# STRUCTURAL PROPERTIES AND ESTIMATION OF DELAY SYSTEMS

by

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## ABSTRACT

Two areas in the theory of delay systems are studied: structural properties and their applications to feedback control, and optimal linear and nonlinear estimation. First, we study the concepts of controllability, stabilizability, observability and detectability. The property of pointwise degeneracy of linear time-invariant delay systems is then considered. Necessary and sufficient conditions for three dimensional linear systems to be made pointwise degenerate by delay feedback are obtained, while sufficient conditions for this to be possible are given for higher dimensional linear systems. These results are then applied to obtain solvability conditions for the minimum time output zeroing control problem by delay feedback. Next, we turn our attention to optimal linear and nonlinear estimation. A representation theorem is given for conditional moment functionals of general nonlinear stochastic delay systems. From this, stochastic differential equations are derived for conditional moment functionals satisfying certain smoothness properties. We then give a complete solution to the estimation problem for general linear delay systems. Stability properties of the infinite-time linear optimal control system with quadratic cost and the optimal linear filter for systems without delays in the observations are studied. When appropriate structural properties hold, the optimal control system and the optimal filter are both shown to be asymptotically stable. Finally, the cascade of the optimal filter and the optimal control system is shown to be asymptotically stable as well.

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# CHAPTER 1

# INTRODUCTION

## 1.1 Mathematical Description of Delay Systems

The research reported in this thesis deals with certain aspects in the theory of delay systems. Delay systems constitute a class of hereditary systems, dynamical systems whose future behavior depends on past events in a fundamental way. Mathematically, the simplest delay system can be described by the following differential equation

$$\dot{x}(t) = f[x(t), x(t-\tau), t] \tag{1.1}$$

where $x(t) \in R^n$, and $\tau$, a positive constant, is a delay. A moment's reflection shows that in order to determine the behavior of the system for $t \geq t_o$, one must specify not only $x(t_o)$, but the function $x(\theta)$ on the interval $[t_o-\tau, t_o]$. If we define the function

$$x_t: [-\tau, 0] \rightarrow R^n$$

by

$$x_t(\theta) = x(t+\theta) \qquad \theta\varepsilon[-\tau, 0] \tag{1.2}$$

we see that knowledge of the function $x_\sigma$ is necessary and sufficient to determine the behavior of the system for $t \geq \sigma$. Thus $x_t$ is the true state of the system (1.1), and being an element of some function space defined on the interval $[-\tau, 0]$, it is infinite dimensional. This is a fundamental difference between delay systems and finite dimensional ordinary differential systems. By a slight abuse of language, we shall still call $x(t)$ the "state" of the system (1.1), and call $x_t$ the "complete state."

Equation (1.1) is often called a differential-difference equation of retarded type [1]. From the control theory point of view, we can have a controlled delay differential equation

$$\dot{x}(t) = f[x(t), x(t-\tau), u(t), u(t-\tau), t] \tag{1.3}$$

If the system trajectory is not directly measured, but is observed through output variables z(t), one can have systems with delays in the observations

$$z(t) = h[x(t), x(t-\tau), t] \tag{1.4}$$

In actual applications, a combination of the above basic types of equations may, of course, arise. Furthermore, these equations can also be generalized to include systems with several delays $\tau_i$, with time-varying delays $\tau(t)$, or with delays $\tau(x(t))$ depending on the state.

So far, we have described delay systems which have a finite number of delays. A more general type of controlled delay systems is described by the retarded functional differential equation [2]

$$\dot{x}(t) = f[x_t, u(t), t] \tag{1.5}$$

where $x_t$ is defined by (1.2). This includes differential equations with point delays (1.1) and equations with distributed delays

$$\dot{x}(t) = \int_{-\tau}^{0} f[x(t+\theta)]d\theta + u(t) \tag{1.6}$$

While even more general types of delay systems can be defined [3], they will not play a role in this thesis and hence will not be discussed.

Finally, delay systems may also be subject to random disturbances. One formulation of a stochastic delay equation is given by

$$dx(t) = f[x(t), x(t-\tau), t]dt + dw(t) \qquad (1.7)$$

where $w(t)$ is a Wiener process, and the equation is understood in the Ito sense. Usually associated with the stochastic delay equation (1.7) is an observation process

$$dz(t) = h[x(t), x(t-\tau), t]dt + dv(t) \qquad (1.8)$$

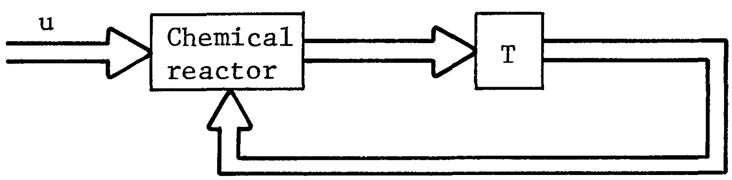
As in the deterministic case, the function $x_t$ plays an important role in the theory of stochastic delay systems.

## 1.2 Dynamical Processes Modeled by Delay Systems

Euler was the first mathematician to study delay differential equations [4]. While there were a number of mathematical investigations of such equations after Euler, notably Volterra [5], the basic mathematical foundations for delay systems were established in the nineteen forties, fifties and early sixties [1], [6]. Since then, the theory of delay systems and, more generally, that of hereditary systems have been the subject of intensive investigation. While it has long been known by control theorists that delay systems are the appropriate models for a wide variety of process control systems [7], recent applications of system theory have shown that some biological, ecological, economical and social processes also take the form of delay systems [8], [9]. This has generated a great deal of interest in various aspects of the theory of delay systems in the control as well as mathematical community. We shall discuss some examples of such processes.

Example 1. Chemical Control Processes

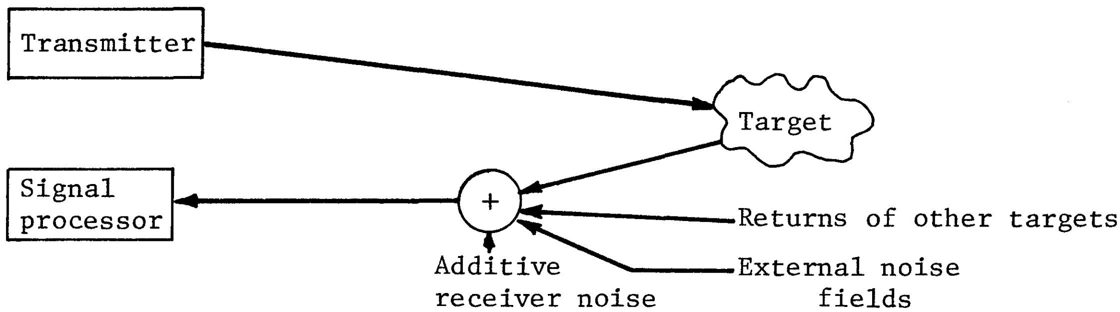
A basic configuration in many chemical process control systems is the following [10]:




Here u usually represents a control of the feed rate of raw materials, T represents a transformation which changes the relative composition of the reactants. A typical example in refineries is that T is composed of a cascade of a reaction cooler, a decanter, and a distillation column. The recycle loop feeds back a certain percentage of the raw materials and some byproducts of the chemical reaction. This recycling introduces a significant time delay, often of the order of a few minutes, into the system. To build a suitable mathematical model for the analysis of such chemical control plants, a delay system must be used.

Example 2. Estimation in Radar-Sonar Problems

We consider the simplest model of a radar system [11]:




Because of the distance of the target, the received signal is a delayed (possibly distorted) version of the transmitted signal. If we assume the target is a slowly fluctuating point target, the reflection is linear, and there are K interfering targets, the received signal can be written in the form [11]

$$r(t) = \sqrt{2}\ \mathrm{Re}\ \{[\sqrt{E_t} \sum_{i=0}^{K} \tilde{b}_i \tilde{f}(t-\tau_i) e^{j\omega_i t} + \tilde{n}(t)] e^{j\omega_c t}\}$$

where $\tilde{f}(t)$ is the complex envelope of the transmitted signal, $E_t$ the energy of the signal, $\omega_c$ the carrier frequency, $\tilde{n}(t)$ the complex representation of an additive Gaussian noise process, and for the $i^{th}$ target, $\tilde{b}_i$ summarizes the amplitude information, $\tau_i$ is the round trip delay time, and $\omega_i$ is the Doppler shift frequency. These multiple-target situations are often referred to as communication over multi-path channels. A basic estimation problem for these radar-sonar systems is to find optimal estimates for the quantities $\tau_i$ and $\omega_i$, $i = 0,\ldots, K$, since these give useful information about the ranges and velocities of the various targets. This is clearly an important application of the estimation theory of stochastic delay systems.

Example 3. Modeling of Epidemics

Delay differential equations have often been used in the modeling of epidemics. Let $x_1(t)$, $x_2(t)$, $x_3(t)$, and $x_4(t)$ denote the number of susceptible but unexposed individuals; infective individuals, removed individuals, and exposed but not infective individuals respectively. Since the infective individuals are those who were infected some time earlier, this introduces delays in the dynamics. The following equation has been proposed for the spread of measles [12]:

$$\dot{x}_1(t) = -\beta(t)x_1(t)[2\gamma + x_1(t-14) - x_1(t-12)] + \gamma$$

where $\beta(t)$ is a proportionality coefficient that is seasonally dependent, $\gamma$ is the (constant) rate at which individuals enter the population, and a unit of time is a day.

A more complex model of controlled epidemics has also been proposed [13]. Letting $u_1$ represent active immunization, $u_2$ passive immunization, the model proposed is

$$\dot{x}_1(t) = -\beta x_1(t)x_2(t) - x_1(t)u_1(t-\tau) - x_1(t)u_2(t)$$

$$\dot{x}_2(t) = a(t) - r(t) - x_2(t)u_2(t)$$

$$\dot{x}_3(t) = r(t)$$

$$\dot{x}_4(t) = k(t)\beta x_1(t)x_2(t)$$

Here $\beta$ is an "effective contact" rate, $a(t)$ the normalized arrival rate of infectives, $r(t)$ the normalized removal rate of infectives, and $\tau$ is a delay time.

Many more examples of processes modeled by delay, or more generally, hereditary systems can be found in [8] and [9].

## 1.3 Outline of Thesis

From the above discussion, it is clear that delay systems arise in many practical applications. This provides ample justification for studying the theory of delay systems in depth. There is a large literature on the existence and uniqueness of solutions to delay differential equations without control and their qualitative properties [2], [14]. However, from the system theory point of view, a large number of system-theoretic concepts remain to be explored and examined. In this thesis, we study two areas of interest: structural properties of delay systems and their applications, and the estimation of stochastic delay systems. In the first area, we are interested in seeing what are the roles played by the structural properties in control and estimation problems, and how properties peculiar to linear delay systems can be exploited to solve control problems not handled by finite dimensional linear systems. In the second area, we are motivated by the desire to develop an estimation theory suitable for multipath communication problems such as that described in Example 2 of this chapter. Of course, these by no means exhaust the theoretical implications or the practical applications of the ideas and results developed in our work. An outline of the thesis is given below.

In Chapter 2, we study the concepts of controllability, stabilizability, observability, and detectability in connection with delay systems. These are properties of great interest in any study of dynamical systems. While they merit an in-depth study in their own right, our interest in them stems mainly from their relevance to the stability properties of the linear optimal control system with quadratic cost and the linear optimal filter. The various notions of controllability and observability are therefore examined and compared from the standpoint of these two problems. The results in this chapter will be used in a crucial way in Chapter 6.

While Chapter 2 deals with standard system-theoretic concepts, in Chapter 3 we study a property which is peculiar to delay systems: the notion of pointwise degeneracy. This property has been studied previously by various authors [15] - [20] as an interesting aspect in the theory of delay systems. Our viewpoint, on the other hand, is to apply it to the control of linear systems. This motivates the development of a useful characterization of systems which can be made pointwise degenerate by delay feedback. These results serve as the foundation for the construction of delay feedback controllers.

In Chapter 4, we discuss the potential applications of the pointwise degeneracy property in delay feedback control for the minimum time output zeroing problem for linear systems. The results of Chapter 3 are then applied to obtain conditions under which the control problem can be solved. The sensitivity of such a control system under perturbations of its parameters is also studied.

Starting with Chapter 5, we turn our attention to the filtering problem for stochastic delay systems. We first give a general discussion and show how previous results are inadequate for solving the filtering problem for delay systems. We next give a representation theorem for conditional moment functionals of nonlinear stochastic delay systems. Under suitable conditions, stochastic differential equations for the conditional moment functionals can be derived from the representation theorem. We then specialize these results to linear systems and give a complete solution to the filtering problem in this case. The similarities as well as differences between the optimal linear filter for delay systems and that for ordinary differential systems are discussed.

In Chapter 6, we study the stability properties of the linear optimal delay system with quadratic cost and the linear optimal filter. When appropriate system-theoretic properties of controllability, stabilizability, observability, and detectability hold, asymptotic stability of these optimal systems can be established. The striking duality between control and estimation is also demonstrated. This gives what we believe is the first instance other than the ordinary linear differential systems case where the control and the filtering problems can both be satisfactorily solved.

Finally, we summarize our findings in Chapter 7 and suggest some directions for future research. We shall see that many more theoretical problems associated with delay systems remain to be solved. It is hoped that the research reported in this thesis will serve as a useful step in establishing a full-fledged theory for these systems.

# CHAPTER 2

# STRUCTURAL PROPERTIES OF LINEAR TIME-INVARIANT DELAY SYSTEMS

## 2.1 Introduction

In this chapter, we study four of the basic structural properties of linear time-invariant delay systems: controllability, stabilizability, observability, and detectability. In contrast to finite dimensional linear time-invariant systems, where simple necessary and sufficient for these properties to hold are known [21], [22], the situation in delay systems is considerably more complicated. For example, there is more than one meaningful notion of controllability or observability, and the one that is more appropriate depends on the application we have in mind. The property that has been studied most extensively is that of pointwise controllability [23] - [28] (also known as relative controllability or Euclidean space controllability). Necessary and sufficient conditions have been given for this property to hold, and various algebraic criteria have been devised [23] - [28]. Stabilizability and functional controllability have also been studied by some authors [23], [25], [29] - [34], but the results are far from complete. On the other hand, observability and detectability have hardly been touched upon in the literature [29]. Our motivation in studying these properties is to see how they can be utilized to solve the stability problems for the linear optimal control system and linear optimal filter (see Chapter 6). The appropriate notions of controllability and observability for these problems turn out to be different from previously given definitions, and we shall explore the relations of these various notions. We make no claim to completeness or depth of our investigations here. In fact, a great deal of work is needed before these structural properties of delay systems can be clarified.

## 2.2 Controllability

We shall be concerned with the controllability properties of the system

$$\dot{x}(t) = Ax(t) + Bx(t-\tau) + Cu(t)$$
$$x(\theta) = \phi(\theta) \qquad \theta\varepsilon[-\tau,0] \tag{2.1}$$

where A, B, C are nxn, nxn, and nxm matrices respectively. We shall denote the solution to (2.1) by $x(t,\phi,u)$. We shall also write $x_t(\phi,u)$ for the complete state of the system (2.1) due to the control u, and it is defined by

$$x_t(\phi,u)(\theta) = x(t+\theta,\phi,u) \qquad \theta\varepsilon[-\tau,0].$$

As indicated in Chapter 1, the state space of a delay system is a function space. Several choices for the state space are possible. The most common one is the space of continuous functions $C([-\tau,0];R^n)$ equipped with the sup norm which we will abbreviate by $\mathscr{C}$[2]. Other choices are $M^2$ (which is isomorphic to $R^n x L^2$) [29] and $W_2^{(1)}$, the Sobolev space of absolutely continuous function on $[-\tau,0]$ with derivatives in $L_2[-\tau,0]$ [33]. Each has its own advantages and disadvantages. For a discussion of this, see [35]. In this thesis, we will usually take $\mathscr{C}$ as our state space. Occasionally, we will discuss other spaces when $\mathscr{C}$ proves to be inappropriate for the problem at hand or when other choices shed light on the problem under consideration.

Since the state space is a function space, there are more than one notion of controllability that makes sense. The first systematic study of controllability for delay systems is that by Kirillova and Churakova [23] and is concerned with pointwise controllability. For simplicity, we will take our set of admissible controls U to be all $R^m$-valued piecewise continuous functions on some interval [0,T], which we shall denote by $PC([0,T]; R^m)$. The same basic arguments can be used for $U = L_2([0,T]; R^m)$, the space of $R^m$-valued square integrable functions on [0,T].

<u>Definition 2.1</u> The delay system (2.1) is called pointwise controllable if for any $\phi$ and any $x_1 \in R^n$, there exists a time T and an admissible control u on [0,T] such that $x(T,\phi,u) = x_1$. We also speak of pointwise controllability on [0,T] or at time T if the terminal time T is fixed.

By the variation of constants formula [2], [36], the solution to (2.1) can be written as

$$x(t,\phi,u) = \Phi(t,0)\phi(0) + \int_{-\tau}^{0} \Phi(t,s+\tau)B\phi(s)ds + \int_{0}^{t} \Phi(t,s)Cu(s)ds \tag{2.2}$$

where $\Phi(t,s)$ is the fundamental matrix associated with (2.1) and satisfies

$$\frac{d}{dt}\Phi(t,s) = A\Phi(t,s) + B\Phi(t-\tau,s)$$

$$\Phi(s,s) = I$$

$$\Phi(t,s) = 0 \qquad t < s \tag{2.3}$$

We will also write (2.2) as

$$x(t,\phi,u) = x(t,\phi,0) + F_t u$$

where for any fixed $t \geq 0$, $F_t : U \to R^n$

$$F_t u = \int_0^t \Phi(t,s)Cu(s)ds$$

A straightforward argument shows that the following is true [37], [26].

Lemma 2.1 System (2.1) is pointwise controllable if and only if there exists a time T such that $\mathscr{R}(F_T) = R^n$ where $\mathscr{R}(X)$ of an operator X denotes the range. An equivalent condition is

$$\text{rank} \int_0^T \Phi(T,s)CC'\Phi'(T,s)ds = n \tag{2.4}$$

In contrast to finite dimensional linear differential systems, the fundamental matrix $\Phi(t,s)$ defined by (2.3) can be singular [17], [35]. Furthermore, pointwise controllability to any point in $R^n$ is not equivalent to pointwise controllability to the null vector in $R^n$. To distinguish these cases, we make the following

Definition 2.2 The delay system (2.1) is called pointwise null controllable if for any $\phi$, there exists a time T and an admissible control u on [0,T] such that $x(T,\phi,u) = 0$.

It is clear that pointwise controllability implies pointwise null controllability. The converse, however, is not true. To discuss necessary and sufficient conditions for pointwise null controllability, we need the notions of pointwise completeness and pointwise degeneracy.

Definition 2.3 System (2.1) is said to be pointwise complete if for each t, there exists a set of initial functions $\phi_i^t \in \mathscr{C}$, $i = 1,\dots,n$ such that the vectors $x(t,\phi_i^t,0)$ $i = 1,\dots n$ form a basis for $R^n$. If the system is not pointwise complete, i.e., if there exists a proper subspace V of $R^n$ such that at some time t, $x(t,\phi,0) \subseteq V$ for all $\phi \in \mathscr{C}$, the system is said to be pointwise degenerate (at time t).

The relevance of these notions is shown by (see [26])

Lemma 2.2 If the system (2.1) is pointwise complete, then the condition

$$\text{rank} \int_0^T \Phi(T,s)CC'\Phi(T,s)ds = n$$

is necessary and sufficient for pointwise null controllability at time T.

The point is that if the system is pointwise degenerate, we do not need $\mathscr{R}(F_T) = R^n$ for pointwise null controllability at T. We only need $\mathscr{R}(F_T) \supset \{x(T,\phi,0): \phi \in \mathscr{C}\}$, which is a proper subspace of $R^n$. The notion of pointwise degeneracy has other useful applications, and we shall be studying it in much more depth in Chapters 3 and 4.

The condition (2.4) can be replaced by a number of algebraic criteria [23], [26], [28]. These can be thought of as generalizations of the well-known controllability results for finite dimensional linear systems. While pointwise controllability is useful in linear control problems with target sets in Euclidean space [35], [38], it is too weak for other problems. For control problems with target sets in function space, we need a much stronger notion of controllability. Here, following Banks et al., [33], we adopt the space $W_2^{(1)}$ as our state space and $L_2$ as our set of admissible controls U. The reason for switching the state space to $W_2^{(1)}$ is that for $u \in L_2$ and $\phi \in W_2^{(1)}$, eq. (2.1) implies that $\dot{x}(t) \in L_2$. Hence $W_2^{(1)}$ is a natural choice for the state space when the control space is $L_2$. For further discussions on this point, see [33].

<u>Definition 2.4</u> The system (2.1) is said to be functionally controllable if for any functions $\phi$ and $\psi \in W_2^{(1)}$, there exists a time T and a control $u \in L_2[0,T]$ such that $x_T(\phi,u) = \psi$. If the terminal time T is fixed, we say that the system is functionally controllable on [0,T].

The following result of Banks, et al., [33] shows that functional controllability on [0,T] where $T > \tau$ imposes extremely stringent conditions on the system which are rarely met in practice.

<u>Lemma 2.3</u> The system (2.1) is functionally controllable on [0,T] where $T > \tau$ if and only if rank C = n.

Functional controllability is too strong for many control problems of interest, for example, the stability properties of linear optimal feedback systems and linear optimal filters. We now develop a new notion of controllability which is applicable to these problems. We revert to $\mathscr{C}$ and piecewise continuous functions as our state space and control space.

Let us first define the function

$$W_T: PC([0,T];\ R^n) \to C([T-\tau,T];\ R^n)$$

by

$$(W_T v)(t) = \int_0^T \Phi(t,s)v(s)ds \qquad t \in [T-\tau,T]$$

Similarly, define

$$W_T C:\ U(= PC([0,T];\ R^m)) \to C([T-\tau,T];\ R^n)$$

by

$$(W_T Cu)(t) = \int_0^T \Phi(t,s)Cu(s)ds \qquad t\in[T-\tau,T]$$

Let us denote the space of all $R^n$-valued functions $\psi$ on $[t,t+\tau]$, which are of bounded variation on $[t,t+\tau]$ and continuous from the left on $(t,t+\tau)$, by $B_o([t,t+\tau];\ R^n)$. We equip $B_o([t,t+\tau];\ R^n)$ with the total variation norm on $[t,t+\tau]$, denoted by $\mathrm{Var}_{[t,t+\tau]}$. For $\psi \in B_o([t,t+\tau];\ R^n)$, $\phi \in C([t-\tau,t];\ R^n)$, define the bilinear form $\langle\psi,\phi\rangle_t$ by (see [2])

$$\langle\psi,\phi\rangle_t = \psi'(t)\phi(t) + \int_{t-\tau}^t \psi'(s+\tau)B\phi(s)ds \tag{2.5}$$

We shall refer to $\langle\psi,\phi\rangle_t$ as the hereditary product at time t. We adopt the following

Definition 2.5 The system (2.1) is said to be controllable on $[0,T]$ if there exists no $\psi \in B_0([T,T+\tau]; R^n)$ such that $\langle\psi,\phi\rangle_T = 0$ for all $\phi \in \overline{\mathcal{R}(W_T C)}$, but that $\langle\psi,\phi_1\rangle_T \neq 0$, for some $\phi_1 \in \overline{\mathcal{R}(W_T)}$.

This definition of controllability may not be very well-motivated at the moment since $PC([0,T]; R^n)$ is not the space of controls. It is obtained strictly as the dual notion of observability, which we will discuss in section 2.4. The physical meaning of $\mathcal{R}(W_T C)$ is clear and corresponds to functions which can be attained at T by suitable control. On the other hand, it is not clear to what $\mathcal{R}(W_T)$ corresponds. However, this definition of controllability is easily seen to be stronger than pointwise controllability and weaker than functional controllability. For if we take t = T, and take $\psi$ to be such that $\psi(T) \neq 0$, $\psi(s) = 0$, $T < s \leq T + \tau$, we get that

$$\langle\psi,\phi\rangle_T = \psi'(T)\phi(T)$$

Controllability then says that there exists no $\psi(T)$ such that $\psi'(T)\int_0^T \Phi(T,s)v(s)ds \neq 0$, for some $v \in PC([0,T]; R^n)$, but that

$$\psi'(T)\int_0^T \Phi(T,s)Cu(s)ds = 0,$$

for all $u \in PC([0,T]; R^m)$. Since $\Phi(T,s) = e^{A(T-s)}$ for $s \in [T-\tau,T]$, for arbitrary $x_1 \in R^n$, we can define

$$\begin{aligned} v(t) &= e^{A'(T-t)}\left[\int_{T-\tau}^{T} e^{A(T-t)}e^{A'(T-t)}dt\right]^{-1}x_1 & t\in[T-\tau,T] \\ &= 0 & t\in[0,T-\tau) \end{aligned}$$

This yields $\int_0^T \Phi(T,s)v(s)ds = x_1$ and shows that $\{\int_0^T \Phi(T,s)v(s)ds:$ $v \in PC([0,T];\ R^n)\} = R^n$. If $\{\int_0^T \Phi(T,s)Cu(s)ds\colon u \in PC([0,T];\ R^m)\}$ is a proper subspace of $R^n$, then we can find $\psi(T)$ such that

$$\psi'(T)\int_0^T \Phi(T,s)Cu(s)ds = 0,$$

for all $u \in PC([0,T];\ R^m)$. Choosing v such that $\int_0^T \Phi(T,s)v(s)ds = \psi(T)$, we see that system (2.1) will not be controllable. Hence controllability implies $\mathscr{R}(F_T) = R^n$ which is precisely pointwise controllability. On the other hand, if rank C = n, then $\overline{\mathscr{R}(W_T)}$ is clearly the same as $\overline{\mathscr{R}(W_TC)}$, from which controllability follows. Hence functional controllability implies controllability.

Finally, there is a fourth notion of controllability, that of approximate functional controllability.

Definition 2.6 System (2.1) is called approximately functionally controllable if for any $\phi$ and $\psi \in \mathscr{C}$, there exists a time T and a sequence $\{u_n\}$ in U such that $x_T(\phi,u_n)$ converges to $\psi$.

Again approximate functional controllability is weaker than functional controllability, but stronger than pointwise controllability. Its relation to our definition of controllability is not known at present. Its use in delay system problems seems to be rather limited. However, it is one of the standard notions of controllability in infinite dimensional linear systems [39], and we have stated it for completeness.

## 2.3 Stabilizability

We start with the definitions of stability and asymptotic stability of unforced delay systems.

Consider the system

$$\dot{x}(t) = Ax(t) + Bx(t-\tau)$$
$$x(\theta) = \phi(\theta) \qquad \theta \in [-\tau,0] \tag{2.6}$$

where $\phi \in \mathscr{C}$.

We shall denote the solution to (2.6) by $x(t,\phi)$ and the complete state of (2.6) by $x_t(\phi)$.

Definition 2.7 System (2.6) is said to be stable if for every $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that $|\phi| < \delta$ implies the solution of (2.6) satisfies $|x_t| < \varepsilon$, for all $t \geq 0$.

Definition 2.8 System (2.6) is said to be asymptotically stable if it is stable, and there is a $\delta_o$ such that $|\phi| < \delta_o$ implies the solution of (2.6) satisfies $\lim_{t\to\infty} |x_t(\phi)| = 0$.

It is known that asymptotic stability of linear delay systems is equivalent to exponential stability [2], [36]. In other words, (2.6) is asymptotically stable if and only if there are positive constants K and $\alpha$ such that $|x_t(\phi)| \leq Ke^{-\alpha t}|\phi|$, $\quad t \geq 0$, for all $\phi \in \mathscr{C}$.

We are now ready to give the definition of stabilizability.

Definition 2.9 System (2.1) is said to be stabilizable if there exists a matrix function $L: [-\tau,0] \to R^{n\times n}$, of bounded variation on $[-\tau,0]$, such that the closed-loop system

$$\dot{x}(t) = Ax(t) + Bx(t-\tau) + C \int_{-\tau}^{0} dL(\theta)x(t+\theta) \tag{2.7}$$

is asymptotically stable. We also say (A,B,C) is stabilizable to mean this definition.

As in the control of finite dimensional linear systems, stabilizability is crucial for the well-posedness of infinite-time control problems for delay systems (see Chapter 6). Necessary and sufficient conditions for stabilizability can be given in terms of controllability of a finite dimensional system obtained by projecting the complete state $x_t$ onto the generalized eigenspaces associated with the eigenvalues with nonnegative real parts [30], [31]. This result is difficult to use, however, because it is necessary to compute the eigenvalues and find a basis for the generalized eigenspaces associated with them. One would like to have conditions expressed explicitly in terms of the system parameters A, B and C. Since the decomposition of the complete state $x_t$ has been given in detail by Hale [2], this appears to be a hopeful, but still open task.

In finite dimensional systems, controllability is a sufficient condition for stabilizability. Since the controllability condition is very easy to check, this is an extremely useful sufficient condition. It is important to see, therefore, if some notion of controllability implies stabilizability in infinite dimensional linear systems. This is an aspect of the theory of infinite dimensional linear systems that has received quite a bit of attention recently [40], [41]. We have seen in the above that for delay systems, there is indeed a notion of controllability that is necessary and sufficient for stabilizability. This notion, however, does not correspond to any of those discussed in section 2.2. It is worthwhile to give a few remarks concerning the relationships between those notions and that of stabilizability.

It is clear that functional controllability is sufficient for stabilizability, since functional controllability enables us to move $x_t$ to the zero function in finite time. Approximate functional controllability is also sufficient since this certainly implies that finite dimensional projections are controllable. On the other hand, pointwise controllability is not sufficient for stabilizability. An example illustrating this has been given by Morse [42]. The relationship between controllability in the sense of Definition 2.5 and stabilizability is not known.

## 2.4 Observability

Here we are concerned with the system

$$\dot{x}(t) = Ax(t) + Bx(t-\tau)$$

$$x(\theta) = \phi(\theta) \qquad \theta \, \varepsilon[-\tau,0] \tag{2.8}$$

$$z(t) = Cx(t) \qquad t \geq 0 \tag{2.9}$$

We would like to know when it is possible to deduce the trajectory of the system $x(t)$, $0 \leq t \leq T$, from the observations $z(t)$, $0 \leq t \leq T$.

In contrast to controllability, pointwise observability has very little meaning. This is because knowledge of $x(\sigma)$ is in general not sufficient in determining the evolution of the system for $t \geq \sigma$. We shall define two notions of observability.

Definition 2.10 The system (2.8) - (2.9) is said to be observable if for any $\phi$, $z(t) = 0$, $t \geq 0$, implies $x(t) = 0$, $t \geq 0$. It is observable on $[0,T]$ if $z(t) = 0$, $0 \leq t \leq T$, implies $x(t) = 0$, $0 \leq t \leq T$.

Since (2.8) - (2.9) are time-invariant, an equivalent definition is: $z(t) = 0$, $t \geq s$, implies $x(t) = 0$, $t \geq s$, any $s \geq 0$. It is useful to rephrase Definition 2.10 in terms of the initial function $\phi$.

Definition 2.10' The system (2.8) - (2.9) is said to be observable if there exists no function $\phi \in \mathscr{C}$ such that $z(t) = 0$, $t \geq 0$, and such that $x(t)$ is not identically zero for $t \geq 0$.

Let us express Definition 2.10' in terms of certain operators. By the variation of constants formula

$$z(t) = C\Phi(t,0)\phi(0) + \int_{-\tau}^{0} C\Phi(t,s+\tau)B\phi(s)ds \tag{2.10}$$

Define the operator $M_T : \mathscr{C} \to PC([0,T];\ R^n)$ by

$$(M_T\phi)(t) = \Phi(t,0)\phi(0) + \int_{-\tau}^{0} \Phi(t,s+\tau)B\phi(s)ds \tag{2.11}$$

for any $t \in [0,T]$. Similarly, define $CM_T : \mathscr{C} \to PC([0,T];\ R^m)$ by

$$(CM_T\phi)(t) = C\Phi(t,0)\phi(0) + \int_{-\tau}^{0} C\Phi(t,s+\tau)B\phi(s)ds \tag{2.12}$$

for any $t \in [0,T]$. Then the system (2.8) - (2.9) is observable on $[0,T]$ if and only if

$$N(M_T) = N(CM_T) \tag{2.13}$$

where $N(X)$ of an operator $X$ denotes the nullspace of $X$. We will now prove the following duality result which relates the notions of controllability and observability.

Theorem 2.1 The delay system

$$\dot{x}(t) = Ax(t) + Bx(t-\tau) + Cu(t)$$

$$x(\theta) = \phi(\theta) \qquad \theta \in [-\tau,0] \tag{2.14}$$

is controllable on $[0,T]$ if and only if its hereditary adjoint system [2], [36]

$$\dot{y}(t) = -A'y(t) - B'y(t+\tau)$$

$$y(\theta) = \psi(\theta) \qquad \theta \in [T,T+\tau],\ \psi \in B_o([T,T+\tau];\ R^n) \tag{2.15}$$

$$z(t) = C'y(t) \tag{2.16}$$

is observable on $[0,T]$.

Remark We shall usually simply say (2.15) - (2.16) is the adjoint or dual system to (2.14). Of course, the adjoint system (2.15) evolves backwards in time starting at $t = T$.

Proof: Applying the variation of constants formula to the adjoint system [2], [36], we obtain, for $t \in [0,T]$

$$y(t) = \Phi'(T,t)\psi(T) + \int_T^{T+\tau} \Phi'(s-\tau,t)B'\psi(s)ds$$

$$\triangleq (H_T^*\psi)(t) \tag{2.17}$$

Hence observability of the adjoint system is equivalent to

$$N(C'H_T^*) = N(H_T^*)$$

Relative to the hereditary product, the adjoint of $H_T^*$ is given by

$$H_T: PC([0,T];\ R^n) \to C([T-\tau,T];\ R^n)$$

$$\langle H_T^*\psi,\ v\rangle_{pc} = \langle\psi,\ H_Tv\rangle_T, \text{ where } \langle u,v\rangle_{pc} = \int_0^T u'(t)v(t)dt.$$

We calculate

$$\langle H_T^*\psi,\ v\rangle_{pc} = \int_0^T \psi'(T)\Phi(T,t)v(t)dt + \int_0^T \int_T^{T+\tau} \psi'(s)B\Phi(s-\tau,t)v(t)dsdt$$

$$= \psi'(T) \int_0^T \Phi(T,t)v(t)dt + \int_T^{T+\tau} \psi'(s)B \int_0^T \Phi(s-\tau,t)v(t)dtds$$

$$= \langle\psi,\ H_Tv\rangle_T$$

Hence $(H_Tv)(t) = \int_0^T \Phi(t,s)v(s)ds \qquad t\varepsilon[T-\tau,T]$

The reader may note here that $H_T = W_T$, with $W_T$ defined in section 2.2.

Similarly, the adjoint of $C'H_T^*$ is given by

$$H_TC: PC([0,T];\ R^m) \to C([T-\tau,T];\ R^n)$$

$$(H_TCv)(t) = \int_0^T \Phi(t,s)Cv(s)ds \qquad t\varepsilon[T-\tau,T]$$

(Note that $H_TC = W_TC$). Now $\psi \varepsilon N(H_T^*)$ if and only if

$$\langle H_T^*\psi,\ v\rangle_{pc} = 0 \quad \text{for all } v \varepsilon PC([0,T];\ R^n),$$

i.e., $\psi$ is in the annihilator of $\overline{\mathscr{R}(H_T)}$ relative to the hereditary product. Hence the condition $N(H_T^*) = N(C'H_T^*)$ says that there exists no $\psi$ such that

$$\langle\psi,\phi\rangle_T = 0 \quad \forall\ \phi \in \overline{\mathscr{R}(H_T C)}$$

but $\quad \langle\psi, \phi_1\rangle_T \neq 0$ for some $\phi_1 \in \overline{\mathscr{R}(H_T)}$

which is precisely the condition required for controllability of the original system.

This duality result will be used in a crucial way in Chapter 6. Since it is based on the use of the hereditary product, it is not the same as the functional analytic duality between the various operators and spaces involved. It would be interesting to see what duality result would be obtained if we use functional analytic adjoints for delay systems, as expounded in Hale [2].

Another definition of observability which is stronger than Definition 2.10 is that of strong observability.

Definition 2.11 The system (2.8) - (2.9) is said to be strongly observable if $z(t) = 0$, $t \geq 0$ implies the initial function $\phi \equiv 0$.

Clearly, if the system is strongly observable, it is observable. The converse is not true. Indeed, the matrix B must necessarily be nonsingular if strong observability is to hold. Otherwise we can choose an initial function $\phi \in \mathscr{C}$ such that $\phi(0) = 0$ and $0 \neq \phi(s) \in N(B)$, $s \in [-\tau,0)$. This yields $z(t) = 0$, $t \geq 0$, but $\phi \not\equiv 0$.

By exactly the same arguments as in Theorem 2.1, we can show that the dual of strong observability is the following notion of controllability: there exists no $\psi \in B_o([T,T+\tau]; R^n)$ such that $\langle\psi, \overline{\mathscr{R}(H_T C)}\rangle_T = 0$. Again B must necessarily be nonsingular. This may be taken as still another notion of controllability if desired.

## 2.5 Detectability

As in finite dimensional linear systems, this is basically the dual notion of stabilizability.

Definition 2.12 The system (2.8) - (2.9) is said to be detectable if there exists a matrix function $K: [-\tau,0] \to R^{nxm}$, of bounded variation on $[-\tau,0]$, such that

$$\dot{x}(t) = Ax(t) + Bx(t-\tau) + \int_{-\tau}^{0} dK(\theta)z(t+\theta) \tag{2.18}$$

is asymptotically stable.

Theorem 2.2 The delay system (2.8) - (2.9) is detectable if and only if the adjoint system

$$\dot{y}(t) = 'A'y(t) - B'y(t+\tau) - C'u(t) \tag{2.19}$$

is stabilizable (again this system runs backward in time).

Proof: Stabilizability of (2.19) requires the existence of a matrix function $K': [-\tau,0] \to R^{mxn}$, of bounded variation on $[-\tau,0]$, such that

$$-\dot{y}(t) = A'y(t) + B'y(t+\tau) + C' \int_{-\tau}^{0} dK'(\theta)y(t-\theta) \tag{2.20}$$

is asymptotically stable. The right hand side of (2.20) defines a Stieltjes integral of the form

$$\int_{-\tau}^{0} d\eta'(\theta)y(t-\theta)$$

$$\text{where } \eta'(\theta) = \begin{cases} C'K'(\theta) & \theta \geq 0 \\ -A'+C'K'(\theta) & -\tau < \theta < 0 \\ -A'-B'-C'K'(\theta) & \theta \leq -\tau \end{cases}$$

The hereditary adjoint system to (2.20) is given by [2]

$$\dot{x}(t) = \int_{-\tau}^{0} d\eta(\theta)x(t+\theta) \tag{2.21}$$

Substituting for $\eta(\theta)$, we get that (2.21) is

$$\dot{x}(t) = Ax(t) + Bu(t-\tau) + \int_{-\tau}^{0} dK(\theta)Cx(t+\theta)$$

$$= Ax(t) + Bx(t-\tau) + \int_{-\tau}^{0} dK(\theta)z(t+\theta) \tag{2.22}$$

Since the stability properties of the adjoint system are the same as those for the original system [2], stability of (2.20) is equivalent to stability of (2.22). Hence the theorem follows.

