DESIGN OF RADAR/SONAR TRAJECTORY ESTIMATION FILTERS FOR THE TRACKING OF MANEUVERING VEHICLES

by

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ABSTRACT

Vehicle trajectory estimation filters for smoothing noisy radar or sonar measurements require a definition of the tracked vehicle's acceleration process. Due to the unpredictable length and type of accelerations executed by maneuvering vehicles, presently used techniques model maneuvers as colored acceleration noise with fixed correlation. Several adaptive schemes have appeared in recent literature involving the use of two or more fixed correlation filters with a maneuver detector for switching between filters. This thesis concerns itself with a continously adaptive correlation coefficient filter and the inherent nonlinear filtering problems associated with its implementation. Monte Carlo simulations were performed to provide data on the performance of several approximations to the optimal adaptive (nonlinear) filter. Equations are also derived for the direct calculation of fixed correlation coefficient Kalman filter performance under mismatched correlation conditions.

THESIS SUPERVISOR: Alan S. Willsky
TITLE: Assistant Professor of Electrical Engineering

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CHAPTER I

INTRODUCTION

1.1 The Need for Filtering

Radar and sonar systems designed to determine the position and velocity of vehicles normally can obtain only limited information with which to determine the desired quantities. Some of these limitations are caused by the following:

1. Typically, such a tracking system is normally confronted with the task of monitoring several vehicles simultaneously. This situation results in a limited rate at which information about the individual vehicles can be collected.

2. Although most systems measure position directly in terms of range, bearing, and often elevation/depression angles, these measurements suffer inaccuracies which are mostly a result of the dispersive effects of the medium in which the measurements are taken. For example, a pulse of acoustic energy transmitted by a sonar in water is severly distorted by refraction due to temperature gradients in the ocean and elongated when reflected from an object whose shape causes multiple reflections. Range estimates based on the total travel time will be in error due to the ambiguity in the arrival time of the elongated pulse.

3. The vehicles being tracked are normally controlled by human pilots capable of performing nearly arbitrary maneuvers that are only limited by the physical boundaries in depth or altitude and the available acceleration capability of the vehicle. This is the most important point in this discussion.

The above limitations thus require relatively sophisticated estimation procedures to provide accurate estimates of position and velocity from the available data. Typical approaches to this problem are available in the literature and include linear regression techniques [5], Kalman, Weiner, and alpha-beta filters [2] and adaptive filters [6,4,11]. These techniques mainly have been studied with regard to their performance in reducing uncertainties due to measurement errors. The central issue considered here is how to cope with the third type of uncertainty mentioned above, i.e. the vehicle being tracked can execute maneuvers in and unpredictable way.

The model we consider involves an acceleration correlation coefficient that is assumed to be unknown. We will develop methods for trajectory estimation that involve the estimation, of this parameter. Since the assumption of an unknown parameter introduces a system nonlinearity, the methods used here for correlation estimation may be viewed as a step toward the development of viable on-line nonlinear filtering and system identification techniques. It is hoped that the specific results generated by this research will suggest further more general techniques for other problems of this type.

1.2 Background

Early techniques treated the maneuver problem in a naive fashion. A typical assumption [7] was that the heading (direction of motion) and speed of the vehicle being tracked remained constant. This model is predicated on the fact that most piloted vehicles (aircraft, ships and submarines) generally follow straight line constant speed trajectories. This model is also an attractive candidate for implementation since standard filtering algorithms such as Kalman filters and linear regression techniques can be applied simply. While constant course-constant speed might be the typical situation in some applications it leaves much to be desired in collision avoidance situations, during aircraft holding pattern maneuvers near airports, and in weapon guidance system applications where maneuvers are the rule rather than the exception.

The constant course-constant speed equations are:

$$dx/dt = v_x \tag{1.1}$$

$$dy/dt = v_y \tag{1.2}$$

$$dv_x/dt = 0 \tag{1.3}$$

$$dv_y/dt = 0 \tag{1.4}$$

where x and y are the cartesian position coordinates and v_x and v_y are the velocity components in the x and y directions respectively. Since the typical measurements are range and bearing (the polar coordinates of vehicle position; see Figure 1), a nonlinear measurement equation arises:

$$z_r = (x^2 + y^2)^{\frac{1}{2}}$$

$$z_b = \arctan(y/x)$$

with z_r and z_b the range and bearing measurements.

An obvious extension to this model is to add an acceleration parameter, resulting in a more complicated filtering algorithm. Singer [1] has suggested such a tracking filter that estimates position, velocity and acceleration in each spatial dimension (e.g. x, y, range, or bearing) under consideration. In this model the acceleration is considered as noise driving a system described by Newton's law. Several versions of this model have appeared in the later literature [2,5,11] and the major assumptions and equations are presented here.

The Singer model postulates that a maneuver can be viewed as a random acceleration which may be correlated between measurements. This correlation will exist if the time required to perform the maneuvers is longer than the measurement interval. If the data sampling rate is comparable to or larger than the maneuver time, the maneuver can be considered to be a white stochastic process. In the latter case the model equations are:

$$dc/dt = v_c \tag{1.7}$$

$$dv_c/dt = w(t) \tag{1.8}$$

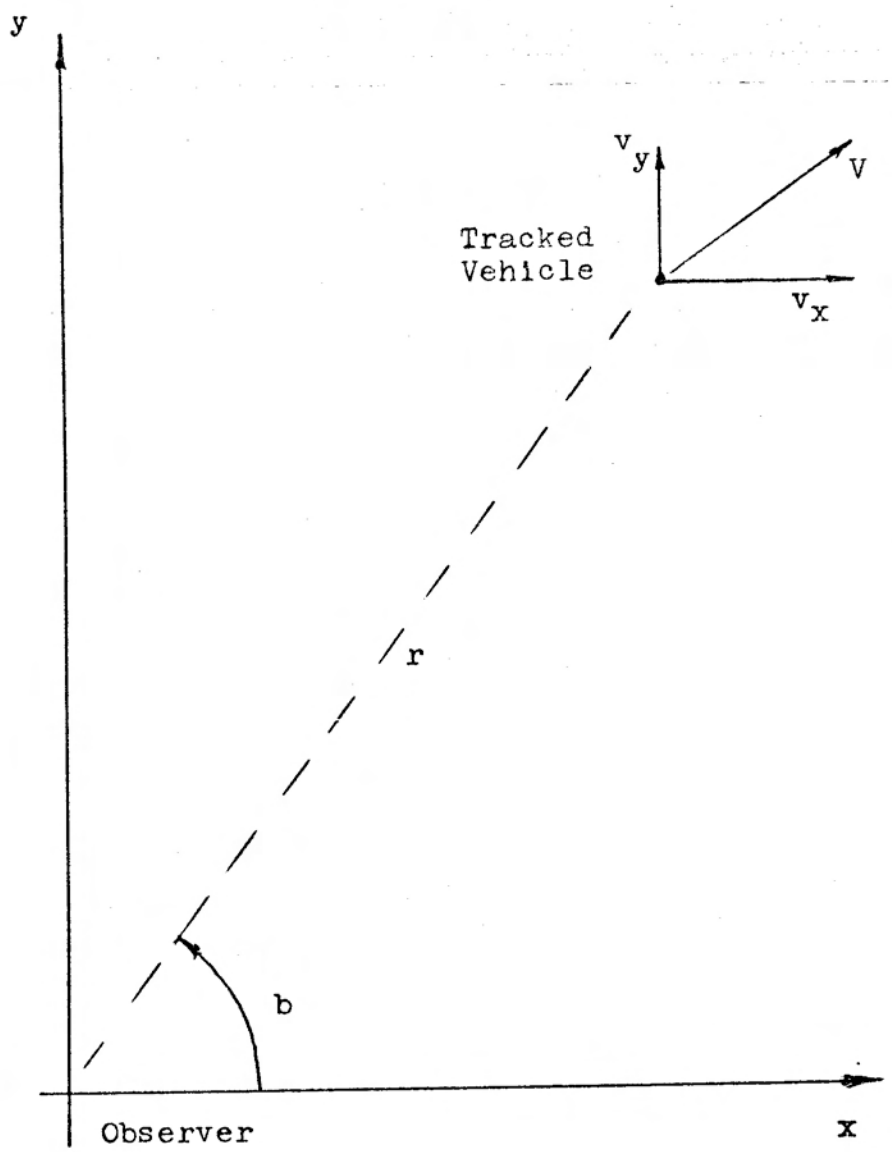

Figure 1. Typical Coordinate System Under Consideration

where w(t) is a white noise driving function with known variance and c(t) is the measured polar or cartesian coordinate.

Note that a set of these equations exist for each spatial dimension under consideration, i.e. the estimation of the position in range and bearing would require a four state model with two white noise inputs. Also note that if the coordinate states (c) are chosen to be range and bearing, a linear measurement equation and a nonlinear dynamic model results. One can also take the range-bearing dynamic equations to be linear [1] in nature, although this is clearly an approximation since the bearing rate is coupled to the tangential velocity in a nonlinear fashion. Studies [1,11] have shown that this is not a serious assumption in most cases since the noise can adapt somewhat to the nonlinearity and at long ranges from the observation platform this coupling has little effect.

As pointed out by Singer, acceleration noise is usually non-white, since maneuvers can typically last from 5 to 60 seconds in aircraft monitoring situations where data rates are in the order of milliseconds to 2 seconds. This situation can be handled by state augmentation techniques. The new model becomes:

$$dc/dt = v_c \qquad (1.9)$$

$$dv_c/dt = a_c \qquad (1.10)$$

$$da_c/dt = p_c a_c + w(t) \quad (1.11)$$

$$z = c + u(t) \quad (1.12)$$

where a_c is the acceleration of the c spatial coordinate and p_c and the strength of w are derived from the following assumed correlation model for the target acceleration:

$$r(T) = E\{a_c(t)a_c(t+T)\} = 2\sigma \exp(p_c|T|)$$

$$p_c \leq 0$$

E(.) denotes the expectation operator

A noisy measurement z of each spatial coordinate c is available. The noise process u(t) is assumed to be white with known variance. The physical significance of p_c is as the reciprocal of the maneuver time constant. The variance (q) of the target acceleration is derived from an assumed probability density of target maneuvers. Applying Weiner-Kolmogorov [3] theory to this correlation model, the strength of the white noise input w(t) is $-2p_c q$.

The motivation for using these more elaborate models is that constant course-constant speed models not only produce poor estimates during maneuvers but may not converge to new estimates even if a new constant course and speed are maintined [4]. The maneuver oriented filters in contrast, track well during maneuvers and converge consistently. The price paid for these advantages is higher filter complexity and slower convergence when constant course and speed are valid assumptions. Solutions to this decreased performance problem have been treated by several authors. One such approach [5,9] is to use a constant course and speed model with a measurement set limited to some fixed number of most recent observations. This solves the divergence problem but also degrades the full measurement history solution.

A second technique [6,11] postulates use of an adaptive scheme to use the simpler constant course and speed model except when maneuvers are detected, at which time either the white or colored acceleration noise models are used. Attention centers here on the maneuver detection scheme which is usually a test of the whiteness of the measurement residuals of the constant course-constant speed Kalman filter.

1.3 Overview

In this thesis we describe an investigation of the effects of incorrect modeling in the design of trajectory estimation filters for radar or sonar systems which track vehicles capable of performing unpredictable maneuvers. A nonlinear model is proposed which allows for an unknown correlation coefficient in an assumed colored acceleration noise model, in an attempt to compensate for the lack of a priori knowledge of a suitable value. A nonlinear parameter estimation filter is designed using this model and tested using Monte Carlo simulations.

Chapter II describes and motivates the proposed nonlinear dynamics model on which the filtering algorithm will be based. In Chapter III, we derive vector equations for computing the exact r.m.s. estimation errors incurred using a filter derived from a linear model which differs from the actual linear dynamics and measurement situation. These results are then applied to a trajectory estimation model [1] with constant correlation coefficient acceleration noise. Chapter IV describes the design of a nonlinear parameter estimation filter based on the model proposed in Chapter II. Chapter V describes the results of Monte Carlo simulations used to test the nonlinear filter and discusses the problems encountered when attempting to perform parameter estimation over the full dynamic range of interest. Conclusions and some recommendations for future study are summarized in Chapter VI.

CHAPTER II

SYSTEM DYNAMICS MODEL UNDER CONSIDERATION

The adaptive schemes [6,11] are in the author's opinion the most promising candidates for providing error performance improvements. At this writing, however, the most elaborate filter in the literature is the Singer colored noise formulation (the maneuver detection scheme in [11] switches between two such filters with different correlation coefficients). Several deficiencies still exist for this filter even if the vehicle being tracked is maneuvering. These mainly involve the choice of parameters used in the correlation model. Unless the tracking system is designed for a very limited class of vehicles, little information is available to establish a nominal set of correlation and variance parameters for the acceleration noise process. It is conceivable that a vehicle might maneuver continuously if it were piloted by some guidance device. In this case the adaptive filters described in [11] could only respond with the mismatched parameters chosen for the colored noise Singer model, and thus filter performance could be degraded.

The problem investigated here is the estimation of the correlation parameter p by the following augmented version of the Singer acceleration model:

$$dx = v\ dt \tag{2.1}$$

$$dv = a\ dt \tag{2.2}$$

$$da = (c+p)a\ dt + [-2(c+p)q]^{\frac{1}{2}}\ dw_1(t) \tag{2.3}$$

$$dp = -bp\ dt + \sigma\ dw_2(t) \tag{2.4}$$

where the $\{w_i\}$ are independent Brownian Motion processes with:

$$E\{\ w_i(t)\ \ w_i(s)\} = \min(t,s) \tag{2.5}$$

and c is a stability constant chosen to keep $E\{a^2(t)\}$ bounded.

Discrete noisy measurements of the form

$$z(kT) = x(kT) + u(kT) \tag{2.6}$$

were used. The measurement error u(kT) is assumed to be a white gaussian sequence with:

$$E\{u(k)u(j)\} = R\ \delta_{kj} \tag{2.7}$$

and independent from $\{w_i(t)\}$.

Note that (2.1)-(2.4) are nonlinear dynamic equations (due to the product of p and a) and any estimation techniques based on this model are essentially suboptimal nonlinear filters. The stability parameter c was included to provide stability control to keep $E\{a^2(t)\}$ bounded (see Appendix B). This model considers the acceleration correlation (p) as a state which can be changed by the unknown $w_2(t)$ process.

The attempt to estimate p is motivated by the fact that although p may remain constant over long periods, little may be known a priori to establish its value, since different values may be desirable depending on the behavior of each vehicle being tracked. A great deal of variation in p for a particular vehicle is possible depending on the existence and length of a vehicle maneuver. The maneuver detection filters of [6], [4], and [11] use two constant p filters and switch between these based on a filter residual test. A reasonable choice might be the use of a constant velocity filter $(c+p=-\infty, q=0)$ prior to maneuver detection and a constant acceleration filter $(c+p=0)$ after detection. Large and small values of $c+p$ could also be used. In contrast, the model described by (2.1)-(2.4) attempts to choose the value of p over a continuous range. If the estimation of the correlation parameter does not significantly degrade filter performance during changes of this piecewise constant value, the filter performance could be significantly improved once this "self matching" has approximated the true correlation value. If such a filter could be constructed, the filter could adapt to slowly maneuvering craft as well as to nearly instantaneous maneuvers between sample measurements.

An important test case studied is the performance of the Singer filter when the actual acceleration noise has a constant but mismatched correlation. These results are used to establish the performance improvement provided by the nonlinear filter. Exact error variance for the Singer filter can be calculated under these conditions as shown in Chapter III. This information should also be useful when choosing correlation values for maneuver dectection filters.

Since each spatial coordinate is estimated by a separate filter, the adaptive scheme investigated here could allow for different correlations in different directions. This was not considered in Singer [2] since a constant value of p is chosen for all situations. In most situations, no a priori information is available preferring maneuvers in a given direction so the correlation parameters were chosen equal. It is expected that an adaptive filter would be initialized this way but could adjust to the case in which they are different.

