad (4.1.4a)$$

$$u' = (u'(0),\ldots,u'(N-1),v') \qquad (4.1.4b)$$

$$\mathcal{P} = \begin{bmatrix} -A & E & 0 & \ldots\ldots\ldots\ldots & 0 \\ 0 & -A & E & 0 \ldots\ldots\ldots & 0 \\ \vdots & & & & \vdots \\ 0\ldots\ldots\ldots\ldots\ldots\ldots 0 & -A & E \\ V_i & 0\ldots\ldots\ldots\ldots\ldots 0 & V_f \end{bmatrix}$$

(4.1.5a)

$$\mathcal{B} = \text{diag } (B,\ldots,B,\ I)$$

(4.1.5b)

Similarly, (4.1.2) can be expressed as

$$y = \mathcal{C}x + r$$

(4.1.6)

where

$$y' = [y'(1),\ y'(2),\ldots,y'(N-1),y_b']$$

(4.1.7a)

$$r' = [r'(1),\ r'(2),\ldots,r'(N-1),r_b']$$

(4.1.7b)

$$\mathcal{C} = \begin{bmatrix} 0 & C & 0\ldots\ldots 0 & 0 \\ 0 & 0 & C & & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & C & 0 \\ W_i & 0 & 0\ldots\ldots 0 & W_f \end{bmatrix}$$

(4.1.7c)

Also, the covariances of u in (4.1.4b) and r in (4.1.7b) are given by

$$\mathcal{Q} = \text{diag}(I,\ldots,I,Q)$$

(4.1.8a)

$$\mathcal{R} = \text{diag } (R,\ldots,R,\Pi)$$

(4.1.8b)

Our problem, then is to estimate x given y. In [3] we have approached this problem using the method of complementary processes. Here, we shall approach this problem via the maximum likelihood philosophy. Let us rewrite (4.1.3) and (4.1.6) as one observation of the unknown vector x as follows

$$\begin{bmatrix} \mathcal{P} \\ \mathcal{C} \end{bmatrix} x = \begin{bmatrix} 0 \\ y \end{bmatrix} + \begin{bmatrix} \mathcal{B}u \\ -r \end{bmatrix}$$

(4.1.9)

then it is not difficult to see that maximizing $p\left(\begin{bmatrix} 0 \\ y \end{bmatrix} | x\right)$ is equivalent to minimizing

$$J = (1/2)[u'Q^{-1}u + r'\mathcal{R}^{-1}r] \qquad (4.1.10)$$

subject to constraint (4.1.9). We can solve for r as a function of y and x using (4.1.9) and thus the optimization problem to be solved is the following:

minimize

$$J = (1/2)[u'Q^{-1}u + (y-Cx)'\mathcal{R}^{-1}(y-Cx)] \qquad (4.1.11)$$

subject to the constraint

$$\mathcal{P}x = \mathcal{B}u. \qquad (4.1.12)$$

This is a standard problem which can be solved using the Lagrange multiplier technique. Let

$$H = J + \lambda'(\mathcal{P}x-\mathcal{B}u). \qquad (4.1.13)$$

Then

$$\frac{\delta H}{\delta x}\bigg|_{x=\hat{x}} = \mathcal{C}'\mathcal{R}^{-1}(\mathcal{C}\hat{x}-y) + \mathcal{P}'\hat{\lambda} = 0 \qquad (4.1.14)$$

$$\frac{\delta H}{\delta u}\bigg|_{u=\hat{u}} = 2Q^{-1}\hat{u} - \mathcal{B}'\hat{\lambda} = 0 \qquad (4.1.15)$$

which implies that

$$\hat{u} = Q\mathcal{B}'\hat{\lambda}. \qquad (4.1.16)$$

Equation (4.1.16) and (4.1.12) imply that

$$\mathcal{P}\hat{x} - \mathcal{B}Q\mathcal{B}'\hat{\lambda} = 0 \qquad (4.1.17)$$

and (4.1.14) implies that

$$\mathcal{C}'\mathcal{R}^{-1}\mathcal{C}\hat{x} + \mathcal{P}'\hat{\lambda} = \mathcal{C}'\mathcal{R}^{-1}y. \qquad (4.1.18)$$

Expressions (4.1.17)-(4.1.18) can be expressed as follows

$$\begin{bmatrix} \mathcal{P} & -\mathcal{B}Q\mathcal{B}' \\ \mathcal{C}'\mathcal{R}^{-1}\mathcal{C} & \mathcal{P}' \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{\lambda} \end{bmatrix} = \begin{bmatrix} 0 \\ \mathcal{C}'\mathcal{R}^{-1}y \end{bmatrix}. \qquad (4.1.19)$$

The matrix on the left-hand side of (4.1.19) can be shown to be invertible as follows. Since $\mathcal{S}$ is invertible (thanks to the well-posedness condition), we need only show that the Schur complement

$$D = \mathcal{S}' + \mathcal{C}'\mathcal{R}^{-1}\mathcal{C}\mathcal{S}^{-1}\mathcal{B}Q\mathcal{B}' \tag{4.1.20}$$

of the (1,1) block is invertible. Note that

$$D(\mathcal{S}')^{-1} = I + ML \tag{4.1.21}$$

where

$$M = \mathcal{C}'\mathcal{R}^{-1}\mathcal{C} \geq 0 \tag{4.1.22a}$$

and

$$L = \mathcal{S}^{-1}\mathcal{B}Q\mathcal{B}'(\mathcal{S}')^{-1} \geq 0. \tag{4.1.22b}$$

The invertibility of D then follows from the fact that ML cannot have negative eigenvalues.[1]

Equation (4.1.21) defines a well-posed TPBVDS, but to obtain the most illuminating form of this system requires a permutation of the equations and variables in (4.1.21). Specifically, let

$$\hat{\lambda}' = [\hat{\lambda}'(1),\ldots,\hat{\lambda}'(N),\hat{\lambda}'(0)] \tag{4.1.23}$$

(the reason for our particular choice of labeling of components in (4.1.23) will be made clear shortly), then it is straightforward to verify that (4.1.19) is equivalent to

$$\mathcal{F}\xi = \eta, \tag{4.1.24}$$

where

$$\xi' = [(\hat{x}'(0), \hat{\lambda}'(0)), (\hat{x}'(1), \hat{\lambda}'(1)),\ldots,(\hat{x}'(N),\hat{\lambda}'(N))] \tag{4.1.25a}$$

---

[1] Suppose $MLv=\lambda v$. Then $v'L'MLv=\lambda v'L'v$, so that $\lambda=(v'L'MLv'/(v'L'v)\geq 0$.

$$\eta' = \left[\begin{bmatrix} 0 \\ W_i{}'\Pi^{-1}y_b \end{bmatrix}' \cdot \begin{bmatrix} 0 \\ C'R^{-1}y(1) \end{bmatrix}' \cdots \begin{bmatrix} 0 \\ C'R^{-1}y(N-1) \end{bmatrix}' \cdot \begin{bmatrix} 0 \\ W_f{}'\Pi^{-1}y_b \end{bmatrix}' \right]$$

(4.1.25b)

$$\mathcal{F} = \begin{bmatrix} \mathcal{V}_{11} & \mathcal{E} & 0 & 0\ldots\ldots0 & \mathcal{V}_{12} \\ 0 & -\mathcal{A} & \mathcal{E} & 0\ldots\ldots0 & 0 \\ 0 & 0 & -\mathcal{A} & \mathcal{E}\ldots\ldots0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & -\mathcal{A} & \mathcal{E} \\ \mathcal{V}_{21} & 0 & 0 & 0\ldots\ldots0 & \mathcal{V}_{22} \end{bmatrix}$$

(4.1.26)

with

$$\mathcal{E} = \begin{bmatrix} E & -BQB' \\ 0 & -A' \end{bmatrix}, \quad \mathcal{A} = \begin{bmatrix} A & 0 \\ -C'R^{-1}C & -E' \end{bmatrix}$$

(4.1.27)

$$\mathcal{V}_{11} = \begin{bmatrix} -A & 0 \\ W_i{}'\Pi^{-1}W_i & V_i{}' \end{bmatrix}, \quad \mathcal{V}_{12} = \begin{bmatrix} 0 & 0 \\ W_i{}'\Pi^{-1}W_f & 0 \end{bmatrix}$$

(4.1.28a)

$$\mathcal{V}_{21} = \begin{bmatrix} V_i & -Q \\ W_f{}'\Pi^{-1}W_i & V_f{}' \end{bmatrix}, \quad \mathcal{V}_{22} = \begin{bmatrix} V_f & 0 \\ W_f{}'\Pi^{-1}W_f & E' \end{bmatrix}$$

(4.1.28b)

Comparing the form of $\mathcal{F}$ in (4.1.26) to that of $\mathcal{G}$ in (4.1.3) we see that (4.1.24) is almost a standard TPBVDS except for the top row of equation (4.1.24), i.e. the fact that $\mathcal{V}_{11}$ in (4.1.26) appears rather than $-\mathcal{A}$ and that $\mathcal{V}_{12}$ is present at all. This is a consequence of the discrete nature of the time index and the intrinsic asymmetry of the model (4.1.1)-(4.1.2). We can, however, reduce these equations to a standard TPBVDS by means of a basic technique in the analysis of boundary-value systems. Specifically, we can think of (4.1.24) as a TPBVDS with boundary values consisting of

$(\hat{x}'(0), \hat{\lambda}'(0))'$ and $(\hat{x}'(N), \hat{\lambda}'(N))'$. Because of the well-posedness of (4.1.24) it is possible to eliminate some of the variables from (4.1.24) by solving for them in terms of the remaining variables. More specifically, it is possible to move the boundary values <u>inward</u> by eliminating boundary values at one end of the interval, the other, or both. One can iterate this process, and in fact this type of recursion forms the basis for a notion of state for boundary value systems (see Section 2.2). For our purposes here, however, we need only to consider a single step of this type.

Specifically, the invertibility of $\mathcal{F}$ implies that

$$\begin{bmatrix} \mathcal{V}_{11} \\ \mathcal{V}_{21} \end{bmatrix}$$

has full column rank and thus that we can eliminate $(\hat{x}'(0), \hat{\lambda}(0))'$ as follows. We construct matrices M_1 and M_2 such that $[M_1 \ M_2]$ has full row rank and

$$[M_1 \ M_2] \begin{bmatrix} \mathcal{V}_{11} \\ \mathcal{V}_{21} \end{bmatrix} = 0. \qquad (4.1.29)$$

If we then premultiply (4.1.24) by the following full-rank matrix

$$\begin{bmatrix} 0 & I & 0 & \dots & 0 & 0 \\ 0 & 0 & I & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & I & 0 \\ M_1 & 0 & 0 & \dots & 0 & M_2 \end{bmatrix}$$

we obtain a TPBVDS of a form exactly as in (4.1.1). Specifically, this computation yields

$$\mathcal{E} \begin{bmatrix} \hat{x}(k+1) \\ \hat{\lambda}(k+1) \end{bmatrix} = \mathcal{A} \begin{bmatrix} \hat{x}(k) \\ \hat{\lambda}(k) \end{bmatrix} + \begin{bmatrix} 0 \\ C'R^{-1}y(k) \end{bmatrix}, \qquad k=1,\dots,N-1 \qquad (4.1.30a)$$

with boundary condition

$$M_1 \mathcal{E} \begin{bmatrix} \hat{x}(1) \\ \hat{\lambda}(1) \end{bmatrix} + [M_1 \mathcal{V}_{12} + M_2 \mathcal{V}_{22}] \begin{bmatrix} \hat{x}(N) \\ \hat{\lambda}(N) \end{bmatrix} = M_1 \begin{bmatrix} 0 \\ W_i' \Pi^{-1} y_b \end{bmatrix} + M_2 \begin{bmatrix} 0 \\ W_f' \Pi^{-1} y_b \end{bmatrix}. \qquad (4.1.30b)$$

By construction we know that this system is well-posed. Also, once we have computed $\hat{x}(k)$, $\hat{\lambda}(k)$, $k=1,\ldots,N$, we can determine the previously eliminated boundary values $\hat{x}(0)$, $\hat{\lambda}(0)$:

$$
\begin{bmatrix} \hat{x}(0) \\ \hat{\lambda}(0) \end{bmatrix} = D\Psi_{11}' \begin{bmatrix} 0 \\ W_i'\Pi^{-1}y_b \end{bmatrix} + \Psi_{21}' \begin{bmatrix} 0 \\ W_f'\Pi^{-1}y_b \end{bmatrix} - \Psi_{11}'\mathcal{E} \begin{bmatrix} \hat{x}(1) \\ \hat{\lambda}(1) \end{bmatrix}
$$
$$
- [-\Psi_{11}\Psi_{12}+\Psi_{21}'\Psi_{22}] \begin{bmatrix} \hat{x}(N) \\ \hat{\lambda}(N) \end{bmatrix}
\tag{4.1.31a}
$$

where

$$
D = [\Psi_{11}'\Psi_{11} + \Psi_{21}'\Psi_{21}]^{-1}.
\tag{4.1.31b}
$$

Note that on examination of (4.1.30) and the form of $\mathcal{E}$ and $\mathcal{A}$ in (4.1.27), we see that what we have derived is a generalization of the Hamiltonian form of the optimal smoother for causal systems (see, e.g. [10]). This immediately suggests the possibility of generalizing methods for solving smoothing equations such as diagonalization of the Hamiltonian dynamics [10] to produce forward and backward recursions. Such an approach is described in the following sections.

Example 4.1.1

Consider the smoothing problem for the following stochastic TPBVDS

$$
\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x(k+1) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k)
\tag{4.1.32a}
$$
$$
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x(0) + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x(N) = v
\tag{4.1.32b}
$$

where Q, the variance of v is equal to I, and the observations are given by

$$y(k) = x(k) + r(k) \tag{4.1.33a}$$

$$y_b = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} x(0) + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} x(N) + r_b. \tag{4.1.33b}$$

where the variance of $r(k)$ is equal to that of r_b and equals to I. Referring

to (4.1.30a), we can see that the optimal estimate $\hat{x}(k)$ satisfies the

dynamics equation

$$\begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \hat{x}(k+1) \\ \hat{\lambda}(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}(k) \\ \hat{\lambda}(k) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} y(k). \tag{4.1.34}$$

To obtain the boundary condition for this system, we need to compute M_1 amd

M_2 defined in (4.1.29). Note that we have

$$\mathscr{Y}_{11} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} , \quad \mathscr{Y}_{12} = 0 \tag{4.1.35a}$$

$$\mathscr{Y}_{21} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} , \quad \mathscr{Y}_{22} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \tag{4.1.35b}$$

We can then compute M_1 and M_2 satisfying (4.1.29):

$$M_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} , \quad M_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & -2 \end{bmatrix}. \tag{4.1.36}$$

The boudary condition for system (4.1.34) can now be computed from (4.1.30b):

$$\begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \hat{x}(1) \\ \hat{\lambda}(1) \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -3 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}(N) \\ \hat{\lambda}(N) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -2 \end{bmatrix} y_b. \tag{4.1.37}$$

The estimate $\hat{x}(k)$ can now be computed by solving the TPBVDS (4.1.34), (4.1.37) using the two-filter solution method.

Example 4.1.2

In this example we introduce the class of cyclic TPBVDS's for which the boundary condition (4.1.1b) takes the special form

$$x(0) = x(N).\qquad(4.1.38)$$

Equivalently we can think of a cyclic system as being defined on [0,N-1] with the boundary condition

$$Ex(0) - Ax(N-1) = Bu(N-1)\qquad(4.1.39)$$

(so that $\mathcal{F}$ in (4.1.5a) is block-circulant).

Consider the smoothing problem for such a system when the boundary observation is

$$y_b = Cx(0) + r_b\qquad(4.1.40)$$

with $\Pi = R$. it is not difficult to check that in this case $\mathcal{F}$ is also block-circulant (i.e. $\mathcal{V}_{11} = \mathcal{V}_{22} = -\mathcal{A}$, $\mathcal{V}_{12} = 0$, $\mathcal{V}_{21} = \mathcal{E}$) so that the smoother is also a cyclic TPBVDS over [0,N-1] (with no need to move the boundary in one step as in (4.1.29)-(4.1.31)). The optimal estimate in this case can be obtained from solving the cyclic TPBVDS

$$\mathcal{E}\begin{bmatrix}\hat{x}(k+1)\\\hat{\lambda}(k+1)\end{bmatrix} = \mathcal{A}\begin{bmatrix}\hat{x}(k)\\\hat{\lambda}(k)\end{bmatrix} + \begin{bmatrix}0\\C'R^{-1}y(k)\end{bmatrix}\quad,\quad k \bmod N\qquad(4.1.41)$$

with $y(0)$ defined as y_b.

Before ending this section we show that the smoothing error $\tilde{x}(k)=x(k)-\hat{x}(k)$ can also be expressed as a part of a boundary value process. Specifically let

$$\tilde{x}' = [\tilde{x}(0)',\ldots,\tilde{x}(N)'] = x' - \hat{x}' \qquad (4.1.42)$$

then using (4.1.12) we can see that

$$\mathcal{G}\tilde{x} = \mathcal{G}x - \mathcal{G}\hat{x}. \qquad (4.1.43)$$

Replacing $\mathcal{G}\hat{x}$ with $\mathcal{B}Q\mathcal{B}'\hat{\lambda}$ (using (4.1.17) and (4.1.19)) in (4.1.43) yields the following expression

$$\mathcal{G}\tilde{x} = \mathcal{B}u - \mathcal{B}Q\mathcal{B}'\hat{\lambda}. \qquad (4.1.45)$$

From (4.1.16), (4.1.18), (4.1.19) and (4.1.42) we get that

$$\mathcal{C}'\mathcal{R}^{-1}\mathcal{C}\tilde{x} - \mathcal{G}'\hat{\lambda} = -\mathcal{C}'\mathcal{R}^{-1}r. \qquad (4.1.46)$$

Finally, by combining (4.1.45) and (4.1.46) into a single equation, we obtain

$$\begin{bmatrix} \mathcal{G} & -\mathcal{B}Q\mathcal{B}' \\ \mathcal{C}'\mathcal{R}^{-1}\mathcal{C} & \mathcal{G}' \end{bmatrix} \begin{bmatrix} \tilde{x} \\ -\hat{\lambda} \end{bmatrix} = \begin{bmatrix} \mathcal{B} & 0 \\ 0 & -\mathcal{C}'\mathcal{R}^{-1} \end{bmatrix} \begin{bmatrix} u \\ r \end{bmatrix}. \qquad (4.1.47)$$

Proceeding in the same way as for (4.1.21), we can show that (4.1.47) is equivalent to

$$\mathcal{E}\begin{bmatrix} \tilde{x}(k+1) \\ -\hat{\lambda}(k+1) \end{bmatrix} = \mathcal{A}\begin{bmatrix} \tilde{x}(k) \\ -\hat{\lambda}(k) \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & C'R^{-1} \end{bmatrix} \begin{bmatrix} u(k) \\ r(k) \end{bmatrix} \qquad (4.1.48)$$

with boundary conditions

$$M_1\mathcal{E}\begin{bmatrix} \tilde{x}(1) \\ -\hat{\lambda}(1) \end{bmatrix} + [M_1\mathcal{V}_{12}+M_2\mathcal{V}_{22}]\begin{bmatrix} \tilde{x}(N) \\ -\hat{\lambda}(N) \end{bmatrix} = M_1\begin{bmatrix} Bu(0) \\ W_i'\,\Pi^{-1}r_b \end{bmatrix} + M_2\begin{bmatrix} v \\ W_f'\Pi^{-1}r_b \end{bmatrix}. \qquad (4.1.49)$$

Examining (4.1.48)-(4.1.49), we see that the evaluation of the covariance of the estimation error $\tilde{x}(k)$ corresponds to the computation of (the upper left-hand block of) the covariance of the TPBVDS (4.1.48)-(4.1.49) driven by white noise $(u'(k), r'(k))$ and with independent boundary conditions (see [16], [3] for a discussion of this type of computation for general TPBVDS's).

In this section, we have shown that the optimal estimate for the TPBVDS (4.1.1)-(4.1.2) can be obtained by computing the solution of the TPBVDS (4.1.30). For this purpose, we can use the two-filter solution for which we have to transform the dynamics of TPBVDS (4.1.30) into the forward-backward stable form. We shall discuss such a transformation in the next section.

4.2-Hamiltonian Diagonalization and Generalized Riccati Equations

In this section, we first consider the problem of Hamiltonian diagonalization, i.e. the problem of transforming the smoother dynamics obtained in the previous section into forward-backward stable form. This block diagonalization is useful for the implementation of the two-filter solution to solve the smoothing problem. In particular, we show that if the system is strongly reachable and observable, the problem of Hamiltonian diagonalization reduces to the problem of obtaining positive definite solutions to some algebraic generalized (or descriptor) Riccati equations. These algebraic generalized Riccati equations are studied and a method for constructing their solutions is proposed.

In the second part of this section, we study further these generalized Riccati equations. In particular, we consider the convergence properties of these equations and obtain weaker conditions for the existence of a solution to these algebraic equations. The results of this part are used in the next section to obtain a direct method for the computation of the smoothed estimate for the TPBVDS (4.1.1)-(4.1.2).

4.2.1-Hamiltonian Diagonalization

In the previous sections we have seen that the optimal smoother for the TPBVDS has the following dynamics

$$\mathcal{E}\begin{bmatrix}\hat{x}(k+1)\\\hat{\lambda}(k+1)\end{bmatrix} = \mathcal{A}\begin{bmatrix}\hat{x}(k)\\\hat{\lambda}(k)\end{bmatrix} + \mathcal{B}y(k) \qquad (4.2.1)$$

and the corresponding smoothing error $\tilde{x}$ satisfies

$$\mathcal{E}\begin{bmatrix} \tilde{x}(k+1) \\ -\hat{\lambda}(k+1) \end{bmatrix} = \mathcal{A}\begin{bmatrix} \tilde{x}(k) \\ -\hat{\lambda}(k) \end{bmatrix} + \tilde{\mathcal{B}}\begin{bmatrix} u(k) \\ r(k) \end{bmatrix} \qquad (4.2.2)$$

where

$$\mathcal{E} = \begin{bmatrix} E & -BB' \\ 0 & -A' \end{bmatrix}, \ \mathcal{A} = \begin{bmatrix} A & 0 \\ -C'R^{-1}C & -E' \end{bmatrix}, \ \mathcal{B} = \begin{bmatrix} 0 \\ C'R^{-1} \end{bmatrix}, \ \tilde{\mathcal{B}} = \begin{bmatrix} B & 0 \\ 0 & C'R^{-1} \end{bmatrix}. \quad (4.2.3)$$

In this section we consider the problem of Hamiltonian diagonalization, i.e. finding a sequence of invertible matrices $\mathcal{F}_k$ and $\mathcal{T}_k$ such that

$$\mathcal{F}_k \mathcal{E} \mathcal{T}_{k+1}^{-1} = \begin{bmatrix} I & 0 \\ 0 & A_k^b \end{bmatrix} \qquad (4.2.4)$$

and

$$\mathcal{F}_k \mathcal{A} \mathcal{T}_k^{-1} = \begin{bmatrix} A_k^f & 0 \\ 0 & I \end{bmatrix} \qquad (4.2.5)$$

for some matrices A_k^f and A_k^b. We shall suppose that the system is strongly reachable and observable.