The remarks following the proof of Theorem 2.1 concerning the use of adjoint systems also applies to this case.

Analogous to the situation for stabilizability, it would be interesting to see what notion of observability implies detectability. Presumably this would be the dual of that notion of controllability which implies stabilizability.

From the above discussion, it is clear that a great deal of work remains to be done in studying the concepts of controllability, stabilizability, observability, and detectability. We have merely formulated definitions which appear to be useful in control and estimation problems, and pointed out some of the relationships between these concepts. Our discussion has not even exhausted the definitions of controllability and observability which have appeared in the literature. Motivated by the algebraic approach to finite dimensional linear systems, Kamen and Morse [42], [43] have given some algebraic definitions of controllability and observability which are useful in realization theory for delay systems. It is not clear what the physical interpretations of these definitions are, or what their relationships to our definitions are. We shall leave this entire area as a subject for future research.

CHAPTER 3

# POINTWISE DEGENERACY OF LINEAR DELAY SYSTEMS

## 3.1 Introduction

In Chapter 2, we studied some structural properties of linear delay systems and introduced the concept of pointwise degeneracy. In this chapter, we shall study the degeneracy property in much more depth, with a view towards applying it to feedback control. The degeneracy property of delay systems has recently attracted the attention of many researchers [15] - [20], partly because of its connections with other system-theoretic concepts (see Chapter 2), and partly because it is peculiar to delay equations and has no counterpart in ordinary differential equations. Popov [15] is the first to make a systematic study of the subject. His fundamental results have laid the foundations from which our results are developed. Since the publication of his paper, many other results have been obtained on pointwise degeneracy. It is fair to say, however, that this peculiar property is still far from being completely understood. Our viewpoint differs somewhat from these previous investigations in that we are primarily interested in applying the property in the construction of delay feedback controls. We do not therefore study pointwise degeneracy as an intrinsic property of delay systems. Rather, we study conditions under which a linear system can be made pointwise degenerate by delay feedback. Since we rely heavily on the techniques developed by Popov, in

Section 3.2, we shall first review his results and others which are germaine to our subsequent development. In Section 3.3, we give a characterization of systems which can be made pointwise degenerate by delay feedback. We shall then explore some of the implications of this characterization. This will prepare the way for the discussions in Chapter 4 where these results are used to obtain delay feedback controls for certain problems in linear system theory.

## 3.2 Some Existing Results on the Pointwise Degeneracy of Linear Delay Systems

We shall be concerned primarily with linear, constant delay systems of the form

$$\dot{x}(t) = Ax(t) + Bx(t-\tau) \tag{3.1}$$

with an initial function $\varepsilon\ \mathscr{C}$. We first give an alternative definition of pointwise degeneracy of Eq. (3.1).

Definition 3.1 The linear delay system (3.1) is called pointwise degenerate if there exist an n-vector $q \neq 0$ and a number $t_1 > 0$ such that every continuous function $x\colon [-\tau, t_1] \to R^n$ satisfying (3.1) in the open interval $(0, t_1)$ satisfies also $q'x(t_1) = 0$. If the system is not pointwise degenerate, it is called pointwise complete.

Notation. We will abbreviate pointwise degenerate by p.d. and pointwise complete by p.c. We will also say $(A, B, q, \tau)$ is p.d. at $t_1$ or $(A, B, \tau)$ is p.c. to mean the above definition. If the time of pointwise degeneracy is of no concern, we will drop the qualification "at $t_1$" from our statements.

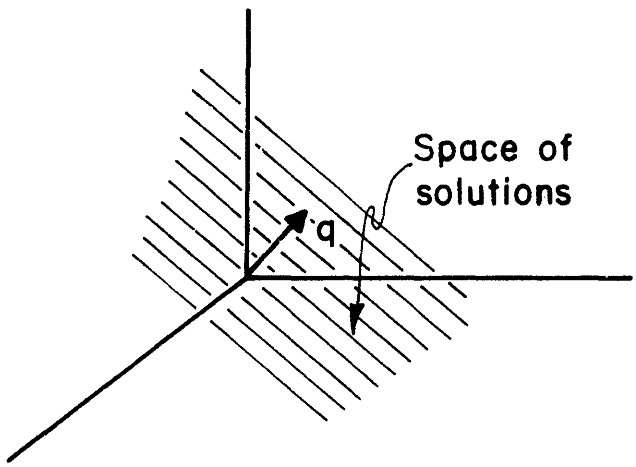
Definition 3.1 is readily seen to be equivalent to Definition 2.3 of pointwise degeneracy. However, Definition 3.1 has proved to be much more useful in obtaining results. We will adopt this definition of pointwise degeneracy from now on.

Let us note the following important consequence of Definition 3.1:

If $(A, B, q, \tau)$ is p.d. at $t_1$, then it will be p.d. on an interval of the form $[t_1, \infty)$, which we will call the degeneracy interval. To see this, take any number $t_2 > t_1$. For any continuous function $\tilde{x}: [-\tau, t_2] \to R^n$ satisfying (3.1) on $(0, t_2)$, define the shifted function $\tilde{x}: [-\tau, t_1] \to R^n$ by $\tilde{x}(t) = x(t + t_2 - t_1)$. Then $\tilde{x}$ is continuous and satisfies (3.1) in $(0, t_1)$. Hence $q'\tilde{x}(t_1) = q'x(t_2) = 0$ and x is p.d. at $t_2$ also.

This simple fact is of central importance in the application of the p.d. property in delay feedback control, as will be explained later. In fact, a stronger result holds for the structure of the degeneracy interval. This will be given in Theorem 3.1.

Definition 3.1 admits a simple geometric interpretation. For each initial function $\phi$, there corresponds a solution of (3.1), which we will denote by $x(t, \phi)$. Then pointwise degeneracy means that there is a fixed vector q such that the solutions $x(t, \phi)$ evolve in a space perpendicular to q, for any choice of $\phi \in \mathscr{C}$.




The first example of a p.d. system was given by Popov [15], which has

$$A = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

$$q' = [1 \quad -2 \quad -1]$$

and $\tau = 1$ with a degeneracy interval of $[2, \infty)$. This example sparked a surge of interest in the p.d. property. By now, much more is known about this example. We shall return to it in Chapter 4.

Most of the basic results concerning pointwise degeneracy have been given by Popov. The following theorems are fundamental.

<u>Theorem 3.1</u> (Popov [15]). Assume that (3.1) is p.d. for the vector q. Then the largest set of points $t_1$ at which (3.1) is p.d. for q is the interval $[\ell\tau, \infty)$, where $\ell$ is the smallest integer with the property

$$q'S(\sigma)^{\ell+1} = 0 \tag{3.2}$$

and $S(\sigma)$ is the nxn polynomial matrix given by

$$(\sigma I-A)S(\sigma) = B\det(\sigma I-A)$$

Moreover, $2 \leq \ell \leq n-1$

<u>Theorem 3.2</u> (Popov [15]). The delay system (3.1) is p.d. for q at time $t_1 > 0$ if and only if there exist an integer $m > 0$, k matrices $P_j$, mxn (k is the largest integer such that $k\tau \leq t_1$), an mxm matrix V, and an m-vector v such that

$$P_1B = 0 \tag{3.3}$$

$$P_jA + P_{j+1}B = VP_j \qquad j = 1,2\ldots k-1 \tag{3.4}$$

$$P_kA = VP_k \tag{3.5}$$

$$v'e^{V\tau}P_1 = 0 \tag{3.6}$$

$$v'e^{V\tau}P_{j+1}-v'P_j = 0 \qquad j = 1,2\ldots,k-1 \tag{3.7}$$

$$v'P_k = q' \tag{3.8}$$

Moreover, if the above quantities exist, one can always choose them so that rank $(P_1 \ldots P_k) = m$.

From these theorems, one can deduce the following corollaries, some of which were obtained by other authors.

Corollary 3.1 (Popov [15]). System (3.1) is p.c. if rank B = 1.

Corollary 3.2. System (3.1) is p.c. if rank B = n.

Proof: If rank B = n, (3.3) implies $P_1 = 0$. Using (3.4) repeatedly, we immediately see that $P_2, \ldots, P_k$ are all equal to zero, and the corollary follows.

This result was apparently established earlier by E.B. Lee.

Corollary 3.3. System (3.1) is p.c. if $n \leq 2$.

Proof: If n = 1, Corollary 3.1 immediately shows that (3.1) is p.c. Assume therefore n = 2. In this case, rank B is either 1 or 2. But then Corollary 3.1 and 3.2 together imply (3.1) is p.c.

This result was apparently established earlier by J.A. Yorke and J. Kato.

Although in the proof of Theorem 3.2, Popov gave a constructive procedure for finding the matrices $P_1, \ldots, P_k$, V, and the vector v, it is so complicated that in applications, Theorem 3.2 will be difficult to use. However, Popov was able to isolate a class of p.d. systems which have a remarkably simple structure. These are the so-called regular pointwise degenerate (r.p.d.) systems.

Definition 3.2. A p.d. system of the form (3.1) is called "regular" if the pair (A, B) is completely controllable and there exists an n-vector q such that the pair (q', A) is completely observable, and (3.1) is p.d. for q in the interval $[2\tau, \infty)$.

For such systems, the following theorem is true.

Theorem 3.3 (Popov [15]). Suppose (3.1) is r.p.d. with respect to q. Then there exists an nxn matrix Z such that

$$ZAZ = Z^2A \tag{3.9}$$

$$q'Z^2 = 0 \tag{3.10}$$

$$q'Z = q'e^{A\tau} \tag{3.11}$$

Equation (3.1) then takes the form

$$\dot{x}(t) = Ax(t) + (AZ-ZA)x(t-\tau) \tag{3.12}$$

(that is, AZ-ZA = B). Conversely, every equation of the form (3.12), in which Z satisfies (3.9) - (3.11), is p.d. for q in $[2\tau, \infty)$ (even though the other conditions for regularity may not be satisfied).

The following corollary can be deduced from Theorem 3.3.

Corollary 3.4. (Popov [15]). If B can be written as AZ-ZA where Z is given by

$$Z = rq'e^{A\tau} \tag{3.13}$$

and r is the solution to the equations

$$q'r = 1 \tag{3.14}$$

$$q'e^{A\tau}r = 0 \tag{3.15}$$

$$q'e^{A\tau}Ar = 0 \tag{3.16}$$

then $(A,B,q,\tau)$ is p.d. on $[2\tau, \infty)$.

Finally, Popov [15] has proved the following important result for 3 dimensional p.d. systems.

Theorem 3.4. Any 3 dimensional p.d. system is regular and can be written in the form (3.12) with Z given by (3.13) - (3.16).

For later reference, we emphasize here that our results on pointwise degeneracy and delay feedback are heavily based on Corollary 3.4 and Theorem 3.4. It is also worth pointing out that while (3.13) - (3.16) are considerably simpler than (3.3) - (3.8), they still involve the calculation of matrix exponentials. Given a triple $(A,q,\tau)$ it is not obvious from (3.14) to (3.16) when a solution for r will exist. Our results given in section 3.3 will be concerned precisely with simple conditions on $(A,q,\tau)$ under which a solution r to (3.14) - (3.16) exists.

Some additional properties of r.p.d. systems can be deduced from (3.9) - (3.12). For example, the Z matrix is in fact unique for regular p.d. systems. To see this, suppose $Z_1$ and $Z_2$ both satisfy (3.9) - (3.12). Then

$$q'(Z_1 - Z_2) = 0$$

Also

$$q'AZ_1 = q'(Z_1A + B) = q'Z_1A + q'B$$
$$= q'e^{A\tau}A + q'B$$

this implies

$$q'A(Z_1 - Z_2) = 0$$

Similarly, it can be shown that

$$q'A^i(Z_1 - Z_2) = 0 \qquad i = 0,\ldots,n-1$$

By observability of $(q', A)$, $Z_1 = Z_2$ and uniqueness follows. It can also be shown that $Z^{n-1} = 0$ and $B^n = 0$ for r.p.d. systems. The significance of these properties are not known at the present time and we shall not pursue them further here.

We have summarized in the above some known results on pointwise degeneracy. We will apply these in our investigations in the next section.

## 3.3 A Criterion for Pointwise Degeneracy for Linear Delay Systems

Let us motivate our investigations by the following control problem (see also Popov [44]). Consider the linear control system

$$\dot{x}(t) = Ax(t) + u(t) \tag{3.17}$$

$$y(t) = q'x(t) \tag{3.18}$$

The objective is to find a linear state feedback law, possibly delayed with a fixed delay time $\tau > 0$, to drive the output $y(t)$ to zero in minimum time for all initial conditions, remaining zero thereafter. We shall call this minimum time output zeroing by delay feedback.

It is obvious that this problem cannot be solved by using instantaneous state feedback

$$u(t) = Kx(t)$$

However, if we can find a matrix B such that $(A,B,q,\tau)$ is p.d. at $2\tau$, then by definition of the p.d. property, $y(t) = 0$ for $t \geq 2\tau$. Furthermore, since $2\tau$ is the smallest instant at which a delay system can be p.d. (see Theorem 3.1), this matrix B will be the solution to our minimum time control problem.

The first question to be examined concerning this approach is to find conditions on the matrix A, the vector q, and the delay time $\tau$, such that a matrix B will exist with the desired properties. Corollary 3.4 tells us that if we can find a solution r to (3.14) - (3.16), then on constructing the Z matrix as in (3.13) and setting $B = AZ - ZA$, we will obtain a p.d. system with degeneracy interval $[2\tau, \infty)$. Furthermore, Theorem 3.4 shows that for 3 dimensional systems, the above construction gives the unique B. Therefore, for 3 dimensional systems an equivalent problem is to find conditions on A, q, and $\tau$ such that (3.14) - (3.16) admit a solution. We will exploit this in the proof of the following theorem.

<u>Theorem 3.5</u>. Given a 3x3 matrix A with eigenvalues $\lambda_1$, $\lambda_2$, $\lambda_3$, a 3-vector q, and a number $\tau > 0$, a matrix B with the property that

$(A,B,q,\tau)$ is p.d. exists if and only if the following conditions are satisfied:

(i) The geometric multiplicity [45] of each distinct eigenvalue of A is one;

(ii) If there is a pair of complex eigenvalues, say $\lambda_2 = \lambda_3^*$, $\lambda_1$ real, then

$$\xi + e^{\eta\tau}(\eta\sin\xi\tau - \xi\cos\xi\tau) \neq 0$$

where $\eta = \lambda_1 - \mathrm{Re}\lambda_2$, $\xi = -\mathrm{Im}\lambda_2$;

(iii) $(q',A)$ is observable.

Before proving Theorem 3.5, we first state a lemma which is completely obvious. This lemma is not restricted to 3 dimensions.

Lemma 3.1. Suppose $(A,B,q,\tau)$ is p.d. then for any nonsingular matrix F, $(FAF^{-1}, FBF^{-1}, F'^{-1}q, \tau)$ is also p.d.

Lemma 3.1 implies that if we are free to choose B such that $(A,B,q,\tau)$ is p.d., there is no loss of generality in assuming A to be in Jordan form. We exploit this in

Proof of Theorem 3.5: Let $q' = [q_1\ q_2\ q_3]$. We will apply the p.d. criterion established in Theorem 3.4. By the above remarks, we may assume, without loss of generality, that A is in Jordan form. Since n=3, the possible Jordan forms are as follows:

(I)

$$J = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$\lambda_i$'s may not be distinct, and there can only be a pair of complex eigenvalues.

(II)

$$J = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}$$

$$\text{or} \quad \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{bmatrix}$$

$\lambda_1$, $\lambda_2$ may not be distinct, but they are necessarily real.

(III)

$$J = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$$

$\lambda$ is necessarily real.

We first note that if there exists a solution r to (3.14) - (3.16), the vectors $q'$, $q'e^{A\tau}$, and $q'e^{A\tau}A$ are necessarily linearly independent (cf. Popov [15], p. 559). Theorem 3.4 then requires that

$$\det \begin{bmatrix} q' \\ q'e^{J\tau} \\ q'e^{J\tau}J \end{bmatrix} \triangleq \det W \neq 0$$

We consider each of these cases separately.

Case I. We require

$$\det W = \det \begin{bmatrix} q_1 & q_2 & q_3 \\ q_1 e^{\lambda_1\tau} & q_2 e^{\lambda_2\tau} & q_3 e^{\lambda_3\tau} \\ q_1\lambda_1 e^{\lambda_1\tau} & q_2\lambda_2 e^{\lambda_2\tau} & q_3\lambda_3 e^{\lambda_3\tau} \end{bmatrix}$$

$$= q_1q_2q_3[(\lambda_3-\lambda_2)e^{(\lambda_2+\lambda_3)\tau}+(\lambda_1-\lambda_3)e^{(\lambda_1+\lambda_3)\tau}+(\lambda_2-\lambda_1)e^{(\lambda_1+\lambda_2)\tau}] \neq 0$$

Clearly if any $q_i = 0$, then det W = 0 and (3.1) cannot possibly be degenerate for this q. We must also not have

$$(\lambda_3-\lambda_2)e^{(\lambda_2+\lambda_3)\tau}+(\lambda_1-\lambda_3)e^{(\lambda_1+\lambda_3)\tau}+(\lambda_2-\lambda_1)e^{(\lambda_1+\lambda_2)\tau} = 0 \qquad (3.20)$$

Dividing throughout by $e^{(\lambda_2+\lambda_3)\tau}$, (3.20) becomes

$$(\lambda_3-\lambda_2) + (\lambda_1-\lambda_3)e^{(\lambda_1-\lambda_2)\tau} + (\lambda_2-\lambda_1)e^{(\lambda_1-\lambda_3)\tau} = 0 \qquad (3.21)$$

Let $\gamma_1 = \lambda_1-\lambda_2$, $\gamma_2 = \lambda_1-\lambda_3$, and let

$$f(\gamma_1,\gamma_2) = (\gamma_1-\gamma_2) + \gamma_2 e^{\gamma_1\tau} - \gamma_1 e^{\gamma_2\tau} \qquad (3.22)$$

Our problem now is reduced to finding conditions under which $f(\gamma_1,\gamma_2) \neq 0$. We first consider the case where all the $\lambda_i$'s are real (so that $\gamma_1$ and $\gamma_2$ are also real). Without loss of generality, we may assume $\gamma_1 \geq 0$ (i.e., $\lambda_1 \geq \lambda_3$). If $\gamma_1 = 0$, we have immediately $f(\gamma_1,\gamma_2) = 0$. Suppose $\gamma_1 > 0$ is fixed. We plot $\gamma_2(e^{\gamma_1\tau}-1)$ and $\gamma_1(e^{\gamma_2\tau}-1)$ as functions of $\gamma_2$. $f(\gamma_1,\gamma_2) = 0$ at the points where these two functions are equal. Note that at $\gamma_2 = 0$, both functions are zero, and at $\gamma_2 = \gamma_1$, they are equal.

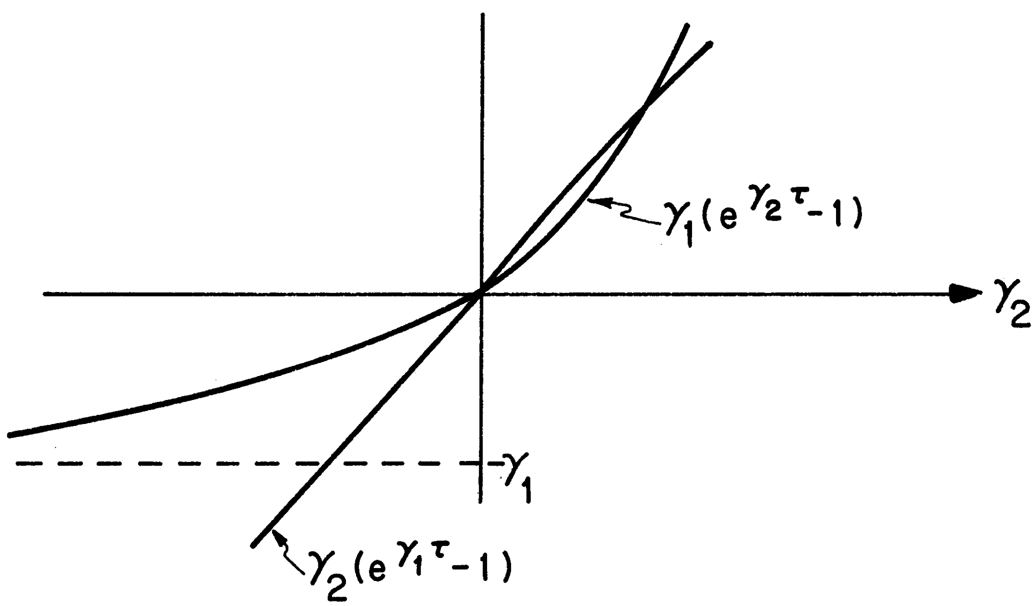
$$\frac{d}{d\gamma_2}[\gamma_2(e^{\gamma_1\tau}-1)] = e^{\gamma_1\tau}-1$$

and $$\frac{d}{d\gamma_2}[\gamma_1(e^{\gamma_2\tau}-1)] = \gamma_1\tau e^{\gamma_2\tau}$$

In the region where $\gamma_2 \leq 0$, $\gamma_1\tau e^{\gamma_2\tau} \leq \gamma_1\tau < e^{\gamma_1\tau}-1$, for $\gamma_1 > 0$. In this region, therefore

$$\gamma_2(e^{\gamma_1\tau}-1) \leq \gamma_1(e^{\gamma_2\tau}-1)$$

We can now see that the graphs of the two functions look like



The functions can only intersect at two points. But we already know that they do intersect at $\gamma_2 = 0$ and $\gamma_2 = \gamma_1$. Hence these are the only two points at which they can be equal. So in the region $\gamma_1 > 0$, $f(\gamma_1,\gamma_2) = 0$ if and only if $\gamma_2 = \gamma_1$ or $\gamma_2 = 0$. We can now conclude that if all the $\lambda_i$'s are real, $f(\gamma_1,\gamma_2) = 0$ if and only if $\gamma_1 = 0$, or $\gamma_2 = 0$, or $\gamma_1 = \gamma_2$. In other words, det W = 0 if and only if either $\lambda_1 = \lambda_2$, or $\lambda_2 = \lambda_3$, or $\lambda_3 = \lambda_1$, or any of the $q_i$'s = 0.

Next we consider the case when there is a pair of complex eigenvalues, say, $\lambda_2 = \lambda_3^*$. This implies $\lambda_1$, $\lambda_2$, and $\lambda_3$ are necessarily distinct. With $\gamma_1$ and $\gamma_2$ defined as before, we have that $\gamma_1 = \gamma_2^*$. Define $\eta = \lambda_1 - \mathrm{Re}\lambda_2$, $\xi = -\mathrm{Im}\lambda_2$ (i.e., $\gamma_1 = \eta + i\xi$). Then

$$
\begin{aligned}
f(\gamma_1,\gamma_2) &= 2i\xi + (\eta-i\xi)e^{(\eta+i\xi)\tau} - (\eta+i\xi)e^{(\eta-i\xi)\tau} \\
&= 2i\xi + e^{\eta\tau}\{(\eta-i\xi)(\cos\xi\tau + i\sin\xi\tau) - (\eta+i\xi)(\cos\xi\tau - i\sin\xi\tau)\} \\
&= 2i[\xi+e^{\eta\tau}(\eta\sin\xi\tau - \xi\cos\xi\tau)] \qquad (3.23)
\end{aligned}
$$

$f(\gamma_1,\gamma_2) = 0$ if and only if $\xi+e^{\eta\tau}(\eta\sin\xi\tau - \xi\cos\xi\tau) = 0$. Thus, if there is a pair of complex eigenvalues, we require precisely condition (ii) of Theorem 3.5, in addition to no $q_i = 0$. This completes the consideration of Case I.

Case II. Since the same arguments hold for either

$$
J = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{bmatrix}
$$

we only discuss the former situation. In this case,

$$e^{J\tau} = \begin{bmatrix} e^{\lambda_1\tau} & \tau e^{\lambda_1\tau} & 0 \\ 0 & e^{\lambda_1\tau} & 0 \\ 0 & 0 & e^{\lambda_2\tau} \end{bmatrix}$$

Thus

$$\det W = \det \begin{bmatrix} q_1 & q_2 & q_3 \\ q_1 e^{\lambda_1\tau} & (q_1\tau+q_2)e^{\lambda_2\tau} & q_3 e^{\lambda_2\tau} \\ q_1\lambda_1 e^{\lambda_1\tau} & (q_1+q_1\lambda_1\tau+q_2\lambda_1)e^{\lambda_1\tau} & q_3\lambda_2 e^{\lambda_2\tau} \end{bmatrix}$$

$$= q_1^2 q_3 e^{\lambda_1\tau}[(\lambda_2\tau-1-\lambda_1\tau)e^{\lambda_2\tau}+e^{\lambda_1\tau}]$$

after some calculations. Hence

$$\det W \neq 0 \text{ if and only if } q_1 \neq 0,\ q_3 \neq 0$$

and $(\lambda_2\tau-1-\lambda_1\tau)e^{\lambda_2\tau}+e^{\lambda_1\tau} \neq 0$.

Let $\gamma = \lambda_1 - \lambda_2$. Then

$$(\lambda_2\tau-1-\lambda_1\tau)e^{\lambda_2\tau}+e^{\lambda_1\tau} = 0$$

if and only if

$$\gamma\tau + 1 = e^{\gamma\tau} \tag{3.24}$$

The only real solution to (3.24) is $\gamma = 0$. Hence det $W \neq 0$ if and only if $q_1 \neq 0$, $q_3 \neq 0$, and $\lambda_1 \neq \lambda_2$. The same arguments show that if

$$J = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{bmatrix}$$

det $W \neq 0$ if and only if $q_1 \neq 0$, $q_2 \neq 0$ and $\lambda_1 \neq \lambda_2$. This completes the consideration of Case II.

Case III. In this case

$$e^J = \begin{bmatrix} e^{\lambda\tau} & \tau e^{\lambda\tau} & \frac{1}{2}\tau^2 e^{\lambda\tau} \\ 0 & e^{\lambda\tau} & \tau e^{\lambda\tau} \\ 0 & 0 & e^{\lambda\tau} \end{bmatrix}$$

$$\det W = \det \begin{bmatrix} q_1 & q_2 & q_3 \\ q_1 e^{\lambda\tau} & (q_1\tau + q_2)e^{\lambda\tau} & (q_1 \frac{\tau^2}{2} + q_2\tau + q_3)e^{\lambda\tau} \\ q_1\lambda e^{\lambda\tau} & (q_1 + \lambda q_1\tau + \lambda q_2)e^{\lambda\tau} & (q_1\tau + q_2 + \lambda q_1 \frac{\tau^2}{2} + \lambda q_2\tau + q_3\lambda)e^{\lambda\tau} \end{bmatrix}$$

$$= q_1^3 \frac{\tau^2}{2} e^{2\lambda\tau}$$

Thus det $W \neq 0$ if and only if $q_1 \neq 0$.

If we now combine all three cases, it is clear that the eigenvalues $\lambda_i$ are required to satisfy conditions (i) and (ii) of Theorem 3.5. Furthermore, if a pair (q', A) is observable, then $(q'T^{-1}, TAT^{-1})$ is also observable for any nonsingular T. Take T such that

$$TAT^{-1} = J$$

It is known that for observability of $(q'T^{-1}, TAT^{-1})$, the components of $q'T^{-1}$ must satisfy precisely the same conditions imposed on the $q_i$'s in the above discussion [46]. Hence observability of (q', A) is also required for the solution of (3.14) - (3.16). Since conditions (i), (ii) and (iii) of Theorem 3.5 are necessary and sufficient for the solvability of (3.14) - (3.16), they are also necessary and sufficient for the existence of the matrix B. The proof is finished.

As an easy corollary of Theorem 3.5, we have

Corollary 3.5. Suppose $n > 3$. Given an nxn matrix A, an n-vector q, and a number $\tau > 0$, if at least three of the eigenvalues satisfy conditions (i) and (ii) of Theorem 3.5, and if (q', A) is observable, then there exists a B such that $(A,B,q,\tau)$ is p.d. on $[2\tau, \infty)$.

Proof: Without loss of generality, we may assume that A is in Jordan form and in fact the first three eigenvalues satisfy conditions (i) and (ii) of Theorem 3.5. By observability of (q', A), the first three components of q satisfy the conditions imposed in the proof of Theorem 3.5. If we now consider a solution r to (3.14) - (3.16) in the form of

$$r = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad (3.25)$$

a little thought shows that the equations to be satisfied for the solvability of $r_1$, $r_2$, and $r_3$ are precisely the same as those in Theorem 3.5. By that theorem, we can solve for $r_1$, $r_2$, and $r_3$ with the assumptions in this corollary. Thus a solution to (3.14) - (3.16) exists in the form (3.25) and the corollary follows.

We make several remarks to clarify the contents of Theorem 3.5 and Corollary 3.5.

Remark 3.1. In contrast to Theorem 3.5, Corollary 3.5 gives only a set of sufficient conditions for the existence of the desired B matrix. The reason for this is that for $n > 3$, solvability of (3.14) - (3.16) constitutes only a set of sufficient conditions, generally not necessary, for the existence of the B matrix. To obtain necessary and sufficient conditions in the style of Theorem 3.5 for $n > 3$ seems to be rather difficult and remains an open problem. Nevertheless, it is worthwhile to make a few comments on how Corollary 3.5 can be extended. For example, if A is 4x4 and has two distinct pairs of complex eigenvalues, $\lambda_1 = \lambda_2^*$, $\lambda_3 = \lambda_4^*$, then condition (ii) of Theorem 3.5 cannot be satisfied since there are no real eigenvalues. However, in this case, the modified condition

$$\xi' + e^{\eta'\tau}(\eta'\sin\xi'\tau - \xi'\cos\xi'\tau) \neq 0$$

where $\xi' = \text{Re}\lambda_1 - \text{Re}\lambda_3$, $\eta' = \text{Im}\lambda_1 - \text{Im}\lambda_3$, can be used instead. We can also relax the requirement that (q', A) be observable. On the other hand, Corollary 3.5 is easy to state and apply, and is sufficient for our purposes later in Chapter 4. We shall therefore be contented with giving Corollary 3.5 as it stands.

Remark 3.2. Theorem 3.5 and Corollary 3.5 can be given the following interpretation:

Given A, q, and $\tau$, and $n \geq 3$, there "almost always" exists a B such that (A,B,q,$\tau$) is p.d. on $[2\tau, \infty)$. This is because observability of (q', A) is a generic property of a pair (q', A) [47] (conditions (i) and (ii) are also generic properties of square matrices). Thus, while pointwise degeneracy of the delay system (3.1) is a singular property, the existence of a B for which $(A,B,q,\tau)$ is p.d. on $[2\tau, \infty)$ is a generic property of A, q, and $\tau$.

Remark 3.3. The singular cases where no B exists are basically related to the eigenvalues of A. For example, if in the 3x3 case, A is symmetric and has two equal eigenvalues, then no B exists regardless of what q and $\tau$ are. Furthermore, if B is obtained as in Corollary 3.4, it will depend on the eigenvalues of A. This remark will be very useful when we apply Theorem 3.5 and Corollary 3.5 to construct delay feedback controllers in the next chapter.

Remark 3.4. If the matrix B is obtained in the manner described in Corollary 3.4, the eigenvalues associated with the delay equation (3.1), i.e., those values of $\lambda$ such that

$$\det(\lambda I-A-e^{-\lambda\tau}B) = 0 \tag{3.26}$$

are precisely the same as the eigenvalues of A. To see this, first note that (3.14) - (3.16) imply $Z^2 = ZAZ = 0$. We obtain successively

$$\begin{aligned}\det(\lambda I-A-e^{-\lambda\tau}B) &= \det[\lambda I-A-e^{-\lambda\tau}(AZ-ZA)] \\ &= \det[\lambda I-A-e^{-\lambda\tau}AZ+e^{-\lambda\tau}ZA+e^{-2\lambda\tau}ZAZ) \\ &= \det[\lambda I+(e^{-\lambda\tau}Z-I)A(e^{-\lambda\tau}Z+I)] \\ &= \det(I-e^{-\lambda\tau}Z)(\lambda I-A)(I+e^{-\lambda\tau}Z) \\ &= \det(\lambda I-A)\end{aligned}$$

By a result of Henry [48], the spectrum of the infinitesimal generator of the semigroup for the delay equation is finite, and the range of the semigroup is finite dimensional. Kappel [20] has proved a number of interesting results for this class of p.d. systems.

Remark 3.5. The peculiar condition (ii) in Theorem 3.5 relates the eigenvalues of A to the delay time $\tau$. Indeed, if $\eta = 0$, $\xi = \frac{2\pi}{\tau}$, $\xi + e^{\eta\tau}(\eta\sin\xi\tau - \xi\cos\xi\tau) = 0$. This suggests that for degeneracy to occur, the period of the natural frequency of A should not "match up" with the delay time. A more precise interpretation of (ii) is not known.

Theorem 3.5 and Corollary 3.5 give relatively simple and transparent conditions for the existence of the B matrix. Only simple calculations are necessary to check the conditions. More important, they give us a hold on when singular situations will arise. As such, they form the basis for the development of delay feedback controllers investigated in the next chapter.

CHAPTER 4

# DELAY FEEDBACK CONTROL OF LINEAR SYSTEMS

## 4.1 Introduction

In the previous chapter, we developed a theorem which allows us, under certain conditions, to assert the existence of a matrix B such that $(A,B,q,\tau)$ is pointwise degenerate. This was motivated by the control problem described at the beginning of section 3.3. In this chapter, we would like to generalize the formulation of the control problem to include an input matrix C. Specifically, we would like to solve the problem of minimum time output zeroing by delay feedback for the linear system

$$\dot{x}(t) = Ax(t) + Cu(t) \tag{4.1}$$

$$y(t) = q'x(t) \tag{4.2}$$

Let us consider the 3 dimensional case. If we just use delay feedback of the form

$$u(t) = Lx(t-\tau)$$

Theorem 3.5 will impose certain conditions on A, q and $\tau$ for the existence of a suitable matrix L. Furthermore, if we construct a matrix B for which $(A,B,q,\tau)$ is p.d., we must also have the range of B included in the range of C in order for L to exist. These are rather severe restrictions. On the other hand, since the eigenvalues of A play such an important role (see remark 3.3), we may be able to solve the problem of minimum time output zeroing if we can modify the eigenvalues of A. This suggests using a feedback law of the form

$$u(t) = Kx(t) + Lx(t-\tau) \tag{4.3}$$

We shall see that this choice of the feedback law enables us to solve the minimum time output zeroing problem for (4.1) and (4.2) under rather mild conditions. We first study the 3 dimensional case, as the required notation is relatively simple, and the techniques used can be readily extended to higher dimensions. In section 4.2, we establish the form of the matrix B for which $(A,B,q,\tau)$ is p.d. in terms of the parameters A, q and $\tau$. We then prove a theorem in section 4.3 which gives necessary and sufficient conditions for the solvability of the minimum time output zeroing problem in 3 dimensions. In fact, the proof of the theorem gives a constructive procedure for finding the desired matrices K and L. For greater clarity and completeness, a summary of the algorithm is given after the proof of the theorem. Next, the techniques are adapted to obtain a solution to the control problem in higher dimensions. Again, we summarize the algorithm for this case at the end of section 4.4. In section 4.5, we study the properties of the feedback control system under perturbation of its parameters. Finally, we give some examples which illustrate the theory.

## 4.2 Construction of the Matrix B in 3 Dimensional Pointwise Degenerate Systems

In the development of delay feedback control using the p.d. property, it is very useful to know the explicit form of B. We shall carry out the necessary calculations in this section. Using these results, we show that in Popov's example (see section 3.2), degeneracy in fact occurs for a unique $\tau$ and a unique q (up to scalar multiples).

Notation. Since all our p.d. systems will be constructed using Corollary 3.4, they will have a degeneracy interval of $[2\tau,\infty)$. From now on, the term p.d. will mean p.d. on the interval $[2\tau,\infty)$.

As in section 3.3, we consider three cases:

(I) A has three real distinct eigenvalues

(II) A has two repeated eigenvalues

and has a Jordan form

$$J = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}$$

$\lambda_1$ and $\lambda_2$ are distinct and necessarily real.

(III) A has three repeated real eigenvalues and has a Jordan form

$$J = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$$

Throughout this section, we assume (q',A) to be observable. By Theorem 3.5, we know that there exists a B such that $(A,B,q,\tau)$ is p.d.

Case I: Let T be the matrix which diagonalizes A, i.e.,

$$T^{-1}AT = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \triangleq \Lambda$$

with $\lambda_i$'s real and distinct. Equations (3.14) to (3.16) can be rewritten as

$$q'TT^{-1}r = 1 \tag{4.5}$$

$$q'Te^{\Lambda\tau}T^{-1}r = 0 \tag{4.6}$$

$$q'Te^{\Lambda\tau}\Lambda T^{-1}r = 0 \tag{4.7}$$

Letting $\alpha' = q'T$, $\eta = T^{-1}r$, we obtain

$$\alpha'\eta = 1 \tag{4.8}$$

$$\alpha' e^{\Lambda\tau}\eta = 0 \tag{4.9}$$

$$\alpha' e^{\Lambda\tau}\Lambda\eta = 0 \tag{4.10}$$

Letting $\beta_i = \alpha_i\eta_i$, $i = 1,2,3$, we can write (4.8) - (4.10) as

$$\beta_1 + \beta_2 + \beta_3 = 1 \tag{4.11}$$

$$\beta_1 e^{\lambda_1\tau} + \beta_2 e^{\lambda_2\tau} + \beta_3 e^{\lambda_3\tau} = 0 \tag{4.12}$$

$$\beta_1\lambda_1 e^{\lambda_1\tau} + \beta_2\lambda_2 e^{\lambda_2\tau} + \beta_3\lambda_3 e^{\lambda_3\tau} = 0 \tag{4.13}$$

Solving for $\beta_1$, $\beta_2$, and $\beta_3$, we obtain

$$\beta_1 = \frac{(\lambda_2-\lambda_3)e^{\lambda_2\tau}}{D} \tag{4.14}$$

$$\beta_2 = \frac{(\lambda_3-\lambda_1)e^{\lambda_1\tau}}{D}$$

$$\beta_3 = \frac{(\lambda_1-\lambda_2)e^{(\lambda_1+\lambda_2-\lambda_3)\tau}}{D} \tag{4.16}$$

where $D = (\lambda_3-\lambda_1)(e^{\lambda_1\tau} - e^{\lambda_2\tau}) + (\lambda_1-\lambda_2)e^{(\lambda_2-\lambda_3)\tau}(e^{\lambda_1\tau} - e^{\lambda_3\tau})$ (4.17)

Note that D is always nonzero since the $\lambda_i$'s are real and distinct (see section 3.3).