In summary, this research was aimed at clarifying whether the noise process associated with p could be well modeled so as to adapt to changes in vehicle behavior. The model we propose has been developed in an attempt to investiage adaptive tracking techniques that are capable of tracking maneuvering vehicles. Due to the product term pa in (2.3) the solution becomes a problem of either system identification (think of p as a parameter) or nonlinear filtering(think of p as a state variable).

CHAPTER III

SENSITIVITY OF THE SINGER FILTER TO CORRELATION COEFFICIENT ERRORS

3.1 Derivation of the Error Variance for the Discrete Kalman Filter with Plant Coefficient Errors

Since the choice of a filter for a given application normally involves a tradeoff between filter complexity and error performance, some indication as to performance sensitivity to correlation coefficient errors seems desirable. It was felt that the performance of the Singer filter would provide a useful basis for comparison with the adaptive scheme investigated. If the measurements are generated from a linear system with different system parameters, the exact error variance of the filter can be derived in a straightforward manner. The following result corresponds to that in [13] for the continuous case. Let the actual system dynamics and measurements be given by:

$$x(k+1) = Ax(k) + w(k+1) \tag{3.1}$$

$$m(k) = Bx(k) + u(k) \tag{3.2}$$

where

x is the n dimensional system state.

A is the nxn dimensional state transition matrix

B is the mxn dimensional measurement matrix.

The driving noise $w(k)$ and measurement noise $u(k)$ are assumed to be white noise vectors with correlation matrices:

$$E\{w(k)w'(j)\} = C\,\delta_{kj} \qquad (3.3)$$

$$E\{u(k)u'(j)\} = D\,\delta_{kj} \qquad (3.4)$$

$$E\{w(k)u'(j)\} = 0 \qquad (3.5)$$

The Kalman Filter used to process these measurements is derived from another system with assumed system matrix F, measurement matrix G, plant noise variance Q and measurement noise variance R. The state estimate $x(k)$ is thus given by:

$$\hat{x}(k+1) = F\hat{x}(k) + K(k+1)\{m(k+1) - GF\hat{x}(k)\} \qquad (3.6)$$

with gains $K(k+1)$ given by:

$$K(k+1) = P(k+1|k+1)\,G'\,R^{-1} \qquad (3.7)$$

$$P(k+1|k+1) = P(k+1|k)$$
$$- P(k+1|k)G'\{GP(k+1|k)G' + R\}^{-1}GP(k+1|k) \qquad (3.8)$$

$$P(k+1|k) = FPF' + Q \qquad (3.9)$$

Defining the estimation error $e(k)$ as:

$$e(k) = \hat{x}(k) - x(k) \qquad (3.10)$$

and combining (3.6) with (3.7), we obtain:

$$e(k+1) = F\hat{x}(k+1) + K(k+1)\{m(k+1) - GF\hat{x}(k)\} - Ax(k) - w(k+1) \qquad (3.11)$$

and using (3.2):

$$e(k+1) = F\hat{x}(k) + K(k+1)\{Bx(k+1) + u(k+1) - GF\hat{x}(k)\}$$
$$- Ax(k) - w(k+1)$$

$$e(k+1) = F(e(k)+x(k)) + K(k+1)BAx(k) + K(k+1)Bw(k+1)$$
$$+ K(k+1)u(k+1) - K(k+1)GF(e(k)+x(k))$$
$$- Ax(k) - w(k+1)$$

$$e(k+1) = (F-K(k+1)GF)e(k) + (F-A + K(k+1)[BA-GF])x(k)$$
$$+(K(k+1)B-I)w(k+1) + K(k+1)u(k+1) \qquad (3.12)$$

Letting

$$M(k+1) = F - K(k+1)GF$$

and

$$N(k+1) = [F-A] + K(k+1)[BA-GF]$$

we have the following set of coupled difference equations:

$$e(k+1) = M(k+1)e(k) + N(k+1)x(k)$$
$$+ [K(k+1)B-I]w(k+1) + K(k+1)u(k+1) \qquad (3.13)$$

$$x(k+1) = Ax(k) + w(k+1) \qquad (3.14)$$

The desired result is the covariance matrix of the estimate error e defined by:

$$E(k) = E(e(k)e'(k)) \qquad (3.15)$$

This can be derived by considering a new augmented system with state:

$$[\ e(k) \mid x(k)\ ]' \qquad (3.16)$$

Dropping the k+1 index on K,M,N for convenience,(3.13) and (3.14) can be written as:

$$\begin{bmatrix} e(k+1) \\ x(k+1) \end{bmatrix} = \begin{bmatrix} M & N \\ 0 & A \end{bmatrix} \begin{bmatrix} e(k) \\ x(k) \end{bmatrix} + \begin{bmatrix} KB-I & K \\ I & 0 \end{bmatrix} \begin{bmatrix} w(k+1) \\ u(k+1) \end{bmatrix} \qquad (3.17)$$

The recursion formula for the error covariance can be expressed directly in the following partitioned form [12,p.90]

$$\begin{bmatrix} E(k+1) & S(k+1) \\ S'(k+1) & X(k+1) \end{bmatrix} = \begin{bmatrix} M & N \\ O & A \end{bmatrix} \begin{bmatrix} E(k) & S(k) \\ S'(k) & X(k) \end{bmatrix} \begin{bmatrix} M' & O \\ N' & A' \end{bmatrix} + \begin{bmatrix} KB-I & K \\ I & O \end{bmatrix} \begin{bmatrix} C & O \\ O & D \end{bmatrix} \begin{bmatrix} (KB-I)' & I \\ K' & O \end{bmatrix} \quad (3.18)$$

$$\left[\begin{array}{c|c} E & S \\ \hline S' & X \end{array}\right]_{k+1} = \left[\begin{array}{c|c} ME(k)M'+MS(k)N'+NS'(k)M'+NX(k)N' & MS(k)A'+NX(k)A' \\ \hline AS'(k)M'+AXM' & AXA' \end{array}\right] + \left[\begin{array}{c|c} (KB-I)C(KB-I)+KDK' & (KB-I)C \\ \hline C(KB-I)' & C \end{array}\right] \quad (3.19)$$

Thus E(k) can be calculated by first computing:

$$X(k+1)= AX(k)A'+C, \quad X(0) = E(\ x(0)x'(0)\ ) \quad (3.20)$$

and then combining this result and the Kalman gain sequence computed with (3.7)-(3.9) to get:

$$S(k+1)=M(k+1)S(k)A'+N(k+1)X(k)A'+[K(k+1)B-I]C \quad (3.21)$$

with

$$S(0) = E\{e(0)x'(0)\}=E\{x(0)x'(0)\}-E\{x(0)x'(0)\}$$
$$= -X(0) \quad (3.22)$$

and finally computing the desired result using:

$$E(k+1) = M(k+1)E(k)M'(k+1) + M(k+1)S(k)N'(k+1)$$
$$+ N(k+1)S'(k)M'(k+1) + N(k+1)X(k)N'(k+1)$$
$$+ [K(k+1)B-I]C[K(k+1)B-I]'+ K(k+1)DK'(k+1) \quad (3.23)$$

with initial condition:

$$E(0) = E\{e(0)e'(0)\} = E\{\hat{x}(0)\hat{x}'(0)\} - E\{\hat{x}(0)x'(0)\}$$
$$+ E\{x(0)x'(0)\} - E\{\hat{x}(0)x'(0)\}$$
$$= X(0|0) \quad (3.24)$$

3.2 Discussion of the Effects of Maneuver Length Mismatch

In this section, we discuss the r. m. s. estimation error performance of a Singer type Kalman filter when the actual correlation length of the acceleration noise differs from that assumed by the filter. We define h to be the correlation coefficient of the actual system and f to be that assumed by the filter. Both the filter and plant under discussion in this section have a constant correlation coefficient and f is used to represent $-(c+p)$ in (2.3) with $b=\sigma=0$ in (2.4). The filter equations and matrix definitions are given in Appendix A. The matrices A and C in (3.1) and (3.2) are defined by using (A.6) and (A.7) with h instead of f. When the measurements are derived from a system which has the same correlation coefficient as the filter, estimation error performance is of course optimal in the mean square sense (since the actual and assumed plant are both linear and identical). The steady state estimation performance under these conditions is described parametrically in [1] as a function of the correlation coefficient (h), maneuver variance (q) and measurement variance (r). The main point of this section is to describe the performance degradation due to a mismatch between the assumed correlation length (1/f) and the actual correlation length (1/h) of the acceleration noise process.

As discussed in [1] and [13], the steady state error covariance matrix (E) in the matched case (h=f) is a function of q,r, the correlation coefficient (h) and the measurement interval (T) such that E/r is a function of q/r, h, and T only. This was also found to be true under mismatched conditions, i.e. E/r is a function only of q/r, the actual correlation time (1/h), the assumed correlation time (1/f) and T. We note that q/r has the useful interpretation as a signal to noise ratio.

To evaluate the extent of performance degradation under mismatched conditions, equations (3.21)-(3.23) were utilized with a constant q/r of.01, 1 and 100 under a variety of correlation coefficient mismatch cases. The upper triangular elements of the steady state error covariance matrix normalized by r for a given q/r and T are shown in Figures 2-4 as a function of both h and f. This data is also listed in Tables 1-3. The values of h and f were chosen to be powers of ten times the measurement interval, except for the h=0 (constant acceleration plant) case. The filter error variance reaches a steady state value, however. The f=0 (constant acceleration filter) cases are not plotted since the estimate errors diverge (except when h=0), as the filter gains approach zero and the filter becomes oblivious to any changes in acceleration.

The h or f equals infinity case corresponds to a zero mean acceleration jitter which corresponds closely to constant velocity filtering. A coefficient of 100T was deemed large enough to allow its use as a reasonable approximation to a constant velocity filter.

The motivation for this study was to:

i. provide useful design data to complement that found in [1]

ii. determine which cases were good candidates for studying performance improvements using the parameter estimation model described in chapter II

iii. to predict the upper and lower performance limits of an adaptive filter when used to filter data from a constant correlation acceleration noise system. The results of Monte Carlo simulations of such a system (filter based on the model (2.1)-(2.4) and the actual plant based on a constant correlation) are presented in Chapter V.

Figures 2-4 illustrate that corresponding matrix elements of E/r exhibit the same general acceleration noise correlation time behavior for a given measurement interval independent of signal to noise ratio (q/r). The intersection of the error variance surfaces with the plane h=f represents the performance of a matched filter as a function of acceleration noise correlation. Along these matched curves the position and velocity variances and covariances (E11,E12,E13,E22,E23) have a single maximum which depends on T and q/r and approach zero as h tends to both zero and infinity. The maximum occurs at larger correlation times (smaller h) as signal to noise decreases. This behavior is also exhibited along the intersection of planes of constant mismatch ratio (h/f=constant) and the variance contour (see Figure 3a for an illustration of constant mismatch ratio curves).

The acceleration variance (E33) behaves differently in that it only approaches zero as h=f tends to zero, and rapidly approaches the maneuver variance(q) for 1/h less than or in the vicinity of the measurement interval. Improvement in this variance is only achieved when the actual noise correlation time exceeds the measurement interval. The steady state variance can even exceed the maneuver variance in some mismatch cases. Such behavior is to be expected since the acceleration state approaches a white sequence (which differs from white noise in that the correlation function approaches a finite spike as the correlation time approaches zero) as h tends to zero. When the correlation time is less than the measurement interval the acceleration sequence appears white to the filter.

The effects of correlation mismatch appear to be much more pronounced when signal to noise ratio is small (e.g. q/r=.01) and when the assumed correlation coefficient is greater than the actual one. The increased sensitivity due to a decrease in signal to noise ratio was not too surprising, since the estimation performance due to filtering (instead of memoryless estimation) is greatest when signal to noise ratio is low. This is best seen by observing E11/r since this ratio can be viewed as a percentage improvement due to filtering. If this ratio is 1, then the position estimates are no better than using single measurement estimates of position. As this ratio approaches zero the filter becomes increasingly effective.

Another important point to note is that for a fixed coefficient filter (i.e. f fixed) it is possible for performance to improve when a plant mismatch ($h \neq f$) occurs. This is not a contradiction, as the improvement never exceeds that gained by using the matched filter for the given plant. However, for a given fixed actual coefficient h, performance (in the r.m.s. sense) always degrades if the assumed coefficient f differs from h.