Let

$$\mathcal{F}_k = \begin{bmatrix} I & AT_k^{-1} \\ A'S_k^{-1} & -I \end{bmatrix} \qquad (4.2.6)$$

$$\mathcal{T}_k = \begin{bmatrix} E & -\Psi_k \\ \Theta_k & E' \end{bmatrix} \qquad (4.2.7)$$

where

$$\Theta_k = A'S_k^{-1}A + C'R^{-1}C \qquad (4.2.8)$$

$$S_k = E\Theta_{k+1}^{-1}E' + BB' \qquad (4.2.9)$$

$$\Psi_{k+1} = AT_k^{-1}A' + BB' \qquad (4.2.10)$$

$$T_k = E'\Psi_k^{-1}E + C'R^{-1}C. \qquad (4.2.11)$$

Then by substituting the expressions in (4.2.6) and (4.2.7) into (4.2.4) and (4.2.5), and using (4.2.8)-(4.2.11), we get

$$A_k^f = AT_k^{-1}E'\Psi_k^{-1} \qquad (4.2.12a)$$

$$A_{k+1}^b = A'S_k^{-1}E\Theta_{k+1}^{-1}. \qquad (4.2.12b)$$

A sequence of matrices Θ_k, S_k satisfying (4.2.8)-(4.2.9) can easily be constructed by arbitrarily choosing a positive definite Θ_N and performing the backward recursion (4.2.8)-(4.2.9). Note that thanks to the strong reachability and observability assumptions (in fact all that is required is that the zero and infinite eigenmodes be strongly reachable and observable[1]), all Θ_k and S_k obtained from this recursion are positive definite. To see this, suppose that S_k is not positive definite but Θ_{k+1} is. Then there exists a vector v such that

$$v'S_k v = v'E\Theta_{k+1}^{-1}E'v + v'BB'v = 0. \tag{4.2.13}$$

But since BB' is positive semi-definite and Θ_{k+1}^{-1} positive definite, we must have that $v'E = v'B = 0$ which contradicts the assumption that the infinite eigenmode is strongly reachable. The positive definiteness of Θ_k can be shown in a similar fashion.

Similarly, Ψ_k and T_k can be constructed by picking a positive definite T_0 and performing the forward recursion (4.2.10)-(4.2.11). Finally, the invertibility of $\mathcal{S}_k$ and $\mathcal{T}_k$ can be checked by noting that their Schur complements of there (1,1) and (2,1) blocks are given respectively by

$$I + A'S_k^{-1}AT_k^{-1} \quad \text{and} \quad \Psi_k + E\Theta_k^{-1}E'$$

which are both invertible[2].

---

[1] As defined in Chapter II, an eigenmode σ is strongly reachable if for some (s,t), $\sigma = s/t$ and $[sE - tA \vdots B]$ has full rank. Similarly, an eigenmode σ is strongly observable if for some (s,t), $\sigma = s/t$ and $[sE' - tA' \vdots C']$ has full rank. These tests of strong reachability and observability do not require that {E,A} be in standard or block standard form.

[2] Note that $\Psi_k > 0$ and $E\Theta_k^{-1}E' \geq 0$, thus $\Psi_k + E\Theta_k^{-1}E' > 0$. Also, $I + A'S_k^{-1}AT_k^{-1} = (T_k + A'S_k^{-1}A)T_k^{-1}$ where $T_k + A'S_k^{-1}A > 0$ and $T_k^{-1} > 0$, thus $I + A'S_k^{-1}AT_k^{-1}$ is the product of two invertible matrices and must be invertible.

We can simplify recursion (4.2.8)-(4.2.9) by eliminating S_k and obtaining a direct recursion for Θ_k:

$$\Theta_k = A'(E\Theta_{k+1}^{-1}E'+BB')^{-1}A + C'R^{-1}C, \qquad (4.2.14a)$$

or eliminating Θ_k and obtaining a direct recursion for S_k:

$$S_k = E(A'S_{k+1}^{-1}A+C'R^{-1}C)^{-1}E' + BB'. \qquad (4.2.14b)$$

Similarly for T_k and Ψ_k, the following forward recursions can be obtained

$$T_{k+1} = E'(AT_k^{-1}A'+BB')^{-1}E + C'R^{-1}C \qquad (4.2.15a)$$

$$\Psi_{k+1} = A(E'\Psi_k^{-1}E+C'R^{-1}C)^{-1}A' + BB'. \qquad (4.2.15b)$$

We shall refer to (4.2.14)-(4.2.15) as generalized Riccati equations. Note that by replacing E with the identity matrix in (4.2.15), we obtain the usual Riccati equations associated with the forward Kalman filter for a causal system. Replacing E with the identity matrix in (4.2.14) yields the usual Riccati equation associated with the backward Kalman filter for a causal system. The Riccati equation (4.2.15b), with $E=I$, is commonly expressed in a different form. Specifically, using the ABCD matrix inversion formula

$$(A+BCD)^{-1} = A^{-1}-A^{-1}B(DA^{-1}B+C^{-1})^{-1}DA^{-1}, \qquad (4.2.16)$$

(4.2.15b) with $E=I$ can be expressed as follows

$$\Psi_{k+1} = A\Psi_kA' - A\Psi_kC'(C\Psi_kC'+R)^{-1}C\Psi_kA' + BB'. \qquad (4.2.17)$$

In the standard Kalman filtering problem, Ψ_k obtained from (4.2.17) or (4.2.15b) with $E=I$, corresponds to the predicted error covariance i.e. the covariance of the estimation error of x(k) based on observations up to k-1. The initial condition for this recursion, i.e. Ψ_0, is just the a priori covariance of x(0). The advantage of using expression (4.2.17) is that Ψ_k need not be necessarily positive definite. For example, we could very well have complete knowledge of the initial state x(0). In this case, Ψ_0 is zero and recursion (4.2.15b) cannot be used whereas, (4.2.17) can be used to update the

predicted error covariance. For the case of TPBVDS's, with E singular, we do not know of any matrix inversion formula that can be used to express (4.4.15b) in a way such that Ψ_k^{-1} does not appear explicitly in the expression. In Section 4.2.2, we present an alternative expression that can be used in the case where Ψ_k is not necessarily positive-definite, and which requires computation of a limit.

Naturally, for the implementation of the two-filter solution for the smoothing problem, we are interested in the case where the sequences of matrices S_k, Θ_k, T_k and Ψ_k can be chosen to be constant sequences and A_k^f and A_k^b are stable. For this, we need to study the algebraic generalized Riccati equations (i.e. (4.2.14)-(4.2.17) without the subscripts k and k+1). We consider two of these equations (the other two are trivially related to these two), namely

$$\Psi = A(E'\Psi^{-1}E + C'R^{-1}C)^{-1}A' + BB'. \qquad (4.2.18a)$$

$$S = E(A'S^{-1}A + C'R^{-1}C)^{-1}E' + BB'. \qquad (4.2.18b)$$

We will show that under the assumption of strong reachability and observability, (4.2.18) have unique positive-definite solutions Ψ and S. In the process we will also construct a method for computing Ψ and S. Our approach is similar to the approach which is used to analyze standard discrete-time Riccati equations [31]. But first we shall investigate some properties of the smoother descriptor dynamics.

Theorem 4.2.1

Let the system be strongly reachable and observable then the smoothing error dynamics $(\mathcal{E}, \mathcal{A}, \tilde{\mathcal{B}})$ is strongly reachable.

Proof

Recall from Section 2.4 that $(\mathcal{E}, \mathcal{A}, \tilde{\mathcal{B}})$ is strongly reachable if and only if $[s\mathcal{E} - t\mathcal{A} \mid \tilde{\mathcal{B}}]$ has full rank for all $(s, t) \neq (0, 0)$. Thus, using expressions (4.2.3) we see that $(\mathcal{E}, \mathcal{A}, \tilde{\mathcal{B}})$ is strongly reachable if and only if

$$\begin{bmatrix} sE - tA & -sBB' & B & 0 \\ tC'R^{-1}C & -sA' + tE' & 0 & C'R^{-1} \end{bmatrix}$$

has full row rank for all $(s, t) \neq (0, 0)$. Multiplying on the right by the invertible matrix

$$\begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & sB' & I & 0 \\ -tC & 0 & 0 & I \end{bmatrix}$$

yields

$$\begin{bmatrix} sE - tA & 0 & B & 0 \\ 0 & -sA' + tE' & 0 & C'R^{-1} \end{bmatrix}$$

from which the theorem follows immediately.

Theorem 4.2.2

If (s_0, t_0) is an eigenmode of the pencil $\{\mathcal{E}, \mathcal{A}\}$ then so is (t_0, s_0).

Proof

Note first that if $(s_0, 0)$ is an eigenmode, i.e. if

$$\det(\mathcal{E}) = \det \begin{bmatrix} E & -BB' \\ 0 & -A' \end{bmatrix} = 0 \qquad (4.2.19)$$

then $(0, s_0)$ is also an eigenmode, i.e.

$$\det(\mathcal{A}) = \det \begin{bmatrix} A & 0 \\ -C'R^{-1}C & -E' \end{bmatrix} = 0. \qquad (4.2.20)$$

Consider then any eigenmode (s_0, t_0) with $s_0, t_0 \neq 0$. The following computation shows that (t_0, s_0) is also an eigenmode:

$$\det(t_0 \mathcal{E} - s_0 \mathcal{A}) = \det(t_0 \mathcal{E}' - s_0 \mathcal{A}') =$$

$$\det(\begin{bmatrix} 0 & (1/t_0)I \\ (1/s_0)I & 0 \end{bmatrix} (t_0 \mathcal{E}' - s_0 \mathcal{A}') \begin{bmatrix} 0 & s_0 I \\ t_0 I & 0 \end{bmatrix}) =$$

$$\det(s_0 \mathcal{E} - t_0 \mathcal{A}) = 0. \qquad (4.2.21)$$

Note that this is the generalization of the usual reciprocal symmetry of Hamiltonian eigenvalues for causal systems.

Theorem 4.2.3

The pencil $\{\mathcal{E}, \mathcal{A}\}$ has no eigenmode on the unit circle if and only if the eigenmodes of $\{E, A\}$ that are on the unit circle are strongly reachable and observable, i.e. $[sE-tA \vdots B]$ and $[sE'-tA' \vdots C']$ have full row rank for all eigenmodes such that $|s/t|=1$.

The following proof is just a generalization of the proof of Theorem 3 in [28] to the descriptor case.

Proof

Assume that

$$\begin{bmatrix} E & -BB' \\ 0 & -A' \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \lambda\begin{bmatrix} A & 0 \\ -C'R^{-1}C & -E' \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \tag{4.2.22}$$

with $|\lambda|=1$. Then

$$Ez_1 - BB'z_2 = \lambda A z_1 \tag{4.2.23a}$$

$$A'z_2 = \lambda C'R^{-1}Cz_1 + \lambda E'z_2, \tag{4.2.23b}$$

so that

$$\lambda^* z_2^H E z_1 - \lambda^* z_2^H BB'z_2 = |\lambda|^2 z_2^H A z_1 \tag{4.2.24a}$$

$$z_2^H A z_1 = \lambda^* z_1^H C'R^{-1}Cz_1 + \lambda^* z_2^H E z_1. \tag{4.2.24b}$$

Since $|\lambda|=1$, the above identities imply that

$$z_1^H C'R^{-1}Cz_1 + z_2^H BB'z_2 = 0. \tag{4.2.25}$$

Noting that both $z_1^H C'R^{-1}Cz_1$ and $z_2^H BB'z_2$ are non-negative we deduce that

$$Cz_1 = B'z_2 = 0. \tag{4.2.26}$$

But (4.2.23) and (4.2.26) imply that

$$(E-\lambda A)z_1 = (A'-\lambda E')z_2 = 0. \tag{4.2.27}$$

Combining (4.2.26) and (4.2.27) yields the following expression

$$\begin{bmatrix} C \\ E-\lambda A \end{bmatrix}z_1 = z_2'[A-\lambda E \vdots B] = 0, \tag{4.2.28}$$

which thanks to the strong reachability and observability of eigenmodes on the unit circle implies that

$$z_1 = z_2 = 0, \tag{4.2.29}$$

which completes this proof.

Note that Theorem 4.2.3 holds in particular when (E,A,B) and (C,E,A) are strongly reachable and observable.

Now we return to the analysis of the algebraic generalized Riccati equations. The following result generalizes one of the well known results concerning the existence of a positive-definite solution to the standard algebraic Riccati equation under reachability and observability assumptions and proposes a method for constructing the solutions.

Theorem 4.2.4

If (E,A,B) and (C,E,A) are strongly reachable and observable, respectively, then (4.2.18) has positive-definite solutions Ψ and S.

Proof

Because of the symmetry, we shall only show the result for (4.2.18a). Since the system is assumed to be strongly reachable and observable, the smoother dynamics

$$\begin{bmatrix} E & -BB' \\ 0 & -A' \end{bmatrix} \begin{bmatrix} \hat{x}(k+1) \\ \hat{\lambda}(k+1) \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C'R^{-1}C & -E' \end{bmatrix} \begin{bmatrix} \hat{x}(k) \\ \hat{\lambda}(k) \end{bmatrix} + \hat{\mathcal{B}}y(k) \qquad (4.2.30)$$

has no eigenmode on the unit circle (in fact from the previous theorem we know that strong reachability and observability of eigenmodes on the unit circle is sufficient). We also know that if σ is an eigenmode so is $1/\sigma$. Thus we can write

$$\begin{bmatrix} E & -BB' \\ 0 & -A' \end{bmatrix} \begin{bmatrix} F \\ G \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C'R^{-1}C & -E' \end{bmatrix} \begin{bmatrix} F \\ G \end{bmatrix} J \qquad (4.2.31)$$

where J is in Jordan form and is strictly stable (its eigenvalues are inside the unit circle) and $\begin{bmatrix} F \\ G \end{bmatrix}$ is formed from the eigenvectors and generalized

eigenvectors of the pencil $\left\{ \begin{bmatrix} A & 0 \\ -C'R^{-1}C & -E' \end{bmatrix}, \begin{bmatrix} E & -BB' \\ 0 & -A' \end{bmatrix} \right\}$ corresponding to

stable eigenmodes. We shall first show that if F is invertible, then GF^{-1} is

real-valued.

Note that the Jordan blocks of J that correspond to complex eigenmodes,

are in complex-conjugate pairs and thus J and J^* (where $*$ denotes complex

conjugate) are similar, i.e. there exists an invertible matrix W such that

$J=W^{-1}J^*W$. Thus by using this fact and taking the complex-conjugate of both

sides of (4.2.31), we obtain

$$\begin{bmatrix} E & -BB' \\ 0 & -A' \end{bmatrix} \begin{bmatrix} F^*W \\ G^*W \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C'R^{-1}C & -E' \end{bmatrix} \begin{bmatrix} F^*W \\ G^*W \end{bmatrix} J \qquad (4.2.32)$$

which implies that the columns of $\begin{bmatrix} F^*W \\ G^*W \end{bmatrix}$ are formed from the eigenvectors and

generalized eigenvectors of the same pencil $\left\{ \begin{bmatrix} A & 0 \\ -C'R^{-1}C & -E' \end{bmatrix}, \begin{bmatrix} E & -BB' \\ 0 & -A' \end{bmatrix} \right\}$

corresponding to stable eigenmodes. Thus the range of $\begin{bmatrix} F^*W \\ G^*W \end{bmatrix}$ must equal that of

$\begin{bmatrix} F \\ G \end{bmatrix}$ which implies that for some invertible matrix V, $\begin{bmatrix} F^*W \\ G^*W \end{bmatrix} V = \begin{bmatrix} F \\ G \end{bmatrix}$, from which we

can deduce that $GF^{-1}=G^*(F^*)^{-1}=(GF^{-1})^*$. Thus, if F is invertible, GF^{-1} is

real-valued.

Continuing with the proof, let us rewrite (4.2.31) as

$$EF-BB'G = AFJ \qquad (4.2.33a)$$

$$A'G = C'R^{-1}CFJ+E'GJ, \qquad (4.2.33b)$$

we get

$$G^HEF = G^HBB'G + G^HAFJ \qquad (4.2.34)$$

$$J^HF^HA'G = J^HF^HC'R^{-1}CFJ+J^HF^HE'GJ, \qquad (4.2.35)$$

where H denotes conjugate-transpose. By adding (4.2.35) to the conjugate-transpose of (4.2.34), we get

$$F^H E'G - J^H F^H E'GJ = G^H BB'G + J^H F^H C'R^{-1}CFJ. \tag{4.2.36}$$

Note that the right-hand side of (4.2.36) is Hermitian positive semi-definite. Equation (4.2.36) is a Lyapunov equation and since J is strictly stable $F^H E'G$ must be also Hermitian positive semi-definite.

Now we would like to show that F is invertible. Suppose that it is not invertible, and let $\mathcal{N}$ represent the null-space of F. Then if v is a vector in $\mathcal{N}$, we have

$$Fv = 0. \tag{4.2.37}$$

By multiplying both sides of (4.2.36) on the left and on the right by v and v^H respectively we get

$$v^H(J^H F^H E'GJ + G^H BB'G + J^H F^H C'R^{-1}CFJ)v = 0. \tag{4.2.38}$$

This implies that

$$B'Gv=0 \tag{4.2.39}$$

$$CFJv = 0 \tag{4.2.40}$$

and by multiplying (4.2.33a) on the right by v and using (4.2.39), we get that

$$AFJv = 0. \tag{4.2.41}$$

Since the system is strongly observable[3], (4.2.40) and (4.2.41) imply that

$$FJv = 0 \tag{4.2.42}$$

and thus $\mathcal{N}$ is J invariant. Then, there must be at least one eigenvector of J in $\mathcal{N}$. Let w be such an eigenvector, and let μ be the corresponding eigenmode. Then,

$$B'Gw = 0. \tag{4.2.43}$$

---

[3] In fact, all we need here is that the zero eigenmodes be strongly observable, i.e., $[C'|A']$ have full rank.

Also from (4.2.33b), we get that

$$A'Gw = E'GJw = \mu E'Gw. \qquad (4.2.44)$$

Combining (4.2.43), (4.2.44) and the fact that the system is strongly reachable[4], we get that

$$Gw = 0. \qquad (4.2.45)$$

But

$$Fw = 0 \qquad (4.2.46)$$

which is a contradiction because $\begin{bmatrix} F \\ G \end{bmatrix}$ cannot then have full rank. Thus we have shown that F is invertible.

Using the fact that F is invertible, we can rewrite (4.2.33b) as

$$(E'GF^{-1} + C'R^{-1}C)FJ = A'G. \qquad (4.2.47)$$

Now we need to show that $(E'GF^{-1}+C'R^{-1}C)$ is positive-definite. The matrix $E'GF^{-1}=(F^{-1})^H(F^HE'G)F^{-1}$ is real-valued and positive semi-definite since, as we have shown, $F^HE'G$ is Hermitian and positive semi-definite, and GF^{-1} is real-valued. Thus all we need to show is that $(E'GF^{-1}+C'R^{-1}C)$ is invertible. To show this we will first show that a vector v satisfies

$$GF^{-1}v = 0 \qquad (4.2.48)$$

if and only if it satisfies

$$Ev = 0. \qquad (4.2.49)$$

First we show the only if part. Let (4.2.49) be true, then

$$(F^{-1})^H G^H Ev = 0 = E'GF^{-1}v \qquad (4.2.50)$$

because $E'GF^{-1}$ is symmetric and real-valued. Multiplying (4.2.36) on left and

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[4] In fact, since $|\mu|<1$, all that is needed is that the stable eigenmodes be strongly reachable.

right by $(F^{-1}v)^H$ and $F^{-1}v$ respectively, we obtain

$$v^H E'GF^{-1}v - (JF^{-1}v)^H F^H E'GJF^{-1}v = (GF^{-1}v)^H BB'GF^{-1}v + (CFJF^{-1}v)^H R^{-1}CFJF^{-1}v.$$

$$(4.2.51)$$

Note that thanks to (4.2.50) the left hand side of (4.2.51) is non-positive on the other hand the right hand side is non-negative and thus both sides must be zero which implies that

$$B'GF^{-1}v = 0. \qquad (4.2.52)$$

Combining (4.2.50), (4.2.52) and the fact that the system is strongly reachable[5], we get that

$$GF^{-1}v = 0 \qquad (4.2.53)$$

which is what we wanted to show.