Next, we construct a matrix $\bar{Z}$.

$$\bar{Z} = \eta\alpha' e^{\Lambda\tau}$$

$$= \frac{1}{D}\begin{bmatrix} (\lambda_2-\lambda_3)e^{(\lambda_1+\lambda_2)\tau} & \frac{\alpha_1}{\alpha_2}(\lambda_2-\lambda_3)e^{2\lambda_2\tau} & \frac{\alpha_3}{\alpha_1}(\lambda_2-\lambda_3)e^{(\lambda_2+\lambda_3)\tau} \\ (\lambda_3-\lambda_1)\frac{\alpha_1}{\alpha_2}e^{2\lambda_1\tau} & (\lambda_3-\lambda_1)e^{(\lambda_1+\lambda_2)\tau} & (\lambda_3-\lambda_1)\frac{\alpha_1}{\alpha_2}e^{(\lambda_1+\lambda_3)\tau} \\ (\lambda_1-\lambda_2)\frac{\alpha_1}{\alpha_2}e^{(2\lambda_1+\lambda_2-\lambda_3)\tau} & (\lambda_1-\lambda_2)\frac{\alpha_2}{\alpha_3}e^{(\lambda_1+2\lambda_2-\lambda_3)\tau} & (\lambda_1-\lambda_2)e^{(\lambda_1+\lambda_2)\tau} \end{bmatrix}$$

And $\bar{B} = \Lambda\bar{Z} - \bar{Z}\Lambda$

$$= \frac{1}{D}\begin{bmatrix} 0 & \frac{\alpha_2}{\alpha_1}(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)e^{2\lambda_2\tau} & \frac{\alpha_3}{\alpha_1}(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)e^{(\lambda_2+\lambda_3)\tau} \\ \frac{\alpha_1}{\alpha_2}(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)e^{2\lambda_1\tau} & 0 & \frac{\alpha_3}{\alpha_2}(\lambda_2-\lambda_3)(\lambda_3-\lambda_1)e^{(\lambda_1+\lambda_3)\tau} \\ \frac{\alpha_1}{\alpha_3}(\lambda_3-\lambda_1)(\lambda_1-\lambda_2)e^{(2\lambda_1+\lambda_2-\lambda_3)\tau} & \frac{\alpha_2}{\alpha_3}(\lambda_3-\lambda_2)(\lambda_1-\lambda_2)e^{(\lambda_1+2\lambda_2-\lambda_3)\tau} & 0 \end{bmatrix} \tag{4.19}$$

Now $\bar{Z}$ is related to the original system parameters by

$$\bar{Z} = \eta\alpha' e^{\Lambda\tau} = T^{-1}rq'TT^{-1}e^{A\tau}T$$

$$= T^{-1}ZT$$

and

$$\bar{B} = \Lambda\bar{Z} - \bar{Z}\Lambda$$

$$= T^{-1}ATT^{-1}ZT - T^{-1}ZTT^{-1}AT$$

$$= T^{-1}BT$$

Thus the desired matrix B is given by

$$B = T\bar{B}T^{-1} \tag{4.20}$$

We note that the range space of $\bar{B}$ is spanned by the vectors

$$\begin{bmatrix} 0 \\ 1 \\ -\frac{\alpha_2}{\alpha_3} e^{(\lambda_2-\lambda_3)\tau} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 0 \\ -\frac{\alpha_1}{\alpha_3} e^{(\lambda_1-\lambda_3)\tau} \end{bmatrix} \tag{4.21}$$

<u>Case II:</u> Let T be the matrix which brings A to its Jordan form, i.e.,

$$T^{-1}AT = J = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}$$

Completely analogous arguments show that the desired matrix B is given by

$$B = T \begin{bmatrix} \eta_2\alpha_1 e^{\lambda_1\tau} & [\eta_2(\alpha_1\tau+\alpha_2)-\eta_1\alpha_1]e^{\lambda_1\tau} & [(\lambda_1-\lambda_2)\eta_1\alpha_3+\eta_2\alpha_3]e^{\lambda_2\tau} \\ 0 & -\eta_2\alpha_1 e^{\lambda_1\tau} & (\lambda_1-\lambda_2)\eta_2\alpha_3 e^{\lambda_2\tau} \\ (\lambda_2-\lambda_1)\eta_3\alpha_1 e^{\lambda_1\tau} & [(\lambda_2-\lambda_1)\eta_3(\alpha_1\tau+\alpha_2)-\eta_3\alpha_1]e^{\lambda_1\tau} & 0 \end{bmatrix} T^{-1}$$

(4.22)

where $\alpha' = q'T$ and the $\eta_i$'s satisfy

$$\eta_1 = \frac{[(\lambda_2-\lambda_1)\tau + \frac{\alpha_2}{\alpha_1}(\lambda_2-\lambda_1)-1]e^{\lambda_2\tau}}{\alpha_1 D} \tag{4.23}$$

$$\eta_2 = \frac{-(\lambda_2-\lambda_1)e^{\lambda_2\tau}}{\alpha_1 D} \tag{4.24}$$

$$\eta_3 = \frac{e^{\lambda_1\tau}}{\alpha_3 D} \tag{4.25}$$

$$\text{with } D = (\lambda_2-\lambda_1)\tau e^{\lambda_2\tau} + e^{\lambda_1\tau} - e^{\lambda_2\tau} \tag{4.26}$$

The range space of $\overline{B}$ is spanned by the vectors.

$$\begin{bmatrix} \eta_2 \\ 0 \\ (\lambda_2-\lambda_1)\eta_3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} \tag{4.27}$$

<u>Case III</u>: Let T be the transformation matrix such that

$$T^{-1}AT = J = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$B = T \begin{bmatrix} \eta_2\alpha_1 e^{\lambda\tau} & [\eta_2(\alpha_1\tau+\alpha_2)-\eta_1\alpha_1]e^{\lambda\tau} & [\eta_2(\alpha_1 \frac{\tau^2}{2} +\alpha_2\tau+\alpha_3)-\eta_1(\alpha_1\tau+\alpha_2)]e^{\lambda\tau} \\ \eta_3\alpha_1 e^{\lambda\tau} & [\eta_3(\alpha_1\tau+\alpha_2)-\eta_2\alpha_1]e^{\lambda\tau} & [\eta_3(\alpha_1 \frac{\tau^2}{2} +\alpha_2\tau+\alpha_3)-\eta_2(\alpha_1\tau+\alpha_2)]e^{\lambda\tau} \\ 0 & -\eta_3\alpha_1 e^{\lambda\tau} & -\eta_3(\alpha_1\tau+\alpha_2)e^{\lambda\tau} \end{bmatrix} T^{-1} \tag{4.28}$$

where $\alpha' = q'T$ and the $\eta_i$'s satisfy

$$\eta_1 = \frac{\alpha_1^2\tau^2 + 2\alpha_1\alpha_2\tau + 2\alpha_2^2 - 2\alpha_1\alpha_3}{\alpha_1^3\tau^2} \tag{4.29}$$

$$\eta_2 = \frac{-2(\alpha_1\tau + \alpha_2)}{\alpha_1^2\tau^2} \tag{4.30}$$

$$\eta_3 = \frac{2}{\alpha_1\tau^2} \tag{4.31}$$

The range space of $\bar{B}$ is spanned by the vectors

$$\begin{bmatrix} \eta_2 \\ \eta_3 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} \tag{4.32}$$

It is important to note that the transformation matrix T will in general depend on A. Thus the expressions for B given above do not show B as an explicit function of A. However, the form that B must assume will be useful in our subsequent analysis.

As an application of the above construction, let us re-examine Popov's example. Using the uniqueness of B, straightforward calculations show that for degeneracy to occur, $\tau$ must be 1, and the q vector must be such that $q_2 = -2q_1$, $q_3 = -q_1$. We can therefore conclude that Popov's example will be p.d. for a unique vector q, up to scalar multiples, and a unique $\tau$.

## 4.3 Delay Feedback Control of 3 Dimensional Linear Systems

In this section, we apply the results developed in sections 3.3 and 4.2 to obtain the solution of the minimum time output zeroing problem using the feedback law (4.3) for the system defined by (4.1) and (4.2). We shall assume throughout this section that the system (4.1) is 3 dimensional. As in every theorem related to pointwise degeneracy, the result is simple and elegant in three dimensions.

Theorem 4.1 For matrices K and L to exist such that the closed-loop system

$$\dot{x}(t) = (A + CK)x(t) + CLx(t-\tau) \tag{4.33}$$

satisfies $q'x(t) = 0$, $t \geq 2\tau$, for all initial conditions, it is necessary and sufficient that rank $C \geq 2$, and (A,C) be controllable.

It is helpful to discuss the conditions of Theorem 4.1 before the proof. This will help to motivate the construction given below. Rank $C \geq 2$ is clearly necessary since otherwise rank $CL \leq 1$ and by Corollary 3.1, (4.33) is always p.c. Controllability of (A,C) is introduced so that we can choose K to shift the eigenvalues of A arbitrarily (of course, it is necessary for another reason; see the necessity proof of Theorem 4.1 below). This enables us to avoid the singular situations described in Theorem 3.5. For example, if C is 3x3 and of full rank, then obviously we can find a matrix K such that the conditions of Theorem 3.5 are satisfied. Furthermore, if we construct a matrix B such that $(A+CK,B,q,\tau)$ is p.d., then since C is nonsingular, there always exists a matrix L such that B = CL. Of more interest is the case where rank C = 2. We can still choose K such that (A+CK) has eigenvalues which satisfy conditions (i) and (ii) of Theorem 3.5. However, it is not clear a priori that there will always exist L such that B = CL, where B is such that $(A+CK,B,q,\tau)$ is p.d. Indeed, most of the work involved in proving Theorem 4.1 is to show we can choose the eigenvalues of A+CK in such a way that the matrix B, constructed as in section 4.2, always lie in the range space of C.

We shall need the following lemmas whose proofs are given in Appendix A.

Lemma 4.1 Let $\alpha_1$, $\alpha_2$, $\alpha_3$ be real numbers with the property $\alpha_2^2 - 4\alpha_1\alpha_3 > 0$. Then the function $f(\lambda) \triangleq (\alpha_1 + \alpha_2\lambda + \alpha_3\lambda^2)e^{\lambda\tau}$ for $\lambda$ real has one maximum and one minimum. Furthermore there exist straight lines of the form $g(\lambda) = \gamma_\beta(\lambda-\beta)$ and $g(\lambda) = \omega$ each of which intersect $f(\lambda)$ at three real distinct points, none of which is a zero of $f(\lambda)$. Here $\beta$ is a finite real number not equal to any of the zeros of $f(\lambda)$, $\gamma_\beta$ is some real number dependent on $\beta$, and $\omega$ is some real constant.

Lemma 4.2 Suppose we are given a 3-vector q, a 3x3 matrix A, and a 3x2 matrix C of full rank, with (A,C) controllable. Suppose $q'c_i \neq 0$ where $c_i$ is the $i^{th}$ column of C. Then there exist matrices K and P such that $P^{-1}(A+CK)P$ is in companion form, $q'P = (\alpha_1 \;\; \alpha_2 \;\; \alpha_3)$ is such that $\alpha_2^2 - 4\alpha_1\alpha_3 > 0$, and that $P^{-1}C$ has as its $i^{th}$ column

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Furthermore, if the other column can be written as

$$\begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$$

with $\beta_1 \neq 0$, then P can be chosen such that $\frac{\beta_2}{\beta_1}$ is not a zero of $\eta(\lambda) \triangleq \alpha_1+\alpha_2\lambda + \alpha_3\lambda^2$.

Proof of Theorem 4.1 (Necessity) That rank $C \geq 2$ is necessary has already been discussed above. If (A,C) is not controllable, (A+CK,C) is also not controllable for any K since constant state feedback has no effect on controllability. Noting that

$$(CL\ ACL\ldots,\ A^{n-1}CL) = (C\ AC\ldots,\ A^{n-1}C)L$$

we see that (A+CK,CL) is also not controllable for any L. Since by Theorem 3.4, all 3 dimensional p.d. systems are regular, this implies that the feedback system (4.33) is p.c. for any K and L. Thus controllability of (A,C) is necessary.

(Sufficiency) By the discussion following the statement of Theorem 4.1, it suffices to consider the case where C is a 3x2 matrix of full rank. We give a constructive procedure for finding the matrices K and L.

Step 1: We may assume, without loss of generality, that $q'c_1 \neq 0$. We choose matrices $K_1$ and $P_1$ such that $P_1^{-1}(A+CK_1)P_1$ is in companion form, $P_1^{-1}C$ is of the form

$$\begin{bmatrix} 0 & \beta_1 \\ 0 & \beta_2 \\ 1 & \beta_3 \end{bmatrix}$$

and $q'P_1 = (\alpha_1\ \alpha_2\ \alpha_3)$ with $\alpha_2^2 - 4\alpha_1\alpha_3 > 0$. By Lemma 4.2, such matrices exist (see the proof of Lemma 4.2 in Appendix A for the constructive procedure for $K_1$ and $P_1$).

Step 2: Choose a matrix $K_2$ such that the matrix

$$A_2 \triangleq P_1^{-1}(A+CK_1)P_1 + P_1^{-1}CK_2$$

has three real distinct eigenvalues $\lambda_1$, $\lambda_2$, and $\lambda_3$. The $\lambda_i$'s are not arbitrary. Their specification will be given later.

Step 3: Use the Vandemonde matrix

$$P_2 = \begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{bmatrix}$$

to diagonalize $A_2$, i.e.,

$$P_2^{-1}A_2P_2 = \Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

If we define $z(t) = P_2^{-1}\ P_1^{-1}\ x(t)$, $v(t) = u(t)-(K_1+K_2P_1^{-1})x(t)$, we see that

$$\dot{z}(t) = P_2^{-1}P_1^{-1}[A + C(K_1+K_2P_1^{-1})]P_1P_2z(t)+P_2^{-1}P_1^{-1}Cv(t)$$

$$= \Lambda z(t) + P_2^{-1}P_1^{-1}Cv(t) \qquad (4.34)$$

Step 4: We now restrict $\lambda_1$, $\lambda_2$, $\lambda_3$ to be such that they are not solutions of the equation

$$\alpha_1 + \alpha_2\lambda + \alpha_3\lambda^2 = 0 \tag{4.35}$$

This implies $q'P_1P_2 = (\eta_1\ \eta_2\ \eta_3)$, where $\eta_i = \alpha_1 + \alpha_2\lambda_i + \alpha_3\lambda_i^2$, has components all nonzero. By Theorem 3.5, we can construct a matrix B such that $(\Lambda, B, q'P_1P_2, \tau)$ is p.d. In fact, the results in section 4.2 show that B is given by

$$B = \frac{1}{D}\begin{bmatrix} 0 & \frac{\eta_2}{\eta_1}(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)e^{2\lambda_2\tau} & \frac{\eta_3}{\eta_1}(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)e^{(\lambda_2+\lambda_3)\tau} \\ \frac{\eta_1}{\eta_2}(\lambda_1-\lambda_1)(\lambda_3-\lambda_1)e^{2\lambda_1\tau} & 0 & \frac{\eta_3}{\eta_2}(\lambda_2-\lambda_3)(\lambda_3-\lambda_1)e^{(\lambda_1+\lambda_3)\tau} \\ \frac{\eta_1}{\eta_3}(\lambda_3-\lambda_1)(\lambda_1-\lambda_2)e^{(2\lambda_1+\lambda_2-\lambda_3)\tau} & \frac{\eta_2}{\eta_3}(\lambda_3-\lambda_2)(\lambda_1-\lambda_2)e^{(\lambda_1+2\lambda_2-\lambda_3)\tau} & 0 \end{bmatrix}$$

where $D = (\lambda_3-\lambda_1)(e^{\lambda_1\tau}-e^{\lambda_2\tau}) + (\lambda_1-\lambda_2)e^{(\lambda_2-\lambda_3)\tau}(e^{\lambda_1\tau}-e^{\lambda_2\tau})$

<u>Step 5</u>: We now show that there exists a 2x3 matrix $L_1$ such that $P_2^{-1}\ P_1^{-1}\ CL_1 = B$. We know from section 4.2 that $\mathscr{R}(B)$, the range space of B, is spanned by the vectors

$$\begin{bmatrix} 0 \\ 1 \\ -\frac{\eta_2}{\eta_3}e^{(\lambda_2-\lambda_3)\tau} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 0 \\ -\frac{\eta_1}{\eta_3}e^{(\lambda_1-\lambda_3)\tau} \end{bmatrix}$$

A necessary and sufficient condition for the existence of $L_1$ is that

$\mathscr{R}(B) \subset \mathscr{R}(P_2^{-1} P_1^{-1} C)$. Since $\dim \mathscr{R}(B) = \dim \mathscr{R}(P_2^{-1} P_1^{-1} C) = 2$, $L_1$ exists if $\mathscr{R}(B) = \mathscr{R}(P_2^{-1} P_1^{-1} C)$. Thus we only need to find a 2x2 nonsingular matrix

$$L_2 = \begin{bmatrix} \ell_{11} & \ell_{12} \\ \ell_{21} & \ell_{22} \end{bmatrix}$$

such that

$$P_2^{-1} P_1^{-1} C \begin{bmatrix} \ell_{11} & \ell_{12} \\ \ell_{21} & \ell_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ -\frac{\eta_2}{\eta_3} e^{(\lambda_2-\lambda_3)\tau} & -\frac{\eta_1}{\eta_3} e^{(\lambda_1-\lambda_3)\tau} \end{bmatrix}$$

The first column of $L_1$ is then given by

$$\frac{1}{D} \frac{\eta_1}{\eta_2} (\lambda_2-\lambda_1)(\lambda_3-\lambda_1) e^{2\lambda_1\tau} \begin{bmatrix} \ell_{11} \\ \ell_{21} \end{bmatrix}$$

. The second column of $L_1$ is given by

$$\frac{1}{D} \frac{\eta_2}{\eta_1} (\lambda_1 -\lambda_2)(\lambda_2-\lambda_3) e^{2\lambda_2\tau} \begin{bmatrix} \ell_{12} \\ \ell_{22} \end{bmatrix}$$

. The last column of $L_1$ is given by

$$\frac{1}{D} \frac{\eta_3}{\eta_2} (\lambda_2-\lambda_3)(\lambda_3-\lambda_1) e^{(\lambda_1+\lambda_3)\tau} \begin{bmatrix} \ell_{11} \\ \ell_{21} \end{bmatrix} + \frac{1}{D} \frac{\eta_3}{\eta_1} (\lambda_1-\lambda_3)(\lambda_2-\lambda_3) e^{(\lambda_2+\lambda_3)\tau} \begin{bmatrix} \ell_{12} \\ \ell_{22} \end{bmatrix} .$$

Using the particular forms of $P_1^{-1}$ and $P_2$ we obtain

$$\begin{bmatrix} \beta_1\ell_{21} & \beta_1\ell_{22} \\ \beta_2\ell_{21} & \beta_2\ell_{22} \\ \ell_{11}+\beta_3\ell_{21} & \ell_{12}+\beta_3\ell_{22} \end{bmatrix} = \begin{bmatrix} 1-\frac{\eta_2}{\eta_3}e^{(\lambda_2-\lambda_3)\tau} & 1-\frac{\eta_1}{\eta_3}e^{(\lambda_1-\lambda_3)\tau} \\ \lambda_2-\lambda_3\frac{\eta_2}{\eta_3}e^{(\lambda_2-\lambda_3)\tau} & \lambda_1-\lambda_3\frac{\eta_1}{\eta_3}e^{(\lambda_1-\lambda_3)\tau} \\ \lambda_2^2-\lambda_3^2\frac{\eta_2}{\eta_3}e^{(\lambda_2-\lambda_3)\tau} & \lambda_1^2-\lambda_3^2\frac{\eta_1}{\eta_3}e^{(\lambda_1-\lambda_3)\tau} \end{bmatrix} \tag{4.36}$$

We consider three cases

Case A. $\beta_1 = 0$.

This requires

$$1 - \frac{\eta_2}{\eta_3}e^{(\lambda_2-\lambda_3)\tau} = 0$$

and

$$1 - \frac{\eta_1}{\eta_3}e^{(\lambda_1-\lambda_3)\tau} = 0$$

These can be rewritten as

$$\eta_1 e^{\lambda_1\tau} = \eta_2 e^{\lambda_2\tau} = \eta_3 e^{\lambda_3\tau} \tag{4.37}$$

Since $\alpha_2^2 - 4\alpha_1\alpha_3 > 0$, by Lemma 4.1, we conclude that there exists a constant $\omega$ such that the line $\lambda = \omega$ intersects $(\alpha_1+\alpha_2\lambda + \alpha_3\lambda^2)e^{\lambda\tau}$

at three real distinct points $\lambda_1$, $\lambda_2$, and $\lambda_3$. These will be the eigenvalues that we choose. We can solve for $\ell_{11}$, $\ell_{12}$, $\ell_{21}$, and $\ell_{22}$ from (4.36) explicitly by

$$\ell_{21} = \frac{\lambda_2-\lambda_3}{\beta_2}$$

$$\ell_{22} = \frac{\lambda_1-\lambda_3}{\beta_2}$$

$$\ell_{11} = \lambda_2^2 - \lambda_3^2 - \beta_3\ell_{21}$$

$$\ell_{12} = \lambda_1^2 - \lambda_3^2 - \beta_3\ell_{22}$$

Case B. $\beta_2 = 0$.

This requires

$$\frac{\eta_1 e^{\lambda_1\tau}}{\lambda_1} = \frac{\eta_2 e^{\lambda_2\tau}}{\lambda_2} = \frac{\eta_3 e^{\lambda_3\tau}}{\lambda_3} \qquad (4.38)$$

By appealing to Lemma 4.1, we conclude that real and distinct $\lambda_1$, $\lambda_2$ and $\lambda_3$ satisfying (4.38) exist. With the eigenvalues thus chosen, the matrix $L_2$ can be obtained from (4.36).

Case C. $\beta_1, \beta_2$ both nonzero.

Completely analogous considerations show that we must have

$$\left(\lambda_1 - \frac{\beta_2}{\beta_1}\right)\frac{e^{-\lambda_1\tau}}{\eta_1} = \left(\lambda_2 - \frac{\beta_2}{\beta_1}\right)\frac{e^{-\lambda_2\tau}}{\eta_2} = \left(\lambda_3 - \frac{\beta_2}{\beta_1}\right)\frac{e^{-\lambda_3\tau}}{\eta_3} \qquad (4.39)$$

Appealing to Lemma 4.1 once again, we conclude that desired eigenvalues $\lambda_i$, $i = 1,2,3$ exist, and the matrix $L_2$ can be evaluated from (4.36).

We have now constructed a p.d. system of the form

$$\dot{z}(t) = \Lambda z(t) + P_2^{-1} P_1^{-1} CL_1 z(t-\tau) \tag{4.40}$$

with $q'P_1P_2z(t) = 0 \qquad t \geq 2\tau$

In terms of the original coordinate system, we have

$$\dot{x}(t) = [A+C(K_1+K_2P_1^{-1})]x(t) + CL_1P_2^{-1}P_1^{-1}x(t-\tau) \tag{4.41}$$

is p.d. for q. The desired feedback matrices are thus $K = K_1+K_2P_1^{-1}$ and $L = L_1P_2^{-1} P_1^{-1}$. The theorem is proved.

For greater clarity and completeness, we summarize the algorithm for constructing the matrices K and L in the three dimensional case. We assume, without loss of generality, that $q'c_1 \neq 0$.

(i) Construct, as in the proof of Lemma 4.2 in Appendix A, $P_1$, $K_1$ such that $P_1^{-1}(A + CK_1)P_1$ is in companion form, $P_1^{-1}C$ is of the form

$$\begin{bmatrix} 0 & \beta_1 \\ 0 & \beta_2 \\ 1 & \beta_3 \end{bmatrix},$$

and $q'P_1 = (\alpha_1\ \alpha_2\ \alpha_3)$ with $\alpha_2^2 - 4\alpha_1\alpha_3 > 0$.

(ii) Let $\eta(\lambda) = \alpha_1 + \alpha_2\lambda + \alpha_3\lambda^2$. If $\beta_1 = 0$, choose three real and distinct numbers $\lambda_1$, $\lambda_2$, and $\lambda_3$ such that $\eta(\lambda_i) \neq 0$, i=1,2,3, and that $\eta(\lambda_1)e^{\lambda_1\tau} = \eta(\lambda_2)e^{\lambda_2\tau} = \eta(\lambda_3)e^{\lambda_3\tau}$. If $\beta_2 = 0$, choose real and distinct $\lambda_1$, $\lambda_2$, and $\lambda_3$ such that $\eta(\lambda_i) \neq 0$, i=1,2,3, and

$$\frac{\eta(\lambda_1)e^{\lambda_1\tau}}{\lambda_1} = \frac{\eta(\lambda_2)e^{\lambda_2\tau}}{\lambda_2} = \frac{\eta(\lambda_3)e^{\lambda_3\tau}}{\lambda_3} .$$

If $\beta_1$ and $\beta_2$ are both nonzero, choose real and distinct $\lambda_1$, $\lambda_2$, $\lambda_3$ such that $\eta(\lambda_i) \neq 0$, i=1,2,3, and

$$\left(\lambda_1 - \frac{\beta_2}{\beta_1}\right) \frac{e^{-\lambda_1\tau}}{\eta(\lambda_1)} = \left(\lambda_2 - \frac{\beta_2}{\beta_1}\right) \frac{e^{-\lambda_2\tau}}{\eta(\lambda_2)} = \left(\lambda_3 - \frac{\beta_2}{\beta_1}\right) \frac{e^{-\lambda_3\tau}}{\eta(\lambda_3)}$$

(iii) Once $\lambda_1$, $\lambda_2$, $\lambda_3$ are chosen, choose a matrix $K_2$ such that the matrix $P_1^{-1}(A + CK_1)P_1 + P_1^{-1}CK_2$ has as its eigenvalues $\lambda_1$, $\lambda_2$, and $\lambda_3$.

(iv) Compute the Vandemonde matrix

$$P_2 = \begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{bmatrix}$$

(v) Calculate the matrix $L_1$ as in step 5 of the proof. In other words, find $L_1$ such that $(P_2^{-1}\ P_1^{-1}(A + CK_1 + CK_2P_1^{-1})P_1P_2$, $P_2^{-1}\ P_1^{-1}\ CL_1$, $P_2'\ P_1'\ q$, $\tau)$ is p.d. The choice of $\lambda_1$, $\lambda_2$, and $\lambda_3$ in (ii) guarantees that $L_1$ exists.

(vi) Finally, the desired feedback matrices are given by

$$K = K_1 + K_2P_1^{-1}$$

and $$L = L_1P_2^{-1}\ P_1^{-1}$$

Note that once the eigenvalues $\lambda_1$, $\lambda_2$, and $\lambda_3$ are chosen, the calculations involved in steps (iii) to (vi) are completely explicit and straightforward. The calculations involved in step (i) are detailed in the proof of Lemma 4.2 in Appendix A. They are all explicit excepting perhaps the choice of $f_1$ and $f_2$. The numbers $f_1$ and $f_2$ are chosen to satisfy $\alpha_2^2 - 4\alpha_1'\alpha_3 > 0$ and $\frac{d_{22}}{d_{12}}$ not being a zero of $\alpha_1' + \alpha_2\lambda + \alpha_3\lambda^2$. A simple search procedure will enable us to find an appropriate $f_1$ and $f_2$. As for step (ii), we need to find eigenvalues $\lambda_1$, $\lambda_2$, $\lambda_3$ which satisfy (4.37), or (4.38), or (4.39). One way of doing this is to plot, as in the proof of Lemma 4.1 in Appendix A, the function $f(\lambda)$ and draw the appropriate straight line to intersect $f(\lambda)$ at three real distinct points. While other methods for solving (4.37) - (4.39) can probably be devised, the computations involved in step (ii) will, unfortunately, be nonexplicit.

Remark 4.1 The proof of Theorem 4.1 is rather clumsy and involved. Since we only assume controllability of (A,C) we only have freedom in assigning the eigenvalues. The lengthy construction shows that we can find a suitable set of eigenvalues so that the matrix B we constructed necessarily lies in the range space of the input matrix C. Certainly a simpler and more elegant proof is desirable. However, because of the nonlinear nature of the problem, a substantial simplification may not be possible.

Remark 4.2 Remark 3.4 shows that the eigenvalues of the delay feedback system (4.41) are precisely those of $[A + C(K_1 + K_2P_1^{-1})]$, and these satisfy (4.44) or (4.45) or (4.46). In general, there are many solutions to these equations (see Appendix A). If we can choose the eigenvalues so that they are negative, we will have obtained an asymptotically stable p.d. system. We have not been able to show as yet that such a choice is always possible. Thus, the closed-loop p.d. system (4.41) may be unstable.

Remark 4.3 We have assumed that the delay time $\tau$ is prescribed in advance and not available as a design parameter. However, if it is to be designed, the smaller $\tau$ is, the larger the elements of the B matrix will be (see eqs. (4.19) and (4.17)). Hence we have a tradeoff in this situation. We shall not investigate this point further.

## 4.4 Delay Feedback Control of Higher Dimensional Linear Systems

Our goal in this section is to extend the feedback control results for 3 dimensional systems to higher dimensions $n > 3$. The method of proof is substantially the same as that of Theorem 4.1. Notationally however, it is much more involved.

Theorem 4.2 For matrices K and L to exist such that the closed-loop system

$$\dot{x}(t) = (A + CK)x(t) + CLx(t-\tau)$$

satisfies $q'x(t) = 0$, $t \geq 2\tau$, for all initial conditions, it is sufficient that rank $C \geq 3$, and (A,C) be controllable.

Proof: Consider the equations

$$q'r = 1 \tag{4.42}$$

$$q'e^{A\tau}r = 0 \tag{4.43}$$

$$q'e^{A\tau}Ar = 0 \tag{4.44}$$

where $q' = (q_1\ q_2 \dots\ q_n)$

$$A = \begin{bmatrix} \lambda_1 & 0 & \cdots & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & \vdots \\ \vdots & 0 & \ddots & & \vdots \\ \vdots & \vdots & & \lambda_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & \lambda_n \end{bmatrix}$$

Suppose at least three of the eigenvalues $\lambda_i$ are real and distinct with the corresponding $q_i$'s nonzero. Without loss of generality, we may take $\lambda_1$, $\lambda_2$, and $\lambda_3$ to be these eigenvalues with $q_1$, $q_2$, $q_3$ nonzero. Under these conditions, (4.42) - (4.44) can be solved to give

$$r_3 = \frac{1}{q_3 w_3} \left\{ \frac{(\lambda_2-\lambda_1)e^{(\lambda_1+\lambda_2)\tau}}{(e^{\lambda_2\tau}-e^{\lambda_1\tau})} - \sum_{i=4}^{n} q_i w_i r_i \right\} \tag{4.45}$$

$$r_2 = -\frac{1}{q_2(e^{\lambda_2\tau}-e^{\lambda_1\tau})} \left\{ e^{\lambda_1\tau} + \sum_{i=3}^{n} q_i r_i (e^{\lambda_i\tau}-e^{\lambda_1\tau}) \right\} \tag{4.46}$$

$$r_1 = 1 - \sum_{i=2}^{n} q_i r_i \tag{4.47}$$

where $$w_i = \frac{(\lambda_i-\lambda_1)e^{\lambda_i\tau}(e^{\lambda_2\tau}-e^{\lambda_1\tau})-(e^{\lambda_i\tau}-e^{\lambda_1\tau})(\lambda_2-\lambda_1)e^{\lambda_2\tau}}{e^{\lambda_2\tau}-e^{\lambda_1\tau}} \tag{4.48}$$

Thus $r_1$, $r_2$ and $r_3$ are completely determined in terms of the 2n-3 "free" variables $\lambda_1, \lambda_2, \ldots, \lambda_n$, and $r_4, r_5, \ldots, r_n$. Constructing $\bar{Z} = rq'e^{A\tau}$ and $\bar{B} = A\bar{Z} - \bar{Z}A$ we obtain

$$\bar{B} = \begin{bmatrix} 0 & r_1(\lambda_1-\lambda_2)q_2e^{\lambda_2\tau} & \cdots & r_1(\lambda_1-\lambda_n)q_ne^{\lambda_n\tau} \\ r_2(\lambda_2-\lambda_1)q_1e^{\lambda_1\tau} & 0 & & \vdots \\ \vdots & r_3(\lambda_3-\lambda_2)q_2e^{\lambda_2\tau} & & \vdots \\ \vdots & \vdots & & r_{n-1}(\lambda_{n-1}-\lambda_n)q_ne^{\lambda_n\tau} \\ r_n(\lambda_n-\lambda_1)q_1e^{\lambda_1\tau} & r_n(\lambda_n-\lambda_2)q_2e^{\lambda_2\tau} & & 0 \end{bmatrix}$$

Note that $\bar{B}$ is of rank 2, and its range space is spanned by the vectors

$$\begin{bmatrix} 0 \\ r_2(\lambda_2-\lambda_1) \\ \vdots \\ r_n(\lambda_n-\lambda_1) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} r_1(\lambda_1-\lambda_2) \\ 0 \\ r_3(\lambda_3-\lambda_2) \\ \vdots \\ r_n(\lambda_n-\lambda_2) \end{bmatrix}$$

We now proceed as in the proof of Theorem 4.1. We need only to consider the case where C is nx3 of full rank. Construct matrices K and $P_1$ such that $P_1^{-1}(A + BK)P_1$ is in companion form with distinct eigenvalues, and

$$P_1^{-1}C = \begin{bmatrix} 0 & d_{12} & d_{13} \\ 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot \\ & \cdot & \cdot \\ 1 & d_{n2} & d_{n3} \end{bmatrix}$$

The restrictions on the eigenvalues will be given later. Next, we diagonalize $P_1^{-1}(A + BK)P_1$ by a Vandemonde matrix

$$P_2 = \begin{bmatrix} 1 & 1 & \cdot\ \cdot\ \cdot & 1 \\ \lambda_1 & \lambda_2 & & \lambda_n \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ \lambda_1^{n-1} & \lambda_2^{n-1} & & \lambda_n^{n-1} \end{bmatrix}$$

so that

$$P_2^{-1}\ P_1^{-1}(A + BK)P_1P_2 \triangleq \Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & & \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \lambda_n \end{bmatrix}$$

$$q'P_1P_2 = (\eta_1 \ldots \eta_n)$$

with $\eta_i = \alpha_1 + \alpha_2\lambda_i + \ldots + \alpha_n\lambda_i^{n-1}$

and $q'P_1 = (\alpha_1 \ldots \alpha_n)$

We now impose the restriction that three of the eigenvalues, say $\lambda_1$, $\lambda_2$ and $\lambda_3$ are real and distinct, and that the resulting $\eta_1$, $\eta_2$, and $\eta_3$ are nonzero. Under these conditions, we may construct a matrix B such that ( ,B,q'$P_1P_2$, ) is p.d. Analogous to step 5 of the proof of Theorem 4.1, we see that we need to find a matrix $L_2$ such that

$$P_1^{-1}CL_2 = P_2 \begin{bmatrix} 0 & r_1(\lambda_1-\lambda_2) \\ r_2(\lambda_2-\lambda_1) & 0 \\ \vdots & r_3(\lambda_3-\lambda_2) \\ & \vdots \\ r_n(\lambda_n-\lambda_1) & r_n(\lambda_n-\lambda_2) \end{bmatrix}$$

$$= \begin{bmatrix} \sum\limits_{i\neq 1} r_i(\lambda_i-\lambda_1) & \sum\limits_{i\neq 2} r_i(\lambda_i-\lambda_2) \\ \sum\limits_{i\neq 1} r_i\lambda_i(\lambda_i-\lambda_1) & \vdots \\ \vdots & \vdots \\ \sum\limits_{i\neq 1} r_i\lambda_i^{n-1}(\lambda_i-\lambda_1) & \sum\limits_{i\neq 2} r_i\lambda_i^{n-1}(\lambda_i-\lambda_2) \end{bmatrix}$$

Let

$$L_2 = \begin{bmatrix} \ell_{11} & \ell_{12} \\ \ell_{21} & \ell_{22} \\ \ell_{31} & \ell_{32} \end{bmatrix}$$

Then

$$P^{-1}CL_2 = \begin{bmatrix} d_{12}\ell_{21}+d_{13}\ell_{31} & d_{12}\ell_{22}+d_{13}\ell_{32} \\ d_{22}\ell_{21}+d_{23}\ell_{31} & d_{22}\ell_{22}+d_{23}\ell_{32} \\ \vdots & \vdots \\ d_{n-1,2}\ell_{21}+d_{n-1,3}\ell_{31} & d_{n-1,2}\ell_{22}+d \\ \ell_{11}+d_{n2}\ell_{21}+d_{n3}\ell_{31} & \ell_{12}+d_{n2}\ell_{22}+d_{n3}\ell_{32} \end{bmatrix}$$

We can solve for $\ell_{11}$ and $\ell_{12}$ independently. Thus we only need to show that we can solve for $\ell_{21}$, $\ell_{22}$, $\ell_{31}$, and $\ell_{32}$ consistently from the equations

$$d_{j2}\ell_{21}+d_{j3}\ell_{31} = \sum_{i\neq 1} r_i\lambda_i^{j-1}(\lambda_i-\lambda_1) \qquad j=1,\ldots,n-1 \qquad (4.50)$$

$$d_{j2}\ell_{22}+d_{j3}\ell_{32} = \sum_{i\neq 2} r_i\lambda_i^{j-1}(\lambda_i-\lambda_2) \qquad j=1,\ldots,n-1 \qquad (4.51)$$

We now eliminate $\ell_{21}$, $\ell_{22}$, $\ell_{31}$, $\ell_{32}$ from (4.50) and (4.51). Using the fact that rank C = 3, a little thought shows that we will obtain 2n-6 equations relating $\lambda_1,\ldots,\lambda_n$, $r_1,\ldots,r_n$. Since the number of "free" variables is 2n-3, we expect that it would be possible to choose the $\lambda_i$'s and $r_i$'s in such a way that all the requirements are fulfilled.