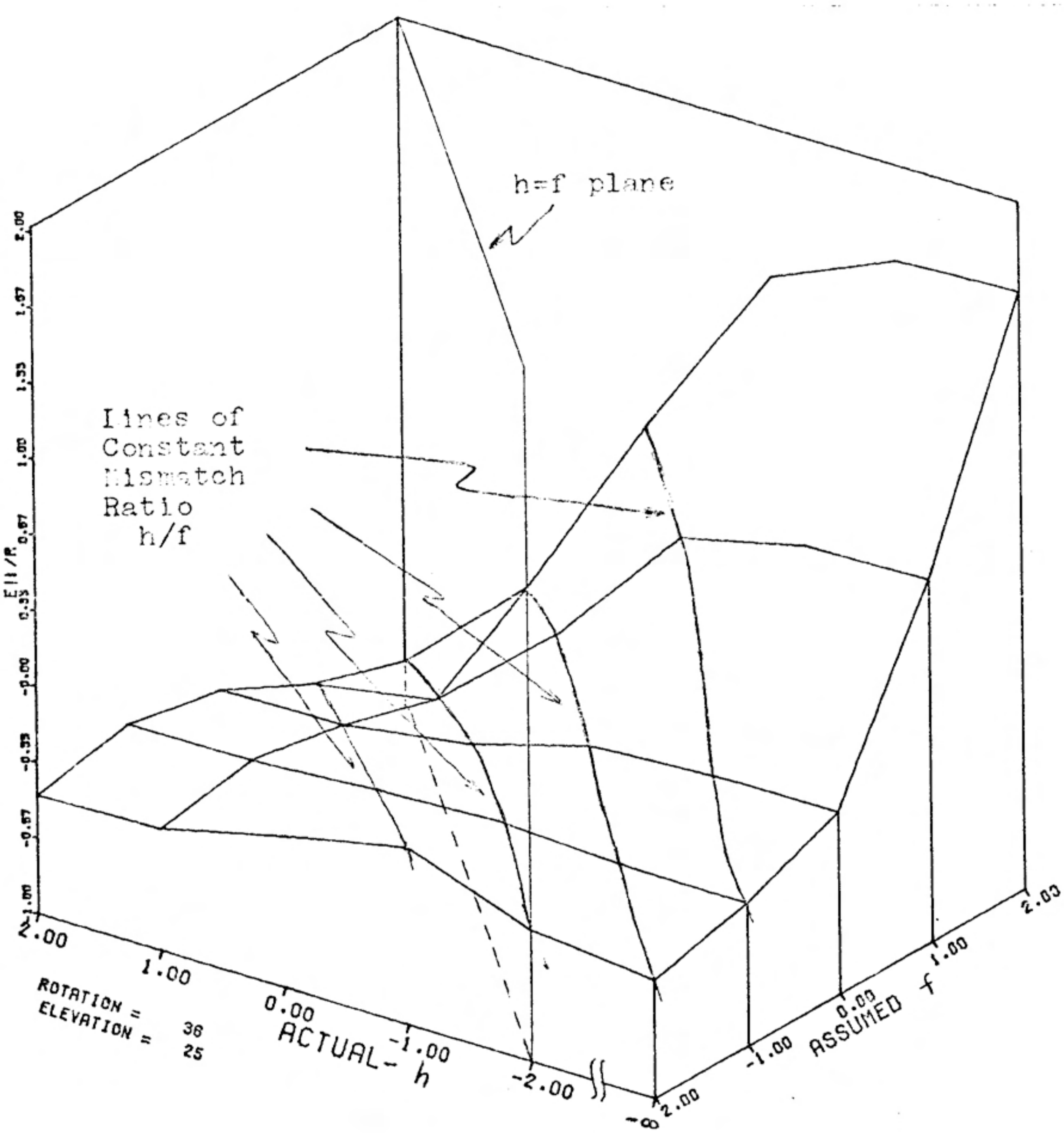

FIGURE 2a. E11/R FOR Q/R=0.01
MEASUREMENT INTERVAL T=1.00

Scales are in $\log_{10}$

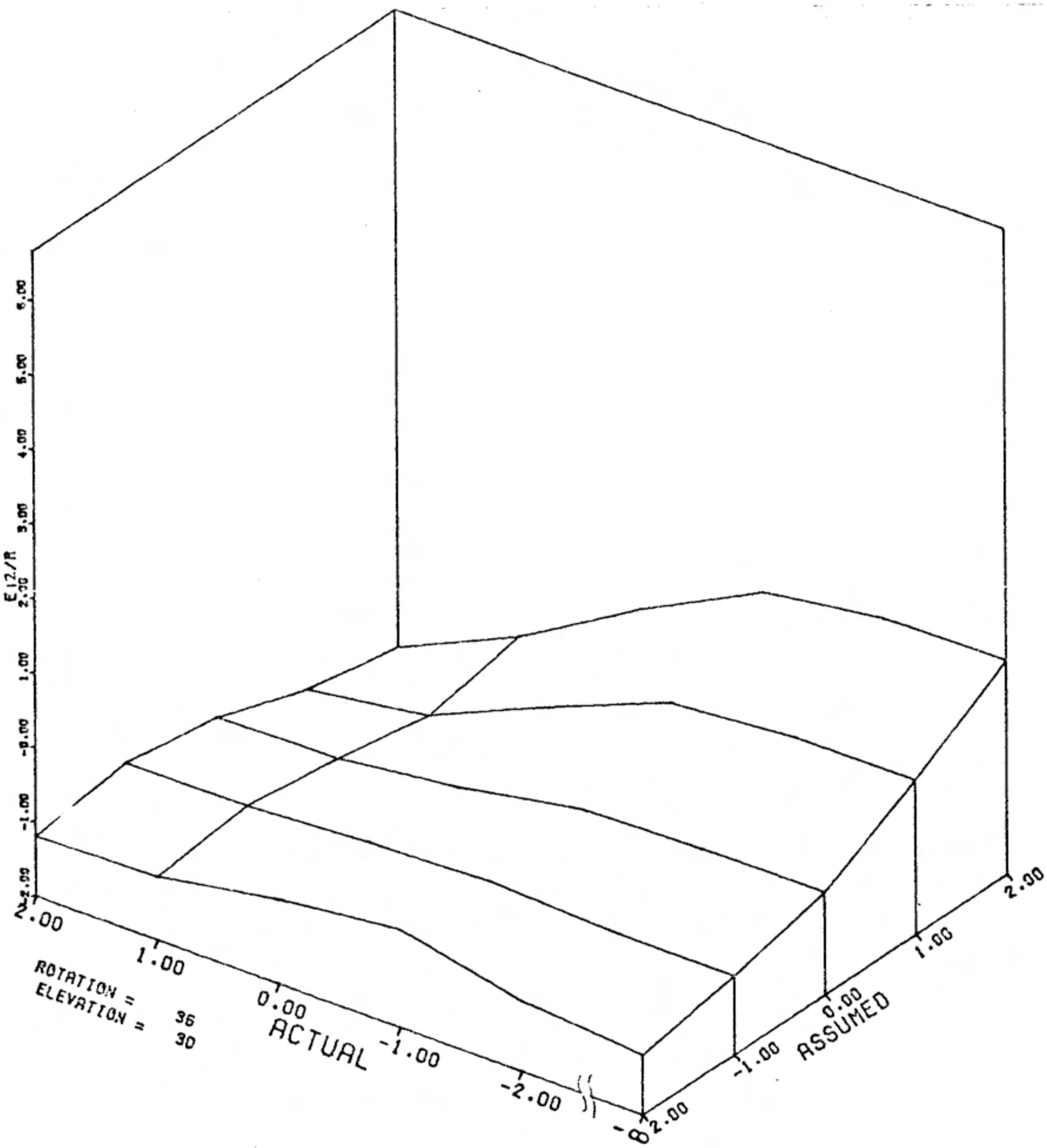

FIGURE 2b. E^{12}/R FOR Q/R=0.01
MEASUREMENT INTERVAL T=1.00

Scales are in $\log_{10}$

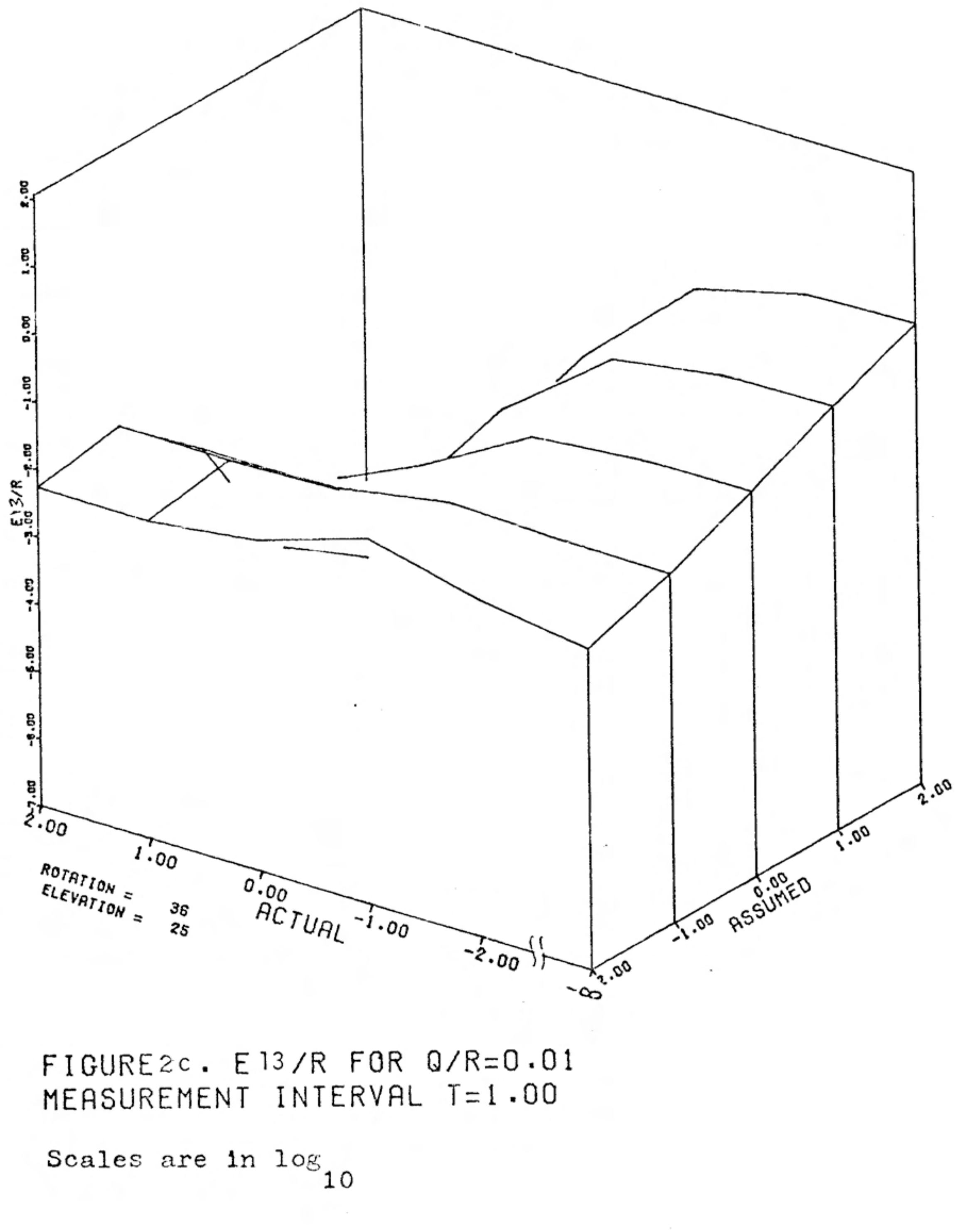

FIGURE2c. E13/R FOR Q/R=0.01
MEASUREMENT INTERVAL T=1.00

Scales are in $\log_{10}$

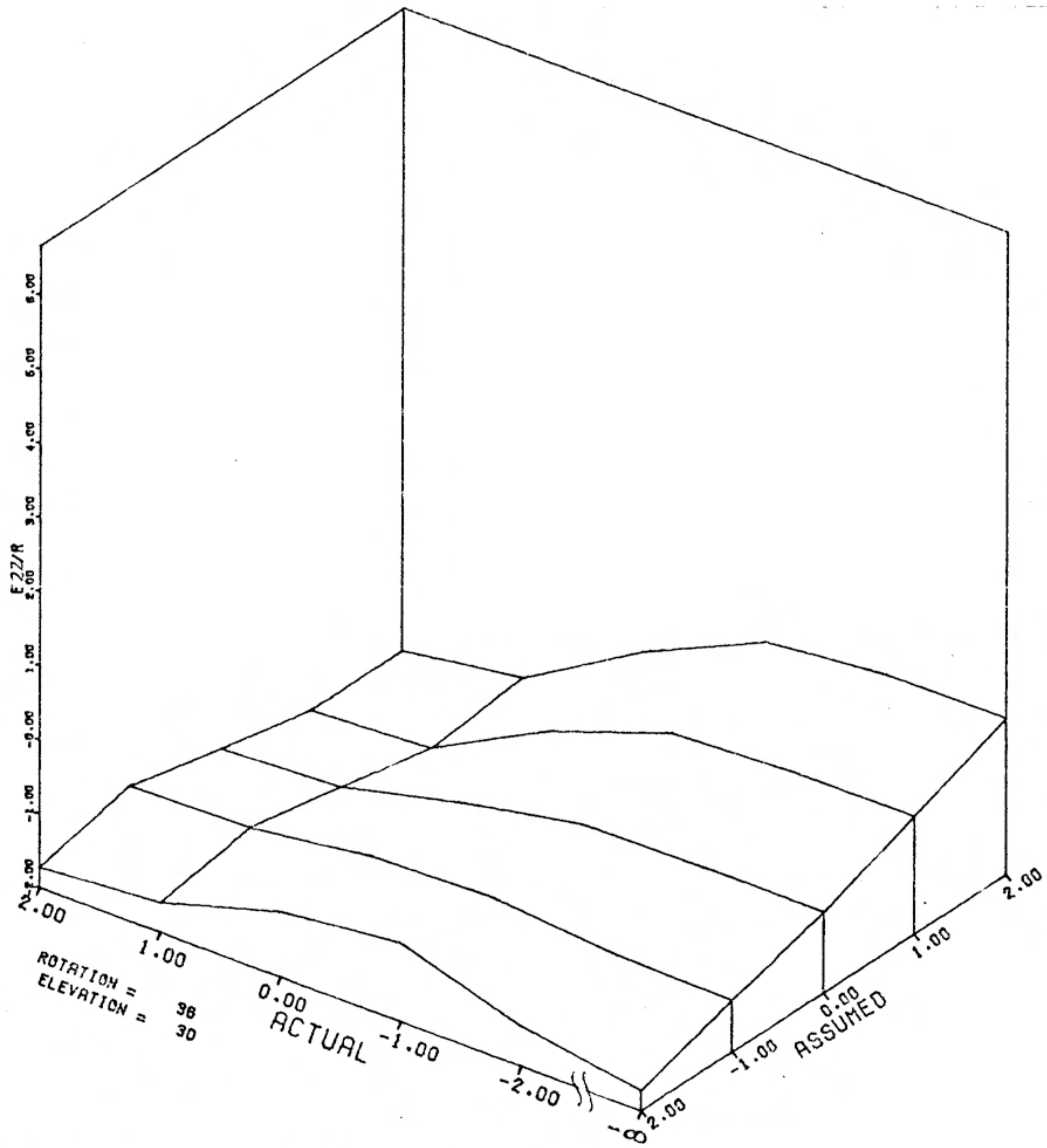

FIGURE 2d. E^{22}/R FOR $Q/R=0.01$
MEASUREMENT INTERVAL $T=1.00$

Scales are in $\log_{10}$