Now we show the if part. Let $\mathcal{M}$ be the null-space of GF^{-1}, and let v be any element v of $\mathcal{M}$, then we want to show that $Ev = 0$. By multiplying (4.2.37) on left and right by $(F^{-1}v)^H$ and $F^{-1}v$ respectively we obtain (4.2.51), both sides of which must be zero thanks to (4.2.48). And so,

$$v^H(JF^{-1})^H(F^H E'G)(JF^{-1})v = 0, \qquad (4.2.54)$$

which since $F^H E'G$ is positive semi-definite, implies that

$$E'GF^{-1}\tilde{A}v = 0 \qquad (4.2.55)$$

where

$$\tilde{A} = FJF^{-1}. \qquad (4.2.56)$$

Also since both sides of (4.2.51) are zero we must have that

$$C\tilde{A}v = 0. \qquad (4.2.57)$$

Since $E'GF^{-1}$ is positive semi-definite and from the only if part, we get that

$$\text{Ker}(E) \subset \text{Ker}(GF^{-1}) \qquad (4.2.58)$$

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[5] All we need here is that the infinite eigenmodes be strongly reachable, i.e., [E¦B] have full rank.

and thus there exists a positive semi-definite matrix Z such that

$$GF^{-1} = ZE. \qquad (4.2.59)$$

By replacing GF^{-1} in (4.2.55) with ZE and using the fact that the null-space of E'ZE and ZE are identical, we get that

$$ZE\tilde{A}v = 0, \qquad (4.2.60)$$

or

$$GF^{-1}\tilde{A}v = 0. \qquad (4.2.61)$$

This of course implies that $\mathcal{M}$ is $\tilde{A}$ invariant. From (4.2.33a) and since $B'GF^{-1}v=0$, which follows the fact $v\in\mathcal{M}=\mathrm{Ker}(GF^{-1})$, we get

$$Ev = A\tilde{A}v \qquad (4.2.62)$$

and so what we need to show is that $A\tilde{A}v=0$ (in fact we will show that $\tilde{A}v=0$). Assume that $\tilde{A}$ has an eigenvector w in $\mathcal{M}$ associated to an eigenvalue μ. Then

$$E\tilde{A}w = \mu Ew = \mu A\tilde{A}w \qquad (4.2.63)$$

so that $\tilde{A}w$ is an eigenvector of $\{E,A\}$. But thanks to (4.2.57),

$$C\tilde{A}w = 0 \qquad (4.2.64)$$

which thanks to (4.2.63) and the strong observability assumption[6] implies that

$$\tilde{A}w = 0. \qquad (4.2.65)$$

which means that μ can only be 0. Thus any eigenvector of $\tilde{A}$ in $\mathcal{M}$ must correspond to a zero eigenvalue. This and the fact that $\mathcal{M}$ is $\tilde{A}$-invariant imply that the eigenvectors and the generalized eigenvectors of $\tilde{A}$ in $\mathcal{M}$ corresponding

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[6] In fact, since $|\mu|<1$, all that is needed is that the unstable eigenmodes be strongly observable.

to the eigenvalue 0, span $\mathcal{M}$.[7] But matrix $\tilde{A}$ cannot have a generalized

eigenvector associated to eigenvalue 0 in $\mathcal{M}$ because then there exists $r \in \mathcal{M}$ such

that

$$\tilde{A}^2 r = 0 \qquad (4.2.66)$$

and

$$\tilde{A} r \neq 0. \qquad (4.2.67)$$

From (4.2.62) and the fact that $r, \tilde{A}r \in \mathcal{M}$, we get that

$$E\tilde{A}r = A\tilde{A}^2 r = 0. \qquad (4.2.68)$$

But from (4.2.57) we get that,

$$C\tilde{A}r = 0 \qquad (4.2.69)$$

so thanks to the strong observability assumption[8],

$$\tilde{A}r = 0 \qquad (4.2.70a)$$

which contradicts (4.2.67). Thus $\mathcal{M}$ is spanned by the eigenvectors of $\tilde{A}$ in $\mathcal{M}$

corresponding to eigenvalue 0. And since $v \in \mathcal{M}$, we must have that

$$\tilde{A}v = 0. \qquad (4.2.70b)$$

Using (4.2.62) we get the desired result that

$$Ev = 0. \qquad (4.2.71)$$

Using the fact that we have now proven that E and GF^{-1} have exactly the

same null space, we have that (4.2.59) holds with $Z > 0$. Also noting that

$$E'GF^{-1} + C'R^{-1}C = E'ZE + C'R^{-1}C = [E' \ C'] \begin{bmatrix} Z & \\ & R^{-1} \end{bmatrix} \begin{bmatrix} E \\ C \end{bmatrix}, \qquad (4.2.72)$$

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[7] Let X be a matrix and S an X-invariant subspace. Then the eigenvectors and their associated generalized eigenvectors of X that are in S, span S. To see this suppose that we are in a coordinate system such that $S = \text{Im} \begin{bmatrix} I \\ 0 \end{bmatrix}$ in which case $X = \begin{bmatrix} T & U \\ 0 & V \end{bmatrix}$. The eigenvectors and generalized eigenvectors of T clearly span S and also they are the eigenvectors and the associated eigenvectors of X in S.

[8] Matrix $[E' \ C']$ has full rank if and only if the infinite eigenmodes of the system are strongly observable.

and using the fact that [E' C'] has full rank[9], we get that

$$E'GF^{-1}+C'R^{-1}C$$

is positive-definite. Now from (4.2.47) and (4.2.33a) we get that

$$(A(E'GF^{-1} + C'R^{-1}C)^{-1}A' + BB')GF^{-1} = E. \qquad (4.2.73)$$

Let

$$\Psi = A(E'GF^{-1} + C'R^{-1}C)^{-1}A' + BB', \qquad (4.2.74)$$

then thanks to the strong reachability assumption (in particular of the zero-eigenmode), Ψ is positive definite and

$$GF^{-1} = \Psi^{-1}E. \qquad (4.2.75)$$

By combining (4.2.75) and (4.2.76), we get

$$\Psi = A(E'\Psi^{-1}E + C'R^{-1}C)^{-1}A' + BB'. \qquad (4.2.76)$$

This, of course, completes this proof.

Note that in the above proof, we did not use the assumption that the eigenmodes of the system that are inside the unit circle are observable (except for the zero eigenmode and in the next section we will see that this is not needed either). Also the assumption that the eigenmodes that are outside the unit circle are reachable was not used. In fact these assumptions are not needed for the existence of Ψ (they are needed to show the existence of S however). All what is needed for the existence of a positive-definite solution Ψ is that the system be <u>forward detectable</u> (i.e. eigenmodes on and outside the unit circle, ∞ included, be strongly observable) and <u>backward stabilizable</u> (i.e. eigenmodes on and inside the unit circle be strongly

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[9] Matrix [E' C'] has full rank if and only if the infinite eigenmodes of the system are strongly observable.

reachable). Backward stabilizability guarantees that the solution to the algebraic generalized Riccati equation (4.2.18a) is positive-definite. For standard systems this is not a major concern because the standard algebraic Riccati equation can have a positive semi-definite solution. On the other hand the condition that is commonly required is that the associated Kalman filter be forward stable for which we need to have <u>forward stabilizability</u> (i.e. eigenmodes on and outside the unit circle be strongly reachable). The following example illustrates the fact that the algebraic generalized (or standard) Riccati equation could have a positive-definite solution even if the system is not forward stabilizable.

Example 4.2.1

Consider the causal system

$$x(k+1) = \alpha x(k) \tag{4.2.77a}$$

$$y(k) = x(k) + r(k) \tag{4.2.77b}$$

where r is a white unit variance Gaussian sequence. By direct calculation, we can show that the generalized algebraic Riccati equation (4.2.18a) has a solution $\Psi = \alpha^2 - 1$ which means that the system has a positive-definite solution if $|\alpha| > 1$. If we use the alternate form (4.2.17) of the Riccati equation, we see that $\Psi = 0$ is also a solution.

We do not worry too much about these conditions in this section since we have a "forward" equation (4.2.18a) and a "backward" equation (4.2.18b). By symmetry, we can show that in order to have a positive definite solution S to (4.2.18b) we must have backward detectability (i.e. eigenmodes on and inside the unit circle are observable) and forward stabilizability and thus if we want to have positive-definite solutions to both algebraic generalized Riccati equations (in fact all 4 algebraic generalized Riccati equations), we must have strong reachability and strong observability. These conditions also guarantee the stability of the forward and backward generalized Kalman filters (this will become clear in the next section). The conditions for existence of a solution to an individual algebraic generalized Riccati equation are studied in the next section.

Example 4.2.2

Consider the generalized Riccati equation (4.2.18a) associated with the TPBVDS of Example 4.1.1. To find the solution to this equation, we need to find the eigenvectors and generalized eigenvectors of the pencil

$$\left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \right\}$$

corresponding to stable eigenmodes. In this case there is only one stable eigenmode which is zero. The eigenvector and the generalized eigenvector associated to eigenmode 0 are

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

Now we can use expression (4.2.75) with F=I and $G=\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ to obtain Ψ:

$$\Psi = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}. \tag{4.2.78a}$$

It can be verified that Ψ is indeed a solution of the algebraic generalized Riccati equation (4.2.18a).

Theorem 4.2.5

Let (E,A,B) and (C,E,A) be strongly reachable and observable respectively and let Θ, S, Ψ, and T denote positive-definite solutions to the algebraic generalized Riccati equations. Then matrices

$$A^f = AT^{-1}E'\Psi^{-1} \tag{4.2.79a}$$

$$A^b = A'S^{-1}E\Theta^{-1}. \tag{4.2.79b}$$

are strictly stable.

Proof

Let us perform the following change of coordinates on (4.2.2):

$$\begin{bmatrix} \gamma(k) \\ \delta(k) \end{bmatrix} = \mathcal{T} \begin{bmatrix} \tilde{x}(k) \\ -\hat{\lambda}(k) \end{bmatrix} \tag{4.2.80}$$

where

$$\mathcal{T} = \begin{bmatrix} E & -\Psi \\ \Theta & E' \end{bmatrix} \tag{4.2.81a}$$

and premultiply (4.2.2) by

$$\mathcal{F} = \begin{bmatrix} I & AT^{-1} \\ A'S^{-1} & -I \end{bmatrix}. \tag{4.2.81b}$$

The result is the following

$$
\begin{bmatrix} I & 0 \\ 0 & A^b \end{bmatrix} \begin{bmatrix} \gamma(k+1) \\ \delta(k+1) \end{bmatrix} = \begin{bmatrix} A^f & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \gamma(k) \\ \delta(k) \end{bmatrix} + \begin{bmatrix} B & AT^{-1}C'R^{-1} \\ A'S^{-1}B & -C'R^{-1} \end{bmatrix} \begin{bmatrix} u(k) \\ r(k) \end{bmatrix}. \quad (4.2.82)
$$

Thus γ and δ satisfy the following recursions

$$
\delta(k) = A^b \delta(k+1) - A'S^{-1}Bu(k) + C'R^{-1}r(k) \quad (4.2.83)
$$

$$
\gamma(k+1) = A^f \gamma(k) + Bu(k) + AT^{-1}C'R^{-1}r(k). \quad (4.2.84)
$$

Note that thanks to the strong reachability of $(\mathcal{E}, \mathcal{A}, \widetilde{\mathcal{B}})$, $\delta(k)$ and $\gamma(k)$ are strongly reachable. Now consider the Lyapunov equation associated to (4.2.83) and (4.2.84):

$$
P_\delta - A^b P_\delta A^{b\cdot} = A'S^{-1}BB'S^{-1}A + C'R^{-1}C \quad (4.2.85)
$$

$$
P_\gamma - A^f P_\gamma A^{f\cdot} = BB' + AT^{-1}C'R^{-1}CT^{-1}A'. \quad (4.2.86)
$$

Thanks to the strong reachability of (4.2.83) and (4.2.84) all we need to show is that (4.2.85) and (4.2.86) have positive definite solutions. Indeed by direct calculation we can show that

$$
P_\delta = \Theta \ , \quad P_\gamma = \Psi \quad (4.2.87)
$$

are solutions which proves the theorem.

Theorem 4.2.6

If (E,A,B) and (C,E,A) are strongly reachable and observable respectively, then (4.2.18a) and (4.2.18b) have unique positive-definite solutions Ψ and S.

Proof

We show the result only for (4.2.18a). Let Ψ_1 and Ψ_2 be two positive solutions of (4.2.18a). Also let

$$T_i = E'\Psi_i^{-1}E + C'R^{-1}C \quad \text{for } i=1,2. \tag{4.2.88a}$$

Then

$$\Psi_i = AT_i^{-1}A' + BB' \quad \text{for } i=1,2. \tag{4.2.88b}$$

Now let

$$\Delta\Psi = \Psi_1 - \Psi_2 \tag{4.2.89a}$$

and

$$\Delta T = T_1 - T_2. \tag{4.2.89b}$$

From (4.2.88b), and (4.2.89), we get that

$$\Delta\Psi = -AT_1^{-1}\Delta T T_2^{-1}A' \tag{4.2.90}$$

and

$$\Delta T = -E'\Psi_1^{-1}\Delta\Psi\Psi_2^{-1}E. \tag{4.2.91}$$

By combining (4.2.90) and (4.2.91), we get

$$\Delta\Psi = AT_1^{-1}E'\Psi_1^{-1}\Delta\Psi\Psi_2^{-1}ET_2^{-1}A'. \tag{4.2.92}$$

Equation (4.2.92) can be expressed as

$$[I - (AT_1^{-1}E'\Psi_1^{-1})\oplus(\Psi_2^{-1}ET_2^{-1}A')']\underline{\Delta\Psi} = 0 \tag{4.2.93}$$

where $\oplus$ denotes the Kronecker product and $\underline{\Delta\Psi}$ is the vector obtained from the entries of $\Delta\Psi$ by lexicographic ordering. Matrices $AT_1^{-1}E'\Psi_1^{-1}$ and $(\Psi_2^{-1}ET_2^{-1}A')'$ are strictly stable (see Theorem 4.2.5) and so their Kronecker product is stable. This implies that (4.2.93) has a unique trivial solution 0. Thus, (4.2.92) has unique solution $\Delta\Psi=0$. This completes the proof of Theorem 4.2.6.

In this section, we have shown how the smoother dynamics can be transformed into forward-backward stable form (assuming strong reachability and observability) so that the two-filter solution can be used to solve the smoother TPBVDS derived in the previous section. The two-filter solution has the advantage of being composed of time-invariant filters, however, it has the disadvantage of requiring the computation of a posteriori correction term (see Appendix A). In Section 4.3 another method for smoothing the TPBVDS's is proposed. This method is closely related to the generalized Riccati equations obtained above, which we study further in the next section. In particular, we shall be interested in obtaining conditions under which the time-varying generalized Riccati equations converge.

However, before ending this section, remember that in Section 3.3.4 we derived the optimal smoother for a stochastically stationary TPBVDS (see equation (3.3.88)) and showed that the estimation error variance can be obtained from solving the following generalized Lyapunov equation

$$\mathcal{E}_s \Pi \mathcal{E}_s' - \mathcal{A}_s \Pi \mathcal{A}_s' = \mathcal{P}_s \tilde{\mathcal{B}}_s \tilde{\mathcal{B}}_s' \mathcal{P}_s - (I - \mathcal{P}_s) \tilde{\mathcal{B}}_s \tilde{\mathcal{B}}_s' (I - \mathcal{P}_s)' \qquad (4.2.94a)$$

where

$$\mathcal{C}_s = \mathcal{C}\mathcal{T}^{-1}, \ \mathcal{E}_s = \mathcal{T}\mathcal{E}\mathcal{T}^{-1}, \ \mathcal{A}_s = \mathcal{T}\mathcal{A}\mathcal{T}^{-1}, \ \tilde{\mathcal{B}}_s = \mathcal{T}\tilde{\mathcal{B}}, \ \mathcal{P}_s = \mathcal{E}_s^\infty \qquad (4.2.94b)$$

and where $\mathcal{T}$ and $\mathcal{T}$ must be chosen so that $\{\mathcal{E}_s, \mathcal{A}_s\}$ is in forward-backward stable form. We also showed that the variance matrix P_e of the smoothing error $\tilde{x}$ is then given by

$$P_e = \mathcal{C}_s \Pi \mathcal{C}_s'. \qquad (4.2.95)$$

In the previous chapter, however, we did not have an expression for $\mathcal{T}$ and $\mathcal{T}$; we do now. Note that transformation (4.2.94b) is just the Hamiltonian diagonalization and thus matrices $\mathcal{T}$ and $\mathcal{T}$ are given by (4.2.81a) and

(4.2.81b), respectively. Also $\mathscr{E}_s$, $\mathscr{A}_s$ and $\tilde{\mathscr{B}}_s$, all appear in (4.2.82) and the projection matrix $\mathscr{P}_s$ is clearly $\begin{bmatrix} I & \\ & 0 \end{bmatrix}$. Thus the generalized Lyapunov equation (4.2.94a) can be expressed as

$$\begin{bmatrix} I & 0 \\ 0 & A^b \end{bmatrix} \Pi \begin{bmatrix} I & 0 \\ 0 & A^{b\cdot} \end{bmatrix} - \begin{bmatrix} A^f & 0 \\ 0 & I \end{bmatrix} \Pi \begin{bmatrix} A^{f\cdot} & 0 \\ 0 & I \end{bmatrix} =$$

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} BB'+AT^{-1}C'R^{-1}CT^{-1}A' & BB'S^{-1}A'-AT^{-1}C'R^{-2}C \\ AS^{-1}BB'-C'R^{-2}CT^{-1}A' & A'S^{-1}BB'S^{-1}A+C'R^{-1}C \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

$$- \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} BB'+AT^{-1}C'R^{-1}CT^{-1}A' & BB'S^{-1}A'-AT^{-1}C'R^{-2}C \\ AS^{-1}BB'-C'R^{-2}CT^{-1}A' & A'S^{-1}BB'S^{-1}A+C'R^{-1}C \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \quad (4.2.96)$$

The (1,1) and (2,2) block entries of (4.2.96) yield equations (4.2.85) and (4.2.86), respectively. The (1,2) and (2,1) block entries of (4.2.96) imply that Π is block diagonal. Thus the solution Π of (4.2.96) is equal to $\begin{bmatrix} \Psi & 0 \\ 0 & \Theta \end{bmatrix}$ where Ψ and Θ are solutions to the algebraic generalized Riccati equations. Since $P_e = \mathscr{E}_s \Pi \mathscr{E}_s'$, (4.2.95) implies

$$P_e = [I \quad 0] \mathscr{T}^{-1} \begin{bmatrix} \Psi & 0 \\ 0 & \Theta \end{bmatrix} \mathscr{T}'^{-1} \begin{bmatrix} I \\ 0 \end{bmatrix}. \quad (4.2.97)$$

Noting that

$$\mathscr{T}^{-1} = \begin{bmatrix} \Delta^{-1}E'\Psi^{-1} & \Delta^{-1} \\ -\Gamma^{-1} & \Gamma^{-1}E\Theta^{-1} \end{bmatrix}, \quad (4.2.98)$$

where

$$\Delta = \Theta + E\Psi^{-1}E' \quad (4.2.99a)$$

$$\Gamma = \Psi + E\Theta^{-1}E', \quad (4.2.99b)$$

we see that the smoothing error variance P_e is simply equal to Δ^{-1}.

4.2.2-A Study of Generalized Riccati Equations

In this section, we generalize the existing results concerning the convergence properties of the standard Riccati equation under the detectability and stabilizability conditions to the case of descriptor systems. We shall consider the generalized Riccati equation

$$\Psi_{k+1} = A\{\underset{\epsilon\to0^+}{\text{Lim}}(E'(\Psi_k+\epsilon Q)^{-1}E+C'R^{-1}C)^{-1}\}A' + BB' \qquad (4.1.100)$$

where Q is any positive semi-definite matrix for which (4.2.100) is defined --i.e., $\Psi_k+\epsilon Q$ is invertible for $\epsilon>0$-- with the assumption only that the infinite eigenmodes of (C,E,A) are strongly observable. We will show that the limit in (4.2.100) exists and moreover, that it does not depend on the particular choice of Q. But before that, note that if Ψ_k is positive-definite, then (4.2.100) reduces to (4.2.15b). Thus (4.2.100) is a generalization of (4.2.15b) which allows us to use positive semi-definite matrices in the recursion.

Thus, we must show that the following limit exists and is independent of the choice of Q (as long as $Q\geq0$ and $\Psi_k+\epsilon Q$ is invertibe for $\epsilon>0$):

$$F_k = \underset{\epsilon\to0^+}{\text{Lim}}T_k^{-1}(\epsilon) \qquad (4.2.101a)$$

where

$$T_k(\epsilon) = E'(\Psi_k+\epsilon I)^{-1}E+C'R^{-1}C. \qquad (4.2.101b)$$

This result follows from the following lemma:

Lemma 4.2.1

Let L be a full column rank matrix and let X be a positive semi-definite matrix, then

$$Z = \underset{\epsilon\to0^+}{\text{Lim}}(L'(X+\epsilon Y)^{-1}L)^{-1}, \qquad (4.2.102)$$

where Y is a positive semi-definite matrix such that $X+\epsilon Y$ is invertible for $\epsilon>0$, exists and is independent of the particular choice of Y.

Proof

See Lemma 4.3.1.

To apply this lemma to (4.2.101), simply let

$$L = \begin{bmatrix} E \\ C \end{bmatrix}, \quad X = \begin{bmatrix} \Psi_k & 0 \\ 0 & R \end{bmatrix} \qquad (4.2.103)$$

and

$$Y = \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}. \qquad (4.2.104)$$

It is then easy to see that

$$Z = F_k. \qquad (4.2.105)$$

In what follows, for simplicity, we shall assume that Q equals I.

Theorem 4.2.7

Let (C,E,A) and (E,A,B) be forward detectable and stabilizable. Then for any $\Psi_0 \geq 0$, as k goes to infinity, the sequence Ψ_k satisfying (4.2.100) converges to a positive semi-definite matrix Ψ exponentially fast.

We shall prove this result by extending the approach used in the non-descriptor case [31]. First, we will show that Ψ_k as k goes to infinity is bounded above. Then, we will show that for a particular initial condition Ψ_0, Ψ_k is increasing, thus guaranteeing the existence of a limit for Ψ_k as k goes to infinity. This result is then extended to the case of an arbitrary initial condition Ψ_0. Finally, we show that the convergences is exponentially fast.

Before starting the proof, we need the following result:

Lemma 4.2.2

Let (C,E,A) be forward detectable and let R be any positive-definite matrix then there exists a positive-definite matrix ψ such that the matrix $\hat{A}$ given by

$$\hat{A} = A\tau^{-1}E'\psi^{-1} \tag{4.2.106}$$

where

$$\tau = E'\psi^{-1}E + C'R^{-1}C \tag{4.2.107}$$

has all of its eigenvalues inside the unit circle.

Proof

First, note that forward detectability of (C,E,A) means that all the eigenmodes of the pencil {E,A} outside the unit circle and, in particular, the infinite eigenmodes, are strongly observable. Thus, [E' C'] has full rank, which, since ψ is assumed to be positive-definite, implies that τ is positive-definite and thus expression (4.2.106) is well defined.

We shall first prove Lemma 4.2.2 in the case where (C,E,A) is strongly observable. Let G be a matrix such that (E,A,G) is strongly reachable. Then from Theorem 4.2.4 we know that the generalized Riccati equation

$$\Psi = A(E'\Psi^{-1}E+C'R^{-1}C)^{-1}A' + GG' \tag{4.2.108}$$

has a positive-definite solution Ψ and from Theorem 4.2.5 we know that

$$A^f = AT^{-1}E'\Psi^{-1} \tag{4.2.109}$$

where

$$T = E'\Psi^{-1}E + C'R^{-1}C \tag{4.2.110}$$

is strictly stable. Clearly then we can take ψ to be Ψ, in which case τ becomes T and $\hat{A}$ becomes A^f.

Now suppose that the (C,E,A) is not strongly observable. Let us first consider the special case where the system matrices have the following special structure

$$E = \begin{bmatrix} \tilde{E}_{11} & 0 \\ 0 & I \end{bmatrix}, \quad A = \begin{bmatrix} \tilde{A}_{11} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}, \quad C'R^{-1}C = \begin{bmatrix} \tilde{H}'\tilde{H} & 0 \\ 0 & 0 \end{bmatrix} \quad (4.2.111)$$

where (H,E_{11},A_{11}) is strongly observable and $\tilde{A}_{22}$ is strictly stable. We shall show later that any forward detectable (C,E,A) can be put into the form (4.2.111). Since $(\tilde{H},\tilde{E}_{11},\tilde{A}_{11})$ is strongly observable, from the first part, we know that there exist matrices ψ_{11} and τ_{11} such that

$$\tau_{11} = \tilde{E}_{11}'\psi_{11}^{-1}\tilde{E}_{11} + \tilde{H}'\tilde{H} \quad (4.2.112)$$

is positive-definite and such that

$$\hat{A}_{11} = \tilde{A}_{11}\tau_{11}^{-1}\tilde{E}_{11}'\psi_{11}^{-1} \quad (4.2.113)$$

is stable.