The simplest way to see this is to examine the special case where $d_{j2} = 0$ for all j except j = m, and $d_{j3} = 0$ for all j except j = k. This gives rise to equations of the form

$$\sum_{i\neq 1} r_i\lambda_i^{j-1}(\lambda_i-\lambda_1) = 0 \qquad j=1,\ldots,n-1,\ j\neq m,k \qquad (4.52)$$

and $$\sum_{i\neq 2} r_i \lambda_i^{j-1}(\lambda_i-\lambda_2) = 0 \qquad \begin{matrix} j=1,\ldots,n-1 \\ j\neq m,k \end{matrix} \tag{4.53}$$

If we take $\lambda_1$, $\lambda_2$, and $\lambda_3$ as fixed real distinct constants, these give rise to 2n-6 equations in the 2n-6 variables $\lambda_4,\ldots, \lambda_n$, $r_4,\ldots, r_n$. Eliminate the variables $r_4,\ldots,r_n$ in the above equations. This yields n-3 equations in the variables $\lambda_4,\ldots, \lambda_n$. Next we note that these equations are invariant under permutation of the indices in $\lambda_i$. Hence if we reduce these n-3 equations into a single equation involving $\lambda_i$ for some i, say of the form $f(\lambda_i) = 0$, then $f(\lambda_j) = 0$, $j=4,\ldots n$ also. However, the form of equations (4.52) and (4.53) implies that f is an exponential polynomial [1] and it is known [1] that the zeros of these functions are distributed symmetrically about the real axis with asymptotic distribution separated at a certain distance. Since there is an infinite number of zeros of these exponential polynomials, it is possible to choose $\lambda_4,\ldots, \lambda_n$ such that they form a symmetric set with the properties $(\alpha_1+\alpha_2\lambda_i+\ldots+ \alpha_n\lambda_i^{n-1}) \neq 0$, $\lambda_i$'s distinct $i=1,\ldots,n$, and $\lambda_1$, $\lambda_2$, $\lambda_3$ real.

With such a choice, the conditions for the existence of the matrix $L_2$ are satisfied and we can construct, from $L_2$, a matrix $L_1$ such that $P_2^{-1} P_1^{-1} CL_1 = B$. While we have only discussed the case where $d_{j2} = 0$, $j = 1,\ldots, n-1$, $j \neq m$ and $d_{j3} = 0$, $j = 1,\ldots, n-1$, $j \neq k$, the other cases can be treated in exactly the same way. We can eliminate variables from (4.50) and (4.51) in a similar fashion to that discussed above and show that the desired $\lambda_1$, $\lambda_2,\ldots, \lambda_n$ exist. The details are therefore omitted. The rest of the proof now proceeds in exactly the same way as that of Theorem 4.1.

We summarize here the algorithm for constructing the matrices K and L in the higher dimensional case. Again, we assume $q'c_1 \neq 0$.

(i) Choose $P_1$, $K_1$ such that $P_1^{-1}(A + CK_1)P_1$ is in companion form, $P_1^{-1}C$ is of the form

$$\begin{bmatrix} 0 & d_{12} & d_{13} \\ 0 & d_{22} & d_{23} \\ \vdots & \vdots & \vdots \\ 0 & \vdots & \vdots \\ 1 & d_{n2} & d_{n3} \end{bmatrix}, \text{ and } q'P_1 = (\alpha_1\ \alpha_2 \dots\ \alpha_n)$$

(ii) Choose eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ such that (4.50) - (4.51) can be solved consistently for $\ell_{ij}$, i = 1,2,3, j = 1,2. Such a choice is possible by the proof of Theorem 4.2.

(iii) Choose $K_2$ such that the matrix $P_1^{-1}(A + CK_1)P_1 + P_1^{-1}CK_2$ has as its eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$.

(iv) Compute the Vandemonde matrix

$$P_2 = \begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_n \\ \vdots & \vdots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \lambda_n^{n-1} \end{bmatrix}$$

(v) Calculate the matrix $L_1$ from $\ell_{ij}$, i = 1,2,3, j = 1,2 such that

$(P_2^{-1}\ P_1^{-1}(A + CK_1 + CK_2P_1^{-1})P_1P_2,\ P_2^{-1}\ P_1^{-1}CL_1,\ P_2'P_1'\ q,\tau)$ is p.d.

(vi) The matrices K and L are given by

$$K = K_1 + K_2P_1^{-1}$$

$$L = L_1P_2^{-1}P_1^{-1}$$

Except for step (ii), all the calculations involved in the algorithm are explicit. To choose the appropriate eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ in step (ii), we can first eliminate $\ell_{21}, \ell_{22}, \ell_{31}, \ell_{32}, r_1, r_2, \ldots, r_n$ from (4.50) - (4.51). This elimination involves only algebraic operations. We are then left with n-3 coupled transcendental equations involving exponential polynomials in the variables $\lambda_1, \lambda_2, \ldots, \lambda_n$. For fixed, real and distinct $\lambda_1, \lambda_2, \lambda_3$, an iterative procedure is now needed to find the appropriate values for $\lambda_4, \lambda_5, \ldots, \lambda_n$. We have not had time to investigate the numerical aspects involved in solving these transcendental equations. However, quite a lot of work has been done on the solutions of transcendental equations involving exponential polynomials. The reader may, for example, consult [1] and the references therein.

Remark 4.4 Theorem 4.2 gives only a sufficient condition for the solvability of the feedback problem. It is not possible to obtain necessary and sufficient conditions by our approach, since for $n > 3$, solvability of (4.42) - (4.44) is not a necessary condition for p.d.

Remark 4.5 It may be possible to prove Theorem 4.2 under the assumption that rank $C \geq 2$. A similar argument will then show that we have 2n-4 equations in 2n-3 variables. However, the algebraic difficulties in showing that these equations can be consistently solved for suitable eigenvalues $\lambda_1, \ldots, \lambda_n$ are much more formidable.

## 4.5 Sensitivity of Pointwise Degenerate Systems

In the last two sections, we have given conditions under which a delay feedback system will have its output identically zero after $2\tau$ units of time. If these results are to be used in practice, we must ensure that the feedback system will behave reasonably well under perturbation of its parameters. Since we usually know which particular combination of state variables serves as the output, it is reasonable to assume that the output vector q is fixed, and we shall do so. There are two questions of interest. Suppose for a fixed set of parameters $A_o$, $C_o$, $\tau_o$, there exist matrices $K_o$ and $L_o$ which give rise to a p.d. system $(A_o + C_oK_o, C_oL_o, q, \tau_o)$. This yields a (nonunique) "design" function $f: (A, C, \tau) \mapsto (K,L)$, and a "performance" function $g(t)$: $(A,C,\tau,f(A_o, C_o, \tau_o)) \mapsto q'x(t)$ for $t \geq 2\tau$. Continuity of f at $(A_o, C_o, \tau_o)$ implies that the delay feedback design procedure is well-posed. Similarly, continuity of $g(t)$ at $(A_o, C_o, \tau_o, K_o, L_o)$ implies that small variations in the parameter values give rise to a small degradation in the system performance. Note that the continuity properties of the functions f and $g(t)$ are related to those of $f_1$ and $g_1(t)$, where $f_1$: $(A,\tau) \mapsto B$ with $(A, B, q, \tau)$ p.d., and $g_1(t)$: $(A, \tau, f_1(A_o, \tau_o)) \mapsto q'x(t)$, for $t \geq 2\tau$. This is because in our construction of the matrices K and L, we first construct a matrix $B_1$ such that $(A + CK, B_1, q, \tau)$, is p.d., and then we construct a matrix L such that $B_1 = CL$. If C is the identity matrix and K can be taken to be zero, then the properties of the functions f and g(t) reduce to those of $f_1$ and $g_1(t)$. We shall only study the behavior of $f_1$ and $g_1(t)$. This section contains some simple results in this direction.

For the study of $f_1$, we will assume that A has distinct eigenvalues. This is the case of most interest to us since in our proof of Theorems 4.1 and 4.2, we assign distinct eigenvalues to the matrix A + CK. We shall concern ourselves only with perturbations of the form $A + \varepsilon A_1 \triangleq A(\varepsilon)$ and $\tau + \varepsilon\tau_1 = \tau(\varepsilon)$. The following perturbation result for matrices is known [49].

<u>Lemma 4.3</u> If $\lambda$ is a simple eigenvalue of A, then for sufficiently small $|\varepsilon|$, there is an eigenvalue $\lambda(\varepsilon)$ of $A(\varepsilon)$ with a power series expansion

$$\lambda(\varepsilon) = \lambda + \varepsilon\lambda^{(1)} + \varepsilon^2\lambda^{(2)} + \ldots$$

and there is a corresponding eigenvector $x(\varepsilon)$ with a power series expansion

$$x(\varepsilon) = x + \varepsilon x^{(1)} + \ldots$$

where x is the eigenvector of A corresponding to $\lambda$.

A procedure for computing the $\lambda^{(i)}$'s and $x^{(i)}$'s from A and $A_1$ may be found in [49]. We can now state

<u>Theorem 4.3</u> Suppose A has n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$. The matrix $B(\varepsilon) = f_1[A(\varepsilon), \tau(\varepsilon)]$ has a power series expansion in $\varepsilon$ for sufficiently small $|\varepsilon|$

$$B(\varepsilon) = B + \varepsilon B^{(1)} + \ldots$$

where $B = f_1(A,\tau)$

Proof: Since A has distinct eigenvalues, a transformation T that diagonalizes A consists of the n eigenvectors of A. By Lemma 4.3, the eigenvectors of $A(\varepsilon)$ can be expanded in a power series in $\varepsilon$ for sufficiently small $|\varepsilon|$. We can therefore also expand $T(\varepsilon)$ and $T^{-1}(\varepsilon)$ in a power series. This implies $q'T(\varepsilon)$ can be expanded in a power series. Equations (4.45) - (4.49) show that $\bar{B}(\varepsilon)$ can again be expanded in a power series in $\varepsilon$. Finally, since $B(\varepsilon) = T(\varepsilon)\bar{B}(\varepsilon)T^{-1}(\varepsilon)$, $B(\varepsilon)$ can also be expanded in a power series in $\varepsilon$ for sufficiently small $|\varepsilon|$.

Next, we consider the behavior of $g_1(t)$ again under perturbations of the form $A(\varepsilon) = A + \varepsilon A_1$, $\tau(\varepsilon) = \tau + \varepsilon\tau_1$.

Theorem 4.4 Suppose the system

$$\dot{x}(t) = Ax(t) + Bx(t-\tau) \tag{4.54}$$

is p.d. with respect to q for $t \geq 2\tau$. Then the system

$$\dot{x}_\varepsilon(t) = A(\varepsilon)x_\varepsilon(t) + Bx_\varepsilon(t-\tau(\varepsilon)) \tag{4.55}$$

will have, for each $t \geq 2\tau$,

$$q'x_\varepsilon(t) \triangleq y_\varepsilon(t) = \varepsilon\gamma_1(t) + \varepsilon^2\gamma_2(t) + \ldots$$

Proof: Let $z(t) = P_1 x_\varepsilon(t-(k-1)\tau(\varepsilon)) + \ldots + P_k x_\varepsilon(t)$ where $P_1, \ldots, P_k$ satisfy Popov's condition (3.3) - (3.8). Then

$$\dot{z}(t) = P_1A(\varepsilon)x_\varepsilon(t-(k-1)\tau(\varepsilon))+P_1Bx_\varepsilon(t-k\tau(\varepsilon))$$

$$+ \ldots + P_kA(\varepsilon)x_\varepsilon(t) + P_kBx_\varepsilon(t-\tau(\varepsilon))$$

$$= Vz(t) + P_1A_1x_\varepsilon(t-(k-1)\tau(\varepsilon))+ \ldots +\varepsilon P_kA_1x_\varepsilon(t)$$

Thus,

$$v'z(t) = v'e^{V\tau(\varepsilon)}z(t-\tau(\varepsilon))$$

$$+ \varepsilon\int_{t-\tau(\varepsilon)}^{t} e^{V(t-s)}\{P_1A_1x_\varepsilon(s-(k-1)\tau(\varepsilon))$$

$$+ \ldots + P_kA_1x_\varepsilon(s)\}ds$$

$$\triangleq v'e^{V\tau(\varepsilon)}z(t-\tau(\varepsilon)) + \varepsilon\gamma_\varepsilon(t) \qquad (4.56)$$

Using equations (3.6) - (3.8) in Chapter 3,

$$v'z(t) - v'e^{V\tau(\varepsilon)}z(t-\tau(\varepsilon))$$

$$= v'z(t)-v'e^{V\tau}z(t-\tau(\varepsilon))-v'(e^{V\tau(\varepsilon)}-e^{V\tau})z(t-\tau(\varepsilon))$$

$$= q'x_\varepsilon(t)-v'(e^{V\tau(\varepsilon)}-e^{V\tau})z(t-\tau(\varepsilon)) \qquad (4.57)$$

Combining (4.61) and (4.62), we get

$$q'x_\varepsilon(t) = v'(e^{V\tau(\varepsilon)}-e^{V\tau})z(t-\tau(\varepsilon)) + \varepsilon\gamma_\varepsilon(t) \qquad (4.58)$$

The right hand side of (4.58) is precisely of the form $\varepsilon\gamma_1(t) + \varepsilon^2\gamma_2(t)+\ldots$, proving the theorem.

Remark 4.6 Theorem 4.4 does not give any estimate of the magnitudes of $\gamma_1(t)$, $\gamma_2(t)$, etc. Furthermore, no claim is made on the uniform behavior in t of the output $y_\varepsilon(t)$. In fact, if the system (4.55) is unstable, $\gamma_1(t), \gamma_2(t)$, etc., which are really derived from $x_\varepsilon(t)$, can become large as t gets large. However, if the original system (4.54) is asymptotically stable, then for small enough $|\varepsilon|$, the perturbed system (4.55) is also asymptotically stable [2]. In that case, $\gamma_1(t)$, $\gamma_2(t)$, etc., will go to zero as t gets large.

To a certain extent, the above remark limits the applicability of the delay feedback approach we have explored so far. If the parameters of the system are perturbed, the errors in the output may grow, and hence the performance may become unacceptable after some time. However, if in the proofs of Theorems 4.1 and 4.2, we can choose the $\lambda_i$'s such that $\mathscr{R}e\lambda_i < 0$, then the resulting feedback p.d. system will be asymptotically stable. In that case, the above mentioned problem does not arise. Whether we can always find feedback matrices K and L such that the resulting closed-loop system is asymptotically stable in addition to being p.d. remains an open problem.

## 4.6 Some Examples

We give two examples which illustrate the use of delay feedback in certain problems in linear system theory.

Example 1: "Deadbeat" output control.

Consider the system

$$\dot{x}(t) = Ax(t) + Cu(t)$$

where

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$y(t) = q'x(t)$$

with $q' = (2 \quad -2 \quad 1)$

The problem is to drive $y(t)$ to zero in minimum time by delay feedback control of the form

$$u(t) = Kx(t) + Lx(t-1)$$

Let us note that $(A,C)$ is controllable and rank $C = 2$. Also $A$ violates condition (i) of Theorem 4.1. Thus we cannot solve this problem by using only delay feedback control (i.e., $K = 0$).

One convenient choice for $K$ is

$$K = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

then

$$A + CK = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

and a suitable B is

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 2e & 0 & 0 \\ 0 & -2e & 0 \end{bmatrix}$$

If we take $L = \begin{bmatrix} 0 & -2e & 0 \\ 2e & 0 & 0 \end{bmatrix}$

we see that CL = B.

With these choices, one can easily verify that (A + CK, CL, q, 1) is p.d. The output y(t) has been controlled to zero and will remain zero for all $t \geq 2$.

The problem that is in a sense "dual" to the deadbeat control problem is that of a "deadbeat" observer. Consider the system

$$\dot{x}(t) = Ax(t)$$

$$y(t) = Cx(t)$$

We would like to reconstruct the state of the system. The usual approach is, under the assumption of observability, to construct an observer of the form

$$\dot{\hat{x}}(t) = A\hat{x}(t) + K[y(t) - C\hat{x}(t)]$$

such that the estimation error $e(t) = x(t) - \hat{x}(t)$ goes to zero asymptotically. The theory developed here suggests the use of a delay observer of the form

$$\dot{\hat{x}}(t) = A\hat{x}(t) + K[y(t)-C\hat{x}(t)] + L[y(t-\tau)-C\hat{x}(t-\tau)]$$

such that a linear combination of the error, $q'e(t)$, is driven to zero in minimum time. If we can observe all but one component of the error, $q'e(t)$, is driven to zero in minimum time. If we can observe all but one component of the error vector, this will enable us to reconstruct the estimation error and hence the state of the system in finite time. While finding necessary and sufficient conditions for the existence of such a delay observer is an open problem at the present, we can give an example illustrating the technique involved.

Example 2: "Deadbeat" observer.

Consider the system

$$\dot{x}(t) = Ax(t)$$

$$y(t) = Cx(t)$$

where

$$A = \begin{bmatrix} -1 & 2 & 0 \\ 2 & -2 & -1 \\ 1 & 2 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

(A,C) is observable and rank C = 2. Since $x_1$ and $x_2$ are directly observed, we need only estimate $x_3$. Using the usual reduced order observer, we can asymptotically reconstruct the state by a 1 dimensional observer.

Here we shall use delay feedback. Consider an observer of the form

$$\dot{\hat{x}}(t) = A\hat{x}(t) - K[y(t)-C\hat{x}(t)] - L[y(t-1) - C\hat{x}(t-1)] \qquad (4.59)$$

Then the estimation error $e(t) = x(t) - \hat{x}(t)$ satisfies

$$\dot{e}(t) = (A + KC)e(t) + LCe(t-1) \qquad (4.60)$$

We want to find K and L such that (4.60) is p.d. One such choice is

$$K = \begin{bmatrix} 1 & 0 \\ -2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$L = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then

$$A + KC = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$LC = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

Equation (4.60) becomes precisely Popov's example, and it satisfies

$$e_1(t) - 2e_2(t) - e_3(t) = 0 \qquad t \geq 2$$

But $x_3(t) = \hat{x}_3(t) + e_3(t)$

$$= \hat{x}_3(t) + e_1(t) - 2e_2(t) \qquad \text{for } t \geq 2$$

$$= \hat{x}_3(t) + y_1(t) - \hat{x}_1(t) - [y_2(t) - 2\hat{x}_2(t)] \qquad \text{for } t \geq 2$$

We know $\hat{x}_1(t)$, $\hat{x}_2(t)$, and $\hat{x}_3(t)$ from the solution of the observer equation (4.59). Hence $x_3(t)$ is reconstructed after 2 units of time.

The tradeoff between using the delay feedback observer (4.59) and the usual reduced order observer is clear. Delay feedback allows us to determine the state exactly after 2 units of time. However, the delay feedback observer must be 3 dimensional.

CHAPTER 5

# OPTIMAL FILTERING OF LINEAR AND NONLINEAR STOCHASTIC DELAY SYSTEMS

## 5.1 Introduction

After studying various deterministic aspects of the theory of delay systems in the previous chapters, we turn our attention to stochastic problems. In this chapter, we study the filtering problem for stochastic delay systems of the form

$$dx(t) = f(x_t,t)dt + H'(t)dw_1(t)$$
$$x(\theta) = x_o(\theta) \qquad \theta\varepsilon[-\tau,0] \tag{5.1}$$

$$dz(t) = h(x_t,t)dt + N(t)dw_2(t)$$
$$z(s) = w_2(s) = 0 \qquad s \leq 0 \tag{5.2}$$

We assume the reader to be familiar with the basic elements of stochastic processes [50], and we simply collect the basic definitions here for future reference. All stochastic processes are defined relative to a given probability space $(\Omega,\mathscr{S},P)$ and on an interval of the form $[0,T]$. The system process $x(t)$ takes values in $R^n$, the observation process $z(t)$ in $R^p$. For simplicity, we take $w_1(t)$ and $w_2(t)$ to be standard Wiener processes in $R^m$ and $R^p$ respectively, completely independent of each other. The initial function $x_o$ is taken to be some random function on $[-\tau,0]$, completely independent of $w_1(t)$ and $w_2(t)$. The maps f and h are functionals, possibly nonlinear, defined on $\mathscr{C}x[0,T]$. $H(t)$ and $N(t)$ are mxn and pxp matrix-valued continuous functions respectively. Furthermore $N(t)$ is assumed to be symmetric and positive definite.

We shall write $dw(t) = H'(t)dw_1(t)$ and $dv(t) = N(t)dw_2(t)$. Define $Q(t) = H'(t)H(t)$, $R(t) = N^2(t)$. With this notation, $w(t)$ is an "unnormalized" Wiener process in $R^n$, with

$$\text{cov}[w(t);\ w(s)] = \int_0^{\min(t,s)} Q(u)du$$

Similarly, $v(t)$ is an "unnormalized" Wiener process in $R^p$, with $\text{cov}[v(t);\ v(s)] = \int_0^{\min(t,s)} R(u)du$. This notation will be used throughout this chapter.

The basic filtering problem is to estimate some function $\phi$ of $x(t)$ given the observations $z(s)$, $0 \leq s \leq t$. It is well-known that the optimal estimate with respect to a large class of criteria, for example minimum mean square error, is the conditional expectation $E\{\phi[x(t)]|z^t\}$, where $z^t$ denotes the $\sigma$-algebra generated by the observations $z(s)$, $0 \leq s \leq t$. We shall also write $E\{\phi(x(t))|z^t\}$ as $\hat{\phi}[x(t)]$ or $E^t\{\phi[x(t)]\}$. Our objective in this chapter is to obtain formulas for $\hat{\phi}[x(t)]$ in the case of delay systems.

The nonlinear filtering problem has been extensively studied by many authors. In particular, Fujisaki, et al., [51] have given a stochastic differential equation for the evolution of $\hat{\phi}[x(t)]$ for rather general stochastic systems, which includes our delay model. Specifically they showed that for the delay system (5.1) and (5.2)

$$d\hat{\phi}[x(t)] = \widehat{\mathscr{L}_t\phi[x(t)]}dt + [\widehat{\phi[x(t)]h'(x_t,t)}-\hat{\phi}[x(t)]\hat{h}'(x_t,t)]\cdot$$

$$R^{-1}(t)[dz(t)-\hat{h}(x_t,t)dt] \qquad (5.3)$$

where $\mathscr{L}_t$ is a differential operator. If there are no delays in the system, the unknown terms on the right hand side of (5.3) are of the form $\widehat{\phi[x(t)]h[x(t),t]}$ and, assuming suitable differentiability conditions, one can write another stochastic differential equation for $\widehat{\phi[x(t)]h[x(t),t]}$, leading, in general, to a countably infinite-order system of moment equations. While there is no known method of analyzing this infinite set of equations, at least the way in which they arise is clear. One can thus use approximation techniques to study these equations, and develop suboptimal filtering schemes.

Unfortunately, when there are delays in the system, the above procedure does not go through. The reason is that one of the terms on the right hand side of (5.3) is $\widehat{\phi[x(t)]h(x_t,t)}$, and since h is a functional on the segment $x_t$, it is not clear how one can develop a stochastic differential equation for $\widehat{\phi[x(t)]h(x_t,t)}$ using (5.3). These difficulties motivated the development of a representation theorem for a functional of $x_t$. Using this representation, we derive stochastic differential equations for certain functionals f and h. The linear case with Gaussian distributions is then studied in detail. Stochastic differential equations generating the conditional mean as well as partial differential equations for the conditional covariance are derived. This will prepare the way for the filter stability discussions in the next chapter. We will also discuss some of the basic difficulties in deriving stochastic differential equations for general nonlinear functionals $\phi(x_t)$.

## 5.2 Existence, Uniqueness and Other Properties of Solutions of Stochastic Functional Differential Equations

In order for our estimation problem to be well-defined, we need conditions which guarantee existence and uniqueness of solutions to the functional stochastic differential equations (5.1) and (5.2). Since (5.1) can be solved independently of (5.2), we shall be mainly concerned with conditions to be imposed on (5.1). Conditions on (5.2) will be stated at the end of this section.

Existence and uniqueness of solutions to stochastic functional differential equations of the type (5.1) have been studied by Ito and Nisio [52], and Fleming and Nisio [53]. We state their results here. Assume

(A.1) $f(\phi,t)$ is measurable and continuous on $\mathscr{C}\times[0,T]$;

(A.2) there exists a bounded measure $\Gamma$ on $[-\tau,0]$ and a positive constant K such that

$$|f(\phi,t)-f(\psi,t)| \leq K \int_{-\tau}^{0} |\phi(s)-\psi(s)| d\Gamma(s)$$

(A.3) on the interval $[-\tau,0]$, $x(t)$ is continuous w.p.1 with $E|x(\theta)|^4 < \infty, \quad -\tau \leq \theta \leq 0.$

Proposition 5.2.1 Assume (A.1) to (A.3) are satisfied. Then there exists a unique solution to (5.1) which is continuous w.p.1 and has bounded second moment, and

$$E|x(t)|^4 \leq \gamma e^{\gamma t} \text{ for some positive } \gamma < \infty.$$

Furthermore $x_t$ is a Markov process.

The proof of Proposition 5.2.1 can be found in Ito and Nisio [52], or Fleming and Nisio [53], and Kushner [54].

Let us now study some of the properties of linear functional stochastic differential equations which are relevant to our later investigations. We consider the equation

$$dx(t) = a(x_t,t)dt + dw(t) \tag{5.4}$$

where $a(x_t,t)$ is given by the Stieltjes integral

$$a(x_t,t) = \int_{-\tau}^{0} d_\theta A(t,\theta)x(t+\theta)$$

Here, $A(t,\theta)$ is a function on RxR, jointly measurable in $(t,\theta)$, continuous in t, of bounded variation in $\theta$ for each t, with $\operatorname{Var}_{[-\tau,0]} A(t,\cdot) \leq m(t)$, a locally integrable function on $R^n$, where $\operatorname{Var}_{[-\tau,0]}$ means the total variation in $[-\tau,0]$. Furthermore, $A(t,\theta) = 0$ for $\theta \geq 0$, $A(t,\theta) = A(t,-\tau)$ for $\theta \leq -\tau$, and it is continuous from the left in $\theta$ on $(-\tau,0)$. We shall prove a type of variation of constants formula for (5.4). We first prove a lemma which will be used later and may be of some independent interest.

Lemma 5.2.1 Let $G(t,s)$ be a deterministic function on RxR such that $\int_{t_o}^{T} |G(t,s)|^2 ds < \infty$, for all $t\varepsilon[t_o,T]$, $G(t,s)$ is continuously differentiable in t, and that $\int_{t_o}^{T}\int_{t_o}^{T} |\dot{G}(t,s)|^2 dt\, ds < \infty (\dot{G}(t,s) = \frac{\partial}{\partial t} G(t,s))$. Then the Wiener integral $g(t) = \int_{t_o}^{t} G(t,s)dw(s)$ has a differential on $[t_o,T]$

$$dg(t) = \int_{t_o}^{t} \dot{G}(t,s)dw(s)dt + G(t,t)dw(t) \tag{5.5}$$

Proof: Proving (5.5) is equivalent to proving that for any $t_1$, $t_2$ in $[t_o,T]$,

$$g(t_2) - g(t_1) = \int_{t_1}^{t_2}\int_{t_o}^{t} G(t,s)dw(s)dt + \int_{t_1}^{t_2} G(t,t)dw(t) \tag{5.6}$$

By the assumption of square integrability of G(t,s), the Fubini type theorem for Wiener and Lebesgue integrals [50], is valid, and so

$$\int_{t_1}^{t_2}\int_{t_o}^{t} \dot{G}(t,s)dw(s)dt = \int_{t_o}^{t_1}\int_{t_1}^{t_2} \dot{G}(t,s)dt\ dw(s)$$

$$+ \int_{t_1}^{t_2}\int_{s}^{t_2} \dot{G}(t,s)dt\ dw(s)$$

$$= \int_{t_o}^{t_1} G(t_2,s)dw(s) - \int_{t_o}^{t_1} G(t_1,s)dw(s) + \int_{t_1}^{t_2} G(t_2,s)dw(s)$$

$$- \int_{t_1}^{t_2} G(s,s)dw(s)$$

The right hand side of (5.6) now yields

$$\int_{t_o}^{t_2} G(t_2,s)dw(s) - \int_{t_o}^{t_1} G(t_1,s)dw(s)$$

which is precisely the left hand side of (5.6).

Using Lemma 5.2.1, we can easily obtain a variation of constants formula for (5.4). Let $\Phi(t,s)$ be the fundamental matrix associated with (5.4), i.e., that unique matrix which is absolutely continuous in t, essentially bounded in s, and such that

$$\frac{\partial\Phi(t,s)}{\partial t} = \int_{-\tau}^{0} d_\theta A(t,\theta)\Phi(t+\theta,s) \qquad t \geq s$$

$$\Phi(t,s) = \begin{cases} 0 & \text{for } s-\tau \leq t<s \\ I & t=s \end{cases}$$

A discussion of the properties of $\Phi(t,s)$ may be found in Hale [2].

Theorem 5.2.1 The solution $x(t)$ of (5.4) can be written as

$$x(t) = \Phi(t,0)x_o(0)+ \int_{-\tau}^{0} d_\beta\left\{\int_0^\tau \Phi(t,s)A(s,\beta-s)ds\right\}x_o(\beta) + \int_0^t \Phi(t,s)dw(s) \tag{5.7}$$

Proof: The fundamental matrix $\Phi(t,s)$ satisfies the conditions required in Lemma 5.2.1. We may therefore apply Lemma 5.2.1 to the right hand side of (5.7) to obtain

$$dx(t) = \int_{-\tau}^{0} d_\theta A(t,\theta)\Phi(t+\theta,0)x_o(0)dt$$
$$+ \int_{-\tau}^{0} d_\beta\left\{\int_0^\tau \int_{-\tau}^{0} d_\theta A(t,\theta)\Phi(t+\theta,s)A(s,\beta-s)ds\right\}x_o(\beta)$$
$$+ \int_0^t \int_{-\tau}^{0} d_\theta A(t,\theta)\Phi(t+\theta,s)dw(s)dt + dw(t)$$

Applying the unsymmetric Fubini theorem of Cameron and Martin [55], we obtain

$$dx(t) = \int_{-\tau}^{0} d_\theta A(t,\theta)x(t+\theta)dt + dw(t)$$

so that (5.7) is indeed a solution of (5.4).

Remark 5.2.1 Similar variation of constants formula for linear stochastic functional differential equations have been obtained by Lindquist [56]. All are related to the representation of solutions of linear functional differential equations given by Banks [57].

The mean of $x(t)$, $\bar{x}(t)$, can easily be seen to satisfy

$$\frac{d}{dt}\bar{x}(t) = \int_{-\tau}^{0} d_\theta A(t,\theta)\bar{x}(t+\theta)$$

$$\bar{x}(\theta) = \bar{x}_o(\theta) \qquad \theta\varepsilon[-\tau,0] \tag{5.8}$$

whose solution is given by

$$\bar{x}(t) = \Phi(t,0)\bar{x}_o(0) + \int_{-\tau}^{0} d_\beta\left\{\int_0^\tau \Phi(t,s)A(s,\beta-s)ds\right\}\bar{x}_o(\beta) \tag{5.9}$$

As for the covariance associated with the solution of (5.6), we shall be interested not only in the covariance of $x(t)$, i.e., $E\{[x(t)-\bar{x}(t)][x(t)-\bar{x}(t)]'\}$, but also in the covariance "operator" $\Sigma(t,\theta,\xi) = E\{[x(t+\theta)-\bar{x}(t+\theta)][x(t+\xi)-\bar{x}(t+\xi)]'\}$, $-\tau \leq \theta,\ \xi \leq 0$. Heuristically, this corresponds to the covariance of $x_t$. We first compute it and then give it a more precise interpretation in terms of the characteristic functional of the $\mathscr{C}$-valued random variable $x_t$.

Let $e(t) = x(t)-\bar{x}(t)$ then from (5.7) and (5.9), and using the fact that $\Phi(t,s) = 0$, $t < s$, we get that

$$e(t+\theta) = \Phi(t,0)e(0) + \int_{-\tau}^{0} d_\beta\left\{\int_0^\tau \Phi(t+\theta,s)A(s,\beta-s)ds\right\}e(\beta) + \int_0^t \Phi(t+\theta,s)dw(s)$$

Using the independence of $x_o$ and $w(t)$, we get

$$\Sigma(t,\theta,\xi) = E\{e(t+\theta)e'(t+\xi)\}$$

$$= \Phi(t,0)\Sigma(0,0,0)\Phi'(t,0)+ \int_{-\tau}^{0} d_\beta\left\{\int_0^\tau \Phi(t+\theta,s)A(s,\beta-s)d\alpha\right\}$$

$$\cdot\Sigma(0,\beta,0)\Phi'(t,0)$$

$$+ \int_{-\tau}^{0}\Phi(t,0)d_\beta \int_0^\tau \Sigma(0,0,\beta)A'(s,\beta-s)\Phi'(t+\xi,s)ds$$

$$+ \int_{-\tau}^{0} d\alpha \int_{-\tau}^{0} d_\beta\left\{\int_0^\tau \Phi(t+\theta,s)A(s,\beta-s)ds\right\}\Sigma(0,\beta,\alpha)$$

$$\cdot\left\{\int_0^\tau A'(s,\eta-s)\Phi'(t+\xi,s)ds\right\}$$

$$+ \int_0^t \Phi(t+\theta,s)Q(s)\Phi'(t+\xi,s)ds \qquad (5.10)$$

If we view $x_t$ as a $\mathscr{C}$-valued random variable, then its characteristic functional $\phi(y,x_t)$ is defined by

$$\phi(y,x_t) = E \exp i \ \langle y, x_t\rangle$$

where y is an element of $\mathscr{C}^*$, the dual space of $\mathscr{C}$, which is the space of functions of bounded variations on $[-\tau,0]$ with $y(0) = 0$, and $\langle , \rangle$ is the pairing between $\mathscr{C}^*$ and $\mathscr{C}$ defined by

$$\langle y,x\rangle = \int_{-\tau}^{0} dy(\theta)x(\theta)$$

Recognizing that for each t, $\langle y, x_t\rangle = \int_{-\tau}^{0} dy(\theta)x(t+\theta)$ is a Gaussian random variable, we can easily evaluate

$$\phi(y,x_t) = e^{i\langle y,\bar{x}_t\rangle - 1/2\, Q_t(y)} \tag{5.11}$$

where $\bar{x}_t$ is given by (5.9) and $Q_t(y)$ is a quadratic form in y given by

$$Q_t(y) = \int_{-\tau}^{0}\int_{-\tau}^{0} dy(\theta)\Sigma(t,\theta,\xi)dy'(\xi) \tag{5.12}$$

The form of $\phi(y,x_t)$ implies that we may interpret $x_t$ as a $\mathscr{C}$-valued Gaussian random variable for eact t, with $\Sigma(t,\theta,\xi)$ as its covariance operator. These considerations will be useful later in section 5.5 when we study the linear filtering problem for delay systems.

We now state conditions concerning eq. (5.2).

(A.4) $h(\phi,t)$ is measurable and continuous on $\mathscr{C}\times[0,T]$.

(A.5) $\int_0^T E[h(x_t,t)'h(x_t,t)]dt < \infty$

We note that (A.1) - (A.5) are not the weakest assumptions for which our results are valid. However, we have not striven for more generality because this will only introduce technical complications without adding insight into the optimal filter structure for delay systems. From now on, (A.1) to (A.5) will be assumed to hold.

## 5.3 A Representation Theorem for Conditional Moment Functionals

In this section, we derive a representation theorem for the conditional expectation of functionals of the form $\phi(x_t)$ given the observations $z(s)$, $0 \le s \le t$. As before, we shall denote $\sigma\{z(s), 0 \le s \le t\}$ by $z^t$, and we shall call objects of the form $E\{\phi(x_t)|z^t\}$ conditional moment functionals. Our approach makes use of the Girsanov measure transformation technique as given in Wong [58]. Using the representation theorem so derived, stochastic differential equations can be obtained for suitably smooth functionals $\phi$, f, and h.

We first note that we can rewrite (5.2) as

$$N^{-1}(t)dz(t) = N^{-1}(t)h(x_t,t)dt + dw_2(t) \tag{5.13}$$

Clearly $z(s)$, $0 \le s \le t$ and $\int_0^s N^{-1}(\sigma)dz(\sigma) \triangleq z_1(s)$, $0 \le s \le t$

generate the same $\sigma$-algebra. Define $h_1(x_t,t) = N^{-1}(t)h(x_t,t)$. In deriving the representation theorem, we shall use (5.13), i.e., $z_1(t)$ and $h_1(x_t,t)$, rather than (5.2). The reason for introducing $z_1(t)$ and $h_1(x_t,t)$ is that the absolute continuity results given in Lemma 5.3.1 are phrased in terms of standard Wiener processes. Define a new measure $P_o$ on $(\Omega,\mathscr{F})$ by the formula

$$\frac{dP_o}{dP} = \exp\left[-\int_0^T h_1(x_t,t)dw_2(t) - \frac{1}{2}\int_0^T h_1(x_t,t)'h_1(x_t,t)dt\right] \tag{5.14}$$

Lemma 5.3.1 $P_o$ is a probability measure with the following properties:

(a) Under $P_o$, $z_1(s)$, $0 \le s \le T$ has components that are independent standard Wiener processes.

(b) Under $P_o$, the processes $z_1(t)$ $0 \le t \le T$ and $x(t)$, $0 \le t \le T$ are independent.

(c) The restriction of $P_o$ to $\sigma\{x(s), -\tau \le s \le T$ is the same as the corresponding restriction of P.

(d) $P << P_o$ and

$$\frac{dP}{dP_o} = \exp\left[ \int_0^T h_1(x_t,t)'dz_1(t) - \frac{1}{2}\int_0^T h_1'(x_t,t)h_1(x_t,t)dt \right] \tag{5.15}$$

Proof: The conditions imposed on the processes involved give rise to a virtually identical situation to that of Proposition 5.1, Chapter 6 of Wong [58]. The proof there applies without change to our case.

Let us denote integration with respect to P and $P_o$ by the expectation operators E and $E_o$, respectively. The reader should note that the true measure relevant to our estimation problem is P. The measure $P_o$ is strictly a device for obtaining the representation theorem.