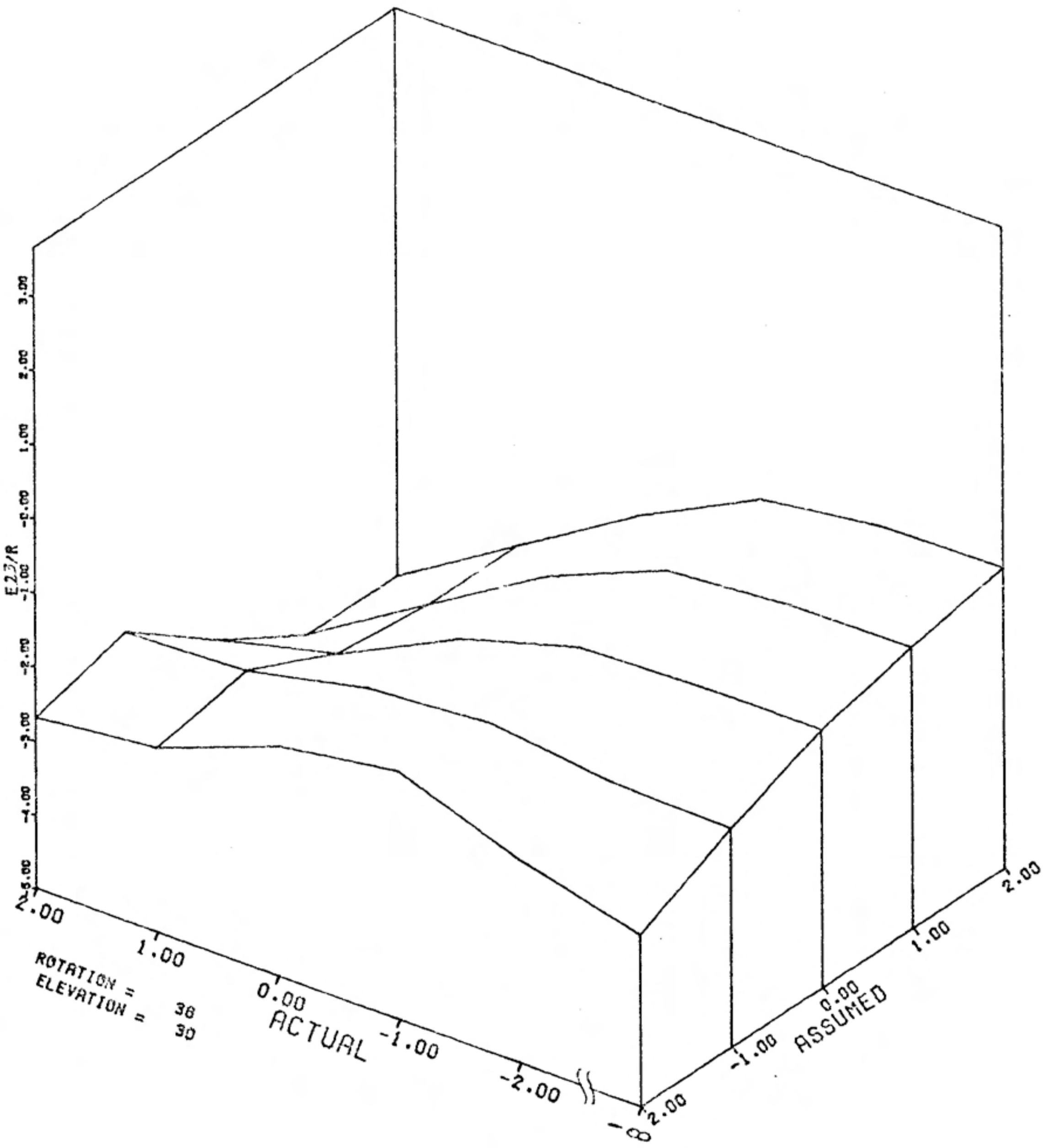

FIGURE 2e. E 23/R FOR Q/R=0.01
MEASUREMENT INTERVAL T=1.00

Scales are in $\log_{10}$

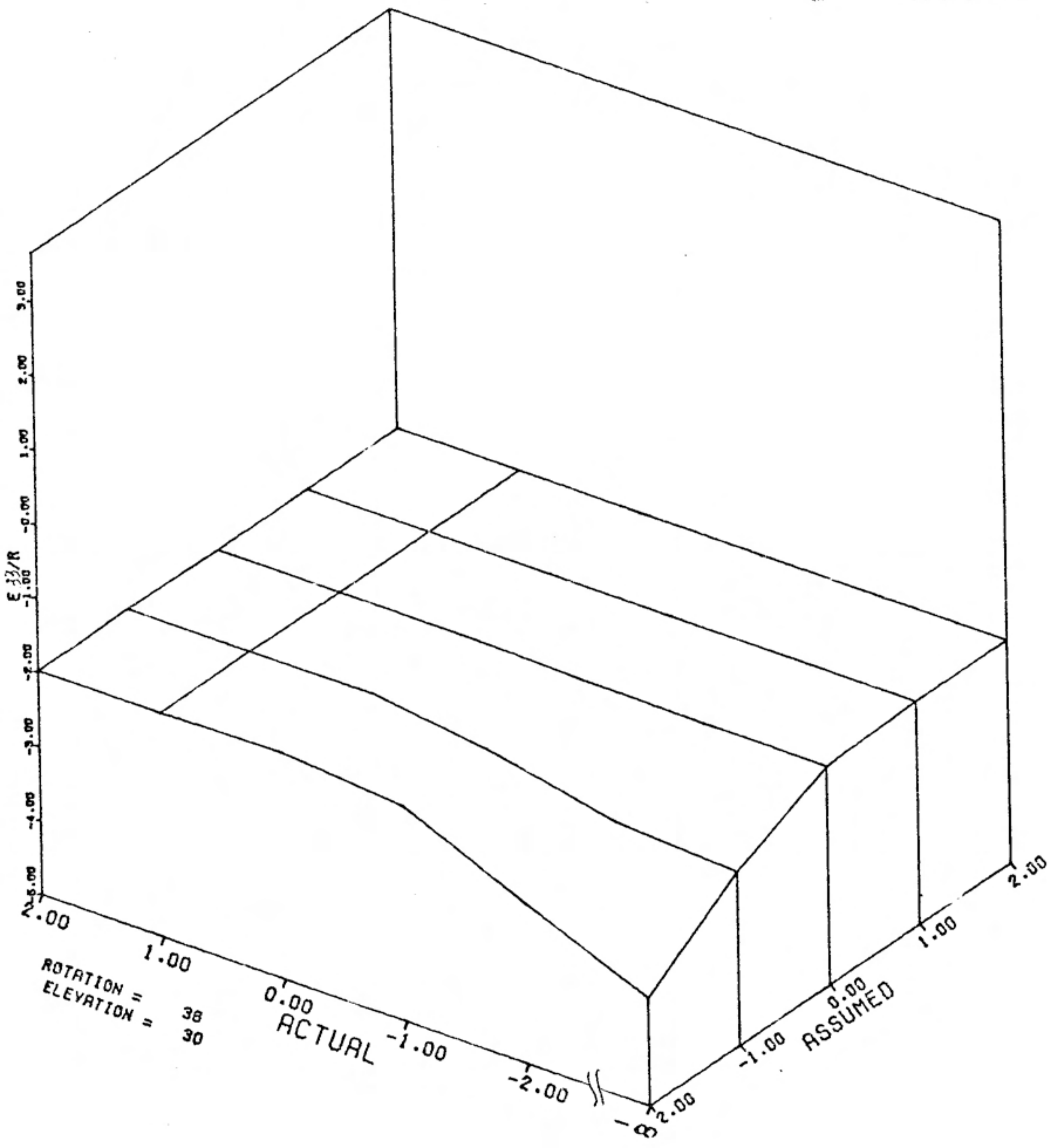

FIGURE 2f. E 33/R FOR Q/R=0.01
MEASUREMENT INTERVAL T=1.00
Scales are in $\log_{10}$

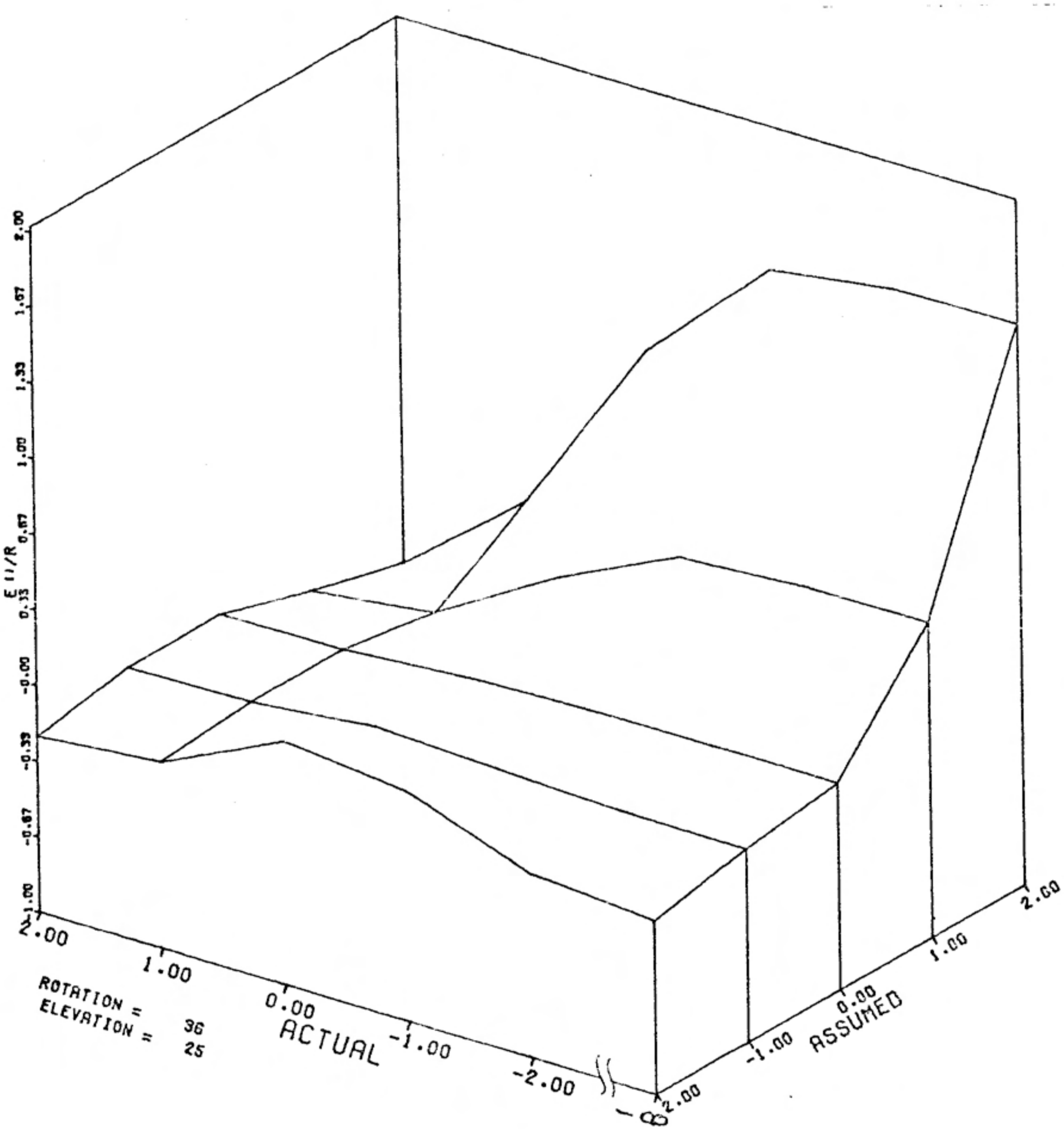

FIGURE 3a. E 11/R FOR Q/R=1.00
MEASUREMENT INTERVAL T=1.00
Scales are in $\log_{10}$

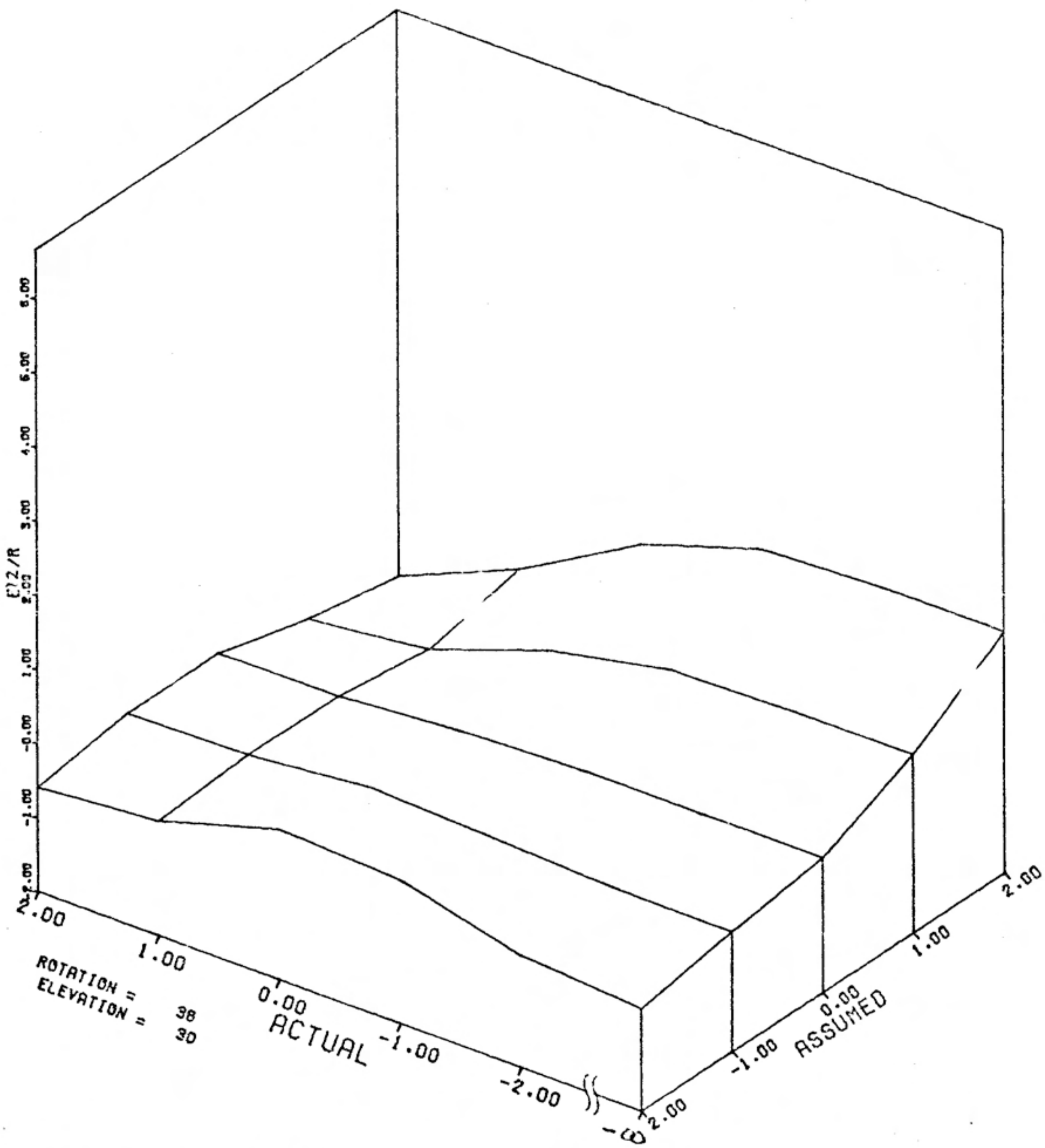

FIGURE 3b. E 12/R FOR Q/R=1.00
MEASUREMENT INTERVAL T=1.00
Scales are in $\log_{10}$