Now let

$$\psi = \begin{bmatrix} \psi_{11} & 0 \\ 0 & \psi_{22} \end{bmatrix} \quad (4.2.114)$$

where ψ_{22} is any positive-definite matrix. Then

$$\hat{A} = \begin{bmatrix} \hat{A}_{11} & 0 \\ \tilde{A}_{21}\tau_{11}^{-1}\tilde{E}_{11}'\psi_{11}^{-1} & \tilde{A}_{22} \end{bmatrix}. \quad (4.2.115)$$

The matrix $\hat{A}$ is stable because $\hat{A}_{11}$ and A_{22} are both stable.

Thus all that remains to be shown is that any forward detectable (C,E,A) can be put into the form (4.2.111). Transforming (C,E,A) into the form (4.2.111) corresponds to separating the strongly observable and unobservable parts of the system. In particular,

there exist invertible matrices L and Q such that

$$LEQ = \begin{bmatrix} E_{11} & 0 \\ E_{21} & E_{22} \end{bmatrix}, \quad LAQ = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}, \quad Q'C'R^{-1}CQ = \begin{bmatrix} H'H & 0 \\ 0 & 0 \end{bmatrix} \quad (4.2.116)$$

for some invertible matrix H (see Section 2.5). If the system is in standard form and O_s denotes the strong observability matrix (C,E,A), Q is any invertible matrix such that

$$O_s Q = [O_s^1 \quad 0] \quad (4.2.117)$$

where O_s^1 has full column rank. In this case L equals Q^{-1}. If {E,A} is not in standard form, the system must be transformed into standard-form first. In this case, L is no longer equal to Q^{-1} but $Q^{-1}(\alpha E + \beta A)^{-1}$ for some α and β such that $\alpha E + \beta A$ is invertible.

Note that thanks to the forward detectability assumption, E_{22} is invertible. Let then

$$\tilde{Q} = \begin{bmatrix} I & 0 \\ -E_{22}^{-1}E_{21} & E_{22}^{-1} \end{bmatrix}. \quad (4.2.118)$$

By direct calculation, we can show that

$$LEQ\tilde{Q} = \begin{bmatrix} E_{11} & 0 \\ 0 & I \end{bmatrix}, \quad LAQ\tilde{Q} = \begin{bmatrix} A_{11} & 0 \\ A_{21} - A_{22}E_{22}^{-1}E_{21} & A_{22}E_{22}^{-1} \end{bmatrix},$$

$$\tilde{Q}'Q'C'R^{-1}CQ\tilde{Q} = \begin{bmatrix} H'H & 0 \\ 0 & 0 \end{bmatrix} \quad (4.2.119)$$

which is the desired form. Thus, the lemma is proved.

Proof of Theorem 4.2.7

The first thing we show is that Ψ_k (obtained from recursion (4.2.100)) is bounded. For this, we shall first show that Ψ_k is bounded above by $\hat{\Psi}_k$

satisfying

$$\hat{\Psi}_0 = \Psi_0 \qquad (4.2.120a)$$

$$\hat{\Psi}_{k+1} = (A\tau^{-1}E'\psi^{-1})\hat{\Psi}_k(A\tau^{-1}E'\psi^{-1})' + A\tau^{-1}C'R^{-1}C\tau^{-1}A' + BB' \qquad (4.2.120b)$$

where ψ is any positive-definite matrix and

$$\tau = E'\psi^{-1}E + C'R^{-1}C. \qquad (4.2.121)$$

We shall prove this result by induction. Note that thanks to (4.2.120a)

$$\hat{\Psi}_0 \geq \Psi_0. \qquad (4.2.122)$$

Now suppose that

$$\hat{\Psi}_k \geq \Psi_k. \qquad (4.2.123)$$

then we have to show that

$$\hat{\Psi}_{k+1} \geq \Psi_{k+1}. \qquad (4.2.124)$$

Let

$$\Psi_{k+1}(\epsilon) = AT_k^{-1}(\epsilon)E'(\Psi_k+\epsilon I)^{-1}(\Psi_k+\epsilon I)(AT_k^{-1}(\epsilon)E'(\Psi_k+\epsilon I)^{-1})' + $$
$$AT_k^{-1}(\epsilon)C'R^{-1}CT_k^{-1}(\epsilon)A' + BB' \qquad (4.2.125a)$$

where $T_k(\epsilon)$ is defined in (4.2.101b). Then some algebra yields

$$\Psi_{k+1} = \lim_{\epsilon \to 0^+}\Psi_{k+1}(\epsilon). \qquad (4.2.125b)$$

Also let

$$\hat{\Psi}_{k+1}(\epsilon) = A\tau^{-1}E'\psi^{-1}(\hat{\Psi}_k+\epsilon I)(A\tau^{-1}E'\psi^{-1})' + A\tau^{-1}C'R^{-1}C\tau^{-1}A' + BB' \qquad (4.2.126a)$$

then

$$\hat{\Psi}_{k+1} = \lim_{\epsilon \to 0^+}\hat{\Psi}_{k+1}(\epsilon). \qquad (4.2.126b)$$

Equation (4.2.123) implies that for some $\Delta \geq 0$,

$$\hat{\Psi}_k = \Psi_k + \Delta. \qquad (4.2.127)$$

Then

$$\hat{\Psi}_{k+1} - \Psi_{k+1} = \lim_{\epsilon \to 0^+}[\hat{\Psi}_{k+1}(\epsilon) - \Psi_{k+1}(\epsilon)]$$

$$= \lim_{\epsilon \to 0^+}[A\tau^{-1}E'\psi^{-1}(\Psi_k+\epsilon I)(A\tau^{-1}E'\psi^{-1})'$$

$$+ A\tau^{-1}C'R^{-1}C\tau^{-1}A' - AT_k^{-1}(\epsilon)E'(\Psi_k+\epsilon I)^{-1}(\Psi_k+\epsilon I)(AT_k^{-1}(\epsilon)E'(\Psi_k+\epsilon I)^{-1})'$$

$$- AT_k^{-1}(\epsilon)C'R^{-1}CT_k^{-1}(\epsilon)A'] + A\tau^{-1}E'\psi^{-1}\Delta(A\tau^{-1}E'\psi^{-1})'. \qquad (4.2.128)$$

Let

$$P(\epsilon) = A\tau^{-1}E'\psi^{-1}(\Psi_k+\epsilon I)(A\tau^{-1}E'\psi^{-1})' + A\tau^{-1}C'R^{-1}C\tau^{-1}A' -$$

$$AT_k^{-1}(\epsilon)E'(\Psi_k+\epsilon I)^{-1}(\Psi_k+\epsilon I)(AT_k^{-1}(\epsilon)E'(\Psi_k+\epsilon I)^{-1})' - AT_k^{-1}(\epsilon)C'R^{-1}CT_k^{-1}(\epsilon)A'.$$

$$(4.2.129)$$

Then (4.2.128) can be expressed as

$$\hat{\Psi}_{k+1} - \Psi_{k+1} = \lim_{\epsilon \to 0^+}P(\epsilon) + A\tau^{-1}E'\psi^{-1}\Delta(A\tau^{-1}E'\psi^{-1})'. \qquad (4.2.130)$$

Since $A\tau^{-1}E'\psi^{-1}\Delta(A\tau^{-1}E'\psi^{-1})'\geq 0$, if we show that $P(\epsilon)\geq 0$, (4.2.124) follows.

Using expressions (4.2.125a) and (4.2.126a), and after some algebraic manipulation we can show that

$$P(\epsilon) = A\tau^{-1}E'V(\epsilon)[\Psi_k+\epsilon I-ET_k^{-1}(\epsilon)E']V(\epsilon)E\tau^{-1}A' \qquad (4.2.131)$$

where

$$V(\epsilon) = (\Psi_k+\epsilon I)^{-1} - \psi^{-1}. \qquad (4.2.132)$$

Thus, if we can show that

$$W(\epsilon) = \Psi_k + \epsilon I - ET_k^{-1}(\epsilon)E' \qquad (4.2.133)$$

is positive semi-definite for $\epsilon>0$, thanks to the fact that $V(\epsilon)$ is symmetric, we obtain the desired result i.e. that $P(\epsilon)\geq 0$. Let us perturb W as follows

$$W(\epsilon,\epsilon') = \Psi_k(\epsilon) - E[T_k(\epsilon)+\epsilon'I]^{-1}E' \qquad (4.2.134)$$

where ϵ' is a small positive number. Then using the ABCD matrix inversion formula and (4.2.101b), (4.2.134) yields the following expression

$$W(\epsilon,\epsilon')^{-1} = (\Psi_k+\epsilon I)^{-1}+(\Psi_k+\epsilon I)^{-1}E(\epsilon'I+C'R^{-1}C)^{-1}E'(\Psi_k+\epsilon I)^{-1} \qquad (4.2.135)$$

which means that $W(\epsilon,\epsilon')^{-1}$ is positive-definite for $\epsilon'>0$. Thus

$$W(\epsilon,\epsilon')^{-1} \geq \Psi_k^{-1}(\epsilon) > 0 \qquad (4.2.136)$$

which implies that

$$W(\epsilon,\epsilon') > 0. \qquad (4.2.137a)$$

Note that since for $\epsilon > 0$ we have that $T_k(\epsilon) > 0$, $W(\epsilon,\epsilon')$ is continuous in ϵ' for $\epsilon' \geq 0$. Thus

$$W(\epsilon) = \lim_{\epsilon' \to 0} W(\epsilon,\epsilon') \geq 0. \qquad (4.2.137b)$$

Thus we have shown that $P(\epsilon) \geq 0$ which implies (4.2.124). Thus for all k,

$$\hat{\Psi}_k \geq \Psi_k. \qquad (4.2.138)$$

Note that (4.2.120b) can also be expressed as follows

$$\hat{\Psi}_{k+1} = \hat{A}\hat{\Psi}_k\hat{A}' + A_T^{-1}C'R^{-1}C_T^{-1}A' + BB' \qquad (4.2.139)$$

where

$$\hat{A} = A_T^{-1}E'\psi^{-1}. \qquad (4.2.140)$$

Thanks to Lemma 4.2.2, ψ can be chosen so that $\hat{A}$ is stable which implies that (4.2.139) is a stable recursion and is bounded and indeed converges. Thus thanks to (4.2.138) Ψ_k is bounded which is the desired result.

The next step in the proof consists in showing that for some choice of initial condition Ψ_0, Ψ_k is monotone increasing and thus establishing the fact that a limit exists. The procedure is inductive, it is shown that Ψ_1 is larger than Ψ_0 and then assuming that Ψ_k is larger than Ψ_{k-1}, it is shown that Ψ_{k+1} is larger than Ψ_k. In particular, let

$$\Psi_0 = 0 \qquad (4.2.141)$$

then clearly

$$\Psi_1 \geq \Psi_0. \qquad (4.2.142)$$

Now suppose that

$$\Psi_k \geq \Psi_{k-1} \tag{4.2.143}$$

then

$$(\Psi_k + \epsilon I)^{-1} \leq (\Psi_{k-1} + \epsilon I)^{-1} \tag{4.2.144}$$

and thus using expression (4.2.100) we can see that

$$\Psi_{k+1} \geq \Psi_k. \tag{4.2.145}$$

Expression (4.2.142), (4.2.143) and (4.2.145) imply that Ψ_k is monotone

increasing. But we have already shown that Ψ_k is bounded which means that if

$$\Psi_0 = 0 \tag{4.2.146}$$

then

$$\lim_{k \to \infty} \Psi_k = \Psi \tag{4.2.147}$$

where Ψ is a positive semi-definite matrix satisfying the algebraic

generalized Riccati equation

$$\Psi = A\{ \lim_{\epsilon \to 0^+} (E'(\Psi + \epsilon I)^{-1} E + C'R^{-1}C)^{-1}\}A' + BB'. \tag{4.2.148}$$

If $\Psi > 0$, (4.2.148) reduces to the algebraic generalized Riccati equation

(4.2.18a).

Thus we have shown that if $\Psi_0 = 0$ then Ψ_k converges to a positive

semi-definite matrix Ψ satisfying (4.2.148).

The next step in the proof consists of extending the convergence result

obtained above to the case of arbitrary initial condition Ψ_0. To do this,

however, we need another result, roughly that the "closed-loop" matrix is

stable under our assumption of forward stabilizability.

Expression (4.2.148) can also be expressed as follows

$$\Psi = \lim_{\epsilon \to 0^+} \{A^f(\epsilon)(\Psi + \epsilon I)A^f(\epsilon)' + K(\epsilon)RK(\epsilon)' + BB'\} \tag{4.2.149}$$

where

$$A^f(\epsilon) = AT^{-1}(\epsilon)E'(\Psi+\epsilon I)^{-1} \tag{4.2.150}$$

$$K(\epsilon) = AT^{-1}(\epsilon)C'R^{-1} \tag{4.2.151}$$

$$T(\epsilon) = E'(\Psi+\epsilon I)^{-1}E + C'R^{-1}C. \tag{4.2.152}$$

Since $T^{-1}(\epsilon)$ converges to some positive semi-definite matrix F as ϵ goes to zero, we can see that the following limit exists

$$\lim_{\epsilon\to 0^+} K(\epsilon) = K = AFC'R^{-1}. \tag{4.2.153}$$

Also note that

$$A^f(\epsilon)(\Psi+\epsilon I) = AT^{-1}(\epsilon)E' \tag{4.2.154a}$$

and

$$A^f(\epsilon)E = A - K(\epsilon)C \tag{4.2.154b}$$

But

$$\Psi+\epsilon I \geq \Psi \geq BB' \tag{4.2.155}$$

which thanks to the forward stabilizability assumption[10], implies that $[\Psi+\epsilon I \; E]$ has full rank. Thus $A^f(\epsilon)$ is completely determined by (4.2.154). In fact, we can express $A^f(\epsilon)$ using (4.2.154) as follows

$$A^f(\epsilon) = [AT^{-1}(\epsilon)+A-K(\epsilon)C]E'[\Psi+\epsilon I+EE']^{-1}. \tag{4.2.156}$$

The limit as ϵ goes to zero of (4.2.156) exists and is given by

$$A^f = \lim_{\epsilon\to 0^+} A^f(\epsilon) = (AF+A-KC)E'(\Psi+EE')^{-1}. \tag{4.2.157}$$

Thus Ψ satisfies the following equation

$$\Psi = A^f\Psi A^{f'} + KRK' + BB' \tag{4.2.158}$$

where

$$A^f E = A - KC. \tag{4.2.159}$$

---

[10] All we need here is that the infinite eigenmodes be strongly reachable, i.e., $[E \mid B]$ have full rank.

Let us show now that A^f is stable. Suppose that λ is an eigenvalue, and

$$vA^f = \lambda v. \tag{4.2.160}$$

Then from (4.2.158) and (4.2.159)

$$(1-|\lambda|^2)v\Psi v^H = vKRK'v^H + vBB'v^H. \tag{4.2.161}$$

If $|\lambda| \geq 1$, then the non-negativity of the right hand side of (4.2.161) implies that both sides must be zero. Thus

$$vB = 0 \tag{4.2.162}$$

and

$$vK = 0 \tag{4.2.163}$$

which thanks to (4.2.159) implies that

$$\lambda vE = vA. \tag{4.2.164}$$

But (4.2.162) and (4.2.164) imply that λ is not strongly reachable, which contradicts the forward stabilizability assumption and thus A^f is stable.

Now we must show that for any arbitrary positive semi-definite Ψ_0, (4.2.147) holds. We shall first prove this result for the case where Ψ_0 is positive-definite. Note that (4.2.100) can be expressed as

$$\Psi_{k+1} = \lim_{\epsilon \to 0^+} \{A_k^f(\epsilon)(\Psi_k + \epsilon I)A_k^f(\epsilon)' + AT_k^{-1}(\epsilon)C'R^{-1}CT_k^{-1}(\epsilon)A' + BB'\}$$

$$\tag{4.2.165}$$

where

$$A_k^f(\epsilon) = AT_k(\epsilon)^{-1}E'(\Psi_k + \epsilon I)^{-1} \tag{4.2.166a}$$

$$T_k(\epsilon) = E'(\Psi_k + \epsilon I)^{-1}E + C'R^{-1}C. \tag{4.2.166b}$$

Note that as always, $T_k^{-1}(\epsilon)$ converges as ϵ goes to zero, and in a manner analogous to that used for $A^f(\epsilon)$ in (4.2.157) we can show that $A_k^f(\epsilon)$ does as well. Thus

$$\Psi_{k+1} = A_k^f\Psi_k A_k^{f'} + K_k RK_k + BB' \tag{4.2.167}$$

where

$$A^f_k = \lim_{\epsilon \to 0^+} A^f_k(\epsilon) = (AF_k + A - K_k C)E'(\Psi_k + EE')^{-1} \qquad (4.2.168a)$$

$$K_k = \lim_{\epsilon \to 0^+} K_k(\epsilon) = AF_k C' R^{-1} \qquad (4.2.168b)$$

and where

$$F_k = \lim_{\epsilon \to 0^+} T^{-1}_k(\epsilon). \qquad (4.2.168c)$$

From (4.2.167) we get

$$\Psi_k = (A^f_{k-1} A^f_{k-2} \cdots A^f_0)\Psi_0 (A^f_{k-1} A^f_{k-2} \cdots A^f_0)' + \text{non-negative terms.} \qquad (4.2.169)$$

But Ψ_k is bounded thus thanks to the fact that Ψ_0 is positive-definite, we get that

$$G_k = A^f_{k-1} A^f_{k-2} \cdots A^f_0 \qquad (4.2.170)$$

is bounded. Now let Ψ be the solution to the algebraic generalized Riccati equation (4.2.148) obtained from (4.2.147), then using expression (4.2.165) and after some algebra we can show that

$$\Psi_{k+1} - \Psi = \lim_{\epsilon \to 0^+} \{A^f(\epsilon)(\Psi_k - \Psi)A^{f'}_k(\epsilon)\} \qquad (4.2.171)$$

$(A^f(\epsilon)$ is defined in (4.2.157)) which means that

$$\Psi_{k+1} - \Psi = A^f(\Psi_k - \Psi)A^{f'}_k. \qquad (4.2.172)$$

Thus

$$\Psi_{k+1} - \Psi = (A^f)^{k+1}(\Psi_0 - \Psi)G'_{k+1}. \qquad (4.2.173)$$

but G_k is bounded and A^f is stable, which mean that Ψ_k converges to Ψ.

To extend this result to the case where Ψ_0 is a positive semi-definite matrix, simply let Ψ^1_k represent the sequence of matrices satisfying the generalized Riccati equation (4.2.100) with $\Psi^1_0 = 0$ and Ψ^2_k the sequence of matrices satisfying the same equation with $\Psi^2_0 \geq \Psi_0$ and $\Psi^2_0 > 0$. Then $\Psi^1_k \leq \Psi_k \leq \Psi^2_k$ (where Ψ_k denotes the solution to (4.2.100) with initial condition Ψ_0) and since Ψ^1_k and Ψ^2_k converge to Ψ, so does Ψ_k.

Finally, we have to show that Ψ_k converges exponentially fast for any initial condition $\Psi_0 \geq 0$. For this, we shall show the result for the case where $\Psi_0 > 0$ and the case where $\Psi_0 = 0$. These results can then be extended to the general case by an argument similar to the one used to show convergence.

Let that Ψ_0 be positive-definite, then G_k is bounded. Thus, since A^f is stable, from (4.2.173) we can deduce that Ψ_k converges to Ψ exponentially fast at a rate determined by the magnitude of the largest eigenvalue of A^f (see Section 4.4 of [31]).

Now let Ψ_0 be zero. If for some k, Ψ_k becomes positive-definite then exponential convergences follows immediately from the result of the previous case. If Ψ_k never becomes positive-definite, using the fact that Ψ_k is monotone-increasing (showed in the first part of this proof), it can be deduced that

$$\text{Im}(\Psi_k) \subset \text{Im}(\Psi_{k+1}), \quad \text{for all } k \geq 0. \tag{4.2.173}$$

Thus for some $j > 0$,

$$\text{Im}(\Psi_k) \subset \text{Im}(\Psi_j), \quad \text{for all } k \geq 0. \tag{4.2.174}$$

Let us now block diagonalize Ψ_j as follows

$$\Psi_j = T' \begin{bmatrix} \Psi_j^{11} & 0 \\ 0 & 0 \end{bmatrix} T \tag{4.2.175}$$

where Ψ_j^{11} is positive-definite. Then, by pre- and post-multiplying (4.2.100) by T and T', respectively, we obtain the following

$$\begin{bmatrix} \Psi_{k+1}^{11} & 0 \\ 0 & 0 \end{bmatrix} = \tilde{A} \{ \lim_{\epsilon \to 0^+} (\tilde{E}' (\begin{bmatrix} \Psi_k^{11} & 0 \\ 0 & 0 \end{bmatrix} + \epsilon I)^{-1} \tilde{E} + C'R^{-1}C)^{-1} \} \tilde{A}' + \tilde{\tilde{B}} \tilde{\tilde{B}}' \tag{4.2.176}$$

where

$$\tilde{E} = TE, \quad \tilde{A} = TA, \quad \tilde{B} = TB. \tag{4.2.177}$$

It is not difficult to see that $\tilde{B}$ must have the following block form

$$\tilde{B} = \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix}. \tag{4.2.178}$$

Using the fact that the infinite eigenmodes of the system are strongly reachable and using an argument similar to the one used in the proof of Lemma 4.2.2, we can show that, by a change of coordinates S, $\tilde{E}$ can be transformed as follows

$$\hat{E} = \tilde{E}S = \begin{bmatrix} \hat{E}_{11} & \hat{E}_{12} \\ 0 & I \end{bmatrix}. \tag{4.2.179}$$

Expression (4.2.176) can then be expressed as

$$\begin{bmatrix} \Psi_{k+1}^{11} & 0 \\ 0 & 0 \end{bmatrix} = \hat{A}\{ \underset{\epsilon \to 0^+}{Lim} (\hat{E}'(\begin{bmatrix} \Psi_k^{11} & 0 \\ 0 & 0 \end{bmatrix} + \epsilon I)^{-1}\hat{E} + C'R^{-1}C)^{-1}\}\hat{A}' + \tilde{B}\tilde{B}' \tag{4.2.180}$$

where

$$\hat{A} = \tilde{A}S. \tag{4.2.181}$$

Using (4.2.180) and the structure of $\hat{E}$, after some algebra, we can show that

$$\hat{A} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ 0 & \hat{A}_{22} \end{bmatrix} \tag{4.2.182}$$

and that if R_{11} denotes the (1,1)-block of $C'R^{-1}C$, then the (1,1) block of (4.2.180) can be expressed as follows

$$\Psi_{k+1}^{11} = \hat{A}_{11}\{ \underset{\epsilon \to 0^+}{Lim} (\hat{E}_{11}'(\Psi_k^{11} + \epsilon I)^{-1}\hat{E}_{11} + R_{11})^{-1}\}\hat{A}_{11}' + \tilde{B}_1\tilde{B}_1'. \tag{4.2.183}$$

It is not difficult to see that $(R_{11}, \hat{E}_{11}, \hat{A}_{11})$ is forward detectable and thus since for some j, Ψ_j^{11} is positive-definite, Ψ_k^{11} converges exponentially fast. Since Ψ_k^{11} is the (1,1)-block of Ψ_k and other blocks are zero, we have shown that Ψ_k converges exponentially fast. This completes the proof of the theroem.