Let us write $\mathscr{F}_t$ for the least $\sigma$-algebra containing $\sigma\{z(s),\ 0 \le s \le t\}$ and $\sigma\{x(s),\ -\tau \le s \le t\}$. It is well-known [59] that the function $L_t$ defined by

$$\begin{aligned} L_t &= \exp\left[ \int_0^t h_1'(x_s,s)dz_1(s) - \frac{1}{2}\int_0^t h_1'(x_s,s)h_1(x_s,s)ds \right] \\ &= \exp\left[ \int_0^t h'(x_s,s)R^{-1}(s)dz(s) - \frac{1}{2}\int_0^t h'(x_s,s)R^{-1}(s)h(x_s,s)ds \right] \end{aligned} \tag{5.16}$$

is a $(\mathscr{F}_t, P_o)$ martingale. Let $\phi$ be a real-valued measurable function on $\mathscr{C}$, with the property that $E|\phi(x_t)| < \infty$. Then we have the following (compare [60])

Lemma 5.3.2 The conditional expectation of $\phi(x_t)$ given $z^t$ can be written as

$$E[\phi(x_t)|z^t] = \frac{E_o[\phi(x_t)L_t|z^t]}{E_o[L_t|z^t]} \quad \text{a.s. } P \tag{5.17}$$

Proof: Since $\frac{dP}{dP_o} = L_T$, it is known [61] that

$$E[\phi(x_t)|z^t] = \frac{E_o[\phi(x_t)L_T|z^t]}{E_o[L_T|z^t]}$$

Using the properties of conditional expectations and the fact that $L_t$ is a $(\mathscr{F}_t, P_o)$ martingale, we can write

$$\begin{aligned} E_o[\phi(x_t)L_T|z^t] &= E_o\{E_o[\phi(x_t)L_T|\mathscr{F}_t]|z^t\} \\ &= E_o\{\phi(x_t)E_o[L_T|\mathscr{F}_t]|z^t\} \\ &= E_o\{\phi(x_t)L_t|z^t\} \end{aligned}$$

Similarly, $E_o[L_T|z^t] = E_o[L_t|z^t]$, and the lemma follows.

In what follows, we shall omit the qualification of almost sure equality for conditional expectations. Such qualifications will always be considered understood when conditional expectations are involved.

An application of the exponential formula [59] to the functional $L_t$ yields

$$L_t = 1 + \int_0^t L_s h'(x_s,s)R^{-1}(s)dz(s) \tag{5.18}$$

We shall recast the numerator of (5.17) into a more convenient form for later calculations. To that end, we make one more assumption:

(A.6) $\int_0^T E|\phi(x_t)h(x_t,t)|^2 dt < \infty$.

From now on, (A.6) will be assumed to hold. We then have

Lemma 5.3.3

$$E_o\left[\int_0^t \phi(x_t)L_s h'(x_s,s)R^{-1}(s)dz(s)\,|\,z^t\right]$$

$$= \int_0^t E_o[\phi(x_t)L_s h'(x_s,s)\,|\,z^t]R^{-1}(s)dz(s) \tag{5.19}$$

Proof: An almost identical lemma is proved in Zakai [62]. His proof can be adapted to our case without difficulty.

Lemma 5.3.4 We can then write the numerator of (5.17) as

$$E_o[\phi(x_t)L_t|z^t] = E_o[\phi(x_t)] + \int_0^t E_o\{E_o[\phi(x_t)|x_s]L_s h'(x_s,s)|z^s\}R^{-1}(s)dz(s) \tag{5.20}$$

Proof: Applying (5.18), we can write

$$E_o[\phi(x_t)L_t|z^t] = E_o[\phi(x_t)(1+\int_0^t L_s h'(x_s,s)R^{-1}(s)dz(s))|z^t]$$

$$= E_o[\phi(x_t)|z^t]+E_o\left[\int_0^t \phi(x_t)L_s h'(x_s,s)R^{-1}(s)dz(s)|z^t\right] \tag{5.21}$$

Since under $P_o$, $x(t)$ and $z(t)$ are completely independent, $E_o[\phi(x_t)|z^t] = E_o[\phi(x_t)]$. Combined with (5.19), we get

$$E_o[\phi(x_t)L_t|z^t] = E_o[\phi(x_t)]+\int_0^t E_o[\phi(x_t)L_s h'(x_s,s)|z^t]R^{-1}(s)dz(s)$$

Now $\int_0^t E_o[\phi(x_t)L_s h'(x_s,s)|z^t]R^{-1}(s)dz(s)$

$$= \int_0^t E_o\{E_o[\phi(x_t)L_s h'(x_s,s)|z^t \vee x^s]|z^t\}R^{-1}(s)dz(s)$$

$$= \int_0^t E_o\{E_o[\phi(x_t)|z^t v x^s]L_s h'(x_s,s)|z^t\}R^{-1}(s)dz(s) \tag{5.22}$$

since $L_s h'(x_s,s)$ is measurable with respect to the $\sigma$-algebra $z^t v x^s$, the smallest $\sigma$-algebra containing $z^t$ and $\sigma\{x(t), -\tau \leq t \leq s\}$. Using the Markov property of $x_t$ and the independence of x(t) and z(t) under $P_o$, we get

$$\int_0^t E_o\{E_o[\phi(x_t)|z^t v x^s]L_s h'(x_s,s)|z^t\}R^{-1}(s)dz(s)$$

$$= \int_0^t E_o\{E_o[\phi(x_t)|x_s]L_s h'(x_s,s)|z^t\}R^{-1}(s)dz(s)$$

$$= \int_0^t E_o\{E_o[\phi(x_t)|x_s]L_s h'(x_s,s)|z^s\}R^{-1}(s)dz(s) \tag{5.23}$$

The lemma follows from (5.21), (5.22) and (5.23).

In order to get the representation for the conditional expectation $E[\phi(x_t)|z^t]$, we need to evaluate the denominator of (5.17). This is done in the following lemma.

<u>Lemma 5.3.5</u> $$E_o(L_t|z^t)^{-1} = 1 - \int_0^t E_o(L_s|z^s)^{-1}E[h'(x_s,s)|z^s]R^{-1}(s)d\nu(s) \tag{5.24}$$

where $\nu(t) = z(t) - \int_0^t E[h(x_s,s)|z^s]ds$ is the innovations.

<u>Proof</u>: Since $L_t = 1 + \int_0^t L_s h'(x_s,s)R^{-1}(s)dz(s)$ Lemma 5.3.3 yields

$$E_o(L_t|z^t) = 1 + \int_0^t E_o[L_s h'(x_s,s)|z^s]R^{-1}(s)dz(s)$$

Applying the Ito differential rule to the function $E_o(L_t|z^t)^{-1}$ gives

$$dE_o(L_t|z^t)^{-1} = -E_o(L_t|z^t)^{-2}E_o(L_t h'(x_t,t)|z^t)R^{-1}(t)dz(t)$$

$$+ E_o(L_t|z^t)^{-3}E_o[L_t h'(x_t,t)|z^t]R^{-1}(t)E_o[L_t h'(x_t,t)|z^t]dt$$

$$= - E_o(L_t|z^t)^{-1}E[h(x_t,t)|z^t]dz(t)$$

$$+ E_o(L_t|z^t)^{-1}E[h'(x_t,t)|z^t]R^{-1}(t)E[h(x_t,t)|z^t]dt$$

$$= - E_o(L_t|z^t)^{-1}E[h(x_t,t)|z^t]R^{-1}(t)d\nu(t) \tag{5.25}$$

Since (5.25) is equivalent to (5.24), the lemma is proved.

From Lemma 5.3.2, we know that $E[\phi(x_t)|z^t]$ is given by the product of the right hand sides of (5.20) and (5.24). In order not to complicate the computations too much, we evaluate one of the terms separately and state it as a lemma.

Lemma 5.3.6 $$E_o(L_t|z^t)^{-1}\int_0^t E_o\{E_o[\phi(x_t)|x_s]L_s h'(x_s,s)|z^s\}R^{-1}(s)dz(s)$$

$$= \int_0^t E^s\{E_o[\phi(x_t)|x_s][h'(x_s,s)-\hat{h}'(x_s,s)]\}R^{-1}(s)d\nu(s)$$

$$+ \int_0^t E_o(L_s|z^s)^{-1}E_o[\phi(x_t)]\hat{h}'(x_s,s)R^{-1}(s)d\nu(s) \tag{5.26}$$

Proof: For convenience, define the functions

$$p(t,s) = E_o\{E_o[\phi(x_t)|x_s]L_s h'(x_s,s)|z^s\}R^{-1}(s)$$

$$q(s) = E_o(L_s|z^s)^{-1}\hat{h}'(x_s,s)R^{-1}(s)$$

Using (5.24), we can write the left hand side of (5.26) as

$$\int_0^t p(t,s)dz(s) - \int_0^t p(t,s)dz(s) \int_0^t q(s)d\nu(s) \qquad (5.27)$$

By properties of Ito integrals [63], (5.27) can be rewritten as

$$\int_0^t p(t,s)dz(s) - \int_0^t \int_0^s p(t,u)dz(u)q(s)d\nu(s)$$

$$- \int_0^t p(t,s) \int_0^s q(u)d\nu(u)dz(s) - \int_0^t p(t,s)q'(s)ds$$

$$= \int_0^t E_o(L_s|z^s)^{-1}p(t,s)dz(s) - \int_0^t p(t,s)q'(s)ds$$

$$- \int_0^t \int_0^s p(t,u)dz(u)q(s)d\nu(s)$$

(where we have used Lemma 5.3.5)

$$= \int_0^t E_o(L_s|z^s)^{-1}p(t,s)d\nu(s) - \int_0^t \int_0^s p(t,u)dz(u)q(s)d\nu(s) \qquad (5.28)$$

Now $-\int_0^t \int_0^s p(t,u)dz(u)q(s)d\nu(s)$

$$= -\int_0^t \{E_o[\phi(x_t)] + \int_0^s p(t,u)dz(u)\}q(s)d\nu(s)$$

$$+ \int_0^t E_o[\phi(x_t)]q(s)d\nu(s)$$

$$= -\int_0^t \{E_o[E_o(\phi(x_t)|x_s)]+\int_0^s E_o[E_o[E_o(\phi(x_t)|x_s v x_u)|x_u]\cdot$$

$$\cdot L_u h'(x_u,u)|z^u]R^{-1}(u)dz(u)\}q(s)d\nu(s)$$

$$+\int_0^t E_o[\phi(x_t)]q(s)d\nu(s)$$

(where we have used the smoothing property of conditional expectations)

$$= -\int_0^t \{E_o[E_o(\phi(x_t)|x_s)]+ \int_0^s E_o[E_o[E_o(\phi(x_t)|x_s)|x_u]L_u h'(s_u,u)|z^u]\cdot$$

$$\cdot R^{-1}(u)dz(u)\}q(s)d\nu(s)$$

$$+ \int_0^t E_o[\phi(x_t)]q(s)d\nu(s)$$

(using the Markov property of $x_t$)

$$= -\int_0^t E^s[E_o(\phi(x_t)|x_s)]E^s[h'(x_s,s)]R^{-1}(s)d\nu(s)$$

$$+ \int_0^t E_o(L_s|z^s)^{-1}E_o[\phi(x_t)]\hat{h}'(x_s,s)R^{-1}(s)d\nu(s) \tag{5.29}$$

(where we have successively used Lemma 5.3.4, the definition of q(s), and Lemma 5.3.2). Combining (5.28) and (5.29) gives the conclusion of the lemma.

We are now ready to state the representation theorem for the conditional moment functionals.

Theorem 5.3.1 Suppose $\phi:\mathscr{C} \to R$ is such that $E|\phi(x_t)| < \infty$, and (A.1) - (A.6) hold. Then we have the following representation for the conditional expectation of $\phi$ given $z^t$:

$$\hat{\phi}(x_t) = E_o[\phi(x_t)] + \int_0^t E^s\{E_o[\phi(x_t)|x_s][h'(x_s,s)-\hat{h}'(x_s,s)]\}\cdot \cdot R^{-1}(s)d\nu(s) \tag{5.30}$$

Proof: From Lemmas 5.3.2 and 5.3.4, we know that

$$\hat{\phi}(x_t) = E_o[\phi(x_t)]E_o(L_t|z^t)^{-1} + \int_0^t E_o\{E_o[\phi(x_t)|x_s]L_s h'(x_s,s)|z^s\}\cdot \cdot R^{-1}(s)dz(s)E_o(L_t|z^t)^{-1} \tag{5.31}$$

From Lemma 5.3.5, we conclude that

$$E_o[\phi(x_t)]E_o(L_t|z^t)^{-1} = E_o[\phi(x_t)] - \int_0^t E_o[\phi(x_t)]E_o(L_s|z^s)^{-1}\hat{h}'(x_s,s)R^{-1}(s)d\nu(s) \tag{5.32}$$

Substituting (5.32) and (5.26) into (5.31) gives (5.30). The proof is completed.

Corollary 5.3.1 The smoothed estimate $E^t[x(t+\theta)]$, $-\tau \leq \theta < 0$, is given by

$$E^t[x(t+\theta)] = E^{t+\theta}[x(t+\theta)] + \int_{t+\theta}^{t} E^s\{E_o[x(t+\theta)|x_s]\cdot [h'(x_s,s)-\hat{h}'(x_s,s)]\}R^{-1}(s)d\nu(s) \tag{5.33}$$

Proof: Using Theorem 5.3.1,

$$E^t[x(t+\theta)] = E_o[x(t+\theta)] + \int_0^t E^s\{E_o[x(t+\theta)|x_s][h'(x_s,s)-\hat{h}'(x_s,s)]\}R^{-1}(s)d\nu(s)$$

$$= E_o[x(t+\theta)] + \int_0^{t+\theta} E^s\{E_o[x(t+\theta)|x_s][h'(x_s,s)-\hat{h}'(x_s,s)]\}\cdot R^{-1}(s)d\nu(s)$$

$$+ \int_{t+\theta}^{t} E^s\{E_o[x(t+\theta)|x_s][h'(x_s,s)-\hat{h}'(x_s,s)]\}R^{-1}(s)d\nu(s)$$

$$= E^{t+\theta}[x(t+\theta)] + \int_{t+\theta}^{t} E^s\{E_o[x(t+\theta)|x_s][h'(x_s,s)-\hat{h}'(x_s,s)]\}\cdot R^{-1}(s)d\nu(s)$$

which is precisely (5.33).

Remark 5.3.1 Theorem 5.3.1 is a generalization of the corresponding representation results for nonlinear filtering of stochastic systems without time delays. While the measure transformation techniques employed here has been used before in connection with systems without delays [58], [62], the present form of the representation appears to be novel even when specialized to systems without delays. The reason for this is that in [58] and [62], the authors were interested in deriving equations for the unnormalized conditional density. They did not, therefore, give explicit formulas for the true conditional moments. In our case, conditional densities for the $x_t$ process do not even make sense. This motivated us to derive the Representation Theorem 5.3.1 for conditional moment functionals. We feel that the present form of the representation is much more convenient to use when we derive stochastic differential equations for conditional moment functionals. Furthermore, it shows clearly the role played by the infinitesimal generator of the Markov process $x_t$. In the next two sections, we shall apply Theorem 5.3.1 to derive stochastic differential equations for the nonlinear and linear filtering problems.

## 5.4 Stochastic Differential Equations for Nonlinear Filtering of Delay Systems

While Theorem 5.3.1 can be thought of as solving our nonlinear filtering problem abstractly, it does not give a recursive solution. That is, for any fixed t, formula (5.30) is valid. However, knowledge of $E[\phi(x_t)|z^t]$ and the observations $z(s)$, $t \leq s \leq t + \Delta$, is not sufficient to determine $E[\phi(x_{t+\Delta})|z^{t+\Delta}]$. In fact, we must completely re-process our past observations. For implementation purposes, one would like to obtain a stochastic differential equation for the evolution of $E[\phi(x_t)|z^t]$. As is expected, this will require certain smoothness conditions on the functional $\phi$. In this section, we shall investigate the following question: under what conditions on $\phi$ can we obtain a stochastic differential equation for $E[\phi(x_t)|z^t]$?

Let us first make a few preliminary calculations. Let $\varepsilon > 0$ be fixed and let $\hat{h}(x_s,s) = E[h(x_s,s)|z^s]$. Then

$$E[\phi(x_{t+\varepsilon})|z^{t+\varepsilon}] - E[\phi(x_t)|z^t]$$

$$= E_o[\phi(x_{t+\varepsilon})-\phi(x_t)] + \int_0^{t+\varepsilon} E\{E_o[\phi(x_{t+\varepsilon})|x_s][h(x_s,s)-\hat{h}(x_s,s)]'|z^s\}\cdot R^{-1}(s)d\nu(s)$$

$$- \int_0^t E\{E_o[\phi(x_t)|x_s][h(x_s,s)-\hat{h}(x_s,s)]'|z^s\}R^{-1}(s)d\nu(s)$$

$$= E_o[\phi(x_{t+\varepsilon})-\phi(x_t)] + \int_0^t E\{E_o[\phi(x_{t+\varepsilon})-\phi(x_t)|x_s]\cdot [h(x_s,s)-\hat{h}(x_s,s)]'|z^s\}R^{-1}(s)d\nu(s)$$

$$+ \int_t^{t+\varepsilon} E\{E_o[\phi(x_{t+\varepsilon})|x_s][h(x_s,s)-\hat{h}(x_s,s)]'|z^s\}R^{-1}(s)d\nu(s) \qquad (5.34)$$

We will be primarily interested in the limit as $\varepsilon \to 0$. The last term in (5.34) is straightforward to evaluate in that case. The difficult calculation is that of evaluating terms of the form

$$E_o[\phi(x_{t+\varepsilon}) - \phi(x_t)|x_s]$$

$$= E_o\{E_o[\phi(x_{t+\varepsilon})-\phi(x_t)|x_t \vee x_s]|x_s\}$$

$$= E_o\{E_o[\phi(x_{t+\varepsilon})-\phi(x_t)|x_t]|x_s\}$$

where we have used the Markov property of $x_t$. The evaluation of $E_o[\phi(x_{t+\varepsilon})-\phi(x_t)|x_t]$ corresponds to the determination of the infinitesimal generator of the Markov process $x_t$.

Let us assume for the time being that the functional $\phi$ is in the domain of the generator, $\mathscr{A}_t$, of the Markov process $x_t$. Then it is true (see Dynkin [64]) that

$$E_o[\phi(x_t)|x_s] - \phi(x_s) = \int_s^t E_o[\mathscr{A}_u\phi(x_u)|x_s]du \qquad (5.35)$$

In this case, we have the following

<u>Theorem 5.4.1</u> Let $\phi$ satisfy the conditions of Theorem 5.3.1. In addition, let $\phi(x_t)$ be in the domain of the infinitesimal generator $\mathscr{A}_t$ of $x_t$ and suppose that

$$\int_0^T E|[\mathscr{A}_t\phi(x_t)]h(x_t,t)|^2 dt$$

Then the functional $\phi(x_t)$ satisfies the stochastic differential equation

$$d\hat{\phi}(x_t) = E[\mathscr{A}_t\phi(x_t)|z^t]dt + E[\phi(x_t)(h(x_t,t)-\hat{h}(x_t,t))'|z^t]R^{-1}(t)d\nu(t) \qquad (5.36)$$

<u>Proof</u>: We start with (5.34). For simplicity, let $e_h(t) = h(x_t,t)-\hat{h}(x_t,t)$. Then

$$\int_0^t E\{E_o[\phi(x_{t+\varepsilon})-\phi(x_t)|x_s]e_h'(s)|z^s\}R^{-1}(s)d\nu(s)$$

$$= \int_0^t E\{E_o\{E_o[\phi(x_{t+\varepsilon})-\phi(x_t)|x_t]|x_s\}e_h'(s)|z^s\}R^{-1}(s)d\nu(s)$$

$$= \int_0^t E\{E_o\{\int_t^{t+\varepsilon} E_o[\mathscr{A}_u\phi(x_u)|x_t]du|x_s\}e_h'(s)|z^s\}R^{-1}(s)d\nu(s)$$

$$= \int_0^t E\{\int_t^{t+\varepsilon} E_o[\mathscr{A}_u\phi(x_u)|x_s]e_h'(s)|z^s\}duR^{-1}(s)d\nu(s)$$

$$= \int_t^{t+\varepsilon}\int_0^t E\{E_o[\mathscr{A}_u\phi(x_u)|x_s]e_h'(s)|z^s\}R^{-1}(s)d\nu(s)du \qquad (5.37)$$

using Lemma 5.3.3.

Next,

$$\int_t^{t+\varepsilon} E\{E_o[\phi(x_{t+\varepsilon})|x_s]e_h'(s)|z^s\}R^{-1}(s)d\nu(s)$$

$$= \int_t^{t+\varepsilon} E\{[\phi(x_s)+\int_s^{t+\varepsilon} E_o[\mathscr{A}_u\phi(x_u)|x_s)]e_h'(s)|z^s\}R^{-1}(s)d\nu(s)$$

$$= \int_t^{t+\varepsilon} E[\phi(x_s)e_h'(s)|z^s]R^{-1}(s)d\nu(s)$$

$$+ \int_t^{t+\varepsilon}\int_s^{t+\varepsilon} E\{E_o[\mathscr{A}_u\phi(x_u)|x_s]e_h'(s)|z^s\}R^{-1}(s)d\nu(s) \qquad (5.38)$$

Finally

$$E_o[\phi(x_{t+\varepsilon})-\phi(x_t)] = E_o\{E_o[\phi(x_{t+\varepsilon}) - \phi(x_t)|x_t]$$

$$= E_o\{\int_t^{t+\varepsilon} E_o[\mathscr{A}_u\phi(x_u)|x_t]du\}$$

$$= \int_t^{t+\varepsilon} E_o[\mathscr{A}_u\phi(x_u)]du$$

$$= \int_t^{t+\varepsilon} E[\mathscr{A}_u\phi(x_u)|z^u]du-$$

$$- \int_t^{t+\varepsilon}\int_0^u E\{E_o[\mathscr{A}_u\phi(x_u)|x_s]e_h'(s)|z^s\}R^{-1}(s)d\nu(s)du \qquad (5.39)$$

using the representation in Theorem 5.3.1. Adding (5.37) - (5.39), we get

$$E[\phi(x_{t+\varepsilon})|z^{t+\varepsilon}] - E[\phi(x_t)|z^t]$$

$$= \int_t^{t+\varepsilon} E[\mathscr{A}_u\phi(x_u)|z^u]du + \int_t^{t+\varepsilon} E[\phi(x_u)e_h'(u)|z^u]R^{-1}(u)d\nu(u)$$

$$+ \int_t^{t+\varepsilon}\int_s^{t+\varepsilon} E\{E_o(\mathscr{A}_u\phi(x_u)|x_s]e_h'(s)|z^s\}duR^{-1}(s)d\nu(s)$$

$$- \int_t^{t+\varepsilon}\int_t^{u} E\{E_o[\mathscr{A}_u\phi(x_u)|x_s]e_h'(s)|z^s\}R^{-1}(s)d\nu(s)du \tag{5.40}$$

Another application of Lemma 5.3.3 shows that the last two terms of (5.40) add to zero. The proof is completed.

Theorem 5.4.1 is a generalization to systems with delays of the usual formula for conditional moments of ordinary diffusion processes. While the form of the stochastic differential equation is exactly the same as that for diffusion processes, there is a subtle difference. In the diffusion process case, x(t) itself is a Markov process. In the delay case, while x(t) still satisfies a stochastic differential equation, it is no longer a Markov process. This special structure of stochastic delay systems is clearly shown in (5.36), where the infinitesimal generator $\mathscr{A}_t$ of $x_t$ plays a crucial role. Thus, in order to apply Theorem 5.4.1, we need to characterize the domain of the infinitesimal generator of the Markov process $x_t$. These are functionals for which the limit

$$\lim_{h \to 0} \frac{1}{h} \{E[\phi(x_{t+h})|x_t] - \phi(x_t)\} \tag{5.41}$$

necessarily exists. This immediately rules out functionals of the form $\phi(x_t) = \phi[x(t+\theta)]$ for some $\theta \in (-\tau,0)$. For in this case,

$$E[\phi(x_{t+h})|x_t] = E\{\phi[x(t+\theta+h)]|x_t\}$$

$$= \phi[x(t+\theta+h)]$$

whenever $-\tau \leq \theta + h \leq 0$. Since the sample path x(t) is not differentiable, the limit in (5.41) does not exist. It is not possible therefore, to derive a stochastic differential equation for a functional of the form $\phi[x(t+\theta)]$ using Theorem 5.4.1.

There are certain special classes of functionals $\phi$ which are in the domain of the generator $\mathscr{A}_t$. These are those considered by Kushner [54]. We shall simply state these results.

<u>Case 1</u>: Suppose the functional $\phi(x_t) = \phi[x(t)]$, and is twice continuously differentiable in its argument, then

$$\mathscr{A}_t\phi[x(t)] = f(x_t,t)'\phi_x[x(t)] + \frac{1}{2}\text{ tr } Q(t)\phi_{xx}(x(t))$$

where $\phi_x$ is the n-vector whose $i^{th}$ component is $\frac{\partial\phi}{\partial x_i}[x(t)]$. Substituting into (5.36) gives

$$d\hat{\phi}[x(t)] = E^t\{f(x_t,t)'\phi_x[x(t)] + \frac{1}{2}\text{ tr } Q(t)\phi_{xx}[x(t)]\}dt$$

$$+ E^t\{\phi[x(t)][h'(x_t,t)-\hat{h}'(x_t,t)]\}R^{-1}(t)d\nu(t) \tag{5.42}$$

This is precisely the formula derived in Fujisaki et al., [51] using a different technique. We shall denote the operator $\mathscr{A}_t$ in this special case by $\mathscr{L}_t$. In particular, the conditional mean $\hat{x}(t)$ satisfies

$$d\hat{x}(t) = E^t[f(x_t,t)]dt + E^t\{x(t)[h'(x_t,t)-\hat{h}'(x_t,t)]\}R^{-1}(t)d\nu(t) \tag{5.43}$$

Equation (5.43) and the formula for the smoothed estimate, (5.33), will be useful in discussing the optimal linear filter.

Case 2: Let $\phi(x_t) = \int_{-\tau}^{0} \psi(\theta)g[x(t+\theta), x(t)]d\theta$ where $\psi$ is continuously differentiable on $[-\tau,0]$ and g is twice continuously differentiable in its second argument. Then

$$\mathscr{A}_t\phi(x_t) = \dot{\psi}(0)g[x(t),x(t)]-\psi(-\tau)g[x(t-\tau),x(t)]$$
$$- \int_{-\tau}^{0} \dot{\psi}(\theta)g[x(t+\theta),x(t)]d\theta + \int_{-\tau}^{0} \psi(\theta)\mathscr{L}_t g[x(t+\theta),x(t)]d\theta \tag{5.44}$$

where $\mathscr{L}_t$ is the operator defined in Case 1 and acts on g as a function of x(t) only.

Case 3: Let $\phi(x_t) = D[F(x_t)]$ where D is a twice continuously differentiable real-valued function, and $F(x_t) = \int_{-\tau}^{0} \psi(\theta)g[x(t+\theta), x(t)]d\theta$ is the type of functional described in Case 2. Then

$$\mathscr{A}_t\phi(x_t) = D_\alpha(\alpha)\Big|_{\alpha=F(x_t)} \mathscr{A}_t F(x_t) + \frac{1}{2} D_{\alpha\alpha}(\alpha)\Big|_{\alpha=F(x_t)} \cdot G$$

where

$$G = \int_{-\tau}^{0}\int_{-\tau}^{0} \psi(\theta)\psi(\eta)\sum_{i,j} g_{\beta_i}[x(t+\theta),x(t)]g_{\beta_j}[x(t+\theta),x(t)]Q_{ij}(t)d\theta d\eta$$

and $g_{\beta_i}$ denotes partial differentiation of g with respect to the $i^{th}$ component of the second argument.

From the above special cases, we can see that basically we need twice continuous differentiability of $\phi$ with respect to the dependence on x(t), and Fréchet differentiability with respect to the dependence on the piece of the trajectory $x_t$. As discussed before, this rules out functionals of the form $\phi[x(t+\theta)],\theta \in [-\tau,0)$. Hence for nonlinear systems with point delays, any attempt in deriving stochastic differential equations for conditional moment functionals will have to face the difficulty of functionals not being in the domain of the generator of the Markov process $x_t$. For example, as in the multipath communication problem mentioned in Chapter 1, if the observation process is of the form

$$dz(t) = \{h_1[x(t)]+h_2[x(t-\tau)]\}dt + dv(t)$$

then for a twice continuously differentiable $\phi[x(t)]$, we get

$$d\hat{\phi}[x(t)] = \widehat{\mathscr{L}_t\phi[x(t)]} + [\widehat{\phi(x(t))h_1(x(t))} + \widehat{\phi(x(t))h_2(x(t-\tau))}$$
$$-\hat{\phi}(x(t))\hat{h}_1(x(t)) - \hat{\phi}(x(t))\hat{h}_2(x(t-\tau))]'R^{-1}(t)d\nu(t) \quad (5.45)$$

If we try to write stochastic differential equations for the unknown quantities on the right hand side of (5.45), we see that this cannot be done for all the unknowns, since $\phi(x(t))h_2(x(t-\tau))$ does not lie in the domain of $\mathscr{A}_t$. Of course, there are many physical problems (for example, radar problems with spread targets [11]) where the observations are of the form $h(x_t) = \int_{-\tau}^{0} \psi(\theta)H[x(t+\theta),x(t)]d\theta$.

Moreover, one can approximate point delays by distributed delays of the above form. This will allow us to write a stochastic differential equation for $\widehat{\phi[x(t)]h(x_t)}$. However, we will then get the unknown $\widehat{\mathscr{A}_t\phi[x(t)]h(x_t)}$ in our equation for $\widehat{\phi[x(t)]h(x_t)}$. If $\psi(-\tau) \neq 0$, $\mathscr{A}_t\phi[x(t)]h(x_t)$ will contain a term with point delay (see Case 2 above), and we are faced with the same problem as before. In general, if the functionals involved are in the domain of $\mathscr{A}_t^i$, $i = 1,\ldots n$, we can write n coupled stochastic differential equations involving the moment functionals, just as in the diffusion process case. It should be clear from the above discussion that this puts rather severe restrictions on the functionals involved. Thus, the study of approximations to the optimal nonlinear filter for delay system remains an important and open problem.

There is, however, one special case where the optimal filter can be completely specified even when there are point delays in the system. This is the linear case with Gaussian distributions and will be treated next.

## 5.5 Optimal Filtering of Linear Stochastic Delay Systems

We consider the system defined by

$$dx(t) = a(x_t,t)dt + dw(t)$$
$$x(\theta) = x_o(\theta) \qquad \theta \in [-\tau,0] \tag{5.46}$$

$$dz(t) = c(x_t,t)dt + dv(t) \tag{5.47}$$

Here $a: \mathscr{C}x[0,T] \to R$

$c: \mathscr{C}x[0,T] \to R$

are continuous linear functionals given by

$$a(x_t,t) = \int_{-\tau}^{0} d_\theta A(t,\theta)x(t+\theta)$$

$$c(x_t,t) = \int_{-\tau}^{0} d_\theta C(t,\theta)x(t+\theta)$$

where $A(t,\theta)$ and $C(t,\theta)$ satisfy the same conditions as those imposed on such functions in section 5.2. Also, we take $x_o$ to be a Gaussian process on $[-\tau,0]$ with $\sup_\theta E|x_o(\theta)|^2 < \infty$. By a similar argument to the case without delays [56], it is readily seen that the conditional distributions of $x(t+\theta)$, for any $\theta \in[-\tau,0]$, given $z(s)$, $0 \leq s \leq t$, is Gaussian. For greater clarity in the subsequent exposition, we shall write $\hat{x}(t|t) = E\{x(t)|z^t\}$ and $\hat{x}(t+\theta|t) = E\{x(t+\theta|z^t\}$, $\theta \in[-\tau,0]$. Using (5.43), we immediately obtain the following stochastic differential equation for the conditional mean

$$d\hat{x}(t|t) = \int_{-\tau}^{0} d_\theta A(t,\theta)\hat{x}(t+\theta|t)dt + \left[ \int_{-\tau}^{0} E^t(x(t)x'(t+\theta|t))d_\theta C'(t,\theta) - \int_{-\tau}^{0} \hat{x}(t|t)x'(t+\theta|t)d_\theta C'(t,\theta)\right]R^{-1}(t)d\nu(t) \tag{5.48}$$

Here the innovations process $\nu(t)$ is given by $z(t)-\int_0^t\int_{-\tau}^0 d_\theta C(s,\theta)\hat{x}(s+\theta|s)ds$ and hence depends, in general, on the smoothed estimates $\hat{x}(s+\theta|s)$, $-\tau \leq \theta \leq 0$, $0 \leq s \leq t$. Define the "smoothed" conditional error covariance as

$$P(t,\theta,\xi) = E^t\{[x(t+\theta)-\hat{x}(t+\theta|t)][x(t+\xi)-\hat{x}(t+\xi|t)]'\}$$

Then (5.48) can be rewritten as

$$d\hat{x}(t|t) = \int_{-\tau}^0 d_\theta A(t,\theta)\hat{x}(t+\theta|t)dt + \int_{-\tau}^0 P(t,0,\theta)d_\theta C'(t,\theta)R^{-1}(t)d\nu(t) \qquad (5.49)$$

To evaluate the unknown terms on the right side of (5.49), we use (5.44) to write the smoothed estimate as

$$\hat{x}(t+\theta|t) = \hat{x}(t+\theta|t+\theta)+\int_{t+\theta}^t E^s\{x(t+\theta)[c'(x_s,s)-\hat{c}'(x_s,s)]\}R^{-1}(s)d\nu(s)$$

$$= \hat{x}(t+\theta|t+\theta)+\int_{t+\theta}^t\int_{-\tau}^0 P(s,t+\theta-s,\xi)d_\xi C(s,\xi)'R^{-1}(s)d\nu(s) \qquad (5.50)$$

for $\theta \in[-\tau,0]$.

An inspection of (5.49) and (5.50) shows that the optimal linear filter is completely characterized by $\hat{x}(t+\theta|t)$, $-\tau \leq \theta \leq 0$, and the "smoothed" error covariance function $P(t,\theta,\xi)$. It remains only to derive appropriate equations for $P(t,\theta,\xi)$. Since the derivations are quite lengthy, we will present them in Appendix B, and state the final result here as

Theorem 5.5.1 The optimal filter for the system (5.46), (5.47) is characterized by the following sets of equations:

(i) Generation of the condition mean $\hat{x}(t|t)$:

$$d\hat{x}(t|t) = \int_{-\tau}^{0} d_\theta A(t,\theta)\hat{x}(t+\theta|t)dt + \int_{-\tau}^{0} P(t,0,\theta)d_\theta C'(t,\theta)R^{-1}(t)d\nu(t) \qquad (5.51)$$

(ii) Generation of the smoothed estimate $\hat{x}(t+\theta|t)$, $-\tau \leq \theta \leq 0$:

$$\hat{x}(t+\theta|t) = \hat{x}(t+\theta|t+\theta)+\int_{t+\theta}^{t}\int_{-\tau}^{0} P(s,t+\theta-s,\xi)d_\xi C(s,\xi)'R^{-1}(s)d\nu(s) \qquad (5.52)$$

(iii) Generation of the smoothed error covariance $P(t,\theta,\xi)$:

$$\frac{d}{dt} P(t,0,0)= \int_{-\tau}^{0} P(t,0,\theta)d_\theta A'(t,\theta)+ \int_{-\tau}^{0} d_\theta A(t,\theta)P(t,\theta,0) - \int_{-\tau}^{0}\int_{-\tau}^{0} P(t,0,\theta)d_\theta C'(t,\theta)R^{-1}(t)d_\xi C(t,\xi)P(t,\xi,0)+Q(t) \qquad (5.53)$$

$$\sqrt{2}\, P_\eta(t,\theta,0)= \int_{-\tau}^{0} P(t,\theta,\xi)d_\xi A'(t,\xi) - \int_{-\tau}^{0}\int_{-\tau}^{0} P(t,\theta,\xi)d_\xi C'(t,\xi)R^{-1}(t)d_\alpha C(t,\alpha)P(t,\alpha,0) \qquad (5.54)$$

$$\sqrt{3}\, P_\sigma(t,\theta,\xi)= -\int_{-\tau}^{0}\int_{-\tau}^{0} P(t,\theta,\beta)d_\beta C'(t,\beta)R^{-1}(t)d_\alpha C(t,\alpha)P(t,\alpha,\xi) \qquad (5.55)$$

where $\eta$ is the unit vector in the (1,-1) direction, $\sigma$ is the unit vector in the (1,-1,-1) direction, and $P_\eta(t,\theta,0)$ and $P_\sigma(t,\theta,\xi)$ are the directional derivatives of $P(t,\theta,0)$ and $P(t,\theta,\xi)$ in the directions $\eta$ and $\sigma$ respectively. The initial conditions are given by

$$\hat{x}(0|0) = \bar{x}_o(0), \quad \hat{x}(\theta|0) = \bar{x}_o(\theta) \qquad \theta \in [-\tau,0]$$

$$P(0,\theta,\xi) = \Sigma_o(\theta,\xi) = E\{[x_o(\theta)-\bar{x}_o(\theta)][x_o(\xi)-\bar{x}_o(\xi)]'\} \tag{5.56}$$

A few comments about Theorem 5.5.1 is in order. Notice the great similarity as well as the striking differences in the form of the solution to the filtering problem for linear delay systems with Gaussian distributions and the corresponding solution for linear systems without delays. In our case, we need to characterize not only the conditional mean $\hat{x}(t|t)$ (as in the non-delay case), but also the smoothed estimate $\hat{x}(t+\theta|t)$, not only the estimation error covariance $P(t,0,0)$, but also the smoothed error covariance function $P(t,\theta,\xi)$. The complexity of the solution is clearly considerably increased when there are delays in the system. It is hoped, however, that the results presented in this and the next chapter will provide a theoretical foundation for the study of implementable filters for delay systems.

Note that our development only shows that $P(t,\theta,\xi)$ is continuous and has directional derivatives. Of course, if $P(t,\theta,\xi)$ were actually differentiable in $(t,\theta,\xi)$, we could rewrite

$$\sqrt{2}\, P_\eta(t,\theta,0) = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial \theta}\right) P(t,\theta,0)$$

and

$$\sqrt{3}\, P_\sigma(t,\theta,0) = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial \theta} - \frac{\partial}{\partial \xi}\right) P(t,\theta,\xi)$$

We would then obtain partial differential equations for $P(t,\theta,\xi)$.

Theorem 5.5.1 completely solves the filtering problem for general linear stochastic delay systems. Notice that although functionals of the form $\phi(x_t) = x(t+\theta)$, $-\tau < \theta < 0$, do not lie in the domain of the generator of $x_t$, we are still able to calculate the directional derivatives of the covariances. This is a consequence of the linearity and Gaussian assumptions.