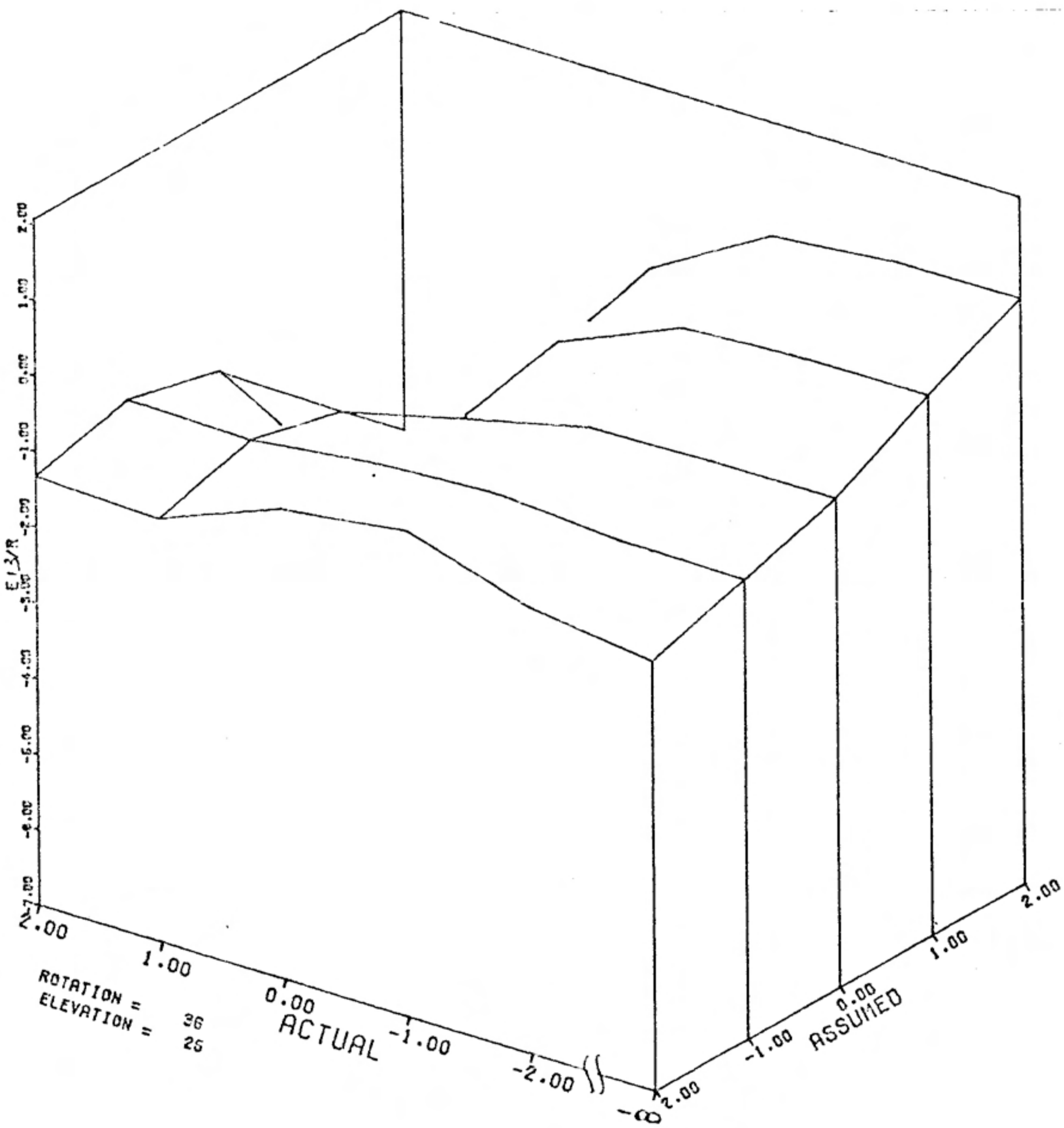

FIGURE 3c. E 13/R FOR Q/R=1.00
MEASUREMENT INTERVAL T=1.00

Scales are in $\log_{10}$

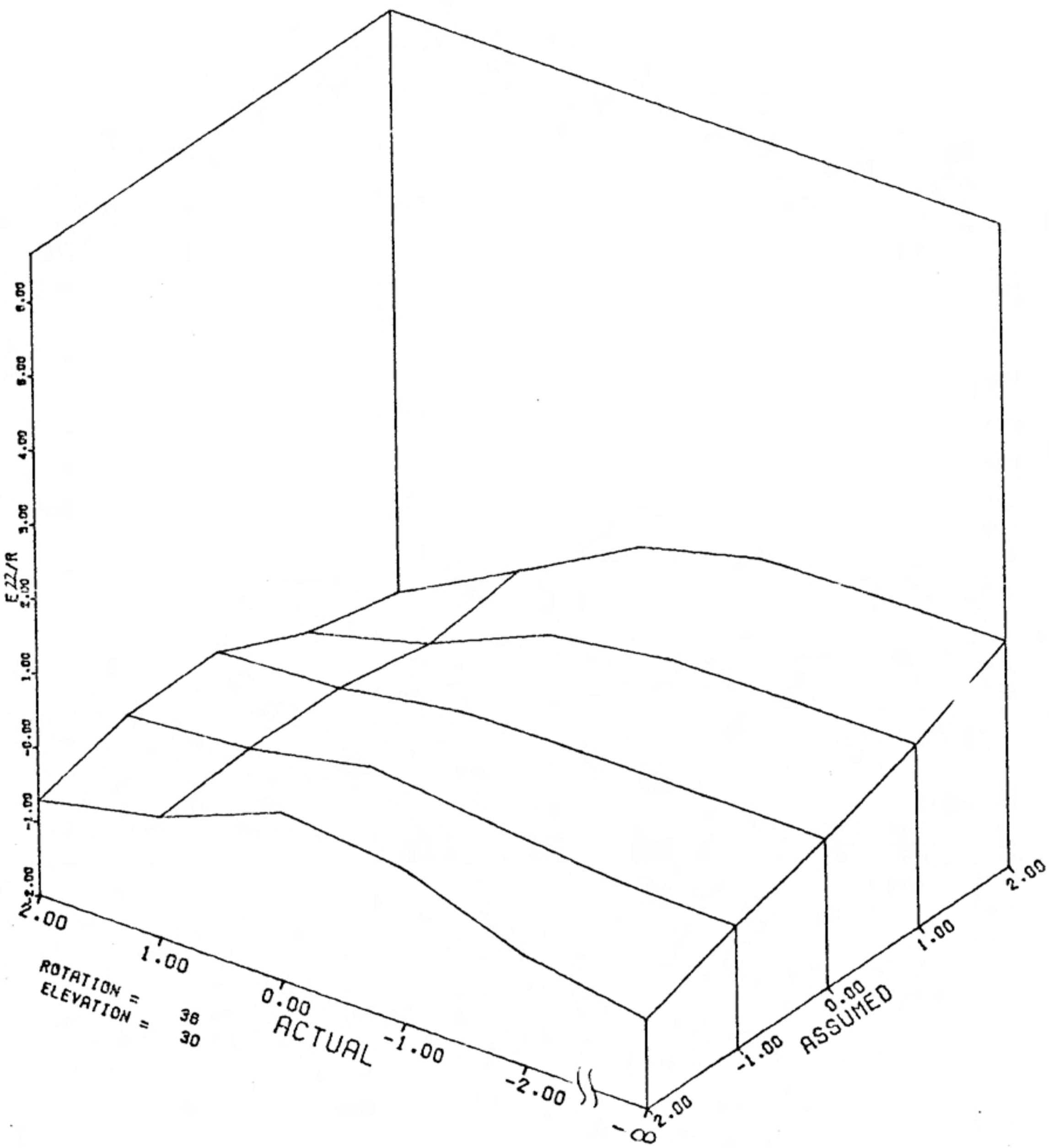

FIGURE 3d. E 22/R FOR Q/R=1.00
MEASUREMENT INTERVAL T=1.00

Scales are in $\log_{10}$

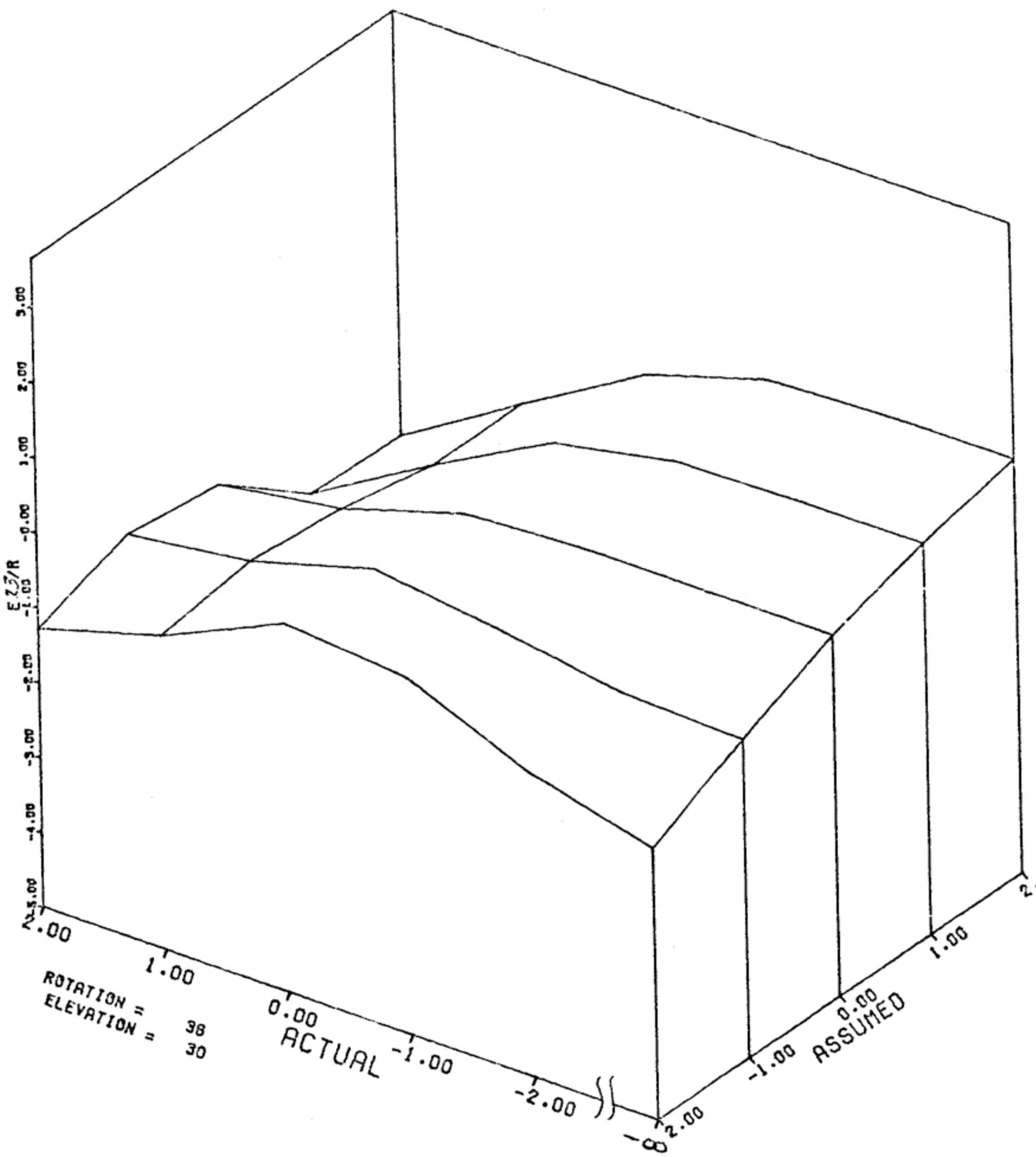

FIGURE 3e. E 23/R FOR Q/R=1.00
MEASUREMENT INTERVAL T=1.00

Scales are in $\log_{10}$

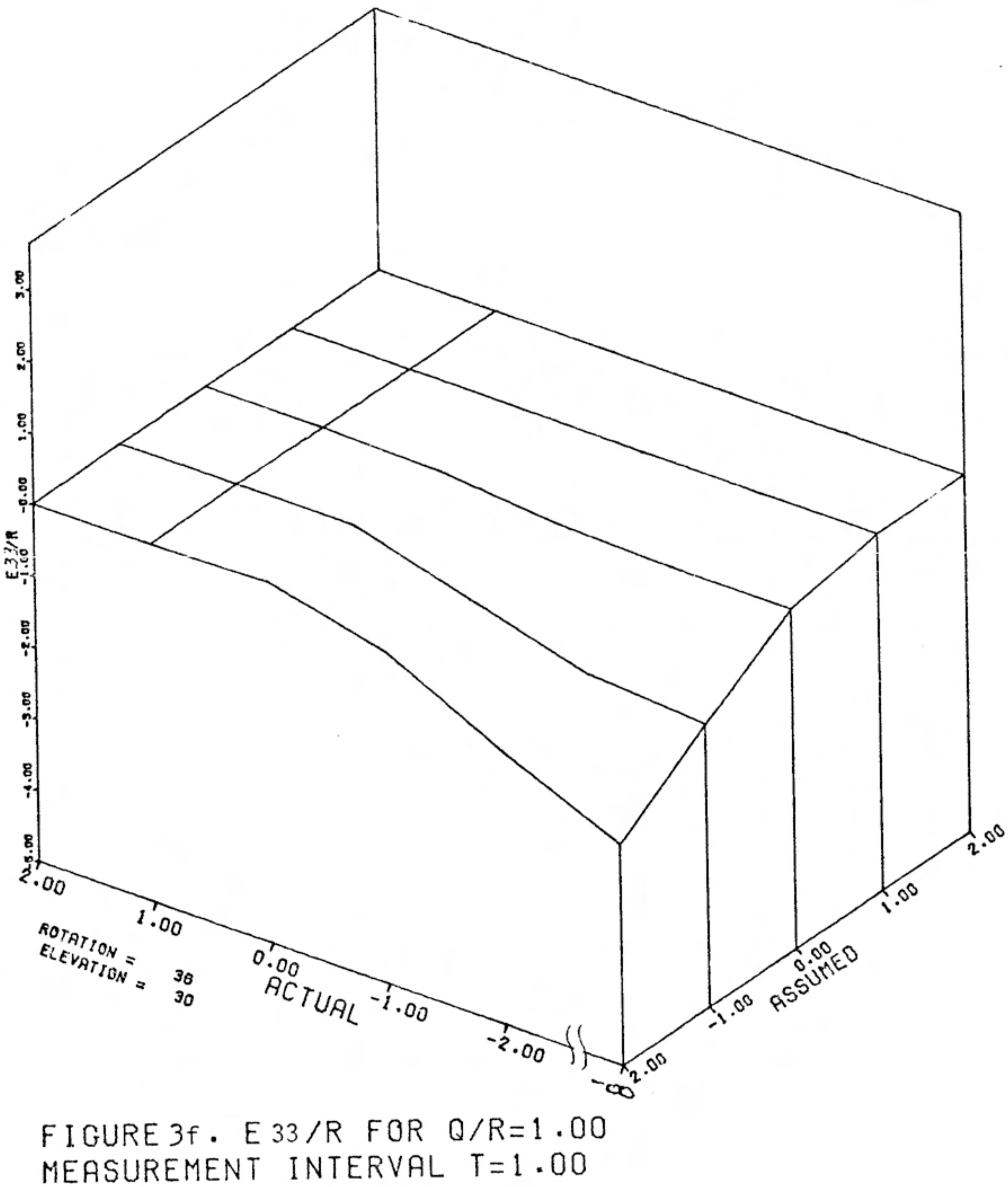

FIGURE 3f. E 33/R FOR Q/R=1.00
MEASUREMENT INTERVAL T=1.00
Scales are in $\log_{10}$

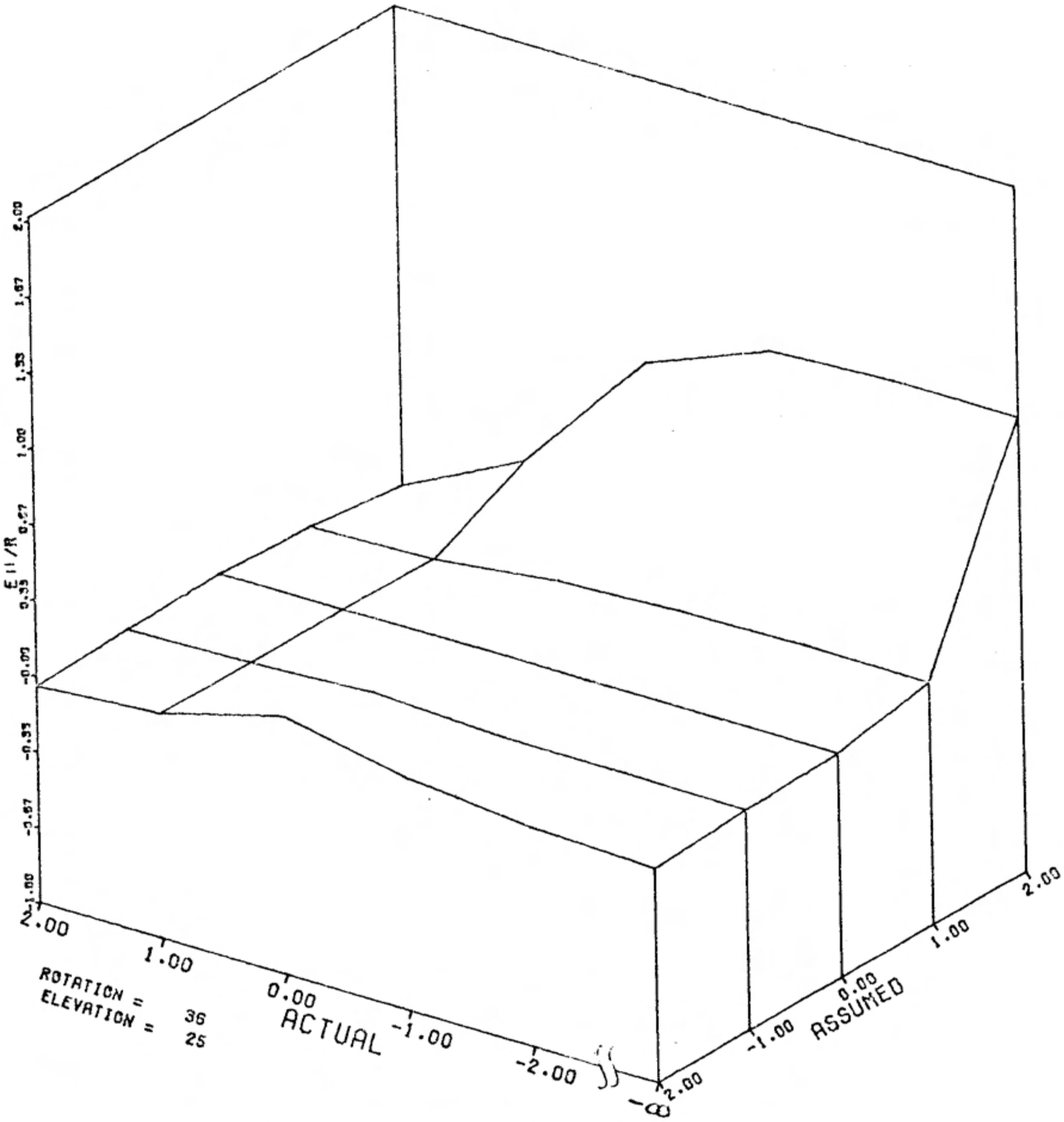

FIGURE 4a. E 11/R FOR Q/R=100.00
MEASUREMENT INTERVAL T=1.00
Scales are in $\log_{10}$