Example 4.2.3

Consider the generalized Riccati equation associated to system (4.1.32)-(4.1.33):

$$\Psi_{k+1} = \lim_{\epsilon \to 0^+} \{ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (\Psi_k + \epsilon I)^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + I)^{-1} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}. \quad (4.2.184)$$

It can easily be verified that regardless of the value of Ψ_0, Ψ_k for $k \geq 1$ equals $\Psi = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$. This is consistent with the result of Example 4.2.2.

In the next section, we interpret Ψ_k as the error variance matrix of some estimation problem and use the matrices A_k^f and T_k to derive a generalization of the Kalman filter. These results are then used to obtain another approach to the smoothing problem for TPBVDS's.

4.3-A Smoothing Algorithm for TPBVDS's

In Section 4.3.2 we present a generalized Kalman filter formulation of descriptor systems. This formulation allows us to associate the solution of the generalized Riccati equation studied in the previous section with the error covariance of a particular estimation problem. This formulation is then used in Section 4.3.2 to generalize the Rauch-Tung-Striebel formulation of the smoother for causal systems to the case of TPBVDS's. Finally, in Section 4.3.3, we examine the limiting behavior of the smoother.

But before starting this study, in the following section, we shall present some of the results concerning the maximum likelihood estimation technique which we will be needing later in this chapter.

4.3.1-Maximum Likelihood Estimation

4.3.1a-Maximum Likelihood versus Bayesian Estimation

Let x be unknown constant parameter vector and let z be an observation of x. Then if $p(z|x)$ denotes the probability density function of z parameterized by x, the maximum likelihood (ML) estimate $\hat{x}$ based on observation z satisfies

$$p(z|\hat{x}_{ML}) \geq p(z|x) \quad \text{for all x.} \tag{4.3.1}$$

In the linear Gaussian case, i.e. when

$$z = Lx + v \tag{4.3.2}$$

where v is a zero-mean, Guassian random vector with variance R, and L a full column-rank matrix, $\hat{x}_{ML}$ can be obtained by noting that

$$\frac{\delta}{\delta x} (\ln[p(z|x)]) \bigg|_{x=\hat{x}_{ML}} = 0. \tag{4.3.3}$$

Since v is Gaussian, so is z and

$$p(z|x) = \alpha \exp[-(z-Lx)'R^{-1}(z-Lx)/2] \qquad (4.3.4)$$

where α is a normalization constant. From (4.3.3) and (4.3.4), we can see that

$$\hat{x}_{ML} = (L'R^{-1}L)^{-1}L'R^{-1}z. \qquad (4.3.5)$$

The error covariance associated to this estimate is given by

$$P_{ML} = \overline{(x-\hat{x}_{ML})(x-\hat{x}_{ML})'} = (L'R^{-1}L)^{-1}. \qquad (4.3.6)$$

To see how the ML estimation method ties in with the Bayesian estimation method, consider the observation (4.3.2) and suppose that x is not an unknown vector but a Gaussian random vector with known mean $\bar{x}$ and variance P_x. Then the Bayesian estimate $\hat{x}_B$ of x based on observation z is

$$\hat{x}_B = P_B(L'R^{-1}z+P_x^{-1}\bar{x}) \qquad (4.3.7)$$

where P_B is the covariance of the estimation error:

$$P_B = \overline{(x-\hat{x}_B)(x-\hat{x}_B)'} = (L'R^{-1}L+P_x^{-1})^{-1}. \qquad (4.3.8)$$

Note that if we let

$$P_x^{-1} = 0, \qquad (4.3.9)$$

the maximum likelihood and the Bayesian estimates and estimation errors are identical.

The maximum likelihood estimation technique can also be used when an a priori estimate of x exists. Specifically, any linear Gaussian Bayesian estimation problem can be formulated as a maximum likelihood estimation problem. Consider the Bayesian problem stated above. This problem can be converted into a maximum likelihood estimation problem if we consider the a priori statistics of x as an extra observation, i.e. consider the following ML

estimation problem

$$\begin{bmatrix} z \\ \bar{x} \end{bmatrix} = \begin{bmatrix} L \\ I \end{bmatrix} x + \begin{bmatrix} v \\ w \end{bmatrix} \qquad (4.3.10)$$

where w is a zero-mean, Gaussian random vector, independent of v and with variance P_x. Applying expression (4.3.5) and (4.3.6) to this problem, we obtain the following

$$\hat{x}_{ML} = P_{ML}(L'R^{-1}z + P_x^{-1}\bar{x}) \qquad (4.3.11)$$

where

$$P_{ML} = (L'R^{-1}L + P_x^{-1})^{-1} \qquad (4.3.12)$$

which are exactly the Bayesian estimate and estimation error covariance (4.3.7) , (4.3.8). Thus it is possible to transform any linear Gaussain Bayesian estimation problem into a ML problem by transforming the a priori estimate of x into an observation.

4.3.1b-The Case of Perfect Observation

In the previous section, we considered the case where R, i.e. the variance of the observation noise, is positive-definite. If R is not invertible, it is clear that (4.3.5) and (4.3.6) cannot be used. In this case, there is a projection of z which is known perfectly, and to obtain the ML estimate, we have to identify this projection.

Consider the ML estimation problem (4.3.2). Let T be a matrix such that

$$TRT' = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{R} \end{bmatrix} \qquad (4.3.13)$$

where $\tilde{R}$ is a positive definite matrix. Then (4.3.2) can be expressed as follows

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} x + \begin{bmatrix} 0 \\ \tilde{v} \end{bmatrix} \qquad (4.3.14)$$

where

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = Tz, \quad \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} = TL, \tag{4.3.15}$$

and $\tilde{v}$ is a zero-mean, Gaussian random vector with invertible variance $\tilde{R}$. Now let S be an invertible matrix such that

$$L_1 S^{-1} = [L_{11} \ 0] \tag{4.3.16}$$

where L_{11} has full column rank. Then if we let

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Sx, \tag{4.3.17}$$

(4.3.14) can be expressed as

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \tilde{v} \end{bmatrix}. \tag{4.3.18}$$

Finally note that since L_{11} has full rank, it has a left inverse. Let L_{11}^{-L} denote a left inverse of L_{11}, then by premultiplying (4.3.18) by

$$W = \begin{bmatrix} I & 0 \\ -L_{21}L_{11}^{-L} & I \end{bmatrix} \tag{4.3.19}$$

we obtain the following

$$\begin{bmatrix} z_1 \\ \tilde{z}_2 \end{bmatrix} = \begin{bmatrix} L_{11} & 0 \\ 0 & L_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \tilde{v} \end{bmatrix}. \tag{4.3.20}$$

where

$$\tilde{z}_2 = z_2 - L_{21}L_{11}^{-L}z_1. \tag{4.3.21}$$

The vector x_1 is the portion of x which is perfectly observed. Clearly,

$$(\hat{x}_1)_{ML} = L_{11}^{-} z_1. \tag{4.3.22}$$

and x_2 can be estimated from the results of the previous section,

$$(\hat{x}_2)_{ML} = (P_2)_{ML} L_{22}' \tilde{R}^{-1} \tilde{z}_2. \tag{4.3.23}$$

where

$$(P_2)_{ML} = (L_{22}' \tilde{R}^{-1} L_{22})^{-1}. \tag{4.3.24}$$

By combining (4.3.22) and (4.3.23), we see that

$$\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}_{ML} = \begin{bmatrix} \bar{L}_{11} & 0 \\ 0 & (L'_{22}\tilde{R}^{-1}L_{22})^{-1}L'_{22}\tilde{R}^{-1} \end{bmatrix} \begin{bmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{bmatrix}. \tag{4.3.25}$$

The ML estimate $\hat{x}_{ML}$ is then given by

$$\hat{x}_{ML} = S^{-1}\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}_{ML}. \tag{4.3.26}$$

The above procedure allows us to compute the ML estimate when R is not invertible. However, it does not allow us to express this estimate in a simple closed from expression. The following result allows us to express this estimate in terms of a limit which, even though it is not useful for computing the ML estimate, it is useful in our analysis (see Section 4.2).

Lemma 4.3.1

Consider the ML estimation problem (4.3.2) with R possibly singular, then

$$P_{ML} = \lim_{\epsilon \to 0^+}(L'(R+\epsilon Q)^{-1}L)^{-1} \tag{4.3.27}$$

$$\hat{x}_{ML} = \lim_{\epsilon \to 0^+}[(L'(R+\epsilon Q)^{-1}L)^{-1}L'(R+\epsilon Q)^{-1}]z \tag{4.3.28}$$

where Q is any positive semi-definite matrix for which the inverses in (4.3.27) and (4.3.28) are defined.

Proof

First note that the lemma holds when R is non-singular. Now suppose that R is singular and assume without loss of generality that (4.3.2) has the following structure (as seen above this can always be achieved by a coordinate

transformation and premultiplication of (4.3.2) by some invertible matrix),

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} L_{11} & 0 \\ 0 & L_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \tilde{v} \end{bmatrix} \tag{4.3.29}$$

where L_{11} has full column rank and $\tilde{v}$ has an invertible variance denoted by $\tilde{R}$. Let

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}. \tag{4.3.30}$$

then, since

$$R = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{R} \end{bmatrix} \tag{4.3.30}$$

and thanks to the assumptions that $Q \geq 0$ and $R + \epsilon Q$ is invertible, we can see that

$$Q_{11} > 0. \tag{4.3.31}$$

Expression B.27 can now be expressed as follows

$$\begin{aligned} P_{ML} &= \lim_{\epsilon \to 0^+} \left(\begin{bmatrix} L_{11} & 0 \\ 0 & L_{22} \end{bmatrix}' \left(\begin{bmatrix} 0 & 0 \\ 0 & \tilde{R} \end{bmatrix} + \epsilon \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \right)^{-1} \begin{bmatrix} L_{11} & 0 \\ 0 & L_{22} \end{bmatrix} \right)^{-1} \\ &= \lim_{\epsilon \to 0^+} \left(\begin{bmatrix} L_{11} & 0 \\ 0 & L_{22} \end{bmatrix}' \begin{bmatrix} \epsilon Q_{11} & \epsilon Q_{12} \\ \epsilon Q_{21} & \tilde{R} + \epsilon Q_{22} \end{bmatrix}^{-1} \begin{bmatrix} L_{11} & 0 \\ 0 & L_{22} \end{bmatrix} \right)^{-1}. \end{aligned} \tag{4.3.32}$$

To evaluate the above expression, we need the following identity (see the Appendix of [27]):

$$\begin{bmatrix} A & D \\ C & B \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} + E\Delta^{-1}F & -E\Delta^{-1} \\ -\Delta^{-1}F & \Delta^{-1} \end{bmatrix} \tag{4.3.33}$$

where $\Delta = B - CA^{-1}D$, $E = A^{-1}D$ and $F = CA^{-1}$. The $(1,1)$-block entry of (4.3.33) can also be expressed as $(A - DB^{-1}C)^{-1}$. Using (4.3.33) with the alternate expression for its $(1,1)$-block entry, we get that

$$\begin{bmatrix} \epsilon Q_{11} & \epsilon Q_{12} \\ \epsilon Q_{21} & \tilde{R} + \epsilon Q_{22} \end{bmatrix}^{-1} =$$

$$\begin{bmatrix} (\epsilon Q_{11} - \epsilon^2 Q_{12}(\tilde{R} + \epsilon Q_{22})^{-1}Q_{21})^{-1} & -Q_{11}^{-1}Q_{12}(\tilde{R} + \epsilon Q_{22} - \epsilon Q_{21}Q_{11}^{-1}Q_{12})^{-1} \\ -(\tilde{R} + \epsilon Q_{22} - \epsilon Q_{21}Q_{11}^{-1}Q_{12})^{-1}Q_{21}Q_{11}^{-1} & (\tilde{R} + \epsilon Q_{22} - \epsilon Q_{21}Q_{11}^{-1}Q_{12})^{-1} \end{bmatrix}. \tag{4.3.34}$$

We can simplify the above expression by separating terms of order ϵ or higher.

The result is

$$\begin{bmatrix} \epsilon Q_{11} & \epsilon Q_{12} \\ \epsilon Q_{21} & \tilde{R} + \epsilon Q_{22} \end{bmatrix}^{-1} = \begin{bmatrix} Q_{11}^{-1}/\epsilon & -Q_{11}^{-1}Q_{12}\tilde{R}^{-1} \\ -\tilde{R}^{-1}Q_{21}Q_{11}^{-1} & \tilde{R}^{-1} \end{bmatrix} + o(\epsilon). \qquad (4.3.35)$$

Thus we get

$$P_{ML} = \underset{\epsilon \to 0^+}{\text{Lim}}\left(\begin{bmatrix} L_{11}'Q_{11}^{-1}L_{11}/\epsilon & -L_{11}'Q_{11}^{-1}Q_{12}\tilde{R}^{-1}L_{22} \\ -L_{22}'\tilde{R}^{-1}Q_{21}Q_{11}^{-1}L_{11} & L_{22}'\tilde{R}^{-1}L_{22} \end{bmatrix} + o(\epsilon)\right)^{-1}$$

$$= \underset{\epsilon \to 0^+}{\text{Lim}} \begin{bmatrix} L_{11}'Q_{11}^{-1}L_{11}/\epsilon & -L_{11}'Q_{11}^{-1}Q_{12}\tilde{R}^{-1}L_{22} \\ -L_{22}'\tilde{R}^{-1}Q_{21}Q_{11}^{-1}L_{11} & L_{22}'\tilde{R}^{-1}L_{22} \end{bmatrix}^{-1}$$

$$\overset{\Delta}{=} \underset{\epsilon \to 0^+}{\text{Lim}} \begin{bmatrix} X/\epsilon & Y \\ Y' & Z \end{bmatrix}^{-1} \qquad (4.3.36)$$

where X and Y are both positive-definite since L_{11} and L_{22} have full rank.

Applying the identity (4.3.33) to (4.3.36) we get

$$P_{ML} = \underset{\epsilon \to 0^+}{\text{Lim}}\left(\begin{bmatrix} \epsilon X^{-1}+\epsilon^2 X^{-1}Y(Z-\epsilon Y'Z^{-1}Y)^{-1}Y'X^{-1} & -\epsilon X^{-1}Y(Z-\epsilon Y'Z^{-1}Y)^{-1} \\ -\epsilon(Z-\epsilon Y'Z^{-1}Y)^{-1}Y'X^{-1} & (Z-\epsilon Y'Z^{-1}Y)^{-1} \end{bmatrix}\right)$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & Z^{-1} \end{bmatrix}. \qquad (4.3.37)$$

Thus,

$$P_{ML} = \begin{bmatrix} 0 & 0 \\ 0 & (L_{22}'\tilde{R}^{-1}L_{22})^{-1} \end{bmatrix}. \qquad (4.3.38)$$

By a simialr argument, expression (4.3.28) yields

$$\hat{x}_{ML} = \begin{bmatrix} (L_{11}'Q_{11}^{-1}L_{11})^{-1}L_{11}'Q_{11}^{-1} & 0 \\ 0 & (L_{22}'\tilde{R}^{-1}L_{22})^{-1}L_{22}'\tilde{R}^{-1} \end{bmatrix} z. \qquad (4.3.39)$$

By noting that

$$(L_{11}'Q_{11}^{-1}L_{11})^{-1}L_{11}'Q_{11}^{-1}$$

is a left inverse of L_{11}, we can see that (4.3.39) is consistent with (4.3.25) and thus the lemma is proved. Note that the non-unicity in the expression (4.3.31) which is due to the fact that Q_{11} can be any positive-definite matrix, is related to the non-unicity of the left inverse of L_{11} when L_{11} is not square, i.e., when redundant, perfect observations are available.

We end this section with the following result, which we will need in the next section:

Lemma 4.3.2

Let x and z be unknown parameters and let $\hat{x}_{ML}$ and $\hat{z}_{ML}$ be their ML estimates based on some observations with ML estimation error variance P_x and P_z, respectively. Then,

(a) the ML estimate of the sum of x and z is the sum of ML estimates of x and z, i.e.,

$$\widehat{(x+z)}_{ML} = \hat{x}_{ML} + \hat{z}_{ML}. \tag{4.3.40}$$

If $\hat{x}_{ML}$ and $\hat{z}_{ML}$ are based on independent observations, then

$$P_{x+z} = P_x + P_z \tag{4.3.41}$$

where P_{x+z} denotes the ML estimation error variance associated to $\widehat{(x+z)}_{ML}$.

(b) Let D be a known constant matrix, then

$$\widehat{(Dx)}_{ML} = D\hat{x}_{ML} \tag{4.3.42}$$

$$P_{Dx} = DP_x D'. \tag{4.3.43}$$

This Lemma follows easily from (4.3.5) and (4.3.6).

4.3.2-Generalized Kalman Filter

Consider the standard Kalman filtering problem for a causal Gauss-Markov process:

$$x(k+1) = Ax(k) + Bu(k), \quad k=0,1,\ldots \qquad (4.3.44a)$$

$$y(k) = Cx(k) + r(k), \quad k=1,2,\ldots\ldots \qquad (4.3.44b)$$

where $u(k)$ and $r(k)$ are independent, white Gaussian sequences with variance I and $R>0$, respectively. The initial state $x(0)$ is also Gaussian with an a priori mean $\bar{x}_0$ and variance F_0. The Kalman filter for this problem consists of sequentially computing the optimal estimate $\hat{x}^f(k)$ of the state $x(k)$ based on observations up to time k. The Kalman filtering equations are of course well known, but we shall rederive them here using a slightly different approach, namely by using the ML formulation, because this method can then be extended to the descriptor case.

In the ML formulation of this problem, we consider $x(k)$ to be an unknown paramemter and convert all the dynamics equations (4.3.44a) and the a priori information on $x(0)$ into observations. The ML estimation problem is then

$$0 = x(k+1) - Ax(k) - Bu(k), \quad k=0,1,\ldots \qquad (4.3.45a)$$

$$y(k) = Cx(k) + r(k), \quad k=1,2,\ldots\ldots \qquad (4.3.45b)$$

$$\bar{x}_0 = x(0) + v \qquad (4.3.46)$$

where v is a Gaussian random vector, independent of u and r, with variance F_0. Here, all of the right hand sides of (4.3.45), (4.3.46) should be considered as measurements, with $-Bu(k)$, $r(k)$ and v playing the roles of measurement noises. A question that arises at this point is that whether $\hat{x}^f(j)$ is the ML estimate of $x(j)$ based on (4.3.46), (4.3.45b) for $1 \leq k \leq j$ and (4.3.45a) for all k, or, (4.3.46), (4.3.45b) for $1 \leq k \leq j$ and (4.3.45a) for $0 \leq k \leq j-1$. The answer is that <u>both</u> of these ML estimates yield the same result. It is straightforward

to check, using the results of Section 4.3.1, that future dynamics, given observations up to the present time, do not supply any information regarding the present state. To see this, consider the "one step in the future" dynamics equation for $x(j)$:

$$0 = x(j+1) - Ax(j) - Bu(j). \qquad (4.3.47)$$

Given observations only up to time j, $x(j+1)$ is completely unknown which clearly implies that (4.3.47) cannot supply any information regarding the value of $x(j)$. Since (4.3.47), with j replaced with $j+1$, does not contain any information regarding $x(j+1)$ either, then by induction we can see that no future dynamic contains information regarding $x(j)$. This, in fact, is closely related to the Markovian nature of the process x in the original formulation of the problem.

Let $\hat{x}^f(j)$ be the ML estimate of $x(j)$ based on (4.3.46), (4.3.45b) up to j and (4.3.45a) up to $j-1$, and let F_j denote the associated error variance. Let us start recursively estimating $x(j)$. For $j=0$, there is no dynamics and no observation equations (4.3.45) and clearly

$$\hat{x}^f(0) \triangleq \hat{x}_{ML}(0) = \bar{x}_0 \qquad (4.3.48a)$$

$$P_{ML}(0) = F_0. \qquad (4.3.48b)$$

At the next step, we have additional observations

$$0 = x(1) - Ax(0) - Bu(0) \qquad (4.3.49a)$$

$$y(1) = Cx(1) + r(1). \qquad (4.3.49b)$$

We claim that previous observations, (at this step there has only been one, namely (4.3.46)) can be summarized into the following equation

$$\hat{x}^f(0) = x(0) + \tilde{x}(0) \qquad (4.3.50)$$

where $\tilde{x}(0)$ is the ML estimation error associated to estimation $\hat{x}^f(0)$ and thus has variance F_0 and is independent of $u(0)$ and $r(1)$. The consistency of this

claim is verified later when we obtain the well known Kalman filtering equations. By combining (4.3.49a) and (4.3.50), and using (4.3.49b) we can formulate the ML estimation problem of $x(1)$ based on past observations, dynamics and the a priori information as follows

$$\begin{bmatrix} A\hat{x}^f(0) \\ y(1) \end{bmatrix} = \begin{bmatrix} I \\ C \end{bmatrix} x(1) + \begin{bmatrix} A\tilde{x}(0){-}Bu(0) \\ r(1) \end{bmatrix}. \tag{4.3.51}$$

Using Lemma 4.3.1, we get

$$F_1 \triangleq P_{ML}(1) = \text{Lim}_{\epsilon \to 0^+}([I \ C'](\begin{bmatrix} AF_0A'{+}BB' & 0 \\ 0 & R \end{bmatrix}{+}\epsilon I)^{-1}\begin{bmatrix} I \\ C \end{bmatrix})^{-1} \tag{4.3.52a}$$

$$\hat{x}^f(1) \triangleq \hat{x}_{ML}(1) =$$

$$\text{Lim}_{\epsilon \to 0^+}([I \ C'](\begin{bmatrix} AF_0A'{+}BB' & 0 \\ 0 & R \end{bmatrix}{+}\epsilon I)^{-1}\begin{bmatrix} I \\ C \end{bmatrix})^{-1}[I \ C'](\begin{bmatrix} AF_0A'{+}BB' & 0 \\ 0 & R \end{bmatrix}{+}\epsilon I)^{-1}\begin{bmatrix} A\hat{x}^f(0) \\ y(1) \end{bmatrix}. $$

$$\tag{4.3.52b}$$

Thus we get that,

$$F_1 = \text{Lim}_{\epsilon \to 0^+}[(\Psi_1{+}\epsilon I)^{-1}{+}C'R^{-1}C]^{-1} = \Psi_1 - \Psi_1C'[C\Psi_1C'{+}R]^{-1}C\Psi_1 \tag{4.3.53a}$$

$$\hat{x}^f(1) = \text{Lim}_{\epsilon \to 0^+}[(\Psi_1{+}\epsilon I)^{-1}{+}C'R^{-1}C]^{-1}[(\Psi_1{+}\epsilon I)^{-1}A\hat{x}^f(0){+}C'R^{-1}y(1)] =$$

$$(I{-}\Psi_1C'[C\Psi_1C'{+}R]^{-1}C)A\hat{x}^f(0) + F_1C'R^{-1}y(1) \tag{4.3.53b}$$

where

$$\Psi_1 = AF_0A' + BB'. \tag{4.3.53c}$$

Now, in anticipation of the extension to descriptor systems, let us write the general update step in a particular way. Specifically, what we claim is that $\hat{x}^f(j)$ is also equal to the ML estimate of $x(j)$ based on the following <u>two</u> "measurements"

$$y(j) = Cx(j) + r(j) \tag{4.3.54a}$$

$$\hat{z}^f(j) = x(j) - \tilde{z}^f(j) \tag{4.3.54b}$$

where $\tilde{z}^f(j)$ is independent of $r(j)$, and has variance Ψ_j. In turn, $\hat{z}^f(j)$ is the

ML estimate of

$$z(j) = Ax(j-1) + Bu(j-1) \tag{4.3.55}$$

based on the two "measurements"

$$\hat{x}^f(j-1) = x(j-1) - \tilde{x}^f(j-1) \tag{4.3.56a}$$

$$0 = u(j-1) + \tilde{u}(j-1) \tag{4.3.56b}$$

where $\tilde{x}^f(j-1)$ is the estimation error associated with $\hat{x}^f(j-1)$ with variance F_j and where $\tilde{u}(j-1)$ is a unit-variance Gaussian vector, independent of $\tilde{x}^f(j-1)$.