Let us make a few remarks on the relationship of our work to past investigations. The linear filtering problem for systems with only point delays was first considered by Kwakernaak [65], whose results are similar to those presented in this section. He restricted his attention to linear filters and followed the approach of Kalman and Bucy [66], and his derivations were formal. In particular, he used partial derivatives liberally when these were not defined. His "proofs", therefore, were far from satisfactory. Lindquist also considered the linear filtering problem in [56]. However, he only gave a complicated integral equation for the filter gain, and did not derive equations for the covariances. His results are thus incomplete and do not display the structure of the filter equations. In particular, they are not suitable for the filter stability investigations that we pursue in Chapter 6. Recently, Mitter and Vinter [67] have studied the linear filtering problem from the viewpoint of stochastic evolution equations. While they did take in account some of the special features of delay systems, they had to exclude point delays from their observation model, an unsatisfactory restriction from various points of view. Furthermore, their derivation of the optimal filtering equations proceeds via a dual control problem. As such, their approach cannot be extended to the nonlinear situation. By contrast, our approach enables us to allow a very general model for the stochastic delay system under consideration, including distributed as well as point delays. Covariance equations have been derived rigorously and the structure of the optimal filter is clearly displayed. We feel, therefore, that our results on linear filtering are the most complete and satisfactory to date.

In the special case where $x_o \equiv 0$, $a(x_t,t) = Ax(t) + Bx(t-\tau)$, $c(x_t,t) = Cx(t)$, Q, R constant matrices, it can be shown by exploiting the connection between linear optimal filtering and optimal control with quadratic criterion that $P(t,\theta,\xi)$ are in fact continuously differentiable in $(t,\theta,\xi)$. When we compare the solutions to the linear optimal control and optimal linear filtering problems in Chapter 6, it will be helpful to use the notation

$$P_o(t) = P(t,0,0) \tag{5.57}$$

$$P_1(t,\theta) = P(t,\theta,0) \tag{5.58}$$

$$P_2(t,\theta,\xi) = P(t,\theta,\xi) \tag{5.59}$$

In this case the optimal filter is given by the equations

$$d\hat{x}(t|t)=A\hat{x}(t|t)+B\hat{x}(t-\tau|t)+P_o(t)C'R^{-1}[dz(t)-C\hat{x}(t|t)dt] \tag{5.60}$$

$$\hat{x}(t-\tau|t)=\hat{x}(t-\tau|t-\tau)+\int_{t-\tau}^{t} P_2(s,t-\tau-s,0)C'R^{-1}[dz(s)-C\hat{x}(s|s)ds] \tag{5.61}$$

$$\hat{x}(\theta|0) = 0 \qquad -\tau \leq \theta \leq 0$$

$$\frac{d}{dt} P_o(t) = AP_o(t)+P_o(t)A'-P_o(t)C'R^{-1}CP_o(t)+Q+BP_1(t,-\tau)+P_1'(t,-\tau)B' \tag{5.62}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial \theta}\right) P_1(t,\theta) = P_1(t,\theta)[A'-C'R^{-1}CP_o(t)]+P_2(t,\theta,-\tau)B' \tag{5.63}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial \theta} - \frac{\partial}{\partial \xi}\right) P_2(t,\theta,\xi) = -P_1(t,\theta)C'R^{-1}CP_1'(t,\xi) \tag{5.64}$$

with $\quad P_o(0) = P_1(0,\theta) = P_2(0,\theta,\xi) = 0 \qquad -\tau \leq \theta,\ \xi \leq 0$

$$P_1(t,0) = P_o(t)$$

$$P_2(t,\theta,0) = P_1(t,\theta)$$

$$P_o(t) = P_o'(t),\ P_2(t,\theta,\xi) = P_2'(t,\xi,\theta) \tag{5.65}$$

We shall be using these equations extensively in the next chapter.

Notice that in this special case, $\hat{x}(t|t)$ depends only on $\hat{x}(s|s)$, $t-\tau \leq s \leq t$, and from (5.60) and (5.61), we can obtain an explicit delay equation for $\hat{x}(t|t)$. This is because we have no delays in the observations. If $c(x_t,t)$ were of the form $C_o x(t) + C_1 x(t-\tau)$, then we would have the following equations

$$d\hat{x}(t|t)=A\hat{x}(t|t)+B\hat{x}(t-\tau|t)+[P_o(t)C_o'+P_1'(t,-\tau)C_1']R^{-1}\cdot$$
$$\cdot[dz(t)-C_o\hat{x}(t|t)dt-C_1\hat{x}(t-\tau|t)dt] \tag{5.66}$$

$$\hat{x}(t-\tau|t)=\hat{x}(t-\tau|t-\tau)+\int_{t-\tau}^{t} [P_2(s,t-\tau-s,0)C_o'+P_2(s,t-\tau-s,-\tau)C_1']R^{-1}\cdot$$
$$[dz(s)-C_o\hat{x}(s|s)ds-C_1\hat{x}(s-\tau|s)ds] \tag{5.67}$$

If we substitute (5.67) into (5.66), we see that $\hat{x}(t|t)$ depends not only on $\hat{x}(s|s)$, $t-\tau \leq s \leq t$, but also on delayed smoothed estimates $\hat{x}(s-\tau|s)$, $t-\tau \leq s \leq t$. The presence of the delayed smoothed estimates prevents us from obtaining a delay equation for $\hat{x}(t|t)$. The additional complications which delayed observations introduce will be discussed further in section 6.7.

It is worth pointing out that in the case where $x_o \equiv 0$, $\hat{x}(t+\theta|t)$, $-\tau \leq \theta \leq 0$, can be expressed as a Wiener integral with respect to the observations $z(s)$, $0 \leq s \leq t$, i.e.,

$$\hat{x}(t+\theta|t) = \int_0^t K(t,\theta,s)\,dz(s) \tag{5.66}$$

for some kernel $K(t,\theta,s)$. To see this, note that $x(t)$ is a Gaussian process. Hence the orthogonal projection of $x(t+\theta)$ onto the Hilbert space spanned by the observations $z(s)$, $0 \leq s \leq t$ is the same as the conditional mean. Since the orthogonal projections can be expressed in the form of the right hand side of (5.66), so can $\hat{x}(t+\theta|t)$. This fact will be used in Chapter 6.

So far, we have studied optimal filtering for stochastic delay systems on a finite time interval. To make the filtering theory of delay systems more complete, we shall study the asymptotic behavior and stability properties of the optimal linear filter in the next chapter.

# CHAPTER 6

# STABILITY OF LINEAR OPTIMAL FILTERS AND CONTROL SYSTEMS

## 6.1 Introduction

In the previous chapter, we derived optimal filter equations for linear and nonlinear stochastic delay differential systems. In this chapter, we study the stability of the linear optimal filter for delay systems. We shall concern ourselves primarily with stochastic systems with delays in the system dynamics, but no delays in the observations. Extensions to the general case, including delays in the observations, will also be discussed. In proving the stability results, we make essential use of the duality between optimal filtering of linear stochastic delay systems and optimal control of linear delay systems with quadratic cost. We shall therefore begin the development with a summary of the known results for optimal control of linear delay systems with quadratic cost, and give some new results on the asymptotic stability of the optimal closed-loop control system. This brings into focus the role played by the concepts of stabilizability and observability. Next, we give a duality theorem which establishes the connection between optimal linear filtering and optimal control of linear delay systems. This enables us to identify the gains of the optimal linear filter with the gains of a "dual" control system in an appropriate way. We can therefore exploit the known convergence of the control gains to infer the convergence of the filter gains. The stability of the optimal filter is then established by constructing a Lyapunov functional. Once the stability of the optimal linear filter has been established, it is relatively straightforward to prove the asymptotic stability of the optimal linear stochastic control system, drawing upon known results on the separation theorem for stochastic delay systems [56], [68].

## 6.2 Optimal Control of Linear Delay Systems with Quadratic Cost

The problem of optimal control of linear delay systems with quadratic cost has received considerable attention in recent years. Various methods [69] - [73] have been devised for solving this problem. The simplest version of the problem can be formulated as follows:

We are given a linear constant delay system of the form

$$\frac{dx(t)}{dt} = Ax(t) + Bx(t-\tau) + Cu(t)$$

$$x(\theta) = \phi(\theta) \qquad \theta \in [-\tau, 0] \tag{6.1}$$

Our choice of function space will be $\mathscr{C}$, although basically the same results hold for other choices, such as $M^2$ [29] , [73]. The admissible control set U is the set of $R^m$-valued $L_2$ functions on [0, T]. The cost functional is given by

$$J_T(u,\phi) = \int_0^T [x'(t)Qx(t) + u'(t)Ru(t)]dt$$

where Q and R are symmetric matrices of appropriate dimensions, $Q \geq 0$, $R > 0$. The objective is to find a control $u \in U$ such that $J_T(u, \phi)$ is minimized. The cases when T is finite or infinite have been studied. We shall briefly survey the known results and then give some extensions.

We start with the case when $T < \infty$. Then it is well-known [69] - [73] that the optimal control can be expressed in feedback form by

$$u^*(t) = -R^{-1}C'K_o(t)x(t) - R^{-1}C' \int_{-\tau}^{0} K_1(t,\theta)x(t+\theta)d\theta \quad (6.2)$$

The feedback gains satisfy the following coupled set of partial differential equations

$$\frac{d}{dt} K_o(t) = -A'K_o(t) - K_o(t)A + K_o(t)CR^{-1}C'K_o(t) - Q - K_1(t,0) - K_1'(t,0) \quad (6.3)$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial \theta}\right) K_1(t,\theta) = -[A'-K_o(t)CR^{-1}C']K_1(t,\theta) - K_2(t,0,\theta) \quad (6.4)$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial \theta} - \frac{\partial}{\partial \xi}\right) K_2(t,\theta,\xi) = K_1'(t,\theta)CR^{-1}C'K_1(t,\xi) \quad (6.5)$$

The boundary conditions are

$$\left.\begin{aligned} &K_o(T) = K_1(T,\theta) = K_2(T,\theta,\xi) = 0 \qquad -\tau \le \theta,\ \xi \le 0 \\ &K_1(t,-\tau) = K_o(t)B \\ &K_2(t,-\tau,\theta) = B'K_1(t,\theta) \end{aligned}\right\} \quad (6.6)$$

Furthermore,

$$K_o(t) = K_o'(t), \qquad K_2(t,\theta,\xi) = K_2'(t,\xi,\theta) \quad (6.7)$$

The optimal cost can be expressed as

$$J_T^*(\phi) = \phi'(0)K_o(0)\phi(0) + \int_{-\tau}^{0} \phi'(0)K_1(0,\theta)\phi(\theta)d\theta$$

$$+ \int_{-\tau}^{0} \phi'(\theta)K_1'(0,\theta)\phi(0)d\theta$$

$$+ \int_{-\tau}^{0}\int_{-\tau}^{0} \phi'(\theta)K_2(0,\theta,\xi)\phi(\xi)d\theta d\xi \qquad (6.8)$$

The integral term in Eq. (6.2) represents the effects of the delay on the optimal control. Thus the optimal control is seen to be given by a linear map K operating on the complete state $x_t$. Similarly, the optimal cost can be thought of as a generalized quadratic form.

We now discuss the infinite-time control problem, i.e., $T = \infty$. For the control problem to be meaningful, we need some condition to ensure that the optimal cost will be finite. It turns out that the system-theoretic concepts of stabilizability and observability introduced in Chapter 2 play a crucial role. The following is the basic result on the convergence of the optimal control law and control gains for the infinite-time problem [32], [72].

Proposition 6.2.1 Assume that system (6.1) is stabilizable. Then the gains $K_o(t)$, $K_1(t,\theta)$ and $K_2(t,\theta,\xi)$, for $t < T$, converge uniformly in t to $K_o$, $K_1(\theta)$, and $K_2(\theta,\xi)$ respectively as $T \to \infty$. The optimal control law for the infinite-time control problem with quadratic cost is given by

$$u^*(t) = -R^{-1}C'K_o x(t) - \int_{-\tau}^{0} R^{-1}C'K_1(\theta)x(t+\theta)d\theta \tag{6.10}$$

where $K_o$, $K_1(\theta)$ and $K_2(\theta,\xi)$ satisfy the following set of equations

$$A'K_o + K_oA - K_oCR^{-1}C'K_o + Q + K_1'(0) + K_1'(0) = 0 \tag{6.11}$$

$$\frac{d}{d\theta} K_1(\theta) = [A' - K_oCR^{-1}C']K_1(\theta) + K_2(0,\theta) \tag{6.12}$$

$$\left(\frac{\partial}{\partial\theta} + \frac{\partial}{\partial\xi}\right) K_2(\theta,\xi) = -K_1'(\theta)CR^{-1}C'K_1(\xi) \tag{6.13}$$

with boundary conditions

$$\begin{aligned} K_1(-\tau) &= K_oB \\ K_2(\theta, -\tau) &= K_1'(\theta)B \end{aligned} \tag{6.14}$$

and symmetry conditions

$$K_o = K_o', \qquad K_2(\theta,\xi) = K_2'(\xi,\theta) \tag{6.15}$$

The optimal cost is given by

$$J_\infty^*(\phi) = \phi'(0)K_o\phi(0) + \int_{-\tau}^{0} \phi'(0)K_1(\theta)\phi(\theta)d\theta + \int_{-\tau}^{0} \phi'(\theta)K_1'(\theta)\phi(0)d\theta + \int_{-\tau}^{0}\int_{-\tau}^{0} \phi'(\theta)K_2(\theta,\xi)\phi(\xi)d\theta d\xi \tag{6.16}$$

Asymptotic stability of the closed-loop system have also been studied previously by Datko [72], and Delfour et al., [32]. In both papers, the assumption that $Q > 0$ was made. Here we relax the assumption to observability.

Let $Q = H'H$ and let $x(t)$ denote the optimal trajectory corresponding to the optimal control $u^*(t)$, i.e., $x(t)$ satisfies

$$\dot{x}(t) = (A-CR^{-1}C'K_o)x(t) + Bx(t-\tau) - \int_{-\tau}^{0} CR^{-1}C'K_1(\theta)x(t+\theta)d\theta \tag{6.17}$$

We then have

Theorem 6.2.1 Assume (A,B,C) is stabilizable and (A,B,H) is observable. Then the optimal control for the infinite time problem is given by (6.10) and the optimal closed-loop system (6.17) is asymptotically stable.

Proof: The first part of the theorem is just Proposition 6.2.1. To prove asymptotic stability of the closed-loop system, consider the Lyapunov functional

$$V(x_t) = x'(t)K_o x(t) + \int_{-\tau}^{0} x'(t)K_1(\theta)x(t+\theta)d\theta + \int_{-\tau}^{0} x'(t+\theta)K_1'(\theta)x(t)d\theta + \int_{-\tau}^{0}\int_{-\tau}^{0} x'(t+\theta)K_2(\theta,\xi)x(t+\xi)d\theta d\xi \tag{6.18}$$

This, however, precisely corresponds to the optimal cost for the infinite-time problem starting at time t with initial function $x_t$. Thus we can write

$$V(x_t) = \int_t^{\infty} [x'(s)Qx(s) + u^{*\prime}(s)Ru^*(s)]ds$$

$V(x_t)$ is clearly nonnegative. We claim that $V(x_t) = 0$ implies $x(t) = 0$:

$V(x_t) = 0$ if and only if

$$u(s) = 0 \qquad s \geq t$$

and $x'(s)Qx(s) = x'(s)H'Hx(s) = 0 \qquad s \geq t$

By observability of (A,B,H), this implies

$$x(s) = 0 \qquad s \geq t$$

Let $U_\ell = \{\phi: V(\phi) < \ell\}$. Then for some $\ell_1 > 0$, $x_t \in U_{\ell_1}$ implies $|x(t)| \leq K$ for some nonnegative constant K. Since $V(x_t) \geq 0$ and $\dot{V}(x_t) = -[x'(t)Qx(t) + u^{*\prime}(t)Ru^*(t)] \leq 0$ we can apply the invariance principle for functional differential equations [74] to conclude that the solutions of (6.17) $x_t$ tends to M, the largest invariant set contained in the set

$$S = \{\phi: \dot{V}(\phi) = 0\}$$

Since $\dot{V}(x_t) = -[x'(t)Qx(t) + u^{*\prime}(t)Ru^*(t)]$

$$\dot{V}(x_t) = 0 \Longrightarrow Hx(t) = 0$$

By observability, the largest invariant set M in S are those solutions of (6.17) for which $|x(t)| = 0$ for $-\infty < t < \infty$. Hence $M = \{0\}$, and the asymptotic stability of (6.17) is proved.

Remark 6.2.1 It is known that asymptotic stability of autonomous linear functional differential equations is equivalent to exponential stability [2]. Hence the optimal closed-loop system for the infinite-time problem is globally asymptotically stable under the conditions of stabilizability and observability.

Corollary 6.2.1 Assume (A,B,C) is stabilizable and $Q > 0$. Then the optimal closed-loop system (6.17) is asymptotically stable. Since $Q > 0$ implies $Q^{1/2} = H > 0$, we immediately have that (A,B,H) is observable and we may invoke Theorem 6.2.1.

Remark 6.2.2 Corollary 6.2.1 was proved earlier by Datko [72] and Delfour et al., [32] using somewhat different techniques.

Corollary 6.2.2 If (A,B,C) is stabilizable and (A,B,H) is strongly observable, then (6.17) is asymptotically stable.

Proof: Follows from Theorem 6.2.1 since strong observability implies observability.

In proving asymptotic stability of the optimal filter, the same basic technique will be used: construction of a suitable Lyapunov functional and an appeal to the invariance principle. The system-theoretic concepts of detectability and controllability are exploited there in a natural way. Before proceeding on that route, we shall first study the duality between optimal estimation and control for linear delay systems.

## 6.3 Duality Between Estimation and Control for Linear Delay Systems

In this section, we extend the duality principle between estimation and control for linear ordinary differential systems to linear delay differential systems. We shall exploit this duality in proving the convergence of the optimal filter gains in the next section. Similar results on duality are given by Lindquist [75], and Mitter and Vinter [67].

Consider the optimal filtering problem (in the minimum mean square error sense) for the stochastic delay system

$$dx(t) = [Ax(t) + Bx(t-\tau)]dt + dw(t)$$
$$x(\theta) = 0 \qquad \theta \leq 0 \tag{6.19}$$

$$dz(t) = [C_o x(t) + C_1 x(t-\tau)]dt + dv(t) \tag{6.20}$$

where $w(t)$ and $v(t)$ are independent Wiener processes with $cov[w(t); w(s)] = Q\min(t,s)$, $cov[v(t); v(s)] = R\min(t,s)$, $Q \geq 0$, and $R > 0$. We know from Chapter 5 that for any vector b, the optimal estimate for $b'x(t)$ can be expressed as $\int_0^t k_t'(s)dz(s)$ for some function $k_t(s)$. We define the "dual" control problem to the optimal filtering problem to be:

$$\text{minimize } J_T(b,u) = \int_0^T [y'(t)Qy(t)+u'(t)Ru(t)]dt \tag{6.21}$$

subject to the constraint

$$\dot{y}(t) = -A'y(t)-B'y(t+\tau)-C_o u(t)-C_1 u(t+\tau) \tag{6.22}$$

with the boundary conditions

$$y(T) = b$$
$$y(s) = 0 \qquad s > T$$
$$u(s) = 0 \qquad s > T$$

As in section 6.2, the admissible controls are $L_2$ functions on [0, T].

The following duality theorem holds.

<u>Theorem 6.3.1</u> Consider the optimal estimation of the quantity $b'x(T)$ in the system (6.19) - (6.20). Let the optimal estimate for $b'x(T)$ be $b'\hat{x}(T|T)$ and let the optimal control for the dual problem (6.21) - (6.22)

be $u_T$. Then $b'\hat{x}(T|T)$ is related to $u_T$ by

$$b'\hat{x}(T|T) = -\int_0^T u_T'(s)dz(s) \tag{6.23}$$

Proof: Consider the function

$$\mathscr{H}(y,x,t) \triangleq y'(t)x(t) + \int_{t-\tau}^{t} y'(s+\tau)Bx(s)ds$$

where $y(t)$ and $x(t)$ are solutions of (6.22) and (6.19) respectively. Applying Ito's differential rule, we see that

$$\begin{aligned} d\mathscr{H}(y,x,t) &= -[y'(t)A+y'(t+\tau)B+u'(t)C_o+u'(t+\tau)C_1]x(t)dt \\ &\quad + y'(t)\{[Ax(t)+Bx(t-\tau)]dt+dw(t)\} \\ &\quad + y'(t+\tau)Bx(t)dt-y'(t)Bx(t-\tau)dt \\ &= -[u'(t)C_o x(t)+u'(t+\tau)C_1x(t)]dt+y'(t)dw(t) \\ &= -u'(t)[dz(t)-C_1x(t-\tau)dt-dv(t)]-u'(t+\tau)C_1x(t)dt+y'(t)dw(t) \end{aligned}$$

In view of the boundary conditions for (6.19) and (6.22), we obtain

$$\begin{aligned} &\mathscr{H}(y,x,T) - \mathscr{H}(y,x,0) \\ &\quad = y'(T)x(T) = b'x(T) \\ &\quad = -\int_0^T u'(t)dz(t) + \int_0^T u'(t)dv(t) \\ &\qquad + \int_0^T y'(t)dw(t) + \int_0^T u'(t)C_1x(t-\tau)dt - \int_0^T u'(t+\tau)C_1x(t)dt \\ &\quad = -\int_0^T u'(t)dz(t) + \int_0^T u'(t)dv(t) + \int_0^T y'(t)dw(t) \end{aligned}$$

Hence

$$E\{b'x(T) + \int_0^T u'(t)dz(t)\}^2 = \int_0^T [y'(t)Qy(t) + u'(t)Ru(t)]dt$$

Minimizing the mean square estimation error is thus the same problem as the dual optimal control problem, and the theorem follows.

Remark 6.3.1 The duality theorem can be extended to more general delay systems such as those with multiple or distributed delays. However, as we shall not need such generality in subsequent developments, we shall not pursue them here. The restriction of $x(\theta) = 0$, for $\theta \leq 0$, is imposed because we want the dual control problem to be one whose solution is known explicitly. If we allow a zero-mean random initial function $x_o$, with $E[x_o(\theta)x_o'(\xi)] = \Sigma_o(\theta,\xi)$, $-\tau \leq \theta$, $\xi \leq 0$, similar arguments show that the dual control problem is to minimize

$$J_T'(b,u) = y'(0)\Sigma_o(0,0)y(0) + \int_{-\tau}^0 y'(\theta+\tau)B\Sigma_o(\theta,0)d\theta y(0)$$

$$+ \int_{-\tau}^0 y'(0)\Sigma_o(0,\theta)B'y(\theta+\tau)d\theta + \int_{-\tau}^0 \int_{-\tau}^0 y'(\theta+\tau)B\Sigma_o(\theta,\xi)B'y(\xi+\tau)d\theta d\xi$$

$$+ \int_0^T y'(t)Qy(t)dt + \int_0^T u'(t)Ru(t)dt$$

In order to use the duality theorem, we would have to first solve the control problem for the cost $J_T'(b,u)$. These are additional complications which do not require any new concepts. To avoid obscuring the main points of our development, we have opted to concentrate on the case $x(\theta) = 0$, $\theta \leq 0$.

We will now restrict our attention to the case where $C_1 = 0$. Denoting $C_o$ by $C$, the system now under consideration is

$$dx(t) = [Ax(t) + Bx(t-\tau)]dt + dw(t) \qquad\qquad (6.24)$$

$$x(\theta) = 0 \qquad -\tau \leq \theta \leq 0$$

$$dz(t) = Cx(t)dt + dv(t) \qquad\qquad (6.25)$$

The case where $C_1 \neq 0$ will be discussed at the end of this chapter since additional difficulties arise. We first recall from Chapter 5 the optimal filter equations for the system defined by (6.24) - (6.25).

$$d\hat{x}(t|t) = [A-P_o(t)C'R^{-1}C]\hat{x}(t|t) + B\hat{x}(t-\tau|t-\tau)$$

$$-B\int_{-\tau}^{0} P_2(t,-\tau,\theta)C'R^{-1}C\hat{x}(t+\theta|t+\theta)d\theta + P_o(t)C'R^{-1}dz(t)$$

$$+B\int_{t-\tau}^{t} P_2(s,t-\tau-s,0)C'R^{-1}dz(s)dt \qquad\qquad (6.26)$$

$$\hat{x}(\theta|0) = 0 \qquad -\tau \leq \theta \leq 0$$

By the variation of constants formula from Chapter 5, we can write the solution to (6.26) as

$$\hat{x}(T|T) = \int_0^T \Phi(T,s)P_o(s)C'R^{-1}dz(s)$$

$$+\int_0^T \Phi(T,s)B\int_{s-\tau}^{s} P_2(u,s-\tau-u,0)C'R^{-1}dz(u)ds$$

(6.27)

where $\Phi(t,s)$ is the fundamental matrix associated with (6.26), and satisfies the matrix differential equation

$$\frac{\partial\Phi(t,s)}{\partial t} = [A-P_o(t)C'R^{-1}C]\Phi(t,s) + B\Phi(t-\tau,s)$$

$$-B\int_{-\tau}^{0} P_2(t,-\tau,\theta)C'R^{-1}C\Phi(t+\theta,s)d\theta$$

$$\Phi(s,s) = I, \quad \Phi(t,s) = 0 \quad t < s \tag{6.28}$$

Applying a Fubini-type theorem for Lebesgue and Wiener integrals to (6.27) [50], we can write

$$\int_0^T \Phi(T,s)B \int_{s-\tau}^{s} P_2(u,s-\tau-u,0)C'R^{-1}dz(u)ds$$

$$= \int_{-\tau}^{T}\int_{u}^{u+\tau} \Phi(T,s)BP_2(u,s-\tau-u,0)C'R^{-1}dsdz(u)$$

$$= \int_0^T\int_0^{\tau} (T,u+\theta)BP_1(u,\theta-\tau)C'R^{-1}d\theta dz(u)$$

using the definitions of $P_2(t,\theta,\xi)$ and $P_1(t,\theta)$, and the fact that $z(u) = 0$ $u < 0$. Hence

$$\hat{x}(T|T) = \int_0^T\{\Phi(T,s)P_o(s)C'R^{-1} + \int_0^{\tau}\Phi(T,s+\theta)BP_1(s,\theta-\tau)C'R^{-1}d\theta\}dz(s) \tag{6.29}$$

This gives an explicit representation of $\hat{x}(T|T)$ in the form

$$\hat{x}(T|T) = \int_0^T K(T,s)dz(s)$$

On the other hand, Theorem 6.3.1 gives a representation of $b'\hat{x}(T|T)$ in the form

$$b'\hat{x}(T|T) = -\int_0^T u_T'(s)dz(s) \tag{6.30}$$

We can therefore compare the representations in (6.29) and (6.30) and identify the control and filter gains appropriately. This will enable us to exploit the known convergence properties of the optimal control gains. We shall do this in the next section and also prove the asymptotic stability of the optimal linear filter.

## 6.4 Stability of the Optimal Filter for Linear Systems with Delays in the Dynamics

In this section, we shall put together the results developed in the previous sections and prove the asymptotic stability of the optimal filter. It is worth noting here that our proof of the convergence of the filter gains is rather indirect, being based on the convergence of the optimal control gains for the dual control problem. However, it is not obvious how a direct argument can be used. Indeed, variational arguments for proving convergence of the control gains do not as yet have a probabilistic interpretation in the filtering context for delay systems. At the present state of our knowledge, it appears that the indirect approach has to be adopted.

We begin now our investigations on the stability of the optimal filter. We first adapt the solution of the optimal control problem for linear delay systems with quadratic cost to solve the dual control problem posed in Eq. (6.21) - (6.22) in the previous section. Throughout this section, we will only consider the case where $C_1 = 0$, $C_o = C$.

We make the change of variables s = T-t (s will be defined this way until further notice). Then Eq. (6.22) becomes

$$-\frac{d}{ds}\, y(T-s) = -A'y(T-s)-B'y(T-s+\tau)-C'u(T-s) \tag{6.31}$$

Define $\bar{y}(s) = y(T-s) = y(t)$

$\bar{u}(s) = u(T-s) = u(t)$

(6.31) becomes

$$\frac{d}{ds}\,\bar{y}(s) = A'\bar{y}(s)+B'\bar{y}(s-\tau)+C'\bar{u}(s)$$
$$\bar{y}(0) = b, \qquad \bar{y}(\theta) = 0, \qquad -\tau \leq \theta < 0 \tag{6.32}$$

(Although the initial function here is not continuous, the results in section 6.2 can be readily extended to cover this case as well [35].) The cost functional $J_T(b,u)$ can be written as

$$J_T(b,u) = \int_0^T [\bar{y}(s)'Q\bar{y}(s) + \bar{u}'(s)R\bar{u}(s)]ds \tag{6.33}$$

The dual control problem has now been cast into the standard form, and the results of section 6.2 can be applied. We can therefore state that the optimal control $\bar{u}^*(s)$ is given by

$$\bar{u}^*(s) = -R^{-1}C\bar{K}_o(s)\bar{y}(s)-\int_{-\tau}^{0} R^{-1}C\bar{K}_o(s,\theta)\bar{y}(s+\theta)d\theta \tag{6.34}$$

where $$\dot{\bar{K}}_o(s) = -\bar{K}_o(s)A'-A\bar{K}_o(s) + \bar{K}_o(s)C'R^{-1}C\bar{K}_o(s)-Q-\bar{K}_1(s,0)-\bar{K}_1'(s,0) \tag{6.35}$$

$$\left(\frac{\partial}{\partial s} - \frac{\partial}{\partial \theta}\right) \overline{K}_1(s,\theta) = -[A-\overline{K}_o(s)C'R^{-1}C]\overline{K}_1(s,\theta)-\overline{K}_2(s,0,\theta) \qquad (6.36)$$

$$\left(\frac{\partial}{\partial s} - \frac{\partial}{\partial \theta} - \frac{\partial}{\partial \xi}\right) \overline{K}_2(s,\theta,\xi) = \overline{K}_1'(s,\theta)C'R^{-1}C\overline{K}_1(s,\xi) \qquad (6.37)$$

with boundary conditions

$$\overline{K}_o(T) = \overline{K}_1(T,\theta) = K_2(T,\theta,\xi) = 0 \quad , \quad -\tau \le \theta,\ \xi \le 0$$

$$\overline{K}_1(s,-\tau) = \overline{K}_o(s)B' \qquad (6.38)$$

$$\overline{K}_2(s,-\tau,\theta) = B\overline{K}_1(s,\theta)$$

and symmetry conditions

$$\overline{K}_o(s) = \overline{K}_o'(s),\ \overline{K}_2(s,\theta,\xi) = \overline{K}_2'(s,\xi,\theta) \qquad (6.39)$$

The closed-loop optimal system is given by

$$\frac{d\overline{y}(s)}{ds} = [A'-C'R^{-1}C\overline{K}_o(s)]\overline{y}(s) + B'\overline{y}(s-\tau)$$

$$-\int_{-\tau}^{0} C'R^{-1}C\overline{K}_1(s,\theta)\overline{y}(s+\theta)d\theta \qquad (6.40)$$

Using the variation of constants formula and the fact that $\overline{y}(s) = 0$ $-\tau \le s < 0$, we get that

$$\overline{y}(s) = \overline{\Phi}(s,0)b \qquad (6.41)$$

where $\overline{\Phi}(s,r)$ is the fundamental matrix associated with (6.40). Now define

$$K_o(t) = \overline{K}_o(s) = \overline{K}_o(T-t)$$

$$K_1(t,\theta) = \overline{K}_1(s,-\theta) = \overline{K}_1(T-t,-\theta)$$

$$K_2(t,\theta,\xi) = \overline{K}_2(s,-\theta,-\xi) = \overline{K}_2(T-t,-\theta,-\xi)$$

The following lemma is now immediate and summarizes the solution of the dual control problem.

<u>Lemma 6.4.1</u> The optimal control for the problem (6.21) - (6.22) is given by

$$u*(t) = -R^{-1}CK_o(t)y(t) - \int_{-\tau}^{0} R^{-1}CK_1(t,-\theta)y(t-\theta)d\theta \tag{6.42}$$

where

$$\dot{K}_o(t) = AK_o(t) + K_o(t)A' - K_o(t)C'R^{-1}CK_o(t) + Q + K_1(t,0) + K_1'(t,0) \tag{6.43}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial \theta}\right) K_1(t,\theta) = [A - K_o(t)C'R^{-1}C]K_1(t,\theta) + K_2(t,0,\theta) \tag{6.44}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial \theta} - \frac{\partial}{\partial \xi}\right) K_2(t,\theta,\xi) = -K_1'(t,\theta)C'R^{-1}CK_1(t,\xi) \tag{6.45}$$

with $\quad K_o(0) = K_1(0,\theta) = K_2(0,\theta,\xi) = 0 \qquad 0 \leq \theta,\ \xi \leq \tau$

$$K_1(t,\tau) = K_o(t)B' \tag{6.46}$$

$$K_2(t,\tau,\xi) = BK_1(t,\xi)$$

and $\quad K_o(t) = K_o'(t),\ K_2(t,\theta,\xi) = K_2'(t,\xi,\theta)$

Furthermore, the optimal closed-loop system satisfies

$$\frac{d}{dt}\, y(t) = -[A'-C'R^{-1}CK_o(t)]y(t) - B'y(t+\tau)$$

$$+ \int_0^{\tau} C'R^{-1}CK_1(t,\theta)y(t+\theta)d\theta$$

$$y(T) = b$$

$$y(t) = 0 \qquad t > T \tag{6.47}$$

Let the fundamental matrix associated with (6.47) be $Y(t,r)$. We then have the following:

<u>Lemma 6.4.2</u> $Y(t,r)$ is related to the matrix $\overline{\Phi}(s,r)$ introduced in (6.41) by

$$Y(t,T-r) = \overline{\Phi}(s,r) \tag{6.48}$$

<u>Proof</u>: By straightforward verification, the above two matrices satisfy the same delay equation as well as boundary conditions. Uniqueness then yields (6.48).

Combining Lemmas 6.4.1 and 6.4.2 and using (6.41), we obtain an alternative formula for the optimal control

$$u^*(t) = -[R^{-1}CK_o(t)Y(t,T) + \int_{-\tau}^{0} R^{-1}CK_1(t,-\theta)Y(t-\theta,T)d\theta]b$$

$$= -[R^{-1}CK_o(t)Y(t,T) + \int_0^{\tau} R^{-1}CK_1(t,\theta)Y(t+\theta,T)d\theta]b \tag{6.49}$$

Substituting (6.49) into (6.30) gives

$$b'\hat{x}(T|T) = b'\left\{ \int_0^T Y'(t,T)K_o(t)C'R^{-1}dz(t) \right.$$

$$\left. + \int_0^T \int_0^{\tau} Y(t+\theta,T)'K_1'(t,\theta)C'R^{-1}d\theta dz(t) \right\}$$

Since b is arbitrary, we obtain

Lemma 6.4.3 The optimal estimate $\hat{x}(T|T)$ for the stochastic delay system (6.24) - (6.25) can be written as

$$\hat{x}(T|T) = \int_0^T \{Y'(t,T)K_o(t)C'R^{-1} + \int_0^T Y(t+\theta,T)'K_1(t,\theta)C'R^{-1}d\theta\}dz(t) \tag{6.50}$$

We are now in a position to compare (6.50) and (6.29). First note that they are of exactly the same form, with $\Phi$, $P_o$, $BP_1$ occurring at the same positions as Y, $K_o$, and $K_1$. One suspects therefore, that

$$\Phi(T,t) = Y(t,T)'$$

$$P_o(t) = K_o(t)$$

and $\quad BP_1(t,\theta-\tau) = K_1'(t,\theta)$

We proceed to confirm this. Define

$$\tilde{P}_2(t,\theta,\xi) = BP_2(t,\theta-\tau,\xi-\tau)B'$$

$$\tilde{P}_1(t,\theta) = P_1'(t,\theta-\tau)B'$$

$$\tilde{P}_o(t) = P_o(t)$$

and recall that $P_o(t)$, $P_1(t,\theta)$, and $P_2(t,\theta,\xi)$ satisfy (5.62) - (5.65). Then it is straightforward to verify that $\tilde{P}_o(t)$, $\tilde{P}_1(t,\theta)$, and $\tilde{P}_2(t,\theta,\xi)$ satisfy exactly the same equations and boundary conditions as $K_o(t)$, $K_1(t,\theta)$, and $K_2(t,\theta,\xi)$. By uniqueness of the optimal control and optimal filter, we may identify

$$P_o(t) = K_o(t)$$

$$P_1'(t,\theta-\tau)B' = K_1(t,\theta)$$

$$BP_2(t,\theta-\tau,\xi-\tau)B' = K_2(t,\theta,\xi) \tag{6.54}$$

$$0 \leq t \leq T \quad , \qquad 0 \leq \theta, \ \xi \leq \tau$$

It remains to prove that $Y(t,T)' = \Phi(T,t)$. Using (6.54), we see that

$$\begin{aligned} K_1(t,-\theta) &= P_1'(t,-\theta-\tau)B' \\ &= P_2(t-\theta,-\tau,\theta)'B' \end{aligned} \tag{6.55}$$

(6.54), (6.55) and (6.47) imply that $Y(t,r)$ is the fundamental matrix of

$$\begin{aligned} \frac{d}{dt}y(t) = &-[A'-C'R^{-1}CP_o(t)]y(t)-B'y(t+\tau) \\ &+\int_{-\tau}^{0} C'R^{-1}CP_2(t-\theta,-\tau,\theta)'B'y(t-\theta)d\theta \end{aligned} \tag{6.56}$$

But (6.56) is precisely the adjoint equation to

$$\begin{aligned} \frac{d}{dt}x(t) = &[A-P_o(t)C'R^{-1}C]x(t) + Bx(t-\tau) \\ &-\int_{-\tau}^{0} BP_2(t,-\tau,\theta)C'R^{-1}Cx(t+\theta)d\theta \end{aligned}$$

and it is known [2], [57] that

$$Y(t,r) = \Phi(r,t)'$$

We summarize this development in

<u>Lemma 6.4.4</u> The optimal filter gains for (6.19) - (6.20) are related to the optimal control gains for the dual problem (6.21) - (6.22) by

$$P_o(t) = K_o(t)$$

$$P_1'(t,\theta-\tau)B' = K_1(t,\theta)$$

$$BP_2(t,\theta-\tau,\xi-\tau)B' = K_2(t,\theta,\xi)$$

$$0 \le t \le T \quad , \qquad 0 \le \theta,\xi \le \tau$$

The fundamental matrix $\Phi(t,r)$ associated with the optimal filter (6.26) is related to that of the optimal control system (6.47) by

$$\Phi(t,r) = Y(r,t)'$$

We are now ready to prove asymptotic stability of the optimal filter for (6.19) - (6.20). Let $Q = H'H$.