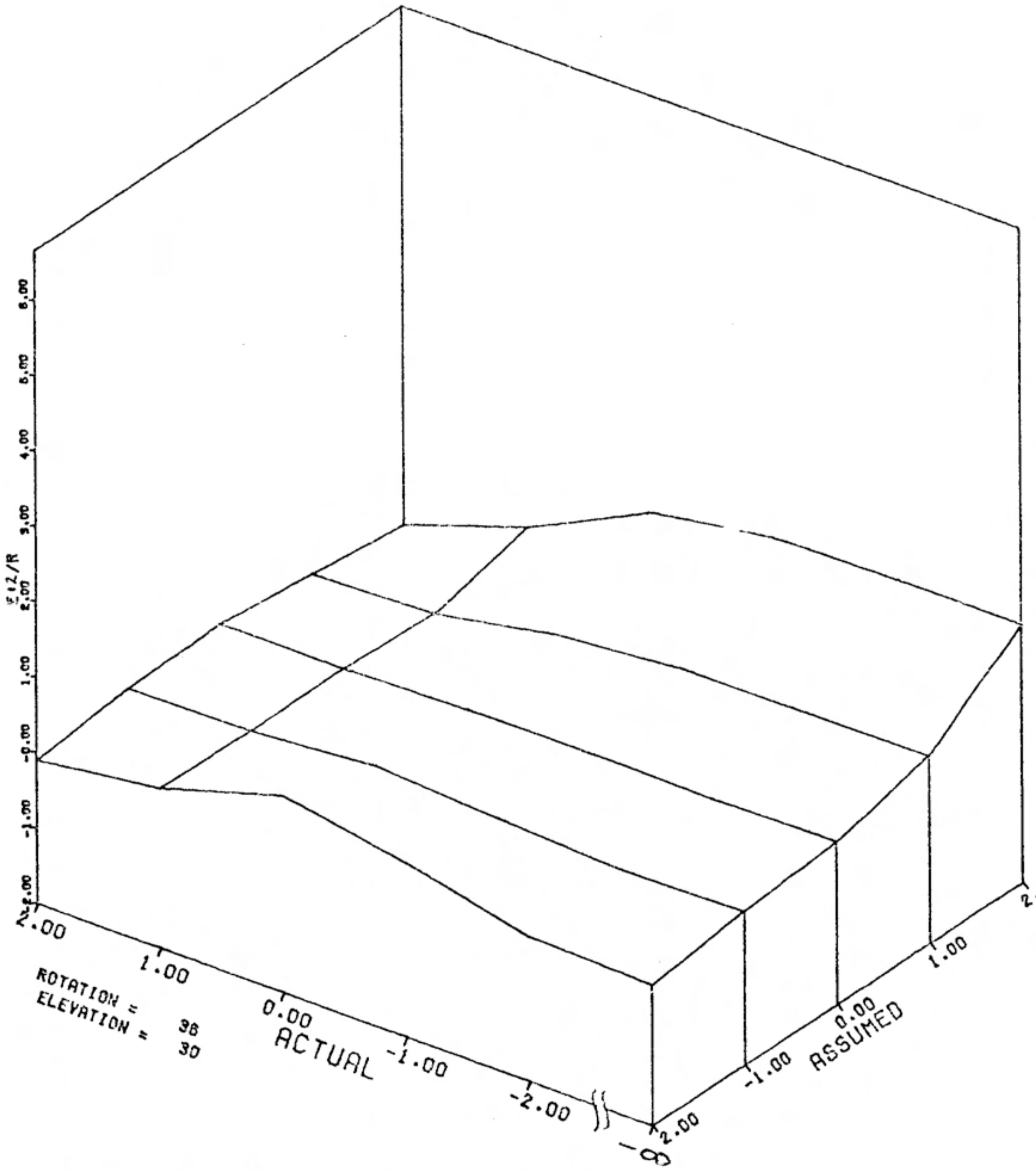

FIGURE 4b. E 12/R FOR Q/R=100.00
MEASUREMENT INTERVAL T=1.00

Scales are in $\log_{10}$

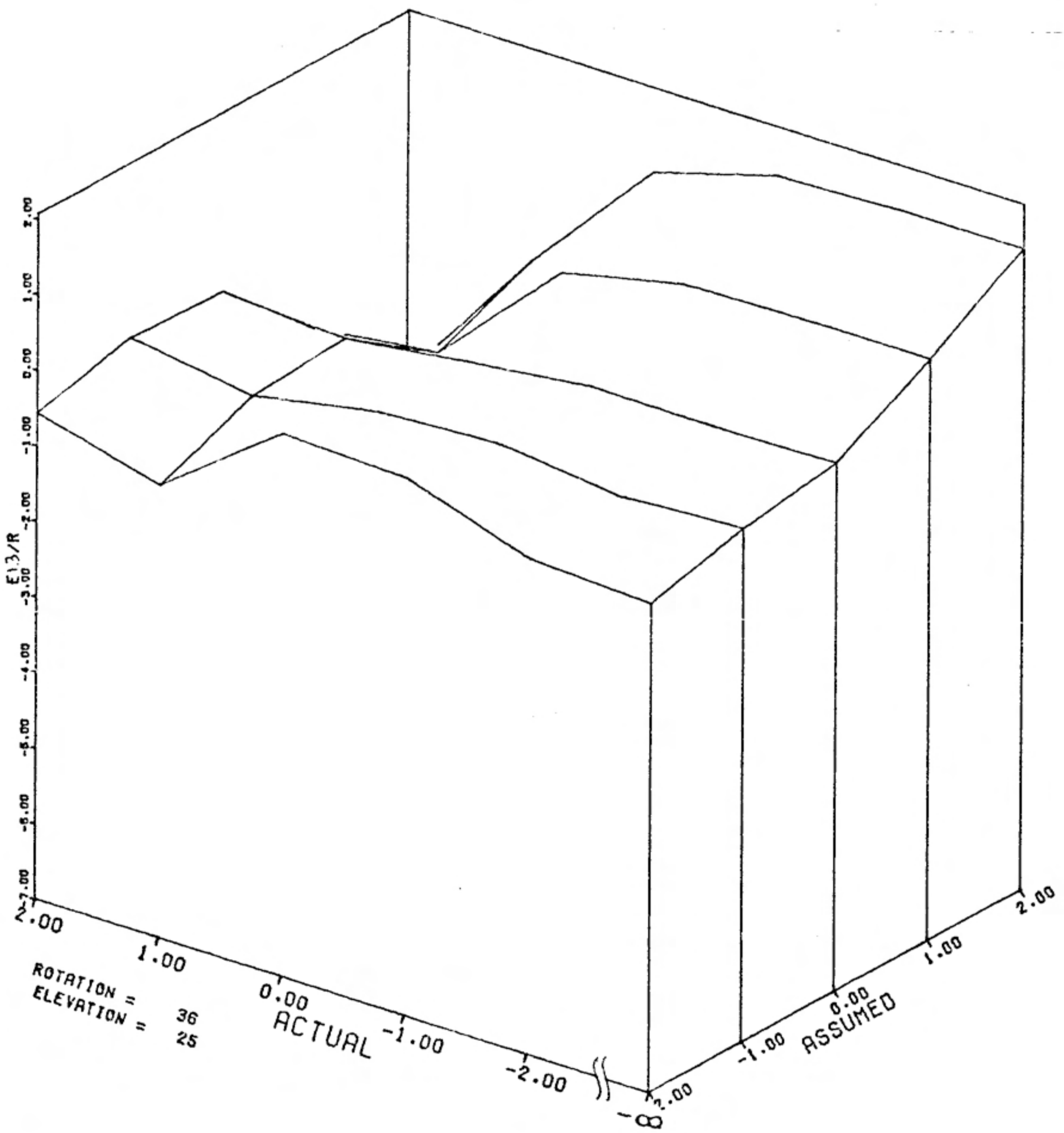

FIGURE 4c. E 13/R FOR Q/R=100.00
MEASUREMENT INTERVAL T=1.00
Scales are in $\log_{10}$

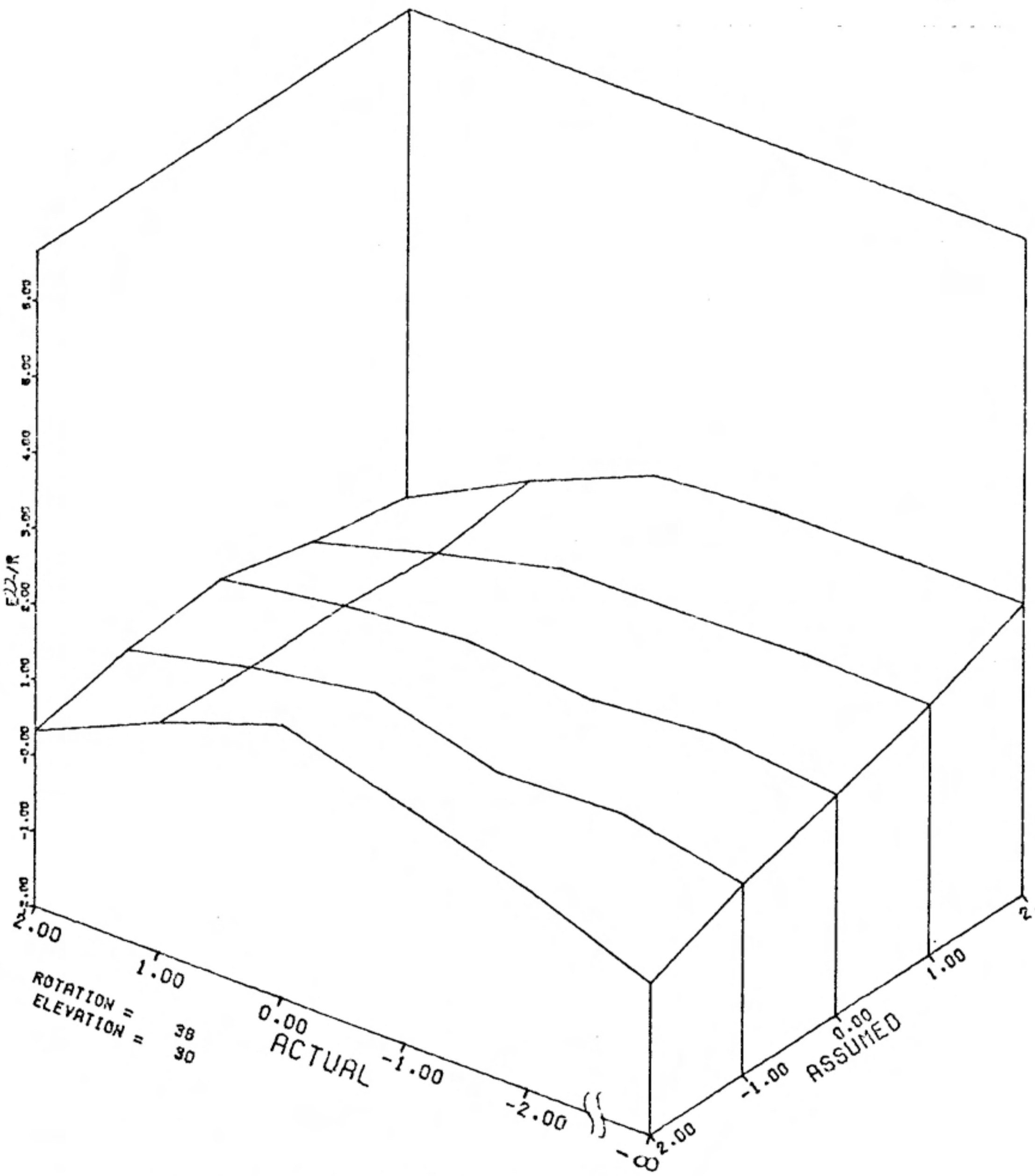

FIGURE 4d. E 22/R FOR Q/R=100.00
MEASUREMENT INTERVAL T=1.00

Scales are in $\log_{10}$

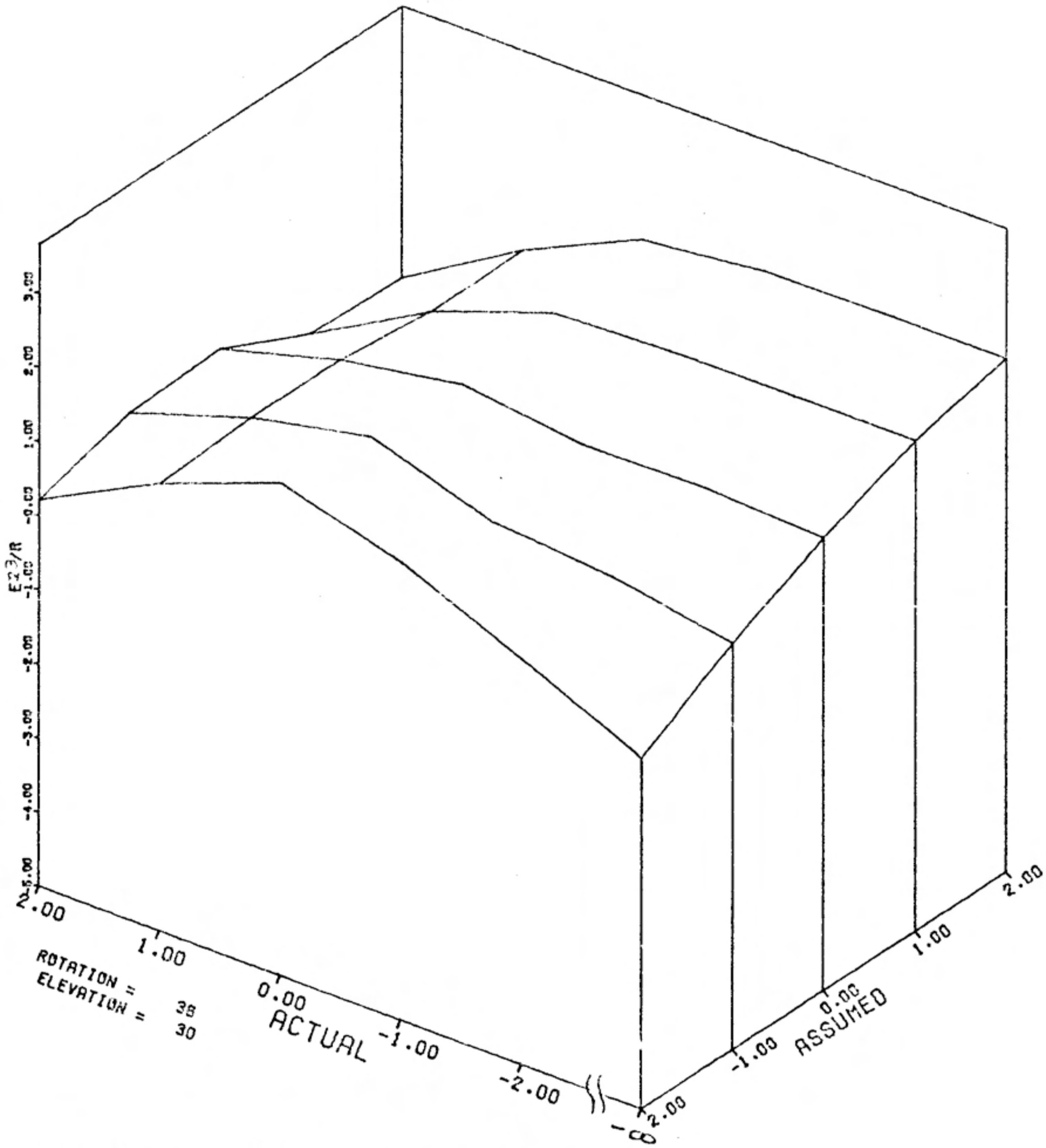

FIGURE 4e. E 23/R FOR Q/R=100.00
MEASUREMENT INTERVAL T=1.00
Scales are in $\log_{10}$