Let us assume this is true. Then, using Lemma 4.3.1 and Lemma 4.3.2, we can solve these two estimation problems. First, we find that

$$\hat{z}^f(j) = A\hat{x}^f(j-1) \tag{4.3.57a}$$

$$\Psi_j = AF_{j-1}A' + BB'. \tag{4.3.57b}$$

Then, we have that

$$\hat{x}^f(j) = \lim_{\epsilon \to 0^+}[(\Psi_j+\epsilon I)^{-1}+C'R^{-1}C]^{-1}[(\Psi_j+\epsilon I)^{-1}\hat{z}^f(j)+C'R^{-1}y(j)] =$$

$$(I-\Psi_jC'[C\Psi_jC'+R]^{-1}C)\hat{z}^f(j) + F_jC'R^{-1}y(j) \tag{4.3.58a}$$

$$F_j = \lim_{\epsilon \to 0^+}[(\Psi_j+\epsilon I)^{-1}+C'R^{-1}C]^{-1} = \Psi_j - \Psi_jC'[C\Psi_jC'+R]^{-1}C\Psi_j. \tag{4.3.58b}$$

Expressions (4.3.57)-(4.3.58) are just the standard Riccati equations. Matrices Ψ_k and F_k are the predicted and the filtered estimation error variances of $x(k)$, and $\hat{x}^f(k)$ is its filtered estimate. It is straightforward to verify that Ψ_k satisfies the (generalized) Riccati equation

$$\Psi_{k+1} = A\{\lim_{\epsilon \to 0^+}((\Psi_k+\epsilon I)^{-1}+C'R^{-1}C)^{-1}\}A' + BB' =$$

$$A\Psi_kA' - A\Psi_kC'(R+C'\Psi_kC)^{-1}C\Psi_k)\Psi_kA' + BB'. \tag{4.3.59}$$

Note that all the simplifications, i.e. elimination of the limits, in the Kalman filter equations (4.3.57)-(4.3.59) is done by using the ABCD matrix inversion formula.

We can extend the above formulation of the optimal filtering problem
(Kalman filter) to the descriptor case. In particular, consider the following
system,

$$Ex(k+1) = Ax(k) + Bu(k), \quad k=0,1,\ldots \tag{4.3.60a}$$

$$y(k) = Cx(k) + r(k), \quad k=1,2,\ldots \tag{4.3.60b}$$

where we have an a priori estimate $\bar{x}(0)$ of $x(0)$ with a corresponding Gaussian
estimation error variance F_0. Sequences $u(k)$ and $r(k)$ are independent, white
and Gaussian, and they are also independent of v. As in the causal case, we
would like to sequentially estimate $x(k)$ based on previous observations.

Throughout this section, we assume that the infinite eigenmodes of the
system are strongly observable, i.e. $[E' \; C']$ has full rank. We shall see that
this condition is needed to guarantee that the estimation error variances
associated with the generalized Kalman filter are finite.

Let us convert this problem into a ML problem. As before, we consider
$x(k)$ to be an unknown paramemter and convert all the dynamics equations
(4.3.60a) and the a priori information on $x(0)$ into observations. The ML
estimation problem is then

$$0 = Ex(k+1) - Ax(k) - Bu(k), \quad k=0,1,\ldots \tag{4.3.61a}$$

$$y(k) = Cx(k) + r(k), \quad k=1,2,\ldots \tag{4.3.61b}$$

$$\bar{x}_0 = x(0) + v \tag{4.3.62}$$

where v is a Gaussian random vector, independent of u and r, with variance F_0.

The difference here with the previous case is that the dynamics
(4.3.61a), $k=j$, does indeed contain information about $x(j)$ when E is not
invertible even if $x(j+1)$ is completely unknown. Specifically, (4.3.61a)
contains information about the projection of $Ax(j)$ which lies in the

null-space of E. Thus, the optimal estimate of x(j) based on observations (4.3.61b), up to j, and observations (4.3.61a) up to j-1, in general differs form the optimal estimate of x(j) based on observations (4.3.61a), up to j, and observations (4.3.61b), for all k. For example consider the case where

$$E = B = 0, \quad A = C = I \tag{4.3.63}$$

with an a priori estimate $\hat{x}^f(0)$ of x(0) with an associated positive-definite error variance F_0. In this case, clearly the only possibility is that x(j)=0 for all j, however, based on the observations (4.3.61a) up to j-1 and observations (4.3.61b) up to j, one can check that

$$\hat{x}^f(j) = y(j). \tag{4.3.64}$$

So in formulating the Kalman filter for descriptor systems, we have to decide what we mean by the filtered estimate $\hat{x}^f(j)$ of x(j). We have decided, for reasons that will become clear later, to define $\hat{x}^f(j)$ as the optimal estimate of x(j) based on y, up to j, and dynamics (observations (4.3.61a)) up to j-1 (and of course the a priori information, i.e. (4.3.62)). We shall denote the error variance associated with $\hat{x}^f(j)$ as F_j.

Thus, the generalized Kalman filtering problem consists of sequentially estimating the sequence of unknown parameters x(j) based on observations y (4.3.61b), up to j, and dynamics equations or observations (4.3.61a), up to j-1 (and the a priori information). It turns out that the steps to follow are very similar to those obtained in the standard causal case.

Clearly, as in the causal case $\hat{x}^f(0)$ is just $\bar{x}_0$ with an associated estimation error variance F_0. What we now claim is that $\hat{x}^f(j)$ can be viewed as the ML estimate based on the following two measurements:

$$y(j) = Cx(j) + r(j) \tag{4.3.65a}$$

$$\hat{z}^f(j) = Ex(j) - \tilde{z}^f(j) \tag{4.3.65b}$$

where $\tilde{z}^f(j)$ is independent of $r(j)$ and has variance Ψ_j. Here, $\hat{z}^f(j)$ is the ML estimate of

$$z(j) = Ax(j-1) + Bu(j-1) \qquad (4.3.66)$$

based on "measurements"

$$\hat{x}^f(j-1) = x(j-1) - \tilde{x}^f(j-1) \qquad (4.3.67a)$$

$$0 = u(j-1) - \tilde{u}(j-1) \qquad (4.3.67b)$$

where $\tilde{x}^f(j-1)$ is the estimation error assciated with $\hat{x}^f(j-1)$ and thus has variance F_{j-1} and $\tilde{u}(j-1)$ is a unit-variance Gaussian vector, independent of $\tilde{x}^f(j-1)$.

Now let proceed as we did in the causal case. Using, lemma 4.3.1 and Lemma 4.3.2, we can see that

$$\hat{z}^f(1) = A\hat{x}^f(0) \qquad (4.3.68a)$$

$$\Psi_1 = AF_0A' + BB'. \qquad (4.3.68b)$$

In general, we get

$$\hat{z}^f(j+1) = A\hat{x}^f(j) \qquad (4.3.69a)$$

$$\Psi_{j+1} = AF_jA' + BB' \qquad (4.3.69b)$$

and

$$\hat{x}^f(j) = \lim_{\epsilon \to 0^+}\{T_j^{-1}(\epsilon)(E'(\Psi_j+\epsilon I)^{-1}\hat{z}^f(j)+C'R^{-1}y(j))\} \qquad (4.3.70a)$$

where $T_j(\epsilon)$ is given by

$$T_j(\epsilon) = E'(\Psi_j+\epsilon I)^{-1}E + C'R^{-1}C. \qquad (4.3.70b)$$

The estimation error variance matrix associated to $\hat{x}^f(j)$ is

$$F_j = \lim_{\epsilon \to 0^+}T_j^{-1}(\epsilon) = \lim_{\epsilon \to 0^+}\{E'(\Psi_j+\epsilon I)^{-1}E+C'R^{-1}C\}. \qquad (4.3.70c)$$

Note that by replacing F_j in (4.3.69b) with the above expression, we get that

$$\Psi_{j+1} = A\{\lim_{\epsilon \to 0^+}(E'(\Psi_j+\epsilon I)^{-1}E+C'R^{-1}C)^{-1}\}A' + BB' \qquad (4.3.71)$$

which is just the generalized Riccati equation (4.2.100) introduced and studied in Section 4.2.

In analogy with the standard Kalman filter, here, $\hat{z}^f$ plays the role of the predicted estimate and $\hat{x}^f$ that of filtered estimate. Note however that $\hat{z}^f(j)$ is not a predicted estimate of $x(j)$ but of $Ex(j)$.

If E equals the identity matrix, (4.3.69)-(4.3.71) reduce to the standard Kalman filtering equations. As stated before, in this case, $x(j)$ and thus the estimate of $x(j)$ is not affected by future dynamics of the system so that we can say that $x^f(j)$ is the best estimate of $x(j)$ based on observations up to time j. Otherwise, we should emphasize that $\hat{x}^f(j)$ is the ML estimate of $x(j)$ based on observations y, up to j, and dynamics, up to $j-1$ (and of course, the initial information).

If we consider the sequence x to be a finite sequence, i.e. if there exists N such that $x(N)$ is the last element of the sequence, then $\hat{x}^f(N)$ is really the ML estimate of $x(N)$ based on all the available information.

At this point, we have not really shown that the generalized Kalman filter yields the estimate that we had in mind (remember that we had to make a claim in deriving the generalized Kalman filter). To show that the generalized Kalman filter does indeed give us the desired estimates, let us reformulate the problem.

Let

$$\mathcal{F}_j \underline{x}(j) = \mathcal{B}_j \underline{u}(j) \tag{4.3.72a}$$

$$\underline{y}(j) = \mathcal{C}_j \underline{x}(j) + \underline{r}(j) \tag{4.3.72b}$$

where $\underline{x}(j)$ is an unknown vector,

$$\underline{x}(j)' = [x(0)', x(1)', \ldots, x(j)']', \tag{4.3.73a}$$

$\underline{y}(j)$ is the vector of observation,

$$\underline{y}(j)' = [\bar{x}_0', y(1)', \ldots, y(j)']', \tag{4.3.73b}$$

$\underline{u}(j)$ is a Gaussian random vector with variance I,

$$\underline{u}(j)' = [u(0)',\ldots,u(j-1)']',\qquad(4.3.73c)$$

$\underline{r}(j)$ is a Gaussian random vector, independent of $\underline{u}(j)$,

$$\underline{r}(j)' = [v',r(1),\ldots,r(j)']',\qquad(4.3.73d)$$

with variance

$$\mathcal{R}_j = \begin{bmatrix} T_0^{-1} & & & \\ R & R & & \\ & & \ddots & \\ & & & R \end{bmatrix}\qquad(4.3.74)$$

and,

$$\mathcal{S}_j = \begin{bmatrix} A & E & & & \\ & -A & E & & \\ & & -A & E & \\ & & & \ddots & \ddots \\ & & & & -A & E \end{bmatrix}\qquad(4.3.75a)$$

$$\mathcal{C}_j = \begin{bmatrix} I & & & \\ & C & & \\ & & C & \\ & & & \ddots & \\ & & & & C \end{bmatrix}\qquad(4.3.75b)$$

$$\mathcal{B}_j = \begin{bmatrix} B & & & \\ & B & & \\ & & B & \\ & & & \ddots & \\ & & & & B \end{bmatrix}.\qquad(4.3.75c)$$

Then consider the maximum likelihood estimate of $\underline{x}(j)$ denoted by

$$\underline{x}(j|j) = [\hat{x}(0|j)',\hat{x}(1|j)',\ldots,\hat{x}(j|j)']'.\qquad(4.3.76)$$

This estimate is given by (see Lemma 4.3.1)

$$\hat{\underline{x}}(j|j) = Z_j\mathcal{C}_j'\mathcal{R}_j^{-1}\underline{y}(j)\qquad(4.3.77)$$

where Z_j is the estimation error variance matrix given by

$$Z_j = \underset{\epsilon \to 0^+}{\text{Lim}}[\mathcal{S}_j'(\mathcal{B}_j\mathcal{B}_j'+\epsilon I)^{-1}\mathcal{S}_j+\mathcal{C}_j'\mathcal{R}_j^{-1}\mathcal{C}_j]^{-1}.\qquad(4.3.78)$$

The limit in (4.3.78) exists if $\begin{bmatrix}\mathcal{S}_j\\\mathcal{C}_j\end{bmatrix}$ has full column rank, which is the case if the infinite eigenmodes of the system are strongly observable. To see why this is true note that, the infinite eigenmodes of the system being strongly

observable means that $\begin{bmatrix} E \\ C \end{bmatrix}$ has full rank. But then there exists an invertible matrix Y such that

$$\Lambda = Y \begin{bmatrix} \mathscr{S}_j \\ \mathscr{C}_j \end{bmatrix} = Y \begin{bmatrix} -A & E & & & \\ & -A & E & \cdots & E \\ I & & C & & \\ & C & & \cdot\cdot C \end{bmatrix} = \begin{bmatrix} 0 & E & & & \\ & 0 & E & \cdots & E \\ I & & C & & \\ & C & & \cdot\cdot C \end{bmatrix}. \qquad (4.3.79)$$

Clearly, Λ is invertible and thus $\begin{bmatrix} \mathscr{S}_j \\ \mathscr{C}_j \end{bmatrix}$ has full rank.

What we claim is that $\hat{x}(j|j)$ is just $\hat{x}^f(j)$ obtained in (4.3.70a). The proof is by induction. First note that for j=0, $\mathscr{S}_j$ has dimension zero and thus (4.3.77)-(4.3.78) yields

$$\hat{x}(0|0) = \hat{x}^f(0). \qquad (4.3.80)$$

Now suppose that

$$\hat{x}(j|j) = \hat{x}^f(j) \qquad (4.3.81)$$

with F_j for error variance. Then we would like to show that

$$\hat{x}(j+1|j+1) = \hat{x}^f(j+1). \qquad (4.3.82)$$

Note that

$$Z_{j+1} = \lim_{\epsilon \to 0} [\mathscr{S}'_{j+1} (\mathscr{B}_{j+1} \mathscr{B}'_{j+1} + \epsilon I)^{-1} \mathscr{S}_{j+1} + \mathscr{C}'_{j+1} \mathscr{R}^{-1}_{j+1} \mathscr{C}_{j+1}]^{-1} \qquad (4.3.83)$$

where

$$\mathscr{S}_{j+1} = \begin{bmatrix} \mathscr{S}_j & 0 \\ (0\ 0\ ..\ 0\ -A) & E \end{bmatrix} \qquad (4.3.84a)$$

$$\mathscr{C}_{j+1} = \begin{bmatrix} \mathscr{C}_j & C \end{bmatrix} \qquad (4.3.84b)$$

$$\mathscr{B}_{j+1} = \begin{bmatrix} \mathscr{B}_j & B \end{bmatrix} \qquad (4.3.84c)$$

$$\mathscr{R}_{j+1} = \begin{bmatrix} \mathscr{R}_j & R \end{bmatrix}. \qquad (4.3.84d)$$

Also, since Z_j is the estimation error variance associated with $\hat{\underline{x}}(j|j)$, its k-th block-diagonal entry, k≤j, equals the estimation error variance associated to the estimate $\hat{x}(k|j)$. In particular, its last block-diagonal

entry corresponds to the estimation error variance associated to $\hat{x}(j|j)$, i.e.,

$$Z_j = \begin{bmatrix} * & * \\ * & F_j \end{bmatrix}$$
(4.3.85)

where $*$ denotes "don't care" entries.

From (4.3.83), using (4.3.84) and (4.3.85), follows that

$$Z_{j+1} = \text{Lim}_{\epsilon\to 0^+}\left[\right.$$

$$\left[\begin{bmatrix} \mathscr{S}_j & 0 \\ (0\ 0..0\ -A) & E \end{bmatrix}'\begin{bmatrix} \mathscr{B}_j\mathscr{B}_j'+\epsilon I & \\ & BB'+\epsilon I \end{bmatrix}^{-1}\begin{bmatrix} \mathscr{S}_j & 0 \\ (0\ 0..0\ -A) & E \end{bmatrix}+\begin{bmatrix}\mathscr{C}_j & C\end{bmatrix}'\begin{bmatrix}\mathscr{R}_j & R\end{bmatrix}^{-1}\begin{bmatrix}\mathscr{C}_j & C\end{bmatrix}\right]^{-1} =$$

$$\text{Lim}_{\epsilon\to 0^+}\left[\right.$$

$$\left[\begin{bmatrix}\mathscr{S}_j'(\mathscr{B}_j\mathscr{B}_j'+\epsilon I)^{-1}\mathscr{S}_j+\mathscr{C}_j\mathscr{R}_j^{-1}\mathscr{C}_j'+\begin{bmatrix}0 & \cdots & 0 & 0 \\ : & & : & : \\ 0 & \cdots & 0 & A'(BB'+\epsilon I)^{-1}A\end{bmatrix}\right.$$
$$\begin{bmatrix}0 \\ : \\ A'(BB'+\epsilon I)^{-1}E\end{bmatrix}$$
$$\left.\begin{bmatrix}0 & \cdots & 0 & E'(BB'+\epsilon I)^{-1}A\end{bmatrix}\right.$$
$$\left.E'(BB'+\epsilon I)^{-1}E\right]^{-1}.$$

(4.3.86)

If we now use the matrix inversion formula (4.3.33), and expressions (4.3.78) and (4.3.85), after some algebra we get that

$$Z_{j+1} = \text{Lim}_{\epsilon\to 0^+}\begin{bmatrix} * & * \\ T_{j+1}^{-1}(\epsilon)[0...0\ \ E'(\epsilon I+BB'+AF_j A')^{-1}A]Z_j(\epsilon) & T_{j+1}^{-1}(\epsilon)\end{bmatrix}$$

(4.3.87)

where $T_{j+1}(\epsilon)$ is defined in (4.3.70b) and

$$Z_j(\epsilon) = [\mathscr{S}_j'(\mathscr{B}_j\mathscr{B}_j'+\epsilon I)^{-1}\mathscr{S}_j+\mathscr{C}_j\mathscr{R}_j^{-1}\mathscr{C}_j]^{-1}.$$
(4.3.88)

Thus,

$$Z_{j+1} = \begin{bmatrix} * & * \\ \text{Lim}_{\epsilon\to 0^+}\{[0...0\ \ T_{j+1}^{-1}(\epsilon)E'(\epsilon I+\Psi_j)^{-1}A]Z_j(\epsilon)\} & F_{j+1}\end{bmatrix}.$$
(4.3.89)

Since both $T_{j+1}^{-1}(\epsilon)E'(\epsilon I+\Psi_j)^{-1}A$ and $Z_j(\epsilon)$ converge as ϵ goes to zero, we get

that

$$Z_{j+1} = \begin{bmatrix} * & * \\ [0 \ldots 0 \ \underset{\epsilon \to 0^+}{\text{Lim}} \{T_{j+1}^{-1}(\epsilon)E'(\epsilon I + \Psi_j)^{-1}A\}Z_j] & F_{j+1} \end{bmatrix}, \quad (4.3.90)$$

and thus from the fact that

$$\hat{\underline{x}}(j+1|j+1) = Z_{j+1}\mathscr{C}_{j+1}'\mathscr{R}_{j+1}^{-1}\underline{Y}(j+1) \quad (4.3.91)$$

and (4.3.77), we get that

$$\hat{x}(j+1|j+1) = \underset{\epsilon \to 0^+}{\text{Lim}}\{T_{j+1}^{-1}(\epsilon)(E'(\Psi_j+\epsilon I)^{-1}A\hat{x}(j|j)+C'R^{-1}y(j))\} \quad (4.3.92)$$

with estimation error variance F_{j+1}. But (4.3.92) is consistent with the

update equation (4.3.70a) and thus

$$\hat{x}(j+1|j+1) = \hat{x}^f(j+1), \quad (4.3.93)$$

which is what we wanted to show.

4.3.3-Smoothing Algorithm for TPBVDS's

In this section we propose a generalization to the case of TPBVDS
(4.1.1)-(4.1.2) of the Rauch-Tung-Striebel formulation of the smoother for
causal systems. We shall first consider the case where the TPBVDS is
stochastically separable and then extend the result to the general case.

For the stochastically separable case, we shall make the following
assumptions:

(a) infinite eigenmodes of the system are strongly observable,

(b) zero eigenmodes of the system are strongly reachable,

(c) the estimation error variance associated with estimating $x(0)$ based on the
boundary conditions and measurements is finite and positive-definite.

We shall express condition (c) explicitly after we define stochastic separability. As we will see, in the non-separable case, we will need somewhat stronger assumptions than (a)-(c).

Definition 4.3.1

The TPBVDS (4.1.1)-(4.1.2) is called stochastically separable if the information available through the boundary condition and boundary observation about $x(0)$ is independent of the information available about $x(N)$ i.e., the estimation error variance matrix P_b associated with the estimation of $\begin{bmatrix} x(0) \\ x(N) \end{bmatrix}$ based on boundary observations (4.1.1b) and (4.1.2b) is block-diagonal.

It is not difficult to see that TPBVDS (4.1.1)-(4.1.2) is stochastically separable if and only if

$$\lim_{\epsilon \to 0^+} (V_i'(Q+\epsilon I)^{-1}V_f + W_i'\Pi^{-1}W_f) = 0. \tag{4.3.94}$$

We can now make explicit our assumption about the information available on $x(0)$ through the boundary informations. The error variance associated with this estimate is just the (1,1)-block entry of P_b (defined in Definition 4.3.1). Thus the assumption is that

$$F_0 = \lim_{\epsilon \to 0^+} (V_i'(Q+\epsilon I)^{-1}V_i + W_i'\Pi^{-1}W_i)^{-1} \tag{4.3.95}$$

is finite (this is the case if and only if $\begin{bmatrix} V_i \\ W_i \end{bmatrix}$ has full rank) and positive-definite. If Q is positive-definite, then F_0 —assuming that it is finite— is also positive-definite. However, F_0 may be positive-definite even

if Q is not. In the following derivation, we will, for simplicity, assume that Q is in fact positive-definite. The extension to the case of Q singular is straightforward although it involves limits as in (4.3.95). At the end of the derivation, we will state the result in the general case, i.e. for possibly singular Q.