Theorem 6.4.1 Suppose (A,B,C) is detectable and (A,B,H') is controllable. Then the gains of the optimal filter converge, and the steady state optimal filter is asymptotically stable.

Proof: From the results of Chapter 2, we know that our hypothesis implies that (A',B',C') is stabilizable and (A',B',H) is observable. By stabilizability, Proposition 6.2.1 shows that the optimal gains for the dual control problem $\overline{K}_o(s)$, $\overline{K}_1(s,\theta)$ and $\overline{K}_2(s,\theta,\xi)$ converge to matrices $\overline{K}_o$, $\overline{K}_1(\theta)$ and $\overline{K}_2(\theta,\xi)$ respectively as $T \to \infty$. The definitions of $K_o(t)$, $K_1(t,\theta)$ and $K_2(t,\theta,\xi)$ now imply that these converge to $K_o$, $K_1(\theta)$, $K_2(\theta,\xi)$ respectively as $t \to \infty$. By Lemma 6.4.4, we can conclude that

$$P_o(t) \to P_o$$

$$BP_1(t,\theta) \to BP_1(\theta) \qquad\qquad \text{as } t \to \infty$$

$$BP_2(t,\theta,\xi)B' \to BP_2(\theta,\xi)B'$$

where (see also section 6.2)

$$AP_o+P_oA'-P_oC'R^{-1}CP_o+Q+BP_1(-\tau)+P_1'(-\tau)B' = 0 \tag{6.57}$$

$$\frac{d}{d\theta} BP_1(\theta) = -BP_1(\theta)[A'-C'R^{-1}CP_o]-BP_2(\theta,-\tau)B' \tag{6.58}$$

$$\left(\frac{\partial}{\partial\theta} + \frac{\partial}{\partial\xi}\right) BP_2(\theta,\xi)B' = BP_1(\theta)C'R^{-1}CP_1'(\xi)B' \tag{6.59}$$

with $P_1(0) = P_o$

$$P_2(\theta,0) = P_1(\theta)$$

$$P_o = P_o', \; P_2(\theta,\xi) = P_2'(\xi,\theta) \tag{6.60}$$

Stability of the steady state filter is then governed by the stability of the equation

$$\frac{d}{dt} x(t) = [A-P_oC'R^{-1}C)x(t) + Bx(t-\tau)$$
$$-B \int_{-\tau}^{0} P_2(-\tau,\theta)C'R^{-1}Cx(t+\theta)d\theta \tag{6.61}$$

Using Lemma 6.4.4 and (6.47), we see that the adjoint to the undriven steady-state filter equation (6.61) is given by

$$\frac{d}{dt} y(t) = -[A'-C'R^{-1}CP_o]y(t) - B'y(t+\tau)$$
$$+ \int_{0}^{\tau} C'R^{-1}CP_1'(\theta-\tau)B'y(t+\theta)d\theta \tag{6.62}$$

Let $s = -t$, $\bar{y}(s) = y(-s) = y(t)$. Then

$$\frac{d}{ds} \bar{y}(s) = (A-C'R^{-1}CP_o)\bar{y}(s) + B'\bar{y}(s-\tau)$$
$$- \int_{0}^{\tau} C'R^{-1}CP_1'(\theta-\tau)B'\bar{y}(s-\theta)d\theta \tag{6.63}$$

We prove that (6.63) is asymptotically stable. Introduce the Lyapunov functional

$$V(\bar{y}_s) = \bar{y}'(s)P_o\bar{y}(s) + \bar{y}'(s) \int_0^{\tau} P_1'(\theta-\tau)B'\bar{y}(s-\theta)d\theta$$

$$+\int_0^{\tau}\bar{y}'(s-\theta)BP_1(\theta-\tau)\bar{y}(s)d\theta + \int_0^{\tau}\int_0^{\tau}\bar{y}'(s-\theta)BP_2(\theta-\tau,\ \xi-\tau)B'$$

$$\cdot\bar{y}(s-\xi)d\theta d \qquad (6.64)$$

Some standard computations show that

$$\dot{V}(\bar{y}_s) = -\Big\{\bar{y}'(s)Q\bar{y}(s) + [R^{-1}CP_o\bar{y}(s) + \int_0^{\tau} R^{-1}CP_1'(\theta-\tau)B'\bar{y}(s-\theta)d\theta]'R$$

$$\cdot[R^{-1}CP_o\bar{y}(s)+\int_0^{\tau} R^{-1}CP_1'(\theta-\tau)B'\bar{y}(s-\theta)d\theta]\Big\} \qquad (6.65)$$

Comparing (6.64) and (6.65) with the situation in Theorem 6.2.1, we see that they are completely analogous. By the same arguments and the observability of (A',B',H), we conclude that (6.63) is asymptotically stable. Let the fundamental matrices associated with (6.61), (6.62) and (6.63) be $\Phi_o(t,r)$, $Y_o(t,r)$ and $\bar{Y}_o(t,r)$ respectively. Then it is easy to see that $Y_o(t,r) = \bar{Y}_o(-t,-r)$. It is known [2] that asymptotic stability of (6.63) is equivalent to the existence of constants $\alpha > 0$, $M \geq 1$ such that

$$||\bar{Y}_o(s,r)|| \leq Me^{-\alpha(s-r)}$$

Thus $||Y_o(t,r)|| = ||\bar{Y}_o(-t,-r)|| \leq Me^{-\alpha(r-t)}$

Since $\Phi_o(t,r) = Y_o'(r,t)$, $||\Phi_o(t,r)|| \leq Me^{-\alpha(t-r)}$

The proof of the theorem is now complete.

Remark 6.4.1 It should be pointed out that the convergence of the filter gains $P_o(t)$, $BP_1(t,\theta)$ and $BP_2(t,\theta,\xi)B'$ do not provide as yet the convergence of the "smoothed" error covariance $P_2(t,\theta,\xi)$. Of course, if B is nonsingular, $BP_2(t,\theta,\xi)B'$ converging will imply that $P_2(t,\theta,\xi)$. It is interesting to note that the nonsingularity of B seems to be related to many questions in the theory of delay equations. For example, it is related to the solution semigroup T(t) being one-to-one [2], the completeness of exponential solutions [76], and the convergence of projection series [77].

Instead of identifying $P_o(t)$, $BP_1(t,\theta)$, and $BP_2(t,\theta,\xi)B'$ with the corresponding control gains as in Lemma 6.4.4, one can directly identify $P_o(t)$, $P_1(t,\theta)$, and $P_2(t,\theta,\xi)$ with the Fredholm kernel $P_c(t,\theta,\xi)$ introduced by Manitius in the optimal control with quadratic cost [71]. The details are straightforward and are left as an amusement for the reader.

## 6.5 Convergence of Estimation Error Covariance Operator

The results of the last section show that under suitable detectability and controllability conditions, the estimation error covariance $P_o(t)$ converges. Our objective in this section is to prove the convergence of the error covariance operator for the steady state optimal filter. If we express the estimation error $e(t|t) = x(t) - \hat{x}(t|t)$ in terms of a stochastic delay equation, its solutions define a trajectory in the space of continuous functions. By the error covariance operator we mean the function

$$\Sigma(t,\theta,\xi) = E\{e(t+\theta|t+\theta)e'(t+\xi|t+\xi)\}, \qquad -\tau \le \theta,\ \xi \le 0.$$

Of course $\Sigma(t,0,0) = P_o(t)$. However, $\Sigma(t,\theta,0) \neq P_1(t,\theta)$, and $\Sigma(t,\theta,\xi) \neq P_2(t,\theta,\xi)$. This is because $\Sigma(t,\theta,0)$ and $\Sigma(t,\theta,\xi)$ are covariances of the filtering estimation error, whereas $P_1(t,\theta)$ and $P_2(t,\theta,\xi)$ are covariances of the smoothing estimation error. While we can show that $\Sigma(t,\theta,\xi)$ converges to a steady-state operator $\Sigma_\infty(\theta,\xi)$, we have not been able to show the convergence of $P(t,\theta,\xi)$, unless B is invertible (see Remark 6.4.1).

It is easy to see that $e(t|t)$ satisfies the stochastic differential equation

$$\begin{aligned} de(t|t) = {} & [(A-P_oC'R^{-1}C)e(t|t) + Be(t-\tau|t-\tau)]dt \\ & + \int_{-\tau}^{0} BP_2(-\tau,\theta)C'R^{-1}Ce(t+\theta|t+\theta)d\theta dt + dw(t) \\ & - P_oC'R^{-1}dv(t) - B\int_{-\tau}^{0} P_2(-\tau,\theta)C'R^{-1}d_\theta v(t+\theta)dt \end{aligned} \tag{6.66}$$

Note that for any fixed t, $v(t+\theta)$ $\theta\varepsilon[-\tau,0]$ is a Brownian motion, and hence the last stochastic integral is well-defined. Again, let $\Phi_o(t,s)$ be the fundamental matrix associated with (6.66). Since (6.66) is an autonomous equation $\Phi_o(t,s) = \Phi_o(t-s,0) \triangleq \Phi_1(t-s)$. Then the variation of constants formula gives

$$e(t|t) = \int_0^t \Phi_1(t-s)dw(s) - \int_0^t \Phi_1(t-s)P_o C'R^{-1}dv(s)$$

$$- \int_0^t \Phi_1(t-s)B \int_{-\tau}^0 P_2(-\tau,\theta)C'R^{-1}d_\theta v(s+\theta)ds \tag{6.67}$$

Using a Fubini-type theorem in Doob [50], the last term in (6.67) can be written as

$$\int_0^t \Phi_1(t-s) \int_{s-\tau}^s P_2(-\tau,\sigma-s)dv(\sigma)ds$$

$$= \int_0^t \int_\sigma^{\min(\sigma+\tau,t)} \Phi_1(t-s)P_2(-\tau,\sigma-s)dsdv(\sigma)$$

$$= \int_0^t \int_0^{\min(t-\sigma,\tau)} \Phi_1(t-\sigma-\beta)P_2(-\tau,-\beta)d\beta dv(\sigma) \tag{6.68}$$

where we have used $v(t) = 0, \quad t \leq 0$. For convenience, we define

$$L(t-s) = \Phi_1(t-s)P_o C'R^{-1} + \int_0^{\min(t-s,\tau)} \Phi_1(t-s-\beta)P_2(-\tau,-\beta)d\beta$$

If we now compute the characteristic functional of the $\mathscr{C}$-valued random variable $e_{t|t}$ $(e_{t|t}(\theta) = e(t+\theta|t+\theta))$, then by exactly the same arguments as in Chapter 5, we get, for $y \in \mathscr{C}^*$,

$$\phi(y,e_{t|t}) = E \exp i < y,e_{t|t}>$$

$$= E \exp\left\{ i \int_{-\tau}^0 dy'(r)\left[\int_0^t \Phi_1(t+r-s)dw(s) - \int_0^t L(t+r-s)dv(s)\right]\right\}$$

using $\Phi_1(t-s) = 0, \quad t < s.$

Since for each fixed t, the exponent is a sum of Gaussian random variables, and w(t) and v(t) are completely independent, we can evaluate the expectation as

$$\exp -\frac{1}{2}\left\{ \int_{-\tau}^{0}\int_{-\tau}^{0} dy'(r) \int_{0}^{t} \Phi_1(t+r-s)Q\Phi_1'(t+\sigma-s)ds\,dy(\sigma) + \int_{-\tau}^{0}\int_{-\tau}^{0} dy'(r) \int_{0}^{t} L(t+r-s)RL'(t+\sigma-s)ds\,dy(\sigma)\right\} \tag{6.69}$$

Therefore, the covariance operator is given by

$$\begin{aligned}\Sigma(t,\theta,\xi) &= \int_0^t \Phi_1(t+\theta-s)Q\Phi_1'(t+\xi-s)ds \\ &\quad + \int_0^t L(t+\theta-s)RL'(t+\xi-s)ds \\ &= \int_0^t [\Phi_1(\theta+s)Q\Phi_1'(\xi+s) + L(\theta+s)RL'(\xi+s)]ds \end{aligned} \tag{6.70}$$

The convergence of the covariance operator $\Sigma(t,\theta,\xi)$ is given by

Theorem 6.5.1 The covariance operator $\Sigma(t,\theta,\xi)$ for the optimal estimation error converges pointwise in $\theta$ and $\xi$ to a stationary covariance operator $\Sigma_\infty(\theta,\xi)$, with

$$\Sigma_\infty(\theta,\xi) = \int_0^\infty \Phi_1(\theta+s)Q\Phi_1'(\xi+s)ds$$

$$+ \int_0^\infty [\Phi_1(\theta+s)P_oC'R^{-1} + \int_0^{\min(\theta+s,\tau)} \Phi_1(\theta+s-\beta)P_2(-\tau,-\beta)d\beta]'$$

$$\cdot R[\Phi_1(\xi+s)P_oC'R^{-1}$$

$$+ \int_0^{\min(\xi+s,\tau)} \Phi_1(\xi+s-\beta)P_2(-\tau,-\beta)d\beta]ds \qquad (6.71)$$

<u>Proof</u>: From (6.70), it is clear that $\Sigma(t,\theta,\xi)$ is a symmetric operator, monotonically increasing in t. (See Remark 6.5.1 following the proof.) Since the optimal filter is asymptotically stable, $||\Phi_1(t-s)|| \leq Me^{-\alpha(t-s)}$ for some $\alpha > 0$, $M \geq 1$. We can therefore make the estimate

$$||\int_0^t \Phi_1(\theta+s)Q\Phi_1(\xi+s)ds||$$

$$\leq \int_0^t ||\Phi_1(\theta+s)Q\Phi_1(\xi+s)||ds$$

$$\leq ||Q|| \int_0^t M^2e^{-\alpha(\theta+s)} e^{-\alpha(\xi+s)}ds$$

$$= ||Q||M^2e^{-\alpha(\theta+\xi)} \left(1 - \frac{e^{-2\alpha t}}{2\alpha}\right)$$

which converges as $t \to \infty$. By a similar argument, the second term in (6.70) is also uniformly bounded in t for each fixed $\theta$ and $\xi$. We can now apply a theorem on the convergence of monotone symmetric operators

[78] to conclude that

$$\lim_{t \to \infty} \Sigma(t,\theta,\xi) = \Sigma_\infty(\theta,\xi)$$

exists pointwise in $\theta$ and $\xi$, and is given by (6.71).

Remark 6.5.1 The filtering error covariance operator $\Sigma(t,\theta,\xi)$ is monotonically increasing in t because we have zero initial estimation error (i.e., the initial condition for the system is deterministic and known, taken to be zero in our formulation). If we have a random initial condition, we may still be able to prove (although we have not done so) that the resulting $\Sigma(t,\theta,\xi)$ converges to $\Sigma_\infty(\theta,\xi)$, since initial conditions are usually "forgotten" as $t \to \infty$. We may note that analogous situations also arise in the filtering problem for systems without delays.

## 6.6 Stochastic Control of Linear Delay Systems and Stability

The preceding development in the stability of the optimal closed-loop control system and the stability of the optimal linear filter suggests that by putting the two together, asymptotic stability of the closed-loop stochastic control system can be obtained. This is of course the case in linear stochastic system without delays. In this section, we examine the corresponding stochastic control system for delay systems and prove a similar result.

We shall first discuss the separation theorem for delay systems as proved by Lindquist [56]. Again we consider the following linear stochastic delay system

$$dx(t) = [Ax(t) + Bx(t-\tau) + Gu(t)]dt + M'dw_1(t)$$
$$x(\theta) = \phi(\theta) \qquad \theta\varepsilon[-\tau,0] \tag{6.72}$$

$$dz(t) = Cx(t)dt + Ndw_2(t) \tag{6.73}$$

$w_1$, $w_2$ are standard Wiener processes, completely independent of each other, and independent of the initial function $\phi$, which is a Gaussian process. The matrix N is assumed to be positive definite. The objective is to choose a control law u in a suitable set of admissible controls such that the cost functional

$$J_T(u,\phi) = E\int_0^T [x'(t)Qx(t) + u'(t)Ru(t)]dt \tag{6.74}$$

is minimized, where $T< \infty$.

Before we define the set of admissible controls, we establish some notation. By the variation of constants formula, we obtain

$$z(t) = \int_0^t C\Phi(s,0)\phi(0)ds + \int_0^t\int_{-\tau}^0 C\Phi(s,\sigma+\tau)B\phi(\sigma)d\sigma ds$$
$$+ \int_0^t\int_0^s C\Phi(s,\sigma)Gu(\sigma)d\sigma ds + \int_0^t\int_0^s C\Phi(s,\sigma)M'dw(\sigma)ds + Nw_2(t)$$

$$\triangleq z_o(t) + \int_0^t\int_0^s \Phi(s,\sigma)Gu(\sigma)d\sigma ds \tag{6.75}$$

Define the set $U_o$ consisting of the class of processes u(t) satisfying the following conditions:

(i) u(t) is measurable with respect to $\sigma\{z(s),\ 0 \le s \le t\}$ for each t;

(ii) For each $u \in U_o$, there exist unique solutions to (6.72) and (6.73);

(iii) $\int_0^T E|u(t)|^2 dt < \infty$

(iv) For each $u \in U_o$, $\sigma\{z(s),\ 0 \le s \le t\} = \sigma\{z_o(s),\ 0 \le s \le t\}$.

In words, (iv) means that by using a control law $u \in U_o$, we cannot gain, from the resulting outputs, more information about the system than that obtained by using no control at all.

We shall take $U_o$ to be the set of admissible controls. The following theorem has been proved by Lindquist [56].

Proposition 6.6.1 The problem of determining $u \in U_o$ so as to minimize (6.74) has the following solution

$$u^*(t) = -R^{-1}G'K_o(t)\hat{x}(t|t) - R^{-1}G' \int_{-\tau}^{0} K_1(t,\theta)\hat{x}(t+\theta|t)d\theta \qquad (6.76)$$

where $K_o(t)$ and $K_1(t,\theta)$ are the optimal gains for the deterministic optimal control problem (section 6.2), and $\hat{x}(s|t)$, $t-\tau \le s \le t$ is the conditional expectation of x(s) given $z(\sigma)$, $0 \le \sigma \le t$ (section 5.5).

Remark 6.6.1 A few comments on the choice of $U_o$ as the set of admissible controls are in order. Let us define the set U to be the class of processes u(t) that are measurable with respect to $\sigma\{z(s),\ 0 \le s \le t\}$ for each t, such that for each $u \in U$, there exist unique solutions to (6.72) and (6.73), and for which $\int_0^T E|u(t)|^2 dt < \infty$.

Clearly $U_o \subset U$. The basic difference between the sets U and $U_o$ is that the requirement $\sigma\{z(s), 0 \leq s \leq t\} = \sigma\{z_o(s), 0 \leq s \leq t\}$ is not imposed on U. However, if we examine the proof of Proposition 6.6.1 in [56], we see that the requirement of $\sigma\{z(s), 0 \leq s \leq t\} = \sigma\{z_o(s), 0 \leq s \leq t\}$, for any u in the set of admissible laws, is crucial. One would really like to prove that the separation theorem holds also for the set U (which is equivalent to showing $U_o = U$) but the truth or falsehood of this is not known at present. We may note that in the separation theorem without delays [79], the conditions (i) to (iv) satisfied by all $u \in U_o$ are obtained by requiring the control laws to be Lipschitz functionals of the past observations. Even in the case without delays, it has not been established that $U_o = U$.

The expression for the optimal cost is given in

Lemma 6.6.1 Corresponding to the optimal control (6.76), the optimal cost J* associated with the stochastic control problem (6.72) - (6.74) is given by

$$J^* = EV(x_o) + \int_0^T \text{tr}M'MK_o(t)dt$$

$$+ \int_0^T \text{tr}\{K_o(t)GR^{-1}G'K_o(t)P_o(t) + \int_{-\tau}^0 K_1'(t,\theta)GR^{-1}G'K_o(t)P_1'(t,\theta)d\theta$$

$$+ \int_{-\tau}^0 K_o(t)GR^{-1}G'K_1(t,\theta)P_1(t,\theta)d\theta + \int_{-\tau}^0\int_{-\tau}^0 K_1'(t,\theta)GR^{-1}G'K_1(t,\xi)\cdot$$

$$\cdot P_2(t,\xi,\theta)d\theta d\xi\}dt \qquad (6.77)$$

where $$V(x_t) = x'(t)K_o(t)x(t) + \int_{-\tau}^{0} x'(t)K_1(t,\theta)x(t+\theta)d\theta$$

$$+ \int_{-\tau}^{0} x'(t+\theta)K_1'(t,\theta)x(t)d\theta + \int_{-\tau}^{0}\int_{-\tau}^{0} x'(t+\theta)K_2(t,\theta,\xi)x(t+\xi)d\theta d\xi \tag{6.78}$$

Proof: See Appendix C.

Remark 6.6.2 Lemma 6.6.1 shows that the optimal cost in the delay case has the same structure as the case without delays: there is a term due to initial conditions, a term due to the noise in the system dynamics, and a term due to the estimation error.

We turn our attention now to the stochastic control system defined by using the steady state version of (6.76):

$$u(t) = -R^{-1}G'K_o\hat{x}(t|t) - R^{-1}G' \int_{-\tau}^{0} K_1(\theta)\hat{x}(t+\theta|t)d\theta \tag{6.79}$$

where $\hat{x}(t|t)$ is generated by the steady state optimal filter. Heuristically, this law is the optimal one for the functional

$$E \int_{0}^{\infty} [x'(t)Qx(t) + u'(t)Ru(t)]dt \tag{6.80}$$

with observations started back at $-\infty$, except that it would give rise to an infinite $J^*$. One must therefore modify the control problem to be one of minimizing

$$\lim_{T\to\infty} \frac{1}{T}\{E \int_{0}^{T} [x'(t)Qx(t) + u'(t)Ru(t)]dt\}$$

or $$\lim_{T\to\infty} \frac{1}{T} \int_0^T [x'(t)Qx(t) + u'(t)Ru(t)]dt$$

This introduces notions of ergodicity and invariant measures into the discussion. Even in the case without delays, the infinite problem has not really been completely resolved. We therefore content ourselves with simply the result on asymptotic stability of the closed-loop system under the law (6.79).

Theorem 6.6.1 Let $Q = H'H$. Assume (A,B,G) is stabilizable, (A,B,M') is controllable, (A,B,C) is detectable, and (A,B,H) is observable. Then the control law

$$u(t) = -R^{-1}G'K_o\hat{x}(t|t)-R^{-1}G' \int_{-\tau}^{0} K_1(\theta)x(t+\theta|t)d\theta,$$

where $\hat{x}(t|t)$ is generated by the steady state filter of Theorem 6.4.1, and $K_o$, $K_1(\theta)$ are generated by the deterministic stationary control law of Theorem 6.2.1, gives rise to an asymptotically stable closed-loop system.

Proof: Stabilizability of (A,B,G) ensures that $K_o$, $K_1(\theta)$, and $K_2(\theta,\xi)$ are well-defined. Observability of (A,B,H) then guarantees that solutions of the system

$$x(t) = (A-GR^{-1}G'K_o)x(t)+Bx(t-\tau) - \int_{-\tau}^{0} GR^{-1}G'K_1(\theta)x(t+\theta)d\theta \tag{6.81}$$

are asymptotically stable (see Theorem 6.2.1). Detectability of (A,B,C) ensures that the steady state filter is well-defined, and controllability of (A,B,M') guarantees that the steady state filter is asymptotically stable. The closed-loop system is defined by the coupled set of equations

$$dx(t) = Ax(t)dt + Bx(t-\tau)dt - GR^{-1}G'K_o\hat{x}(t|t)$$

$$- \int_{-\tau}^{0} GR^{-1}G'K_1(\theta)\hat{x}(t+\theta|t)d\theta + dw(t) \tag{6.82}$$

$$d\hat{x}(t|t) = A\hat{x}(t|t)dt + B\hat{x}(t-\tau|t-\tau)dt$$

$$+B \int_{t-\tau}^{t} P_2(-\tau,s-t)C'(NN')^{-1}[dz(s)-C\hat{x}(s|s)ds]dt$$

$$+P_oC'(NN')^{-1}[dz(t) - C\hat{x}(t|t)dt] \tag{6.83}$$

Expressing $\hat{x}(t|t) = x(t) - e(t|t)$, we get

$$dx(t) = (A-GR^{-1}G'K_o)x(t)dt + Bx(t-\tau)dt$$

$$- \int_{-\tau}^{0} GR^{-1}G'K_1(\theta)x(t+\theta)d\theta + dw(t)$$

$$+GR^{-1}G'K_oe(t|t)+ \int_{-\tau}^{0} GR^{-1}G'K_1(\theta)e(t+\theta|t)d\theta \tag{6.84}$$

and

$$de(t|t) = [A-P_oC'(NN')^{-1}C]e(t|t)dt + Be(t-\tau|t-\tau)dt$$

$$+ \int_{-\tau}^{0} BP_2(-\tau,\theta)C'(NN')^{-1}Ce(t+\theta|t+\theta)d\theta dt$$

$$+ dw(t)-P_oC'R^{-1}dv(t) - B \int_{-\tau}^{0} P_2(-\tau,\theta)C'R^{-1}d_\theta v(t+\theta)dt \tag{6.85}$$

Since (6.85) is decoupled from (6.84), the stability properties of the closed-loop system are precisely those of (6.81) and the steady state optimal filter. Since both of these are asymptotically stable as a consequence of our assumptions, the closed-loop stochastic control system is asymptotically stable as well.

Remark 6.6.3 When the control law (6.79) is applied to the problem of minimizing

$$J_r = \lim_{T\to\infty} \frac{1}{T} E \int_0^T [x'(t)Qx(t) + u'(t)Ru(t)]dt$$

Lemma 6.6.1 shows

$$\begin{aligned} J_r &= \mathrm{tr}M'MK_o \\ &+ \mathrm{tr}\{K_oGR^{-1}G'K_oP_o + \int_{-\tau}^{0} K_1'(\theta)GR^{-1}G'K_oP_1'(\theta)d\theta \\ &+ \int_{-\tau}^{0} K_oGR^{-1}G'K_1(\theta)P_1(\theta)d\theta + \int_{-\tau}^{0}\int_{-\tau}^{0} K_1'(\theta)GR^{-1}G'K_1(\xi)P_2(\xi,\theta)d\xi d\theta\} \end{aligned}$$

This can be thought of as the cost rate associated with the control law (6.79).

## 6.7 Filter Stability for Systems with Delays in The Observations

In the previous sections, we have discussed the stability properties of the optimal filter and control system rather thoroughly for linear stochastic systems with a single delay in the system dynamics. It is relatively straightforward to extend the duality theorem of section 6.3

to cover situations with multiple delays in the state dynamics. Combining this with the results of Delfour, McCalla, and Mitter [32] on optimal control, it should not be difficult to extend our stability results to these systems. What is more interesting is the case in which there are delays in the observations. Since similar difficulties arise for distributed delays as well as point delays, we shall, for convenience, consider systems of the form

$$dx(t) = [Ax(t) + Bx(t-\tau)]dt + dw(t) \tag{6.86}$$

$$x(\theta) = 0 \quad -\tau \leq \theta \leq 0$$

$$dz(t) = [C_o x(t) + C_1 x(t-\tau)]dt + dv(t) \tag{6.87}$$

From the results of Chapter 5, we know that the optimal filter is given by the equations

$$\begin{aligned} d\hat{x}(t|t) = {} & [A\hat{x}(t|t) + B\hat{x}(t-\tau|t)]dt \\ & + [P_o(t)C_o'R^{-1} + P_2(t,0,-\tau)C_1'R^{-1}] \\ & \cdot [dz(t) - C_o\hat{x}(t|t)dt - C_1\hat{x}(t-\tau|t)dt] \end{aligned} \tag{6.88}$$

$$\begin{aligned} \hat{x}(t+\theta|t) = {} & \hat{x}(t+\theta|t+\theta) + \int_{t+\theta}^{t} [P_2(s,t+\theta-s,0)C_o' + P_2(s,t+\theta-s,-\tau)C_1']R^{-1} \\ & \cdot [dz(s) - C_o\hat{x}(s|s)ds - C_1\hat{x}(s-\tau|s)ds] \end{aligned} \tag{6.89}$$

If we apply the same technique as that of section 6.4 to (6.88) - (6.89), we see that

$$d\hat{x}(t|t) = [A\hat{x}(t|t) + B\hat{x}(t-\tau|t-\tau)]dt$$

$$+ B\int_{t-\tau}^{t} [P_2(s,t-\tau-s,0)C_o' + P_2(s,t-\tau-s,-\tau)C_1']R^{-1}$$

$$\cdot[dz(s) - C_o\hat{x}(s|s)ds - C_1\hat{x}(s-\tau|s)ds]dt$$

$$+ [P_o(t)C_o' + P_2(t,0,-\tau)C_1']R^{-1}$$

$$\cdot[dz(t) - C_o\hat{x}(t|t)dt - C_1\hat{x}(t-\tau|t)dt] \tag{6.90}$$

which is not a stochastic delay equation in $\hat{x}(t|t)$. Indeed, it is not possible to express $\hat{x}(t-\tau|t)$ solely in terms of $\hat{x}(s|s)$, $t - \tau \leq s \leq t$. Without this, the technique presented in section 6.4 cannot be applied.

The situation is very similar in the dual control problem, which is to minimize:

$$J_T(b,u) = \int_0^T [y'(t)Qy(t) + u'(t)Ru(t)]dt$$

subject to the constraint

$$\dot{y}(t) = -A'y(t) - B'y(t+\tau) - C_ou(t) - C_1u(t+\tau)$$

$$y(T) = b$$

$$y(s) = 0 \qquad s > T$$

$$u(s) = 0 \qquad s > T$$

Theorem 6.3.1 shows that the optimal estimate of $b'x(T)$ for (6.86) - (6.87) is given by $\int_0^T u_T'(s)dz(s)$, where $u_T(t)$ is the optimal control to the dual problem.

By the time reversal technique of section 6.4, we see that the above control problem is equivalent to an optimal control problem for systems with delays in the control as well as in the state. It is known [80] that this problem is much more complex than the optimal control problem with delays in the state only. To understand the structure of the optimal filter when there are delays in the observations, it is worthwhile to discuss briefly the control problem with delays in both the state and control.

Consider the system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bx(t-\tau) + C_o u(t) + C_1 u(t-\tau) \\ x(\theta) &= \phi(\theta) \qquad \theta \in [-\tau,0] \\ u(\theta) &= 0 \qquad \theta \in [-\tau,0) \end{aligned} \tag{6.91}$$

The problem is to find a control u such that the cost functional

$$J_T(u,\phi) = \int_0^T [x'(t)Qx(t) + u'(t)Ru(t)]dt$$

is minimized. The solution to this problem is given in [80]:

$$\begin{aligned} u(t) = &-R^{-1}C_o'[P_c(t,t,t)x(t) + \int_{t-\tau}^{t} P_c(t,s+\tau,t)Bx(s)ds \\ &+ \int_{t-\tau}^{t} P_c(t,s+\tau,t)C_1u(s)ds \\ &-\gamma(t)R^{-1}C_1'[P_c(t+\tau,t,t)x(t) + \int_{t-\tau}^{t} P_c(t+\tau,s+\tau,t)Bx(s)ds \\ &+ \int_{t-\tau}^{t} P_c(t+\tau,s+\tau,t)C_1u(s)ds] \end{aligned} \tag{6.92}$$

$$\text{where } \gamma(t) = \begin{cases} 0 & \text{if } t \in [T-\tau, T] \\ 1 & \text{otherwise} \end{cases}$$

and $P_c(t,s,r)$ is a type of Fredholm kernel introduced by Manitius [71] and satisfies appropriate partial differential equations.

The optimal control u(t) is not expressed solely in terms of $x_t$, but also in terms of u(s), $t-\tau \leq s < t$. In order to express u(t) solely in terms of x, we can regard (6.92) as an integral equation in u(t). In fact, if we write

$$q(t) = -R^{-1}C_o'[P_c(t,t,t)x(t) + \int_{t-\tau}^{t} P_c(t,s+\tau,t)Bx(s)ds]$$

$$-\gamma(t)R^{-1}C_1'[P_c(t+\tau,t,t)x(t) + \int_{t-\tau}^{t} P_c(t+\tau,s+\tau,t)Bx(s)ds]$$

and

$$N(t,s) = -[R^{-1}C_o'P_c(t,s+\tau,t)C_1 + \gamma(t)R^{-1}C_1'P_c(t+\tau,s+\tau,t)C_1] \tag{6.93}$$

we get

$$u(t) = q(t) + \int_{t-\tau}^{t} N(t,s)u(s)ds \tag{6.94}$$

The properties of $P_c(t,s,r)$ [80] clearly implies that N(t,s) is a Volterra $L_2$-kernel. Substituting (6.94) into itself, we obtain

$$u(t) = q(t) + \int_{t-\tau}^{t} N(t,s)q(s)ds$$

$$+ \int_{t-\tau}^{t} N(t,s) \int_{s-\tau}^{s} N(s,r)u(r)drds$$

$$= q(t) + \int_{t-\tau}^{t} N(t,s)q(s)ds +$$

$$+ \int_{t-2\tau}^{t} \int_{\max(t-\tau,r)}^{\min(r+\tau,t)} N(t,s)N(s,r)dsu(r)dr$$

By induction, it is easily seen that in fact u(t) satisfies a Volterra integral equation of the form

$$u(t) = q(t) + \int_{0}^{t} M(t,s)u(s)ds$$

with M(t,s) an $L_2$-kernel. The solution for the optimal control u in state feedback form can therefore be expressed as

$$u(t) = q(t) + \int_{0}^{t} R(t,s)q(s)ds$$

$$= \int_{-\tau}^{t} d_s F(t,s)x(s) \qquad (6.95)$$

where F(t,s) is an $L_2$-kernel of bounded variation in s. Notice that in state feedback form, the entire past of the state trajectory must be used.

There are two ways to study the stability of the closed-loop system. We can either study the system defined by (6.91) and (6.94) as system of coupled differential and integral equations, or we can substitute (6.95) into (6.91) and study the resulting integrodifferential equation. In any case, the situation is considerably more complex than that without delays in the control. Stability properties of the optimal system are left as a subject for future research.

One can see that a similar situation happens in the optimal filter for systems with delays in the observations. Defining

$$W(t,s) = [P_2(s,t-\tau-s,0)C_o' + P_2(s,t-\tau-s,-\tau)C_1']R^{-1},$$

$$\hat{x}(t-\tau|t) = \hat{x}(t-\tau|t-\tau) + \int_{t-\tau}^{t} W(t,s)[dz(s)-C_o\hat{x}(s|s)ds]$$

$$- \int_{t-\tau}^{t} W(t,s)C_1\hat{x}(s-\tau|s)ds \qquad (6.96)$$

One can view (6.96) as an integral equation in $\hat{x}(t-\tau|t)$. Again the kernel $W(t,s)C_1$ is a deterministic $L_2$-kernel. As before, we can solve $\hat{x}(t-\tau|t)$ in terms of $\hat{x}(s|s)$, $0 \leq s \leq t$. The stability of the optimal filter can thus be studied from either an integrodifferential equation or from a coupled set of differential and integral equations. In view of the duality theorem between estimation and control, the stability of the dual control system should be intimately related to the stability of the optimal filter, in the same sense as the connection discussed in section 6.4.

## 6.8 Perspective

In this chapter, we have studied the stability of the optimal linear control system and the optimal linear filter for systems with delays only in the state dynamics. Since the main results are interconnected in a rather complicated way, it seems worthwhile to recapitulate the various steps involved and put our results in perspective.

We first showed that under the conditions of stabilizability and observability, the infinite-time optimal linear control system with quadratic cost is asymptotically stable. This result not only extends previous work by other investigators, but also shows the important role played by system-theoretic concepts.

In section 6.3, we established a duality theorem which relates the solution of the optimal linear filtering problem to the solution of a dual optimal control problem. This result was motivated by the work of Lindquist [75], and Mitter and Vinter [67], and paved the way for exploiting the results of section 6.2 in studying the stability of the optimal linear filter.

Theorem 6.4.1 in section 6.4 contains the central results of this chapter, and combines the results of Chapter 2 with those of sections 6.2 and 6.3. First, we used the duality theorem to give the precise formulas relating the optimal gains and the fundamental matrix of the optimal filter to those of the corresponding dual optimal control system. This is the content of Lemmas 6.4.1 through 6.4.4. Next, we used the duality results of Chapter 2 to assert that detectability and controllability of the original system corresponds to stabilizability and observability of the dual system. Thus, under the assumptions of detectability and controllability of the original system, we were able to invoke the stability result of section 6.2 to conclude asymptotic stability of the dual optimal control system. Finally, the formulas relating the optimal filter and the dual optimal control system allowed us to conclude asymptotic stability of the optimal filter from that of the dual control system. Once these results were established, it was straightforward to prove, in section 6.5, that the closed-loop stochastic control system is also asymptotically stable. While the analogs of these results for ordinary differential systems are well-known, they are new for linear delay systems, and constitute the first complete extension of the finite dimensional linear-quadratic-Gaussian theory. Furthermore, they bring into focus the importance of the structural properties and duality relations in control and estimation problems for linear delay systems. While our results deal only with a special class of delay systems, we feel that the ideas and techniques will be fundamental to the development of a complete linear-quadratic-Gaussian theory for more general delay systems. We hope that the work reported in this thesis will serve as a stimulus to the development of such a theory.

# CHAPTER 7

## CONCLUSIONS AND SUGGESTIONS FOR FUTURE RESEARCH

In this thesis, we have studied two areas in the theory of delay systems: structural properties and their applications to control in linear time-invariant delay systems, and optimal linear and nonlinear filtering for stochastic delay systems. We summarize our findings here and indicate what we feel are the main contributions of the thesis.

In Chapter 2, we discussed the various notions of controllability, stabilizability, observability, and detectability in connection with linear delay systems. New definitions of controllability and observability were given and some duality results established. While we have not conducted an in-depth study of these concepts, the ideas presented have proved to be very useful in connection with linear optimal control with quadratic cost and optimal linear filtering.

In Chapter 3, we studied the concept of pointwise degeneracy. We completely characterized 3 dimensional systems which can be made pointwise degenerate by delay feedback, and gave sufficient conditions for this to be possible in higher dimensional systems. The conditions obtained are felt to be much simpler and more intuitive than previous results obtained by other investigators.