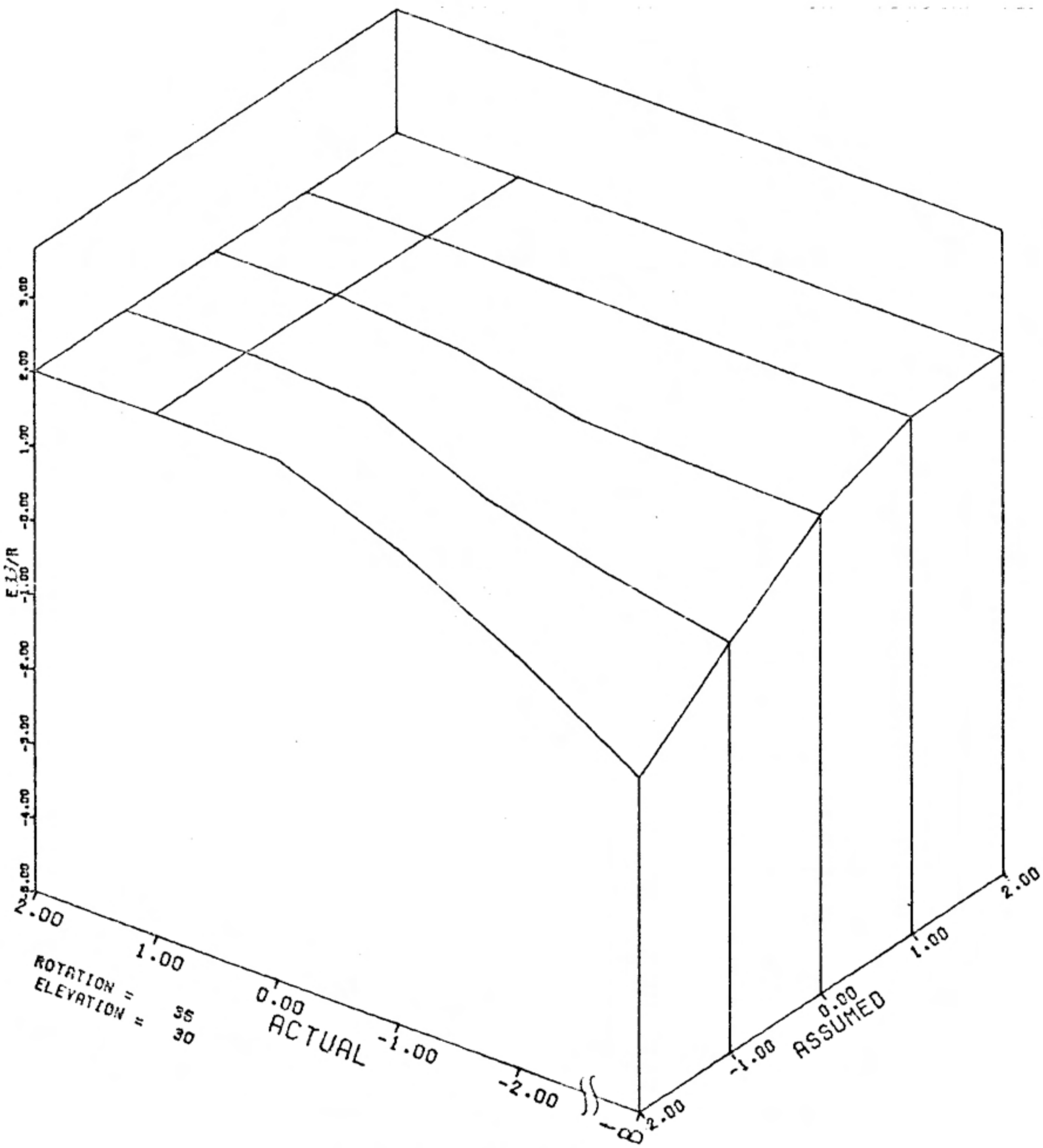

FIGURE 4f. E 33/R FOR Q/R=100.00
MEASUREMENT INTERVAL T=1.00
Scales are in $\log_{10}$

Table 1a. Steady State P11/r as a Function of Assumed and Actual Acceleration Noise Correlation Coefficient for Signal to Noise Ratio(q/r) of 0.01.

ASSUMED(f) \ ACTUAL (h)	100	10	1	0.1	.01	0
100	.155	.47	3.58	23.2	41.8	45.4
10	.209	.258	.739	2.82	3.8	3.93
1	.332	.339	.406	.579	.623	.628
.1	.399	.403	.438	.462	.431	.426
.01	.329	.342	.462	.599	.378	.331
0	.216*	.367*	1.84*	6.39*	.1*	.199

*Unstable solution due to filter gains approaching zero. Values shown for t=40 sec.

Table 1b. Steady State P12/r as a Function of Assumed and Actual Acceleration Noise Correlation Coefficient for Signal to Noise Ratio(q/r) of 0.01.

ASSUMED(f) \ ACTUAL (h)	100	10	1	0.1	.01	0
100	.013	.069	.622	4.09	6.98	7.41
10	.0225	.0385	.193	.863	1.18	1.22
1	.061	.0653	.105	.208	.235	.237
.1	.0964	.0993	.124	.142	.12	.116
.01	.063	.0699	.134	.208	.089	.0639
0	.0242*	.0664*	.476*	1.83*	.523*	.0197

*Unstable solution due to filter gains approaching zero. Values shown for t=40 sec.

Table 1c. Steady State P13/r as a Function of Assumed and Actual Acceleration Noise Correlation Coefficient for Signal to Noise Ratio(q/r) of 0.01.

ACTUAL (h) / ASSUMED(f)	100	10	1	0.1	.01	0
100	$8x10^{-7}$	$8x10^{-5}$	.0077	.244	.577	.65
10	$6x10^{-6}$	$7x10^{-5}$	.0062	.101	.18	.192
1	.0008	.0008	.0045	.0358	.0499	.0513
.1	.0087	.0085	.0098	.0187	.016	.0153
.01	.0054	.0053	.0082	.0267	.013	.0058
0	.0009*	.0017*	.0154*	.142*	.0509*	.0008

*Unstable solution due to filter gains approaching zero. Values shown for t=40 sec.

Table 1d. Steady State P22/r as a Function of Assumed and Actual Acceleration Noise Correlation Coefficient for Signal to Noise Ratio(q/r) of 0.01.

ACTUAL (h) / ASSUMED(f)	100	10	1	0.1	.01	0
100	.0023	.0174	.159	.883	1.44	1.54
10	.0042	.0123	.0838	.315	.382	.386
1	.0187	.0229	.0562	.115	.123	.124
.1	.0372	.0409	.0687	.0773	.0549	.051
.01	.0185	.0247	.0754	.155	.0347	.0187
0	.0044*	.0199*	.164*	.594*	.165*	.0027

*Unstable solution due to filter gains approaching zero. Values shown for t=40 sec.

Table 1e. Steady State P23/r as a Function of Assumed and Actual Acceleration Noise Correlation Coefficient for Signal to Noise Ratio(q/r) of 0.01.

ACTUAL (h) / ASSUMED (f)	100	10	1	0.1	.01	0
100	1×10^{-5}	.001	.0098	.0638	.108	.114
10	.0001	.001	.0095	.0445	.0601	.062
1	.0005	.0014	.0087	.0263	.0304	.0308
.1	.0045	.0053	.0123	.0162	.0103	.0092
.01	.002	.0031	.0125	.0225	.0056	.0021
0	.0003*	.0015*	.0137*	.0599*	.0183*	.0001

*Unstable solution due to filter gains approaching zero. Values shown for t=40 sec.

Table 1f. Steady State P33/r as a Function of Assumed and Actual Acceleration Noise Correlation Coefficient for Signal to Noise Ratio(q/r) of 0.01.

ACTUAL (h) / ASSUMED (f)	100	10	1	0.1	.01	0
100	.01	.01	.01	.01	1	1
10	.01	.01	.01	.01	1	1
1	.01	.01	.01	.0092	.0088	.0088
.1	.0107	.0107	.011	.0064	.0028	.0023
.01	.0102	.0104	.0113	.008	.0014	.0058
0	1.01*	.0101*	.0107*	.0113*	.0027*	.0006

*Unstable solution due to filter gains approaching zero. Values shown for t=40 sec.

Table 2a. Steady State P11/r as a Function of Assumed and Actual Acceleration Noise Correlation Coefficient for Signal to Noise Ratio(q/r) of 1.0.

ASSUMED(f) \ ACTUAL (h)	100	10	1	0.1	.01	0
100	.41	1.11	7.49	24.6	29.2	29.7
10	.53	.61	1.27	2.26	2.45	2.46
1	.71	.72	.78	.82	.82	.82
0.1	.70	.72	.81	.76	.72	.71
.01	.59	.66	1.19	1.03	.65	.59
0	1.87*	17*	164*	620*	166*	.2

*Unstable solution due to filter gains approaching zero. Values shown for t=40 sec.

Table 2b. Steady State P12/r as a Function of Assumed and Actual Acceleration Noise Correlation Coefficient for Signal to Noise Ratio(q/r) of 1.0.

ASSUMED(f) \ ACTUAL (h)	100	10	1	0.1	.01	0
100	.11	.53	4.38	14.7	17.7	18.0
10	.18	.28	1.06	2.28	2.51	2.53
1	.4	.42	.55	.64	.65	.65
0.1	.4	.43	.62	.52	.43	.42
.01	.25	.35	1.06	.86	.33	.24
0	.48*	4.69*	45.6*	181*	50.4*	.0197

*Unstable solution due to filter gains approaching zero. Values shown for t=40 sec.

Table 2c. Steady State P13/r as a Function of Assumed and Actual Acceleration Noise Correlation Coefficient for Signal to Noise Ratio(q/r) of 1.0.

ACTUAL (h) / ASSUMED (f)	100	10	1	0.1	.01	0
100	$6x10^{-5}$	.0058	.44	3.73	5.21	5.42
10	.0009	.0039	.23	1.1	1.35	1.38
1	.0476	.0436	.11	.26	.29	.29
0.1	.0963	.0891	.14	.17	.12	.12
.01	.0462	.0402	.17	.27	.0806	.0487
0	.0083*	.0659*	1.46*	14.1*	5.02*	.0008

*Unstable solution due to filter gains approaching zero. Values shown for t=40 sec.

Table 2d. Steady State P22/r as a Function of Assumed and Actual Acceleration Noise Correlation Coefficient for Signal to Noise Ratio(q/r) of 1.0.

ACTUAL (h) / ASSUMED (f)	100	10	1	0.1	.01	0
100	.0660	.0058	.44	3.73	5.21	5.42
10	.13	.0040	.23	1.10	1.35	1.38
1	.44	.0436	.11	.26	.29	.29
0.1	.42	.0891	.14	.17	.12	.12
.01	.19	.0402	.17	.27	.0806	.0487
0	.18*	1.73*	16.2*	59.2*	16.2*	.0027

*Unstable solution due to filter gains approaching zero. Values shown for t=40 sec.

Table 2e. Steady State P23/r as a Function of Assumed and Actual Acceleration Noise Correlation Coefficient for Signal to Noise Ratio(q/r) of 1.0.

ASSUMED(f) \ ACTUAL (h)	100	10	1	0.1	.01	0
100	.0099	.0989	.87	2.77	3.25	3.3
10	.0109	.0972	.72	1.53	1.66	1.67
1	.097	.17	.54	.67	.67	.66
.1	.14	.23	.67	.4	.21	.18
.01	.0518	.15	.83	.59	.11	.042
0	.0139*	.14*	1.36*	5.98*	1.82*	.0001

*Unstable solution due to filter gains approaching zero. Values shown for t=40 sec.

Table 2f. Steady State P33/r as a Function of Assuned and Actual Acceleration Noise Correlation Coefficient for Signal to Noise Ratio(q/r) of 1.0.

ASSUMED(f) \ ACTUAL (h)	100	10	1	0.1	.01	0
100	1	1	1	1	1	1
10	1	1	1	.98	.97	.97
1	1.03	1.03	.94	.65	.58	.58
.1	1.06	1.08	1.08	.36	.12	.0929
.01	1.01	1.04	1.16	.46	.0668	.0126
0	1*	1.01*	1.07*	1.13*	.27*	$6x10^{-6}$

*Unstable solution due to filter gains approaching zero. Values shown for t=40 sec.

Table 3a. Steady State P11/r as a Function of Assumed and Actual Acceleration Noise Correlation Coefficient for Signal to Noise Ratio(q/r) of 100.

ACTUAL (h) / ASSUMED (f)	100	10	1	0.1	.01	0
100	.814	1.51	5.86	9.50	10	10
10	.914	.947	1.10	1.17	1.19	1.17
1	.967	.971	.981	.977	.979	.976
0.1	.940	.965	1.03	.959	.955	.939
.01	.897	.995	1.38	1.06	.931	.887
0	168*	1683*	16373*	61953*	16555*	.199

*Unstable solution due to filter gains approaching zero. Values shown for t=40 sec.

Table 3b. Steady State P12/r as a Function of Assumed and Actual Acceleration Noise Correlation Coefficient for Signal to Noise Ratio(q/r) of 100.

ACTUAL (h) / ASSUMED (f)	100	10	1	0.1	.01	0
100	.61	2.2	13	24	26	26
10	.814	1.02	2.1	2.93	2.95	3.06
1	1.22	1.21	1.34	1.4	1.33	1.4
0.1	1.09	1.17	1.61	1.25	1.02	1.11
.01	.789	1.3	4.04	2.03	.827	.733
0	47*	468*	4560*	18100*	5040*	.0197

*Unstable solution due to filter gains approaching zero. Values shown for t=40 sec.

Table 3c. Steady State P13/r as a Function of Assumed and Actual Acceleration Noise Correlation Coefficient for Signal to Noise Ratio(q/r) of 100.

ACTUAL (h) / ASSUMED (f)	100	10	1	0.1	.01	0
100	.0019	.179	9.10	27	30	30
10	.0336	.0528	1.97	4.47	4.85	4.91
1	.490	.335	.616	.961	.917	.995
0.1	.559	.311	.634	.780	.511	.634
.01	.271	.0964	1.53	1.3	.364	.301
0	.763*	8.53*	146*	1410*	502*	.0008

*Unstable solution due to filter gains approaching zero. Values shown for t=40 sec.

Table 3d. Steady State P22/r as a Function of Assumed and Actual Acceleration Noise Correlation Coefficient for Signal to Noise Ratio(q/r) of 100.

ACTUAL (h) / ASSUMED (f)	100	10	1	0.1	.01	0
100	1.66	11	52	69	72	70
10	2.69	7.62	20	20	23	20
1	5.35	9.64	13	9.00	13	8.16
0.1	3.94	9.38	17	6.11	7.32	3.43
.01	2.10	11	40	14	4.42	1.04
0	17*	173*	1620*	5920*	1620*	.0027

*Unstable solution due to filter gains approaching zero. Values shown for t=40 sec.

Table 3e. Steady State P23/r as a Function of Assumed and Actual Acceleration Noise Correlation Coefficient for Signal to Noise Ratio(q/r) of 100.

ACTUAL (h) / ASSUMED (f)	100	10	1	0.1	.01	0
100	.994	9.37	52	80	83	83
10	1.13	8.98	34	40	41	41
1	4.41	13	23	14	15	13
.1	3.79	13	29	8.22	5.98	3.05
.01	1.6	11	43	15	2.91	.534
0	1.39*	14*	136*	598*	182*	.0001

*Unstable solution due to filter gains approaching zero. Values shown for t=40 sec.

Table 3f. Steady State P33/r as a Function of Assumed and Actual Acceleration Noise Correlation Coefficient for Signal to Noise Ratio(q/r) of 100.

ACTUAL (h) / ASSUMED (f)	100	10	1	0.1	.01	0
100	100	100	100	99	99	99
10	100	100	96	91	90	90
1	103	106	77	35	31	28
.1	104	111	91	18	7.16	3.4
.01	101	106	103	25	3.43	3.23
0	100*	101*	107*	113*	27*	6×10^{-6}

*Unstable solution due to filter gains approaching zero. Values shown for t=40 sec.

CHAPTER IV

APPROXIMATIONS TO THE NONLINEAR FILTERING PROBLEM

4.1 Approximations Based on the Evolution of a Finite Set of Moments

The design of a filter based on the model of (2.1)-(2.6) requires a careful investigation of the propagation of noise through the nonlinear dynamics of (2.1)-(2.4). The formal solution of the optimal nonlinear filter (in the mean square sense) requires the evaluation of the conditional mean of the state based on the available measurement set. A very general solution is available [10],[12] for problems which can be expressed in the vector form:

$$dy(t) = F[y(t),t]\,dt + Q[y(t),t]\,dw(t) \tag{4.1}$$

$$z(kt) = G[kT]y(kT) + u(kT) \tag{4.2}$$

The proposed model can be expressed in this form by using:

$$y(t) = [x \mid v \mid a \mid p]' \tag{4.3}$$

$$F[y] = [v \mid a \mid (c+p)a \mid -b]' \tag{4.4}$$

$$Q(y) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \{-(c+p)a\}^{1/2} & 0 \\ 0 & 0 \end{bmatrix} \tag{4.5}$$

$$G = [1\ 0\ 0\ 0] \tag{4.6}$$

It should be emphasized here that optimality is only implied when the actual system dynamics match the system model.