In Section 4.1, we showed that solving the smoothing problem for TPBVDS (4.1.1) is equivalent to solving the linear system (4.1.24). This system can be expressed as follows

$$\Omega\rho = \omega \tag{4.3.96}$$

where

$$\Omega = \begin{bmatrix} V_i' & W_i'\Pi^{-1}W_i & -A' & & & & & & & 0 & W_i'\Pi^{-1}W_f \\ 0 & A & BB' & -E & & & & & & 0 & 0 \\ & & E' & C'R^{-1}C & -A' & & & & & & \\ & & & A & BB' & -E & & & & & \\ & & & & E' & C'R^{-1}C & -A' & & & & \\ & & & & & & \cdots\cdots\cdots & & & & \\ & & & & & & & A & BB' & -E \\ V_f' & W_f'\Pi^{-1}W_i & & & & & & & E' & W_f'\Pi^{-1}W_f \\ -Q & V_i & & & & & & & 0 & V_f \end{bmatrix}$$

$$\tag{4.3.97}$$

$$\rho' = [\hat{\lambda}(0)',\hat{x}(0)',\hat{\lambda}(1)',\hat{x}(1)',\ldots,\hat{\lambda}(N)',\hat{x}(N)'] \tag{4.3.98a}$$

$$\omega' = [(W_i'\Pi^{-1}y_b)',0,(C'R^{-1}y(1))',0,\ldots,(C'R^{-1}y(N-1))',0,(W_f'\Pi^{-1}y_b)',0]. \tag{4.3.98b}$$

In Section 4.1, we eliminated (by moving in a boundary condition) $\hat{\lambda}(0)$ and $\hat{x}(0)$ from this system of equations and showed that what remains has a TPBVDS structure. Here, we shall only eliminate $\hat{\lambda}(0)$. We can eliminate $\hat{\lambda}(0)$ by premultiplying (4.3.96) by

$$\begin{bmatrix} I & 0 & \cdots\cdots & 0 & 0 & V_i'Q^{-1} \\ 0 & I & 0 & \cdots\cdots\cdots & & 0 \\ & & I & & & \\ & & & I & & \\ & & & & \vdots & \\ 0 & \cdots\cdots & 0 & I & 0 & 0 \\ 0 & 0 & \cdots\cdots & 0 & I & V_f'Q^{-1} \end{bmatrix}$$

which yields the system

$$\rho^* = \Omega^* \omega^* \tag{4.3.99}$$

where

$$\Omega^* = \begin{bmatrix} T_0 & -A' & & & & & & & V_{if} \\ A & BB' & -E & & & & & & \\ & E' & C'R^{-1}C & -A' & & & & & \\ & & A & BB' & -E & & & & \\ & & & E' & C'R^{-1}C & -A' & & & \\ & & & & \cdots\cdots\cdots & & & \\ & & & & & \cdots\cdots\cdots & & \\ 0 & & & & & & A & BB' & -E \\ V_{fi} & & & & & & & E' & \otimes_N \end{bmatrix} \tag{4.3.100}$$

$$\rho^{*\,'} = [\hat{x}(0)', \hat{\lambda}(1)', \hat{x}(1)', \ldots, \hat{\lambda}(N)', \hat{x}(N)'] \tag{4.3.101a}$$

$$\omega^{*\,'} = [(W_i'\Pi^{-1}y_b)', 0, (C'R^{-1}y(1))', 0, \ldots, (C'R^{-1}y(N-1))', 0, (W_f'\Pi^{-1}y_b)'] \tag{4.3.101b}$$

with

$$T_0 = W_i'\Pi^{-1}W_i + V_i'Q^{-1}V_i \tag{4.3.102}$$

$$\otimes_N = W_f'\Pi^{-1}W_f + V_f'Q^{-1}V_f \tag{4.3.103}$$

$$V_{if} = W_i'\Pi^{-1}W_f + V_i'Q^{-1}V_f \tag{4.3.104}$$

$$V_{fi} = W_f'\Pi^{-1}W_i + V_f'Q^{-1}V_i. \tag{4.3.105}$$

Matrices T_0 and $\otimes_N$ are information matrices associated with estimating respectively $x(0)$ and $x(N)$ based on boundary information. Thanks to the

stochastic separability assumption,

$$V_{if} = V_{fi} = 0. \qquad (4.3.106)$$

In this case, the system (4.3.99) becomes block tri-diagonal and can be solved by simple Gaussian elimination techniques. We can either start upper block diagonalizing from the left side of Ω^* and simultaneously lower block diagonalizing from the right side, thus obtaining a generalization of the forward-backward filter formulation for causal systems [34], or we can start upper block diagonalizing Ω^* from the left all the way to the end and solve for $\hat{x}(N)$ and then run backward solving for the other unknowns by substitution, and thus obtaining a generalization of the Rauch-Tung-Striebel method. We consider here the latter approach. Thus we first upper block diagonalize the following system

$$\begin{bmatrix} T_0 & -A' & & & & & & \\ A & BB' & -E & & & & & \\ & E' & C'R^{-1}C & -A' & & & & \\ & & A & BB' & -E & & & \\ & & & \cdots & \cdots & & & \\ & & & & \cdots & \cdots & & \\ & & & & & \cdots & \cdots & \\ & & & & & A & BB' & -E \\ & & & & & & E' & \Theta_N \end{bmatrix} \begin{bmatrix} \hat{x}(0) \\ \hat{\lambda}(1) \\ \hat{x}(1) \\ \hat{\lambda}(2) \\ \vdots \\ \hat{\lambda}(N) \\ \hat{x}(N) \end{bmatrix} = \begin{bmatrix} W_i'\Pi^{-1}y_b \\ 0 \\ C'R^{-1}y(1) \\ 0 \\ \vdots \\ 0 \\ W_f'\Pi^{-1}y_b \end{bmatrix}$$

$$(4.3.107)$$

Since T_0 is invertible (thanks to the assumption that $\begin{bmatrix} V_i \\ W_i \end{bmatrix}$ has full rank), we can premultiply (4.3.107) by

$$\begin{bmatrix} T_0^{-1} & 0 & & \cdots & & 0 \\ -AT_0^{-1} & I & & & & \\ 0 & 0 & I & & & \vdots \\ & & & \ddots & & \\ & & & \ddots & & 0 \\ 0 & & \cdots & & 0 & I \end{bmatrix}$$

and obtain

$$\begin{bmatrix} I & -T_0^{-1}A' & & & & & & \\ 0 & \Psi_1 & -E & & & & & \\ & E' & C'R^{-1}C & -A' & & & & \\ & & A & BB' & -E & & & \\ & & & & \cdots\cdots\cdots & & & \\ & & & & & \cdots\cdots\cdots & & \\ & & & & & & \cdots\cdots\cdots & \\ 0 & & & & & A & BB' & -E \\ & & & & & & E' & \otimes_N \end{bmatrix} \begin{bmatrix} \hat{x}(0) \\ \hat{\lambda}(1) \\ \hat{x}(1) \\ \hat{\lambda}(2) \\ \vdots \\ \\ \\ \hat{\lambda}(N) \\ \hat{x}(N) \end{bmatrix} = \begin{bmatrix} T_0^{-1}W_i'\Pi^{-1}y_b \\ -AT_0^{-1}W_i'\Pi^{-1}y_b \\ C'R^{-1}y(1) \\ 0 \\ \vdots \\ \\ \\ 0 \\ W_f'\Pi^{-1}y_b \end{bmatrix}$$

(4.3.108)

In the next step we get

$$\begin{bmatrix} I & -T_0^{-1}A' & & & & & & \\ 0 & \Psi_1 & -E & & & & & \\ & 0 & T_1^{-1} & -A' & & & & \\ & & A & BB' & -E & & & \\ & & & & \cdots\cdots & & & \\ & & & & \cdots\cdots & & & \\ & & & & \cdots\cdots & & & \\ & & & & & A & BB' & -E \\ & & & & & & E' & \otimes_N \end{bmatrix} \begin{bmatrix} \hat{x}(0) \\ \hat{\lambda}(1) \\ \hat{x}(1) \\ \hat{\lambda}(2) \\ \vdots \\ \\ \\ \hat{\lambda}(N) \\ \hat{x}(N) \end{bmatrix} = \begin{bmatrix} T_0^{-1}W_i'\Pi^{-1}y_b \\ -AT_0^{-1}W_i'\Pi^{-1}y_b \\ \Psi_1^{-1}AT_0^{-1}W_i'\Pi^{-1}y_b + C'R^{-1}y(1) \\ 0 \\ \vdots \\ \\ \\ 0 \\ W_f'\Pi^{-1}y_b \end{bmatrix}$$

(4.3.109)

and at the final step we get

$$\begin{bmatrix} I & -T_0^{-1}A' & & & & & & \\ 0 & \Psi_1 & -E & & & & & \\ & 0 & I & -T_1^{-1}A' & & & & \\ & & 0 & \Psi_2 & -E & & & \\ & & & & \cdots\cdots & & & \\ & & & & \cdots\cdots & & & \\ & & & & \cdots\cdots & & & \\ & & & & & 0 & \Psi_N & -E \\ & & & & & & E' & \Theta_N \end{bmatrix} \begin{bmatrix} \hat{x}(0) \\ \hat{\lambda}(1) \\ \hat{x}(1) \\ \hat{\lambda}(2) \\ \vdots \\ \\ \hat{\lambda}(N) \\ \hat{x}(N) \end{bmatrix} = \begin{bmatrix} x^f(0) \\ -z^f(1) \\ x^f(1) \\ -z^f(2) \\ \vdots \\ \\ -z^f(N) \\ W_f'\Pi^{-1}y_b \end{bmatrix} \qquad (4.3.110)$$

where matrices T and Ψ, and vectors x^f and z^f are parameters of the generalized Kalman filter, defined in the previous section, with $x^f(0) = T_0^{-1}W_i'\Pi^{-1}y_b$. Thus the first stage of the algorithm consists in applying a generalized Kalman filter. The last predicted estimate obtained, i.e. $z^f(N)$, can be used to solve for $\hat{x}(N)$ using the two last block rows of (4.3.110) as follows

$$\hat{x}(N) = (E'\Psi_N^{-1}E + \Theta_N)^{-1}(E'\Psi_N^{-1}z^f(N) + W_f'\Pi^{-1}y_b). \qquad (4.3.111)$$

Now that we have solved for $\hat{x}(N)$, we can start the second phase of the algorithm, i.e. back substitution. We have

$$\begin{bmatrix} I & -T_0^{-1}A' & & & & & & \\ 0 & \Psi_1 & -E & & & & & \\ & 0 & I & -T_1^{-1}A' & & & & \\ & & 0 & \Psi_2 & -E & & & \\ & & & & \cdots\cdots & & & \\ & & & & & I & -T_{N-1}^{-1}A' & \\ & & & & & 0 & \Psi_N & -E \\ & & & & & & 0 & I \end{bmatrix} \begin{bmatrix} \hat{x}(0) \\ \hat{\lambda}(1) \\ \hat{x}(1) \\ \hat{\lambda}(2) \\ \vdots \\ \hat{x}(N-1) \\ \hat{\lambda}(N) \\ \hat{x}(N) \end{bmatrix} = \begin{bmatrix} x^f(0) \\ -z^f(1) \\ x^f(1) \\ -z^f(2) \\ \vdots \\ x^f(N-1) \\ -z^f(N) \\ \hat{x}(N) \end{bmatrix}, \qquad (4.3.112)$$

so we can solve for $\hat{\lambda}(N)$ as follows

$$
\begin{bmatrix}
I & -T_0^{-1}A' & & & & & \\
0 & \Psi_1 & -E & & & & \\
& 0 & I & -T_1^{-1}A' & & & \\
& & 0 & \Psi_2 & -E & & \\
& & & & \cdots\cdots & & \\
& & & & I & -T_{N-1}^{-1}A' & \\
& & & & 0 & I & 0 \\
& & & & & 0 & I
\end{bmatrix}
\begin{bmatrix}
\hat{x}(0) \\
\hat{\lambda}(1) \\
\hat{x}(1) \\
\hat{\lambda}(2) \\
\vdots \\
\hat{x}(N-1) \\
\hat{\lambda}(N) \\
\hat{x}(N)
\end{bmatrix}
=
\begin{bmatrix}
x^f(0) \\
-z^f(1) \\
x^f(1) \\
-z^f(2) \\
\vdots \\
x^f(N-1) \\
\Psi_N^{-1}(E\hat{x}(N)-z^f(N)) \\
\hat{x}(N)
\end{bmatrix}
$$

$$(4.3.113)$$

which implies that

$$
\hat{x}(N-1) = x^f(N-1) + T_{N-1}^{-1}A'\Psi_N^{-1}(E\hat{x}(N)-z^f(N)) \tag{4.3.114}
$$

and in general

$$
\hat{x}(k) = x^f(k) + T_k^{-1}A'\Psi_{k+1}^{-1}(E\hat{x}(k+1)-z^f(k+1)). \tag{4.3.115}
$$

For computing the smoothing error we can proceed similarly. Using the expression (4.1.49) we can show that

$$
\begin{bmatrix}
T_0 & -A' & & & & & \\
A & BB' & -E & & & & \\
& E' & C'R^{-1}C & -A' & & & \\
& & A & BB' & -E & & \\
& & & & \cdots\cdots & & \\
& & & & \cdots\cdots & & \\
& & & & \cdots\cdots & & \\
& & & & A & BB' & -E \\
& & & & & E' & \Theta_N
\end{bmatrix}
\begin{bmatrix}
\tilde{x}(0) \\
-\hat{\lambda}(1) \\
\tilde{x}(1) \\
-\hat{\lambda}(2) \\
\vdots \\
-\hat{\lambda}(N) \\
\tilde{x}(N)
\end{bmatrix}
=
\begin{bmatrix}
W_i'\Pi^{-1}r_b+V_i'Q^{-1}v \\
-Bu(0) \\
C'R^{-1}r(1) \\
-Bu(1) \\
\vdots \\
-Bu(N-1) \\
W_f'\Pi^{-1}r_b+V_f'Q^{-1}v
\end{bmatrix}
$$

$$(4.3.116)$$

where $\tilde{x}(k)$ is the smoothing error, i.e.

$$
\tilde{x}(k) = x(k) - \hat{x}(k). \tag{4.3.117}
$$

The covariance error P_k is given by

$$
P_k = \overline{\tilde{x}(k)\tilde{x}(k)'}. \tag{4.3.118}
$$

Following the same upper block diagonalization procedure that we used to

compute $\hat{x}(N)$ we can compute $\tilde{x}(N)$ and show that

$$P_N = (E'\psi_N^{-1}E + \circledast_N)^{-1}. \tag{4.3.119}$$

Then, by back-substitution we get

$$P_k = T_k^{-1} + T_k^{-1}A'\psi_{k+1}^{-1}[EP_{k+1}E' - \psi_{k+1}]\psi_{k+1}^{-1}AT_k^{-1}. \tag{4.3.120}$$

The assumption of invertibility of Q is not necessary for this algorithm

if we are willing to compute limits. A summary of the equations for the

generalized Rauch-Tung-Striebel method (allowing for possibility of singular

Q) is given below:

Initialization of the Generalized Kalman Filter

$$z^f(1) = AF_0W_i'\pi^{-1}y_b \tag{4.3.121a}$$

$$\psi_1 = AF_0A' + BB' \tag{4.3.121b}$$

$$F_0 = \lim_{\epsilon \to 0} (W_i'\pi^{-1}W_i + V_i'(Q+\epsilon I)^{-1}V_i)^{-1} \tag{4.3.121c}$$

Generalized Kalman Filter Update Equations

$$z^f(k+1) = A_k^f z^f(k) + K_k y(k) \tag{4.3.122a}$$

$$\psi_{k+1} = A(E'\psi_k^{-1}E + C'R^{-1}C)^{-1}A' + BB' \tag{4.3.122b}$$

$$A_k^f = A(E'\psi_k^{-1}E + C'R^{-1}C)^{-1}E'\psi_k^{-1} \tag{4.3.122c}$$

$$K_k = A(E'\psi_k^{-1}E + C'R^{-1}C)^{-1}C'R^{-1} \tag{4.3.122d}$$

Backward Smoother Initialization

$$\hat{x}(N) = P_N(E\psi_N^{-1}z^f(N) + W_f'\pi^{-1}y_b) \tag{4.3.123a}$$

$$P_N = \lim_{\epsilon \to 0} (E'\psi_N^{-1}E + W_f'\pi^{-1}W_f + V_f'(Q+\epsilon I)^{-1}V_f)^{-1} \tag{4.3.123b}$$

Backward Smoother Equations

$$\hat{x}(k) = x^f(k) + T_k^{-1}A'\Psi_{k+1}^{-1}(E\hat{x}(k+1)-z^f(k+1)) \qquad (4.3.124a)$$

$$T_k = E'\Psi_k^{-1}E + C'R^{-1}C \qquad (4.3.124b)$$

Smoothing Error

$$P_k = T_k^{-1} + T_k^{-1}A'\Psi_{k+1}^{-1}[EP_{k+1}E'-\Psi_{k+1}]\Psi_{k+1}^{-1}AT_k^{-1}. \qquad (4.3.125)$$

In the causal case, (4.3.121)-(4.3.125) reduces to the well known Rauch-Tung-Striebel algorithm.

Let us comment on the assumption that

$$\begin{bmatrix} V_i \\ W_i^i \end{bmatrix}$$

has full rank. In general, for an acausal system, V_i may or may not by itself have full rank. The presence of a boundary observation to augment this may appear to be artificial, but in fact it is not. In particular, note that while $x(k)$ is defined for k=0 through N, our "interior observations" $y(k)$ are defined only for k=1 through N-1. Thus, a _natural_ model would be to have "boundary measurements"

$$y_b = \begin{bmatrix} y(0) \\ y(N) \end{bmatrix} = \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} x(0) \\ x(N) \end{bmatrix} + \begin{bmatrix} r(0) \\ r(N) \end{bmatrix} \qquad (4.3.126)$$

so that our condition becomes one that

$$\begin{bmatrix} V_i \\ C \end{bmatrix}$$

has full rank. Thus, while we still must make an assumption of this type, it is at least reasonable to assume that we do have boundary measurements. Obviously if $\begin{bmatrix} V_f \\ W_f \end{bmatrix}$ has full rank, we can do the analogous operations in the reverse direction.

If $\begin{bmatrix} V_i \\ C \end{bmatrix}$ doesn't have full rank, one might consider the information filter form of our filters. The development of these equations hasn't been done but should be straightforward. Note finally that if neither Q nor $\begin{bmatrix} V_i \\ W_i \end{bmatrix}$ are full rank, then part of $x(0)$ may be known perfectly and part of it not at all. Such a problem, of course, could also arise in the causal case, in which case one would need some hybrid Kalman filter and information filter in order to propagate finite-valued quantities.

4.3.3—Case of Non-separable TPBVDS

To apply this solution method to the case of TPBVDS's that are not stochastically separable we can use the fact that TPBVDS (4.1.1)-(4.1.2) is equivalent to a stochastically separable TPBVDS of twice the dimension and defined over an interval of length one half the length of the original TPBVDS. Specifically, let

$$s(k) = \begin{bmatrix} x(k) \\ x(N-k) \end{bmatrix}. \qquad (4.3.127)$$

Then (4.1.1) can be expressed as

$$\begin{bmatrix} E & 0 \\ 0 & A \end{bmatrix} s(k+1) = \begin{bmatrix} A & 0 \\ 0 & E \end{bmatrix} s(k) + \begin{bmatrix} B & 0 \\ 0 & -B \end{bmatrix} \begin{bmatrix} u(k) \\ u(N-k-1) \end{bmatrix} \qquad (4.3.128a)$$

with boundary condition

$$\begin{bmatrix} V_i & V_f \\ 0 & 0 \end{bmatrix} s(0) + \begin{bmatrix} 0 & 0 \\ I & -I \end{bmatrix} s(N/2) = \begin{bmatrix} v \\ 0 \end{bmatrix} \qquad (4.3.128b)$$

if N is even or

$$\begin{bmatrix} V_i & V_f \\ 0 & 0 \end{bmatrix} s(0) + \begin{bmatrix} 0 & 0 \\ -A & E \end{bmatrix} s[(N-1)/2] = \begin{bmatrix} v \\ Bu[(N-1)/2] \end{bmatrix} \qquad (4.3.128c)$$

if N is odd. Expression (4.1.2) becomes

$$\begin{bmatrix} y(k) \\ y(N-k) \end{bmatrix} = \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix} s(k) + \begin{bmatrix} r(k) \\ r(N-k) \end{bmatrix} \qquad (4.3.129a)$$

with boundary observation

$$\begin{bmatrix} y_b \\ y(N/2) \end{bmatrix} = \begin{bmatrix} W_i & W_f \\ 0 & 0 \end{bmatrix} s(0) + \begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix} s(N/2) + \begin{bmatrix} r_b \\ r(N/2) \end{bmatrix} \qquad (4.3.129b)$$

if N is even and

$$\begin{bmatrix} y_b \\ y[(N-1)/2] \\ y[(N+1)/2] \end{bmatrix} = \begin{bmatrix} W_i & W_f \\ 0 & 0 \\ 0 & 0 \end{bmatrix} s(0) + \begin{bmatrix} 0 & 0 \\ C & 0 \\ 0 & C \end{bmatrix} s(N/2) + \begin{bmatrix} r_b \\ r[(N-1)/2] \\ r[(N+1)/2] \end{bmatrix} \qquad (4.3.129c)$$

if N is odd.

It is not difficult to see that (4.3.128)-(4.3.129) is stochastically separable and thus we can use the Rauch-Tung-Striebel method derived previously. In this case, the assumptions of infinite eigenmode observability and zero eigenmode reachability of our 2n-dimensional model are equivalent to the assumptions of observability and reachability of both zero and infinite eigenmodes of the original system. Furthermore, the finite variance of our estimation error associated with the estimate of s(0) based on boundary observations now becomes

$$P_b = \lim_{\epsilon \to 0} \left[\begin{bmatrix} V_i & V_f \\ W_i & W_f \end{bmatrix}' \left(\begin{bmatrix} Q & 0 \\ 0 & \Pi \end{bmatrix} + \epsilon I \right)^{-1} \begin{bmatrix} V_i & V_f \\ W_i & W_f \end{bmatrix} \right]^{-1} \qquad (4.3.130)$$

exists and is positive-definite. A necessary and sufficient condition for existence is that $\begin{bmatrix} V_i & V_f \\ W_i & W_f \end{bmatrix}$ must have full column rank which means that a finite error variance estimate of both boundary points of our original model can be constructed based solely on the boundary information. The positive-definiteness of P_b implies that no projection of x(0) and x(N) is observed perfectly. If Q is positive-definite, P_b is also but this is not a necessary condition. In what follows, for simplicity, we shall also assume that Q is also positive-definite. If Q is not positive-definite, a limiting expression, like the one used above or in previous sections must be used.