In Chapter 4, we applied the pointwise degeneracy property to the control of linear systems. A complete solution to the minimum time output zeroing problem by delay feedback was given for 3 dimensional linear systems, while sufficient conditions were given for the solvability of the problem in higher dimensions. The solvability conditions are again very simple, although the actual construction of the feedback controllers is rather complicated. Sensitivity of the control system under perturbations of its parameters was also studied and it was shown that "small" perturbations give rise to "small" degradation in the system performance. Some examples were then given to illustrate the theory. All these results are new and show that the concept of delay feedback has potential applications in linear system theory which are worth exploring further.

We began the study of the optimal filtering problem in Chapter 5. The general nonlinear filtering problem was first studied, and a representation theorem for conditional moment functionals derived. The form of the representation is new even when specialized to ordinary differential systems, and can be thought of as solving the nonlinear filtering problem abstractly. We then showed that if the moment functionals were suitably "smooth," stochastic differential equations could be derived. When these results were specialized to the linear case, we obtained a complete solution to the optimal linear filtering problem, generalizing previously known results.

In Chapter 6, we studied the stability properties of the optimal linear filter. The special case where the observations do not involve any delay terms was considered. We showed that there is a duality relation between optimal control with quadratic cost and optimal linear filtering. Under the hypothesis of stabilizability and observability, we showed that the optimal closed-loop system for the infinite time control problem is asymptotically stable. Under the hypothesis of detectability and controllability, we showed that the optimal linear filter is asymptotically stable. Finally, the closed-loop stochastic control system using the deterministic optimal control law and the optimal filter estimate was also shown to be asymptotically stable. This we feel is the first complete extension of the familiar linear-quadratic-Gaussian theory for ordinary differential systems. On the one hand, almost complete analogs of the finite dimensional results have been obtained: the linear feedback nature of the solutions for the optimal control and filtering problems, the Riccati-type partial differential equations, the duality between control and filtering, and the stability results based on system-theoretic concepts. On the other hand, the additional complications due to the presence of delays are also clearly demonstrated by the form of the solutions. These similarities and differences between the finite dimensional theory and the theory developed in this work give a lot of insight into the behavior of linear delay systems.

The results obtained in this thesis suggest that further investigations in the following topics would be fruitful.

1. Study in greater depth the relations between the various notions of controllability, stabilizability, observability, and detectability for delay systems. In particular, algebraic

conditions for checking these properties would be very useful. In addition to the notions discussed in Chapter 2, there are other algebraic formulations of the notions of controllability and observability [42], [43]. It would be very worthwhile to relate these various approaches and obtain a unified theory. This is very much an open area of research.

2. Extend the applications of the pointwise degeneracy property to the construction of delay observers. We have given an example of a "deadbeat" observer. The general theory, however, is lacking, and it would be interesting to see what precise conditions are required. It is suspected that this problem will be related to the properties of the adjoint system.

3. In studying the nonlinear filtering problem, we have shown that in order to derive stochastic differential equations, moment functionals must be in the domain of the infinitesimal generator of the Markov process $x_t$. The characterization of this operator is important in its own right as well as in applying the results of Chapter 5. It is hoped that a detailed characterization of this operator will pave the way for useful approximations to the optimal nonlinear filter.

4. Extend the stability results of Chapter 6 to cover more general models, for example, those with delays in the observations. As discussed in Chapter 6, this will be

intimately related to the control problem with delays in the control. It is not clear what techniques are suitable for this problem. In fact, it is not clear whether the controllability and observability concepts which have proved fruitful need to be reformulated or not. Since many practical systems have delays in the observations, this is an important practical as well as theoretical problem.

## APPENDIX A

## PROOF OF LEMMAS 4.1 AND 4.2

We refer the reader to Chapter 4 for the statement of these lemmas.

Proof of Lemma 4.1: The extrema of the function $f(\lambda) \triangleq (\alpha_1+\alpha_2\lambda+\alpha_3\lambda^2)e^{\lambda\tau}$ occur at

$$\lambda = \frac{-(\alpha_2\tau+2\alpha_3) \pm\sqrt{\alpha_2^2\tau^2+4\alpha_3^2-4\alpha_1\alpha_3\tau^2}}{2\alpha_3\tau}$$

Since $\alpha_2^2-4\alpha_1\alpha_3 > 0$, there are two real extrema for $f(\lambda)$. Also, the zeros of $f(\lambda)$ occur at the zeros of $\alpha_1+\alpha_2\lambda+\alpha_3\lambda^2$, and these are at

$$\frac{-\alpha_2 \pm\sqrt{\alpha_2^2-4\alpha_1\alpha_3}}{2\alpha_3}$$

Again, since $\alpha_2^2-4\alpha_1\alpha_3 > 0$, these give two real zeros of $f(\lambda)$. Let

$$s_1 = \frac{-\alpha_2 +\sqrt{\alpha_2^2-4\alpha_1\alpha_3}}{2\alpha_3}$$

$$s_2 = \frac{-\alpha_2 -\sqrt{\alpha_2^2-4\alpha_1\alpha_3}}{2\alpha_3}$$

Let us perturb $\lambda_i$ by some number $\varepsilon$ and let

$$\eta_\varepsilon(s_i) = \alpha_3(s_i+\varepsilon)^2+\alpha_2(s_i+\varepsilon)+\alpha_1$$

$$= \varepsilon(\alpha_3\varepsilon+2s_i\alpha_3+\alpha_2)$$

Hence $\eta_\varepsilon(s_1) = \varepsilon(\alpha_3\varepsilon + \sqrt{\alpha_2^2 - 4\alpha_1\alpha_3})$

$$\eta_\varepsilon(s_2) = \varepsilon(\alpha_3\varepsilon - \sqrt{\alpha_2^2 - 4\alpha_1\alpha_3})$$

This yields

$$\eta_\varepsilon(s_1) > 0 \quad \text{if } \varepsilon > 0, \text{ and } \varepsilon\alpha_3 > -\sqrt{\alpha_2^2-4\alpha_1\alpha_3}$$

$$< 0 \quad \text{if } \varepsilon > 0, \text{ and } \varepsilon\alpha_3 < -\sqrt{\alpha_2^2-4\alpha_1\alpha_3}$$

$$> 0 \quad \text{if } \varepsilon < 0, \text{ and } \varepsilon\alpha_3 < -\sqrt{\alpha_2^2-4\alpha_1\alpha_3}$$

$$< 0 \quad \text{if } \varepsilon < 0, \text{ and } \varepsilon\alpha_3 > -\sqrt{\alpha_2^2-4\alpha_1\alpha_3}$$

Similarly,

$$\eta_\varepsilon(s_2) > 0 \quad \text{if } \varepsilon > 0, \text{ and } \varepsilon\alpha_3 > \sqrt{\alpha_2^2-4\alpha_1\alpha_3}$$

$$< 0 \quad \text{if } \varepsilon > 0, \text{ and } \varepsilon\alpha_3 < \sqrt{\alpha_2^2-4\alpha_1\alpha_3}$$

$$> 0 \quad \text{if } \varepsilon < 0, \text{ and } \varepsilon\alpha_3 < \sqrt{\alpha_2^2-4\alpha_1\alpha_3}$$

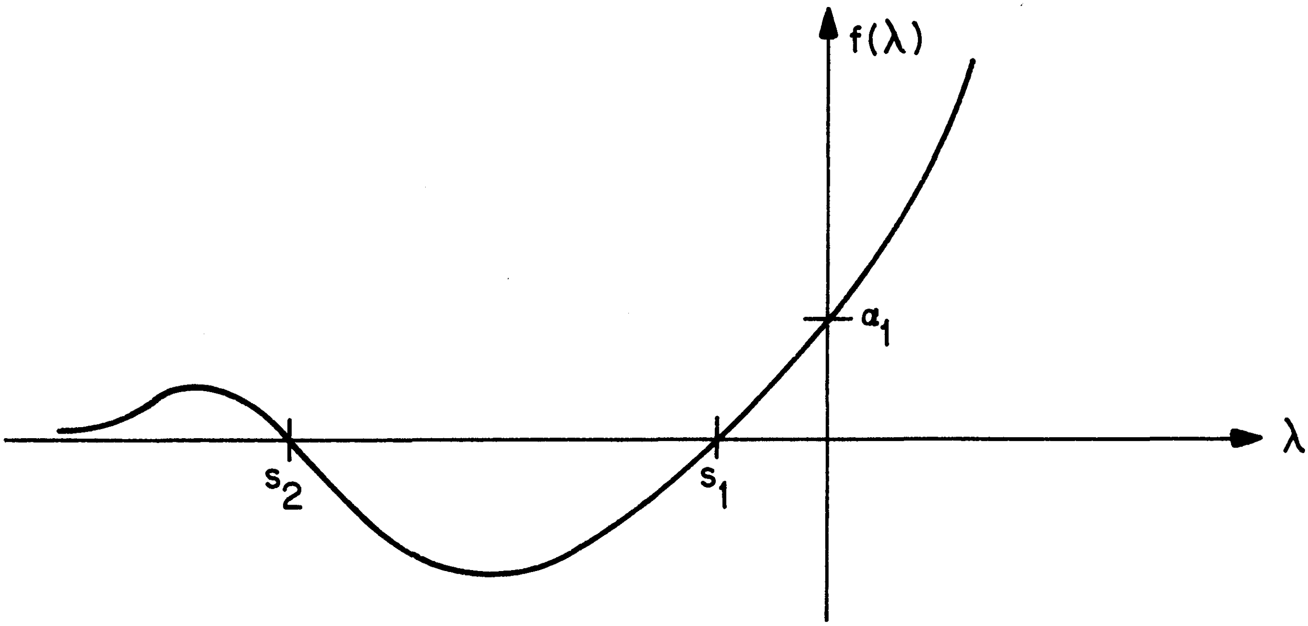
$$< 0 \quad \text{if } \varepsilon < 0, \text{ and } \varepsilon\alpha_3 > \sqrt{\alpha_2^2-4\alpha_1\alpha_3}$$

Let us now examine the graph of the function $f(\lambda)$. There are eight cases, depending on the signs of the $\alpha_i$'s.

Case I: $\alpha_1 > 0$, $\alpha_2 > 0$, $\alpha_3 > 0$. In this case $s_1$ and $s_2$ are both negative. If $\varepsilon > 0$, $\eta_\varepsilon(s_1) > 0$, and if

$$\frac{-\sqrt{\alpha_2^2-4\alpha_1\alpha_3}}{\alpha_3} < \varepsilon < 0, \ \eta_\varepsilon(s_1) < 0.$$

A little thought now shows that the graph of $f(\lambda)$ is of the form




It is obvious that for any finite real $\beta \neq s_1$ or $s_2$, there exists a $\gamma_\beta$ such that the line $\gamma_\beta(\lambda-\beta)$ intersects $f(\lambda)$ at three real distinct values of $\lambda$, none of which equals $s_1$ or $s_2$. There also exists a constant $\omega(\omega > 0$ in this case) such that the line $\lambda=\omega$ intersects $f(\lambda)$ at three real distinct points.

For the other seven cases, entirely similar arguments show that $f(\lambda)$ must have one positive maximum and one negative minimum, and hence the required straight lines exist. We shall omit the details. The lemma is now proved.

Proof of Lemma 4.2: Since C is of full rank, we may assume, without loss of generality, that $q'c_1 = \alpha_3 \neq 0$, where $c_i$ is the $i^{th}$ column of C. Since (A,C) is controllable, we have the following possibilities:

(i) $\{c_1, Ac_1, c_2\}$ are linearly independent

or (ii) $\{c_1, c_2, Ac_2\}$ are linearly independent

Case (i): $\{c_1, Ac_1, c_2\}$ linearly independent.

Construct a matrix $K_1$ as follows:

$$K_1 = [\underline{0}\ e_2\ \underline{0}][c_1\ Ac_1\ c_2]^{-1}$$

where $e_2$ is the second natural Euclidean basis vector in 2 dimensions, and $\underline{0}$ is the null vector. Then

$$(A + CK_1)c_1 = Ac_1$$

$$(A + CK_1)^2 = A^2c_1 + c_2$$

and $(A_1,c_1)$ is controllable, where $A_1 = (A + CK_1)$.

Let the characteristic polynomial of $A_1$ be $p(s) = s^3+p_2s^2+p_1s+p_o$. We can then write

$$c_2 = \beta_1(A_1^2c_1+p_2A_1c_1+p_1c_1) + \beta_2(A_1c_1+p_2c_1) + \beta_3c_1$$

for some $\beta_1$, $\beta_2$, and $\beta_3$. The independence of $c_1$, $Ac_1$, and $c_2$ implies $\beta_1 \neq 0$. The transformation

$$P = [A_1^2c_1\ A_1c_1\ c_1]\begin{bmatrix}1 & 0 & 0\\ p_2 & 1 & 0\\ p_1 & p_2 & 1\end{bmatrix}$$

gives $P^{-1}A_1P$ in companion form and

$$P^{-1}C = \begin{bmatrix}0 & \beta_1\\ 0 & \beta_2\\ 1 & \beta_3\end{bmatrix}$$

Let $q'P = (\alpha_1 \alpha_2 \alpha_3)$. Then

$$\alpha_1 = q'(A^2 c_1 + c_2) + p_2 q' A c_1 + p_1 \alpha_3$$

$$\alpha_2 = q' A c_1 + p_2 \alpha_3$$

$$\alpha_3 = q' c_1$$

If $\alpha_2^2 - 4\alpha_1\alpha_3 > 0$, and $\frac{\beta_2}{\beta_1}$ is not a zero of $\eta(\lambda) = \alpha_1 + \alpha_2\lambda + \alpha_3\lambda^2$, we are done. If not, take a matrix F of the form

$$F = \begin{bmatrix} -f_o & -f_1 & -f_2 \\ 0 & 0 & 0 \end{bmatrix}$$

then $P^{-1}(A+CK_1)P + P^{-1}CF$

$$= P^{-1}(A+CK_1+CFP^{-1})P = P^{-1}A_2P$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -p_o' & -p_1' & -p_2' \end{bmatrix}$$

with $p_i' = p_i + f_i$. Define a matrix $P_1$ by

$$P_1 = P^{-1}[A_2^2 c_1 \; A_2 c_1 \; c_1] \begin{bmatrix} 1 & 0 & 0 \\ p_2' & 1 & 0 \\ p_1' & p_2' & 1 \end{bmatrix}$$

$$\triangleq P^{-1}P_2$$

Then $P_1^{-1}P^{-1}A_2PP_1 = P_2^{-1}A_2P_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -p_o' & -p_1' & -p_2' \end{bmatrix}$

Furthermore $P_1^{-1}P^{-1}C = P_2^{-1}C = \begin{bmatrix} 0 & d_{12} \\ 0 & d_{22} \\ 1 & d_{32} \end{bmatrix}$, for some $d_{12}$, $d_{22}$, $d_{32}$

Now $A_2c_1 = (A+CK_1)c_1+CFP^{-1}c_1$

$$= Ac_1+C \begin{bmatrix} -f_2 \\ 0 \end{bmatrix} = Ac_1-f_2c_1$$

Similarly $A_2^2c_1 = A^2c_1+c_2-f_2Ac_1+(f_2^2-p_2f_2+f_1)c_1$

Choose $f_o$, $f_1$, and $f_2$ to satisfy the properties

(a) $\alpha_2'^2-4\alpha_1'\alpha_3' > 0$, where

$$\alpha_3' = \alpha_3 = q'c_1$$

$$\alpha_2' = q'A_2c_1+p_2'\alpha_3 = q'Ac_1-f_2q'c_1 + (p_2+f_2)\alpha_3 = \alpha_2$$

and $\alpha_1' = q'A_2^2c_1+p_2'q'A_2c_1+p_1'\alpha_3$

$$= q'(A^2c_1+c_2)+p_2q'Ac_1+(f_2^2-f_2-p_2f_2+p_1+2f_1)\alpha_3$$

and

(b) $\dfrac{d_{22}}{d_{12}}$ is not a zero of $\eta'(\lambda) = \alpha_1'+\alpha_2\lambda+\alpha_3\lambda^2$.

A little algebra shows that in fact $d_{12} = \beta_1$, $d_{22} = \beta_2$, and $d_{32} = \beta_1 p_1 + \beta_3 - 2\beta_1 f_1 + 2\beta_1 p_2 f_2$.

Some reflection now shows that appropriate choices for $f_o$, $f_1$, and $f_2$ exist. This completes the consideration of case (i).

Case (ii): $\{c_1, c_2, Ac_2\}$ linearly independent.

We construct a matrix $K_2$ as follows

$$K_2 = [e_2 \ \underline{0} \ \underline{0}][c_1 \ c_2 \ Ac_2]^{-1}$$

Let $A_3 = A+CK_2$. Then $A_3 c_1 = Ac_1 + c_2$ and $A_3^2 c_1 = A^2 c_1 + Ac_2 + CK_2 Ac_1$. Let the characteristic polynomial of $A_3$ be $p(s) = s^3 + p_2 s^2 + p_1 s + p_o$. We can write

$$c_2 = \beta_1 [A_3^2 c_1 + p_2 A_3 c_1 + p_1 c_1] + \beta_2 [A_3 c_1 + p_2 c_1] + \beta_3 c_1$$

$\beta_2$ is clearly nonzero. If $\beta_1 \neq 0$, we carry out the constructions in exactly the same fashion as in case (i). This shows that we can again find appropriate matrices P and K such that $P^{-1}(A+CK)P$, $q'P$, and $P^{-1}C$ have the required structure. The proof of the lemma is completed.

## APPENDIX B

## DERIVATION OF EQUATIONS (5.53) - (5.55)

To derive the appropriate equations for $P(t,\theta,\xi)$, we have to consider three cases:

(i) $0 \le t \le T, \quad -\tau \le \theta < 0, \quad -\tau \le \xi < 0$

(ii) $0 \le t \le T, \quad -\tau \le \theta < 0, \quad \xi = 0$

(this also covers the case $0 \le t \le T$, $\theta = 0$, $-\tau \le \xi < 0$, since clearly $P(t,\theta,\xi) = P'(t,\xi,\theta)$).

(iii) $0 \le t \le T, \quad \theta = \xi = 0.$

Since the derivations are somewhat tedious, we shall give details only for the equation satisfied by $P(t,\theta,\xi)$ for $0 \le t \le T$, $-\tau \le \theta < 0$, $-\tau \le \xi < 0$. To avoid notational complications (since $P(t,\theta,\xi)$ is a matrix, to use Theorem 5.3.1 requires the calculation of $P_{ij}(t,\theta,\xi)$, $1 \le i,j \le n$), we will assume everything is scalar. For convenience we shall write $e_c(x_s,s) = c(x_s,s) - \hat{c}(x_s,s)$

We appeal to Theorem 5.3.1 and (5.50) to write

$$P(t,\theta,\xi) = E^t\{x(t+\theta)x(t+\xi)\} - \hat{x}(t+\theta|t)\hat{x}(t+\xi|t)$$

$$= E_o\{x(t+\theta)x(t+\xi)\} + \int_0^t E^s\{E_o[x(t+\theta)x(t+\xi)|x_s]e_c(x_s,s)]\}$$

$$\cdot R^{-1}(s)d\nu(s)$$

$$-[\hat{x}(t+\theta|t+\theta)+\int_{t+\theta}^{t} E^{s}\{x(t+\theta)e_c(x_s,s)\}R^{-1}(s)d\nu(s)]\ [\hat{x}(t+\xi|t+\xi)$$

$$+\int_{t+\xi}^{t} E^{s}\{x(t+\xi)e_c(x_s,s)\}R^{-1}(s)d\nu(s)]$$

$$= E_o[x(t+\theta)x(t+\xi)] - \hat{x}(t+\theta|t+\theta)\hat{x}(t+\xi|t+\xi)$$

$$+\int_{0}^{t} E^{s}\{E_o[x(t+\theta)x(t+\xi)|x_s]e_c(x_s,s)\}R^{-1}(s)d\nu(s)$$

$$-\hat{x}(t+\theta|t+\theta)\int_{t+\xi}^{t} E^{s}\{x(t+\xi)e_c(x_s,s)\}R^{-1}(s)d\nu(s)$$

$$-\hat{x}(t+\xi|t+\xi)\int_{t+\theta}^{t} E^{s}\{x(t+\theta)e_c(x_s,s)\}R^{-1}(s)d\nu(s)$$

$$-\int_{t+\theta}^{t} E^{s}\{x(t+\theta)e_c(x_s,s)\}R^{-1}(s)d\nu(s)\int_{t+\xi}^{t} E^{s}\{x(t+\xi)e_c(x_s,s)\}R^{-1}(s)d\nu(s) \tag{B.1}$$

It is readily seen from (B.1) that $P(t,\theta,\xi)$ is continuous in $(t,\theta,\xi)$. Similarly for $\varepsilon$ in $(0,\tau)$ such that $-\tau \le \theta + \varepsilon \le 0$, $-\tau \le \xi + \varepsilon \le 0$,

$P(t-\varepsilon,\ \theta+\varepsilon,\ \xi+\varepsilon)$

$$= E_o\{x(t+\theta)x(t+\xi)\}-\hat{x}(t+\theta|t+\theta)\hat{x}(t+\xi|t+\xi)$$

$$+\int_{0}^{t-\varepsilon} E^{s}\{E_o[x(t+\theta)x(t+\xi)|x_s]e_c(x_s,s)\}R^{-1}(s)d\nu(s)$$

$$-\ \hat{x}(t+\theta|t+\theta)\int_{t+\xi}^{t-\varepsilon} E^{s}\{x(t+\xi)e_c(x_s,s)\}R^{-1}(s)d\nu(s)$$

$$-\hat{x}(t+\xi|t+\xi)\int_{t+\theta}^{t-\varepsilon} E^s\{x(t+\theta)e_c(x_s,s)\}R^{-1}(s)d\nu(s)$$

$$-\int_{t+\theta}^{t-\varepsilon} E^s\{x(t+\theta)e_c(x_s,s)\}R^{-1}(s)d\nu(s)\int_{t+\xi}^{t-\varepsilon} E^s\{x(t+\xi)e_c(x_s,s)\}R^{-1}(s)d\nu(s)$$

Hence $P(t,\theta,\xi) - P(t-\varepsilon,\ \theta+\varepsilon,\ \xi+\varepsilon\ )$

$$=\int_{t-\varepsilon}^{t} E^s\{E_o[x(t+\theta)x(t+\xi)|x_s]e_c(x_s,s)\}R^{-1}(s)d\nu(s)$$

$$-\hat{x}(t+\theta|t+\theta)\int_{t-\varepsilon}^{t} E^s\{x(t+\xi)e_c(x_s,s)\}R^{-1}(s)d\nu(s)$$

$$-\hat{x}(t+\xi|t+\xi)\int_{t-\varepsilon}^{t} E^s\{x(t+\theta)e_c(x_s,s)\}R^{-1}(s)d\nu(s)$$

$$-\int_{t+\theta}^{t} E^s\{x(t+\theta)e_c(x_s,s)\}R^{-1}(s)d\nu(s)\int_{t+\xi}^{t} E^s\{x(t+\xi)e_c(x_s,s)\}R^{-1}(s)d\nu(s)$$

$$+\int_{t+\theta}^{t-\varepsilon} E^s\{x(t+\theta)e_c(x_s,s)\}R^{-1}(s)d\nu(s)\int_{t+\xi}^{t-\varepsilon} E^s\{x(t+\xi)e_c(x_s,s)\}R^{-1}(s)d\nu(s) \tag{B.2}$$

The last two terms of (B.2) can be written as

$$-\int_{t-\varepsilon}^{t} E^s\{x(t+\theta)e_c(x_s,s)\}R^{-1}(s)d\nu(s)\int_{t+\xi}^{t} E^s\{x(t+\xi)e_c(x_s,s)\}R^{-1}(s)d\nu(s)$$

$$-\int_{t+\theta}^{t-\varepsilon} E^s\{x(t+\theta)e_c(x_s,s)\}R^{-1}(s)d\nu(s)\int_{t-\varepsilon}^{t} E^s\{x(t+\xi)e_c(x_s,s)\}R^{-1}(s)d\nu(s)$$

$$= -\int_{t-\varepsilon}^{t} E^s\{x(t+\theta)e_c(x_s,s)\}R^{-1}(s)d\nu(s) \int_{t-\varepsilon}^{t} E^s\{x(t+\xi)e_c(x_s,s)\}R^{-1}(s)d\nu(s)$$

$$- \int_{t-\varepsilon}^{t} E^s\{x(t+\theta)e_c(x_s,s)\}R^{-1}(s)d\nu(s) \int_{t+\xi}^{t-\varepsilon} E^s\{x(t+\xi)e_c(x_s,s)\}R^{-1}(s)d\nu(s)$$

$$- \int_{t+\theta}^{t-\varepsilon} E^s\{x(t+\theta)e_c(x_s,s)\}R^{-1}(s)d\nu(s) \int_{t-\varepsilon}^{t} E^s\{x(t+\xi)e_c(x_s,s)\}R^{-1}(s)d\nu(s) \tag{B.3}$$

Substituting (B.3) into (B.2) and using properties of Ito integrals, we get

$P(t,\theta,\xi) - P(t-\varepsilon,\ \theta+\varepsilon,\ \xi+\varepsilon)$

$$= \int_{t-\varepsilon}^{t} E^s\{E_o[x(t+\theta)x(t+\xi)|x_s]e_c(x_s,s)\}R^{-1}(s)d\nu(s)$$

$$-\hat{x}(t+\theta|t-\varepsilon) \int_{t-\varepsilon}^{t} E^s\{x(t+\xi)e_c(x_s,s)\}R^{-1}(s)d\nu(s)$$

$$-\hat{x}(t+\xi|t-\varepsilon) \int_{t-\varepsilon}^{t} E^s\{x(t+\theta)e_c(x_s,s)\}R^{-1}(s)d\nu(s)$$

$$- \int_{t-\varepsilon}^{t} E^s\{x(t+\theta)e_c(x_s,s)\}R^{-1}(s) \int_{t-\varepsilon}^{u} E^u\{x(t+\xi)e_c(x_u,u)\}R^{-1}(u)d\nu(u)d\nu(s)$$

$$- \int_{t-\varepsilon}^{t}\int_{t-\varepsilon}^{s} E^u\{x(t+\theta)e_c(x_u,u)\}R^{-1}(u)E^s\{x(t+\xi)e_c(x_s,s)\}R^{-1}(s)d\nu(u)d\nu(s)$$

$$- \int_{t-\varepsilon}^{t} E^s\{x(t+\theta)e_c(x_s,s)\}R^{-1}(s)E^s\{x(t+\xi)e_c(x_s,s)\}ds$$

$$= \int_{t-\varepsilon}^{t} E^s\{E_o[x(t+\theta)x(t+\xi)|x_s]e_c(x_s,s)\}R^{-1}(s)d\nu(s)$$

$$- \int_{t-\varepsilon}^{t} \hat{x}(t+\xi|s)E^s\{x(t+\theta)e_c(x_s,s)\}R^{-1}(s)d\nu(s)$$

$$- \int_{t-\varepsilon}^{t} \hat{x}(t+\theta|s)E^s\{x(t+\xi)e_c(x_s,s)\}R^{-1}(s)d\nu(s)$$

$$- \int_{t-\varepsilon}^{t} E^s\{x(t+\theta)e_c(x_s,s)\}R^{-1}(s)E^s\{x(t+\xi)e_c(x_s,s)\}ds \qquad (B.4)$$

Since for each $\theta\varepsilon[-\tau,0]$, $x(t+\theta)$ given $z^t$ is Gaussian, the first three terms of (B.4) add to zero, being the third central moment of a Gaussian process. Hence

$$P(t,\theta,\xi) - P(t-\varepsilon, \theta+\varepsilon, \xi+\varepsilon)$$

$$= - \int_{t-\varepsilon}^{t} E^s\{x(t+\theta)e_c(x_s,s)\}R^{-1}(s)E^s\{x(t+\xi)e_c(x_s,s)\}ds \qquad (B.5)$$

Note that there are no random terms in (B.5). Since (B.5) holds for arbitrary $\varepsilon$ in $(0,\tau)$ with $-\tau \leq \theta + \varepsilon \leq 0$, $-\tau \leq \xi + \varepsilon \leq 0$, we can divide by $\frac{1}{\varepsilon}$ and let $\varepsilon$ go to 0. If we let $\sigma$ be the unit vector in the direction $(1, -1, -1)$, then the directional derivative, $P_\sigma(t,\theta,\xi)$ of $P(t,\theta,\xi)$ in the direction $\sigma$ is given by

$$\sqrt{3}\; P_\sigma(t,\theta,\xi) = -E^t\{x(t+\theta)e_c(x_t,t)\}R^{-1}(t)E^t\{x(t+\xi)e_c(x_t,t)\} \tag{B.6}$$

Equation (B.6) is the desired equation for $P(t,\theta,\xi)$, $0 \leq t \leq T$, $-\tau \leq \theta < 0$, $-\tau \leq \xi < 0$. The same technique can be applied to the derivation of the equation for $P(t,\theta,0)$, $0 \leq t \leq T$, $-\tau \leq \theta < 0$, and that for $P(t,0,0)$. We only sketch the steps, omitting the details.

By Theorem 5.3.1,

$$P(t,\theta,0) = E_o[x(t+\theta)x(t)] + \int_0^t E^s\{E_o[x(t+\theta)x(t)|x_s]e_c(x_s,s)\}R^{-1}(s)d\nu(s)$$
$$-\hat{x}(t+\theta|t)\hat{x}(t|t) \tag{B.7}$$

(B.7) implies that $P(t,\theta,0)$ is continuous in $(t,\theta)$. Some calculations yield, for $\varepsilon$ in $(0,\tau)$ such that $-\tau \leq \theta + \varepsilon \leq 0$, $P(t,\theta,0) - P(t-\varepsilon, \theta+\varepsilon, 0)$

$$= \int_{t-\varepsilon}^{\varepsilon} E^s\{x(t+\theta)a(x_s,s)\}ds + \int_{t-\varepsilon}^{t} E^s\{x(t+\theta)x(s)e_c(x_s,s)\}R^{-1}(s)d\nu(s)$$
$$-\hat{x}(t+\theta|t)\hat{x}(t|t)+\hat{x}(t+\theta|t-\varepsilon)\hat{x}(t-\varepsilon|t-\varepsilon) \tag{B.8}$$

The last two terms of (B.8) can be evaluated to give

$$-\hat{x}(t+\theta|t)\hat{x}(t|t)+\hat{x}(t+\theta|t-\varepsilon)\hat{x}(t-\varepsilon|t-\varepsilon)$$

$$= -\int_{t-\varepsilon}^{t} \hat{x}(t+\theta|s)\hat{a}(x_s,s)ds - \int_{t-\varepsilon}^{t} \hat{x}(t+\theta|s)E^s\{x(s)e_c(x_s,s)\}R^{-1}(s)d\nu(s)$$

$$- \int_{t-\varepsilon}^{t} E^s\{x(t+\theta)\hat{x}(s|s)e_c(x_s,s)\}R^{-1}(s)d\nu(s)$$

$$- \int_{t-\varepsilon}^{t} E^s\{x(t+\theta)e_c(x_s,s)\}R^{-1}(s)E^s\{x(s)e_c(x_s,s)\}ds \tag{B.9}$$

Substituting (B.9) into (B.8) and using again the Gaussian property, we see that

$$P(t,\theta,0) - P(t-\varepsilon, \theta+\varepsilon,0)$$

$$= \int_{t-\varepsilon}^{t} E^{s}\{x(t+\theta)e_{a}(x_{s},s)\}ds - \int_{t-\varepsilon}^{t} E^{s}\{x(t+\theta)e_{c}(x_{s},s)\}R^{-1}(s)\cdot$$

$$\cdot E^{s}\{x(s)e_{c}(x_{s},s)\}ds \qquad \text{(B.10)}$$

Again there are no random terms so that $P(t,\theta,0)$ has a directional derivative in the (1,-1) direction. Denoting the unit vector in the (1,-1) direction by $\eta$, we obtain

$$\sqrt{2}\, P_{\eta}(t,\theta,0) = E^{t}\{x(t+\theta)e_{a}(x_{t},t)\}$$

$$-E^{t}\{x(t+\theta)e_{c}(x_{t},t)\}R^{-1}(t)E^{t}\{x(t)e_{c}(x_{t},t)\} \qquad \text{(B.11)}$$

Finally, similar calculations (or using (5.42)) show that $P(t,0,0)$ has a derivative and satisfies

$$\frac{d}{dt} P(t,0,0) = E^{t}\{x(t)e_{a}(x_{t},t)\} + E^{t}\{e_{a}(x_{t},t)x(t)\}$$

$$-E^{t}\{x(t)e_{c}(x_{t},t)\}R^{-1}(t)E\{e_{c}(x_{t},t)x(t)\} + Q(t) \qquad \text{(B.12)}$$

In the vector case, equations identical to those of (B.6), (B.11), and (B.12) hold for $P(t,\theta,\xi)$, $P(t,\theta,0)$, and $P(t,0,0)$. This completes the derivation of equations (5.53) - (5.55). The initial conditions (5.56) follow immediately from the properties of conditional expectations.

APPENDIX C

PROOF OF LEMMA 6.6.1

We apply the Ito differential rule to the function $V(x_t)$ defined in (6.78). We calculate the first and second terms to illustrate the computations involved.

$$
\begin{aligned}
dx'(t)K_o(t)x(t) &= [dx'(t)]K_o(t)x(t)dt \\
&\quad +x'(t)[dK_o(t)]x(t)dt+x'(t)K_o(t)[dx(t)]dt+trM'MK_o(t)dt \\
&= x'(t-\tau)B'K_o(t)x(t)dt+u'(t)G'K_o(t)x(t)dt \\
&\quad +dw'(t)K_o(t)x(t)dt+x'(t)K_o(t)Bx(t-\tau)dt+x'(t)K_o(t)Gu(t)dt \\
&\quad +x'(t)K_o(t)dw(t)-x'(t)Qx(t)dt+x'(t)K_o(t)GR^{-1}G'K_o(t)x(t)dt \\
&\quad -x'(t)K_1'(t,0)x(t)dt-x'(t)K_1(t,0)x(t)dt+trM'MK_o(t)dt
\end{aligned}
$$

$$
\begin{aligned}
d_t \int_{-\tau}^{0} x'(t)K_1(t,\theta)x(t+\theta)d\theta \\
&= d_t[x'(t)\int_{t-\tau}^{t} K_1(t,\sigma-t)x(\sigma)d\sigma] \\
&= \{[x'(t)A'+x'(t-\tau)B'+u'(t)G']dt+dw'(t)\}\int_{-\tau}^{0} K_1(t,\theta)x(t+\theta)d\theta \\
&\quad +x'(t)K_1(t,0)x(t)dt-x'(t)K_1(t,-\tau)x(t-\tau)dt \\
&\quad +x'(t)\int_{t-\tau}^{t} d_tK_1(t,\sigma-t)x(\sigma)d\sigma dt
\end{aligned}
$$

$$= [x'(t)A'+x'(t-\tau)B'+u'(t)G'] \int_{-\tau}^{0} K_1(t,\theta)x(t+\theta)d\theta dt$$

$$+dw'(t) \int_{-\tau}^{0} K_1(t,\theta)x(t+\theta)d\theta+x'(t)K_1(t,0)x(t)dt$$

$$-x'(t)K_1(t,-\tau)x(t-\tau)dt+x'(t) \int_{-\tau}^{0} [\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial \theta}\right) K_1(t,\theta)]x(t+\theta)d\theta dt$$

Similar calculations on the last two terms on the right hand side of (6.78) yields the following expression

$$dV(x_t) = V_1(t)dt+dw'(t)K_o(t)x(t)+x'(t)K_o(t)dw(t)$$

$$+trM'MK_o(t)dt+dw'(t) \int_{-\tau}^{0} K_1(t,\theta)x(t+\theta)d\theta$$

$$+ \int_{-\tau}^{0} x'(t+\theta)K_1'(t,\theta)d\theta dw(t)$$

$$-x'(t)Qx(t)dt-u'(t)Ru(t)dt \qquad (C.1)$$

where $$V_1(t) = [u(t)+R^{-1}G'K_o(t)x(t)+ \int_{-\tau}^{0} R^{-1}G'K_1(t,\theta)x(t+\theta)d\theta]'$$

$$\cdot R[u(t)+R^{-1}G'K_o(t)x(t)+ \int_{-\tau}^{0} R^{-1}G'K_1(t,\xi)x(t+\xi)d\xi] \qquad (C.2)$$

Using the boundary conditions at T for $K_o(t)$, $K_1(t,\theta)$, and $K_2(t,\theta,\xi)$, we see that $V(x_T) = 0$. Therefore

$$\int_0^T [x'(t)Qx(t)+u'(t)Ru(t)]dt$$

$$= V(x_o) + \int_0^T V_1(t)dt + \int_0^T trM'MK_o(t)dt$$

$$+ 2\int_0^T x'(t)K_o(t)dw(t) + 2\int_0^T\int_{-\tau}^0 x'(t+\theta)K_1'(t,\theta)d\theta dw(t)$$

Taking expectations, we get

$$E\int_0^T [x'(t)Qx(t)+u'(t)Ru(t)]dt$$

$$= EV(x_o) + E\int_0^T V_1(t)dt + \int_0^T trM'MK_o(t)dt \qquad (C.3)$$

Now $E\int_0^T V_1(t)dt = \int_0^T EV_1(t)dt$

$$= \int_0^T E\{E[V_1(t)|z^t]\}dt$$

using Fubini's theorem and properties of conditional expectations.

Substituting the control law in (6.76) into (C.2) we get that

$$E[V_1(t)|z^t] = E\{[R^{-1}G'K_o(t)e(t|t)+ \int_{-\tau}^0 R^{-1}G'K_1(t,\theta)e(t+\theta|t)d\theta]'$$

$$\cdot R[R^{-1}G'K_o(t)e(t|t)+ \int_{-\tau}^0 R^{-1}G'K_1(t,\theta)e(t+\theta|t)d\theta]|z^t\}$$

which is precisely

$$\mathrm{tr}\{K_o(t)GR^{-1}G'K_o(t)P_o(t)+\int_{-\tau}^{0} K_1'(t,\theta)GR^{-1}G'K_o(t)P_1'(t,\theta)d\theta$$

$$+\int_{-\tau}^{0} K_o(t)GR^{-1}G'K_1(t,\theta)P_1(t,\theta)d\theta$$

$$+\int_{-\tau}^{0}\int_{-\tau}^{0} K_1'(t,\theta)GR^{-1}G'K_1(t,\xi)P_2(t,\xi,\theta)d\theta d\xi\} \qquad (C.4)$$

$E\{V_1(t)|z^t\}$ is now seen to be a deterministic function and hence equal to $EV_1(t)$. Substituting (C.4) into (C.3) yields the conclusion of the lemma.

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