In this case, the first part of the algorithm which consists of the generalized Kalman filter, essentially estimates x(k) and x(N-k) based on the boundary information, the observations y(j) and the dynamics outside the interval [k,N-k]. Thus the algorithm starts at the boundaries and moves inward towards the center where x(N/2) is estimated. The second part of the algorithm consists of moving outward from the center towards the boundaries.

In the next section, we study the limiting behaviour of the smoother in the general (non-separable) case. In particular, we are interested in the limiting error variance associated with a point at the center of the interval as the length of the interval tends to infinity. Our formulation of the smoother in the general case can easily be used to study this limit, because, the end result of the first stage of our smoothing algorithm, i.e., the generalized Kalman filter, yields the smoothing error variance associated with the center point.

4.3.4-Limiting Smoothing Error

In Chapter II and Chapter III, we studied one notion of extending the domain of a TPBVDS. In particular, we extended the domain by preserving the weighting pattern. There is another notion of extending the domain of a system [2,16] where we consider the boundary conditions as being physical constraints of the problem. In that case, the dynamic equations and the boundary conditions are conserved, simply N changes. Using this notion of extendibility, we have defined a concept of stability called internal stability. A TPBVDS is essentially called internally stable if the

contribution of the boundary value v to the point at the center of the interval as the length of the interval goes to infinity approachs zero.

In this section, we shall consider the limiting behaviour of the smoothing error variance as N goes to infinity while the dynamics and the boundary matrices remain the same.

Let us suppose that the system is strongly reachable and observable, and that N is even. We would like to find an expression for the smoothing error variance $P(N/2)$ of $x(N/2)$ as N goes to infinity.

Note that in applying the generalized Kalman filter to (4.3.128)-(4.3.129), at the end stage, we obtain the optimal estimate of $x(N/2)$ and the associated estimation error. Using expression (4.3.123b), we get that

$$\mathscr{P}(N/2) = \lim_{\epsilon \to 0^+} \left(\begin{bmatrix} E' & 0 \\ 0 & A' \end{bmatrix} \psi_{N/2}^{-1} \begin{bmatrix} E & 0 \\ 0 & A \end{bmatrix} + \begin{bmatrix} 0 & C' \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Pi^{-1} & 0 \\ 0 & R^{-1} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix} + \begin{bmatrix} 0 & I \\ 0 & -I \end{bmatrix} (I/\epsilon) \begin{bmatrix} 0 & 0 \\ I & -I \end{bmatrix} \right)^{-1}$$

$$(4.3.131)$$

where $\mathscr{P}(N/2)$ is the estimation error variance of $s(N/2)$ and $\psi_{N/2}$ is obtained from the following recursion (this is just the generalized Riccati equation for (4.3.128)-(4.3.129))

$$\psi_{k+1} = \begin{bmatrix} A & 0 \\ 0 & E \end{bmatrix} \left(\begin{bmatrix} E' & 0 \\ 0 & A' \end{bmatrix} \psi_k^{-1} \begin{bmatrix} E & 0 \\ 0 & A \end{bmatrix} + \begin{bmatrix} C' & 0 \\ 0 & C' \end{bmatrix} \begin{bmatrix} R^{-1} & 0 \\ 0 & R^{-1} \end{bmatrix} \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix} \right)^{-1} \begin{bmatrix} A' & 0 \\ 0 & E' \end{bmatrix} + \begin{bmatrix} BB' & 0 \\ 0 & BB' \end{bmatrix} .$$

$$(4.3.132)$$

Thanks to strong reachability and observability assumptions, from the results of Section 4.2, we know that

$$\lim_{k \to \infty} \psi_k = \psi.$$

$$(4.3.133a)$$

From (4.3.132) we can deduce that

$$\psi = \begin{bmatrix} \Psi & 0 \\ 0 & S \end{bmatrix}$$

$$(4.3.133b)$$

where Ψ and S are unique solutions of generalized Riccati equations (4.2.18a)

and (4.2.18b). Thus

$$\text{Lim } \mathcal{P}(N/2) = \\ N\to\infty$$

$$\text{Lim}_{\epsilon\to 0^+}(\begin{bmatrix} E' & 0 \\ 0 & A' \end{bmatrix}\begin{bmatrix} \Psi^{-1} & 0 \\ 0 & S^{-1} \end{bmatrix}\begin{bmatrix} E & 0 \\ 0 & A \end{bmatrix}+\begin{bmatrix} 0 & C' \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \Pi^{-1} & 0 \\ 0 & R^{-1} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix}+\begin{bmatrix} 0 & I \\ 0 & -I \end{bmatrix}(I/\epsilon)\begin{bmatrix} 0 & 0 \\ I & -I \end{bmatrix})^{-1} =$$

$$\text{Lim}_{\epsilon\to 0^+}\begin{bmatrix} E'\Psi^{-1}E+C'R^{-1}C+I/\epsilon & -I/\epsilon \\ -I/\epsilon & A'S^{-1}A+I/\epsilon \end{bmatrix}^{-1} . \qquad (4.3.134)$$

The $(1,1)$ block entry of (4.3.134) can be expressed as

$$\text{Lim}_{\epsilon\to 0^+}(E'\Psi^{-1}E+C'R^{-1}C+I/\epsilon-(A'S^{-1}A+I/\epsilon)^{-1}/\epsilon^2)^{-1} = (E'\Psi^{-1}E+C'R^{-1}C+A'S^{-1}A)^{-1}$$

$$(4.3.135)$$

Noting that

$$s(N/2) = \begin{bmatrix} x(N/2) \\ x(N/2) \end{bmatrix} \qquad (4.3.136)$$

we deduce that the variance of the estimation error $P(N/2)$ associated with the optimal estimate of $x(N/2)$ as N goes to infinity is

$$\text{Lim } P(N/2) = (E'\Psi^{-1}E+C'R^{-1}C+A'S^{-1}A)^{-1} = \\ N\to\infty$$

$$(E'\Psi^{-1}E+\Theta)^{-1} = (T+A'S^{-1}A)^{-1} \qquad (4.3.137)$$

where Θ and T are respectively solutions of algebraic generalized Riccati equations (4.2.14a) and (4.2.15a). Thus we see that the solution of generalized Riccati equations provide us directly with the limiting error variance. In the causal case $(E=I)$, these are the formulas that arise in the two-filter form of the optimal smoother. Specifically, in the causal case, the limiting smoothing error P_e satisfies

$$P_e^{-1} = P_f^{-1} + P_b^{-1} \qquad (4.3.138)$$

where P_f is the forward predicted error variance, i.e. Ψ in our notation and P_b the backward filtered error, i.e. Θ^{-1}.

So far, we have shown that under the assumptions of strong reachability and observability, the smoothing error variance of the "state" at the center of the interval does not grow unbounded as we let the length of the interval, N, go to infinity, i.e. the smoother is stable (we have in fact shown more, specifically we have shown that this smoothing error variance converges). The assumptions of strong reachability and observability, however, are not always necessary for the stability of the smoother. Consider for example the case of stable, causal systems. If a causal system is stable, the smoothing error variance does not grow unbounded regardless of whether or not the system is reachable or observable. It is not difficult to see that the same result also holds for TPBVDS's. Specifically, if a TPBVDS is internally stable, then the smoothing error variance associated with its "state" near the center of the interval as N goes to infinity does not grow unbounded (in fact, this error variance is upper bounded by the variance of the "state" at the center of the interval, which due to the stability assumption does not grow unbounded). This result holds even if no boundary observation y_b exists.

If a boundary observation does exist such that $\begin{bmatrix} V_i & V_f \\ W_i & W_f \end{bmatrix}$ has full column rank, then as long as the system has no eigenmode on the unit circle, the smoother is stable. To see why this must be the case, first note that in this case, we can use boundary information alone to construct estimates of the initial and final "states" of the system, respectively denoted by $\hat{x}(0|b)$ and $\hat{x}(N|b)$, having finite error variances based on boundary information. Now let us transform the system into forward-backward stable form

$$C = [C_f \quad C_b], \quad E = \begin{bmatrix} I & 0 \\ 0 & A_b \end{bmatrix}, \quad A = \begin{bmatrix} A_f & 0 \\ 0 & I \end{bmatrix}, \quad B = \begin{bmatrix} B_f \\ B_b \end{bmatrix},$$

$$x(k) = \begin{bmatrix} x_f(k) \\ x_b(k) \end{bmatrix}, \tag{4.3.139}$$

where A_f and A_b are strictly stable and let $\hat{x}_f(k)$ and $\hat{x}_b(k)$ be defined as

follows

$$\hat{x}_f(0) = \hat{x}_f(0|b) \qquad (4.3.140a)$$

$$\hat{x}_f(j+1) = A_f\hat{x}_f(j) + B_fu(j). \qquad (4.3.140b)$$

and

$$\hat{x}_b(N) = \hat{x}_b(N|b) \qquad (4.3.141a)$$

$$\hat{x}_b(j-1) = A_b\hat{x}(j) + B_bu(j-1). \qquad (4.3.141b)$$

If we now consider $\hat{x}_f(k)$ as an estimate of $x_f(k)$, it is not difficult to see that this estimate has a finite error variance that converges as k goes to infinity. Similarly, if we consider $\hat{x}_b(k)$ as an estimate of $x_b(k)$, its associated error variance converges as N goes to infinity. Thus, if we let

$$\hat{x}(k) = \begin{bmatrix} \hat{x}_f(k) \\ \hat{x}_b(k) \end{bmatrix}, \qquad (4.3.142)$$

then $\hat{x}(k)$ is an estimator of $x(k)$ such that its error variance for the "state" at the center of the interval converges as N goes to infinity. This estimator is clearly, in general, not optimal since it only uses the boundary measurements. Thus, the error variance of this estimator bounds the error variance of the optimal smoother and thus the optimal smoother is stable.

The cases considered above are extreme cases: first we showed smoother stability when the system is strongly reachable and observable, then we showed smoother stability when boundary measurements are enough to form a finite variance estimate of the endpoints. An interesting problem to consider, is the general problem of stability, i.e. when the system is only partially observable and reachable, and the boundary measurements provides an estimate of just a projection of x(0) and x(N).

4.4-Conclusion

In this chapter, we have studied the estimation problem for TPBVDS's. First, we have shown that the smoothed estimate of a TPBVDS can be obtained by solving a TPBVDS of twice the dimension of the original system. To solve this TPBVDS, we have proposed using the two-filter method for which we had to study the transformation into forward-backward stable form of the smoother. This transformation, which is a generalization of the Hamiltonian diagonalization for causal systems, has been shown to tie in with the solutions of generalized Riccati equations which have been studied.

Then, we have introduce another approach to the smoothing problem. In particular, we have derived a generalization of the Kalman filter for the case of descriptor systems and used it to derive a generalization of the Rauch-Tung-Striebel formulation of the optimal smoother for TPBVDS's. Finally, we have derived an expression for the limiting error variance of the smoothed estimate x at the center of the interval as the length of the interval tends to infinity and in general considered the problem of smoother stability. We have seen that the solution of our generalized Riccati equations arise in these contexts as well, and in fact these results provide precise probabilistic interpretations for these Riccati equations.

CHAPTER V:

CONCLUSIONS

5.1-Contributions

In this thesis, we have developed a deterministic and a stochastic system theory for two-point boundary-value descriptor systems (TPBVDS's). A part of this work consists of generalization of our previous system-theoretical results for shift-invariant TPBVDS's to the case of arbitrary TPBVDS's (Chapter II). Chapter III contains a realization theory for a special class of TPBVDS's. The results of this chapter are all new. In Chapter IV, an estimation theory is developed for TPBVDS's. In particular, well-known estimation methods for causal systems such as the Kalman filter and the Rauch-Tung-Striebel formulations of the optimal smoother are extended to the case of TPBVDS's.

The major contributions of Chapter II are:

(1)   Derivation of a closed-form expression for the inward process resulting in a complete characterization of the concepts of weak reachabiltiy and observability.

(2)   Characterization of properties of extendibility and input-output extendibility.

(3)   Introduction of the projection matrix in the description of input-output extendible, stationary TPBVDS's. The projection matrix simplifies significantly the description of the contribution of the boundary conditions to the weighting pattern of an input-output extendible, stationary TPBVDS.

(4) Obtaining minimality results for TPBVDS's, and characterizing minimal realizations.

(5) Generalizing the concept of standard-form and developing a modal analysis technique for TPBVDS's.

Chapter III contains a realization theory exclusively for input-output extendible, stationary TPBVDS's. The major contributions of Chapter III are:

(1) Obtaining deterministic realizability conditions and a simple realization technique (not always yielding minimal realizations).

(2) Introduction and characterization of the (s,t)-transform, in particular, development of a factorization theory for rational matrices in s and t including a generalization of the McMillan degree.

(3) Obtaining an expression for the dimension of a minimal deterministic realization using the (s,t)-transform and developing a deterministic realization method using factorization techniques yielding directly minimal realizations.

(4) Introducing the concept of stability for input-output extendible, stationary TPBVDS's and studying its properties.

(5) Obtaining necessary and sufficient conditions for stochastic stationarity in terms of a generalized Lyapunov equation.

(6) Characterization of stochastic extendibility and in particular showing that deterministically minimal, stochastically extendible systems are necessarily stable.

(7) Obtaining necessary and sufficient conditions for stochastic realizability.

(8) Proposing a method for testing minimality of a stochastic realization and an expression for its dimension in terms of the (s,t)-transform of its output covariance.

(9) Showing that, as in the causal case, the stochastic realization problem is equivalent to a spectral factorization problem.

Chapter IV is devoted to a study of optimal filtering for TPBVDS's. The major contributions of this chapter are:

(1) Generalization of the Hamiltonian form of the optimal smoother for causal systems.

(2) Generalization of the notion of Hamiltonian diagonalization involving generalizations of standard Riccati equations.

(3) Development of a theory for generalized Riccati equations paralleling the existing theory for standard Riccati equations.

(4) Formulation and characterization of generalized Kalman filters for descriptor systems.

(5) Generalization of the Rauch-Tung-Striebel smoothing method for causal systems to the case of TPBVDS's.

(6) A study of the limiting behaviour of the error variance matrix associated to the optimal smoothing of TPBVDS's and the relationship to solutions of the generalized Riccati equations.

5.2-Suggestions for Further Research

The following are some open questions and possible topics of future research:

(1) In Chapter II, we have shown that there exists some degree of freedom in the system matrices of minimal TPBVDS's when the system is not strongly reachable and observable. Since, when a stationary system is strongly reachable and observable, it also has the displacement property, a natural question to ask is: can we use the freedom in the specification of a minimal stationary system to transform it into a displacement system?

(2) In Chapter III, we are able to develop a realization theory (as opposed to a partial realization theory) for input-output extendible stationary TPBVDS's because the weighting patterns associated with such systems are defined form $-\infty$ to $+\infty$. Thus, it should be possible to generalize this theory to the case of non-stationary extendible TPBVDS's. The difficulty in this case is that the weighting pattern is a function of two time indices.

(3) In this thesis, we have mostly considered one of the two notions of extendibility, namely extendibility by preserving the weighting pattern. In [1,16], the other notion of extendibility, i.e. extending the system by preserving the boundary conditions, has been shown to yield interesting mathematical structures. In particular, this notion of extendibility allows us to define an internal stability property and relate this property to the existence of positive-definite solutions to generalized Lyapunov equations. There are a number of interesting problems that arise based on this notion of extendibility:

(3a) By extending TPBVDS's this way, i.e. by conserving the boundary conditions, system properties such as stationarity are not conserved. Thus, it would be interesting to study conditions under which all such extensions of a stationary system are stationary, i.e. conditions under which, if (C,V_i,V_f,E,A,B,N) is a TPBVDS in normalized form, then for all M, $(C,\Gamma_M V_i,\Gamma_M V_f,E,A,B,M)$ where

$$\Gamma_M = (V_i E^M + V_f A^M)^{-1}, \qquad (5.2.1)$$

is stationary. We call such a system globally stationary (in general, we propose calling properites that hold for all such extensions of a TPBVDS global).

(3b) It can also be shown that the weakly reachable and observable spaces, even though they keep constant dimension, may rotate as we extend the system. Thus, naturally, we are interested in finding necessary conditions for global weak reachability and observability, and maybe relating these notions to that of global minimality.

(3c) Another interesting problem is the realization problem, both deterministic and stochastic. When we extended TPBVDS's while conserving their weighting patterns, we obtained a single weighting pattern defined everywhere which we used for deterministic realization. Specifically, we considered the problem of realizing a TPBVDS from its (extended) weighting pattern. When we extend TPBVDS's while conserving their boundary conditions, we obtain a sequence of different weighting patterns defined over various intervals. Thus, the realization problem is more complex and can be posed as follows: given a sequence of weighting patterns W_M defined over various intervals, find a TPBVDS such that, if we extend it while conserving its boundary conditions, we obtain TPBVDS's having weighting patterns W_M.

(3d) The stochastic realization problem can be posed in a similar manner, i.e., given a sequence of covariances Λ_M, find a TPBVDS such that the output covariance of its extensions match Λ_M. It is not difficult to verify that stochastic stationarity is conserved when TPBVDS's are extended while conserving their boundary conditions. For this reason, we suspect that the class of stochastically realizable sequences is richer than the one studied in Section 3.3.

(4) In the causal case, the stochastic realization theory ties in very nicely with the estimation problem. In particular, it is shown that the Kalman filter is just the minimum variance realization of the output process, and that the "reversed-time realization" of the backward Kalman filter is the maximum variance realization of the output process (see for example [40]). Similar realtionships may exist in the TPBVDS case.

(5) In Chapter IV, we have obtained a generalization of the Kalman filter and Riccati equations. The expressions associated to this filter, in general, contain limits; this of course is not desirable. There may be two ways of eliminating these limits: the first one is to find equivalent expressions not involving limits, i.e. finding a generalization of the ABCD inversion lemma. The second one is to use another implementation of the filter, for example a generalization of the square-root Kalman filter for causal systems.

(6) The innovation process plays an important role in causal filtering theory. It would be valuable to see whether a similar concept can be found for the TPBVDS case. This may be accomplished by studying how the generalized Kalman filter of the first phase of the generalized Rauch-Tung-Striebel algotrithm, which moves from the boundaries to the center, relates to the inward process z_i. In particular, we may be able to construct an innovations process for z_i. Similarly, an innovations process for z_o may be found.

(7) The problem of system identification is an intersting problem to consider in the case of TPBVDS's. The system theoretical properties studied in this thesis should provide a solid basis for this study.

(8) One of the motivation behind studying TPBVDS's has been the possibility of extending its theory to the multi-dimensional case where boundary conditions arise naturaly in the specifications of problems. So far, we have not considered the multi-dimensional case, however, we suspect that some of our results can be extended to that case. In particular, the notions of inward and outward processes may easily be extended to the multi-dimensioanl case. However, since the size of the boundary in 2D depends on the sze of the domain, the dimension of these processes must change as they move in and out, and thus concepts of reachability, observability, etc., cannot be defined as easily as in the 1D case.

APPENDIX:

THE TWO-FILTER SOLUTION

Adams [46] formulates the general solution of TPBVDS (2.2.1)-(2.2.2) as a linear combination of two stable recursions, one forward and the other backward. His formulation is presented below as it appeared in [46], with only a few changes in the notation.

Since $\{E,A\}$ comprise a regular pencil, there exist nonsingular matrices F and T such that

$$FET^{-1} = \begin{bmatrix} I & 0 \\ 0 & A_b \end{bmatrix} \triangleq \hat{E} \tag{A.1a}$$

and

$$FAT^{-1} = \begin{bmatrix} A_f & 0 \\ 0 & I \end{bmatrix} \triangleq \hat{A} \tag{A.1b}$$

where all eigenvalues of A_f and A_b lie within the unit circle (we assume that the system has no eigen-mode on the unit circle). The above decomposition splits the system into two subsystems:

$$x_f(k+1) = A_f x_f(k) + B_f u(k) \tag{A.2a}$$

and

$$x_b(k) = A_b x_b(k+1) - B_b u(k) \tag{A.2b}$$

where

$$\begin{bmatrix} x_f(k) \\ x_b(k) \end{bmatrix} = Tx(k) \tag{A.3a}$$

and

$$\begin{bmatrix} B_f \\ B_b \end{bmatrix} = FB. \tag{A.3b}$$

Given the above transformation, the boundary condition (2.2.2) takes the form

$$[V_f^i \!\mid\! V_b^i]\begin{bmatrix} x_f(0) \\ x_b(0) \end{bmatrix} + [V_f^f \!\mid\! V_b^f]\begin{bmatrix} x_f(N) \\ x_b(N) \end{bmatrix} = v \tag{A.4}$$

where

$$[V_f^i \vdots V_b^i] = V_i T^{-1} \triangleq \hat{V}_i \tag{A.5a}$$

$$[V_f^f \vdots V_b^f] = V_f T^{-1} \triangleq \hat{V}_f. \tag{A.5b}$$

Define $x_f^0(k)$ as the solution to (A.2a) with zero initial condition and $x_b^0(k)$ as the solution to (A.2b) with zero final condition. Then it is easy to see that

$$x_f(k) = (A_f)^k x_f(0) + x_f^0(k) \tag{A.6a}$$

$$x_b(k) = (A_b)^{N-k} x_b(k) + x_b^0(k). \tag{A.6b}$$

Substituting for $x_f(N)$ and $x_b(0)$ from (A.6a) and (A.6b) into (A.4) and solving for $x_f(0)$ and $x_b(N)$ gives

$$\begin{bmatrix} x_f(0) \\ x_b(N) \end{bmatrix} = (F_N)^{-1} [v - V_f^f x_f^0(N) - V_b^i x_b^0(0)] \tag{A.7}$$

where

$$F_N = \hat{V}_i \hat{E}^N + \hat{V}_f \hat{A}^N. \tag{A.8}$$

Finally, substituting for $x_f(0)$ and $x_b(k)$ from (A.7) and (A.8), it can be shown that the solution to (A.2) is given by

$$\begin{bmatrix} x_f(k) \\ x_b(k) \end{bmatrix} = \hat{E}^{N-k} \hat{A}^k (F_N)^{-1} [v - V_f^f x_f^0(N) - V_b^i x_b^0(0)] + \begin{bmatrix} x_f^0(k) \\ x_b^0(k) \end{bmatrix}. \tag{A.9}$$

Applying the inverse of the transformation in (A.3a), the original process $x(k)$ is recovered by

$$x(k) = T^{-1} \begin{bmatrix} x_f(k) \\ x_b(k) \end{bmatrix}. \tag{A.10}$$

In this way, Adams has constructed a stable forward/backward two filter recursive implementation of the general solution of a TPBVDS. Notice that the invertibility of the matrix F_N is not an issue, since F_N is invertible if the system is well-posed (in fact invertibility of F_N is our test for well-posedness).

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