Unfortunately, the structure of the derived filter is infinite-dimensional and some approximation is needed if a physically realizable estimation system is desired. The dimensionality problem occurs because, when the filter is viewed as a tracking system, the tracker gains for the optimal filter require consideration of the total stochastic description of the system state variables. In the linear dynamics case (F[y]=Fy) this information is contained in the first two moments of the joint probability density functions of the state, i.e. the mean vector and covariance matrix. This fortuitous situation does not hold however when nonlinearities exists in the system. For the parameter estimation proposed here, the covariance differential equations to be derived later in this section involve terms with third order moments. This is further complicated by the fact that the equations describing the evolution of the third order moments involve fourth order moments and so on, thus producing an infinite-dimensional set of equations describing the information needed for optimality. This is typical of all parameter estimation problems unless of course the parameters are perfectly measurable. Typically, solutions to this approximation problem have centered around the truncation of the infinite set of equations in some hopefully effective way. References [10] and [12] provide excellent state of the art treatments of the subject. The most widely used technique, the so-called Extended Kalman Filter (EKF), is the lowest order approximation and easiest to implement. The EKF equations are derived by linearizing the system dynamics about the current estimate and propagating the covariance matrix using linear filtering theory. Fortunately, the EKF performs quite well in a large number of nonlinear problems. Licht [12, pg. 90] points out, however, that caution is in order when the disturbance function Q[y(t)] is in fact state dependent as in (2.3). This was in fact found to be true when the filter equations were derived for the proposed model. The discussion of other techniques will be deferred until after the moment equations are derived. As stated previously, the design of the desired filter requires computation of the conditional moments:

$$m_{k_1 k_2 k_3 k_4}(t|t) = E(x^{k_1} v^{k_2} a^{k_3} p^{k_4}(t) | z(s),\ 0 \leq s \leq t\ ) \qquad (4.7)$$

In the discrete measurement case investigated here, the conditional expectation need only be evaluated at measurement times. In between measurements, the statistics are propagated from initial values obtained at the previous measurement time using the following result of Ito stochastic calculus[12,p.167]. For any twice continuously differentiable real-valued function $\varphi(y)$:

$$E(d\varphi(y)) = E((\partial\varphi/\partial y)'F)dt + (1/2)trE(QQ'(\partial^2\varphi/\partial y^2))dt \quad (4.8)$$

We now proceed to develop the equations which describe the evolution of the central moments of (2.1)-(2.4) using (4.8).

The first moments are derived using $\varphi(y)=y$ and are:

$$d\hat{x}/dt = \hat{v}\ dt \quad (4.9a)$$

$$d\hat{v}/dt = \hat{a}\ dt \quad (4.9b)$$

$$d\hat{a}/dt = \widehat{(c+p)a} = (c\hat{a}+\widehat{pa})dt=(c+\hat{p})\hat{a} + P34 \quad (4.9c)$$

$$d\hat{p}/dt = -b\ \hat{p} \quad (4.9d)$$

The variances are derived using $\varphi(y)=yy'$ and $\varphi(\hat{y}) = \hat{y}\hat{y}'$ and then using:

$$E((x-\hat{x})(x-\hat{x})) = \widehat{xx} - \hat{x}\hat{x} = P11 \quad (4.10a)$$

$$E((x-\hat{x})(v-\hat{v})) = \widehat{xv} - \hat{x}\hat{v} = P12 \quad (4.10b)$$

$$E((x-\hat{x})(a-\hat{a})) = \widehat{xa} - \hat{x}\hat{a} = P13 \quad (4.10c)$$

$$E((x-\hat{x})(p-\hat{p})) = \widehat{xp} - \hat{x}\hat{p} = P14 \quad (4.10d)$$

$$E((v-\hat{v})(v-\hat{v})) = \widehat{vv} - \hat{v}\hat{v} = P22 \quad (4.10e)$$

$$E((v-\hat{v})(a-\hat{a})) = \widehat{va} - \hat{v}\hat{a} = P23 \quad (4.10f)$$

$$E((v-\hat{v})(p-\hat{p})) = \widehat{vp} - \hat{v}\hat{p} = P24 \quad (4.10g)$$

$$E((a-\hat{a})(a-\hat{a})) = \widehat{aa} - \hat{a}\hat{a} = P33 \quad (4.10h)$$

$$E((a-\hat{a})(p-\hat{p})) = \widehat{ap} - \hat{a}\hat{p} = P34 \quad (4.10i)$$

$$E((p-\hat{p})(p-\hat{p})) = \widehat{pp} - \hat{p}\hat{p} = P44 \quad (4.10j)$$

Defining third order central moments for state variables i,j, and k as Sijk (e.g. $S134 = E((x-\hat{x})(a-\hat{a})(p-\hat{p}))$), the variance equations are:

$$dP11/dt = 2\ P12 \quad (4.11a)$$

$$dP12/dt = P13 + P22 \quad (4.11b)$$

$$dP13/dt = (c+\hat{p})P13 + \hat{a}P14 + P23 + S134 \quad (4.11c)$$

$$dP14/dt = -b\ P14 + P24 \quad (4.11d)$$

$$dP22/dt = 2\ P23 \quad (4.11e)$$

$$dP23/dt = (c+\hat{p})P23 + P33 + \hat{a}\ P24 + S234 \quad (4.11f)$$

$$dP24/dt = -b\ P24 + P34 \quad (4.11g)$$

$$dP33/dt = 2((c+\hat{p})P33 + \hat{a}P34 + S334 - g(c+\hat{p})) \quad (4.11h)$$

$$dP34/dt = (c+\hat{p}-b)P34 + \hat{a}\ P34 + S344 \quad (4.11i)$$

$$dP44/dt = -2b\ P44 + \sigma^2 \quad (4.11j)$$

Fortunately, all the third moments do not enter into the covariance equations. Only the S134,S234,S334,S344,S444,S244,and S144 equations require derivation. The need for S144, S244,and S444 arises when deriving the remaining moments. The third central moment expressions require considerable manipulation but are the result of applying (4.8) to the individual terms of:

$$Sijk = E(y_i y_j y_k) + 2E(y_i)E(y_j)E(y_k)$$
$$-E(y_i)E(y_j y_k) - E(y_j)E(y_i y_k) - E(y_k)E(y_i y_j) \quad (4.12)$$

When applying (4.8), several fourth order moments are encountered. As mentioned previously, this process produces an endless set of equations.

At this point we wish to discuss several alternative procedures. Reference [12, pg. 336] suggests two approximation techniques. The first procedure involves truncating all third and higher moments to zero and generates what has been called a truncated second order filter. The second approximation involves substituting fourth order central moments with

$$E((y_i-\hat{y}_i)(y_j-\hat{y}_j)(y_m-\hat{y}_m)(y_n-\hat{y}_n))= \\ P_{jm} P_{in} + P_{jm} P_{im} + P_{mn} P_{ij} \qquad (4.13)$$

which is a jointly gaussian assumption. Note that this substitution would produce a closed set of equations. This approximation is much like the second order assumption; however, the inclusion of third order central moments allows us to consider distributions that have some degree of skewness. We shall call this the "third order filter".

An alternative procedure is suggested in [14] using a technique called the 'cumulants method'. This approach is also considered in [15], [16]. Cumulants are the coefficients of the Taylor series expansion of the logarithm of the characteristic function. The interest in this technique stems from the fact that it is more reasonable to set to zero all cumulants of order higher than some given number than to set to zero all higher order moments. The moments are the Taylor Series coefficients of the characteristic function. A truncation of these coefficients leads to a characteristic function whose inverse transform has derivatives of delta functions and thus is not a well-defined probability density function. Truncation of the cumulants, however, leads to well-behaved functions (see [15]).

For the system being considered here, expressions relating the moments of four jointly distributed random variables to their cumulant series appeared to be quite a difficult bookkeeping task. A computer program was developed to perform the necessary algebraic manipulations, since the expansion to fourth order required several thousand terms. After simplifications however it was found that the following relationship exists between the fourth order cumulants (k_{ijmn}) and the fourth central moments:

$$E\{(y_i-\hat{y}_i)(y_j-\hat{y}_j)(y_m-\hat{y}_m)(y_n-\hat{y}_n)\} = k_{ijmn} + P_{jm}P_{in} + P_{jm}P_{im} + P_{mn}P_{ij} \qquad (4.14)$$

Note that (4.14) is identical to (4.13) except for the additional fourth order cumulant. Setting such cumumlants to zero reduces to the same structure as would be derived using (4.13). It should be pointed out, however, that the cumulants technique could be used to keep all moments up to any desired order by simply assuming that all higher order cumulants are zero. Of course, the computation requirement increases quite quickly as the order of the filter increases. Interpretations of these higher order cumulants filters have yet to be obtained. Licht [10, pg. 86] has found that this approximation provided an effective solution to a plant noise variance estimation problem in which he had encountered difficulties with the EKF approximation discussed earlier in this chapter. In his example, the variance estimate was found to depend on the product of the measurement residual and the covariance between the measured and variance states. The EKF (and also the second order filter) equations produced a zero value for this covariance since the only term driving the covariance evolution equation was a third order moment and measurement residual product. Neglecting third order moments, therefore, produced an algorithm which could not change the variance estimate given measurement information.

An analogous situation occurs for the problem under consideration here. Note that the P14 equation(4.10d) is only driven by P24 which in turn is only driven by P34. Since P14 determines the change in $\hat{p}$ when the Kalman filter conditioning equations are used (see (A.11)), the value of P34 is deemed crucial to estimation performance. The terms driving the P34 equation are a third order moment(S344) and a P44. When $\hat{a}$ is small or P44 is small relative to S344 (assuming considerable skewness exists), neglecting third order moments could be significant.

In view of these results, and the largely increased complexity involved in using higher order approximations, the gaussian technique using (4.13) was chosen for performing the Monte Carlo experiments discused in Chapter V.

Applying (4.12) and (4.13), the remaining equations for the evolution of the third central moments are:

$$dS344/dt = (c+\hat{p}-2b)S344 + 2P34\ P44+\hat{a}\ S444 \qquad (4.15a)$$

$$dS444/dt = -3\ b\ S444 \qquad (4.15b)$$

$$dS334/dt=(2c+2\hat{p}-b)S334+a\ S344+2P34^2+2P44[P33-q] \qquad (4.15c)$$

$$dS234/dt=(c+\hat{p}-b)S234+P23\ P44+P24\ P34+\hat{a}\ S244 \qquad (4,15d)$$

$$dS244/dt = -2\ b\ S244 + S344 \qquad (4.15e)$$

$$dS134/dt = (c+\hat{p}-b)S134 + P14\ P34 + P13\ P44 + S234 + \hat{a}\ S144 \qquad (4.15f)$$

$$dS144/dt = -2\ b\ S144 + S244 \qquad (4.15g)$$

A system block diagram of (4.9), (4.11) and (4.15) is shown in Figure 5.

Additional motivation for the inclusion of third order moments is found in [17]. Geier[17, Appendix B] shows that the density function of the product of two gaussian random variates as in (2.3) is not gaussian and can have considerable skewness. An example similar to the system of (2.3) and (2.4) with $c=b=\sigma=0$ was also discussed in [17,pg. 60]. Geier suggests the use of analyis similar to Appendix B of this thesis in order to evaluate the evolution of the covariance matrix.

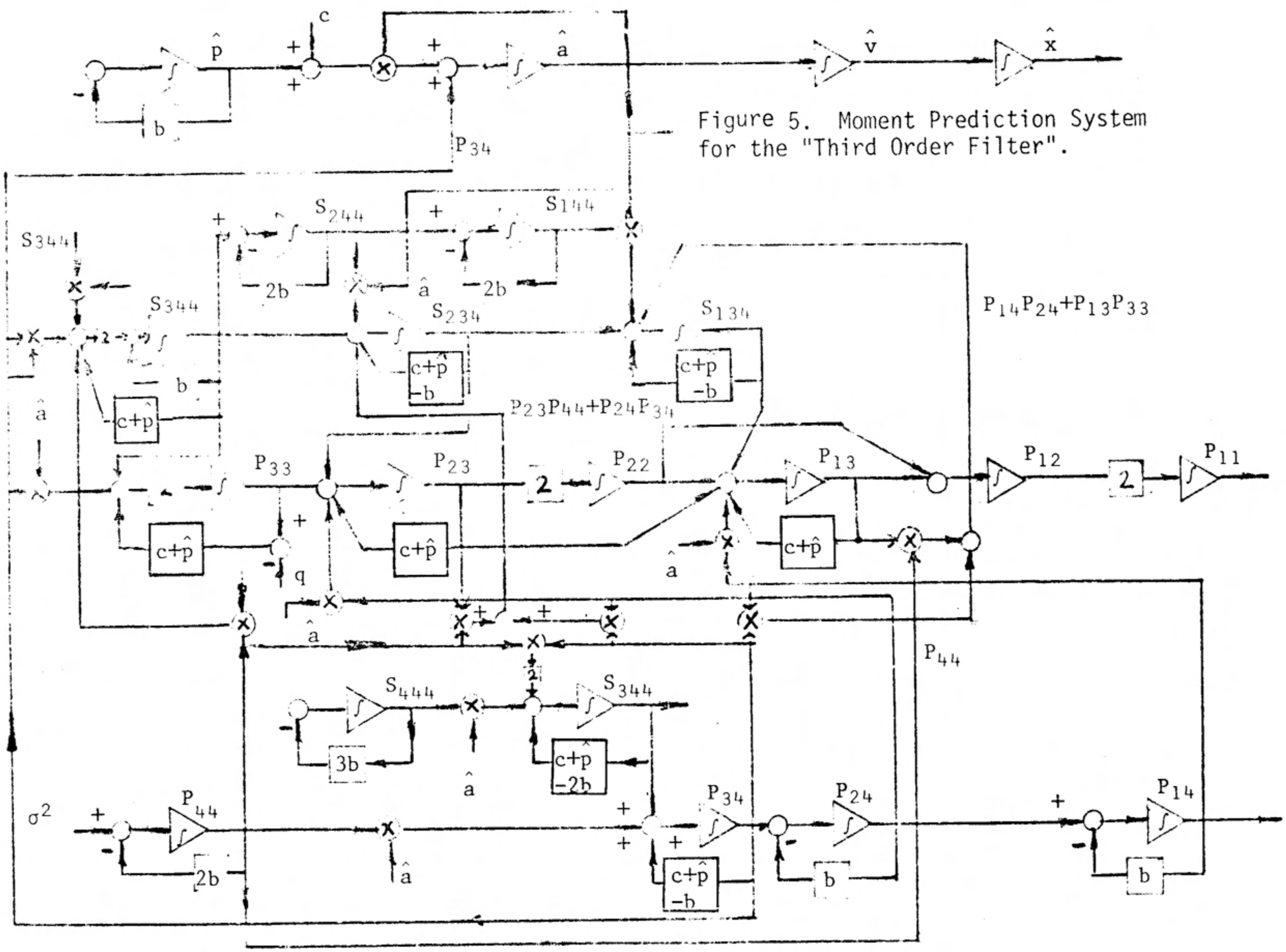

Figure 5. Moment Prediction System for the "Third Order Filter".

Considering the complexity of evaluating (B.16) and the need to perform two additional integration steps to derive the position and velocity variances, the usefulness of this technique is doubtful for other than the simplest problem discussed in [17].

In order to complete the definition of a filtering algorithm, a procedure for conditioning the moments at an observation must be defined. Reference [12, pg. 342] describes a scheme for approximating the relationship between moments prior to an observation and after an observation by assuming the conditional moments to be a power series of the measurement residual (z(kT)-G[y(kT|kT-T)]) at that observation. Because a linear measurement equation was used here, we have chosen a gaussian approximation which reduces to using the Kalman filter conditioning equations when the procedure in [12] is applied. The conditional mean vector is:

$$\hat{y}(k+1|k+1) = \hat{y}(k+1|k) + K(k+1)(z(k+1)-G\hat{y}(k+1|k)) \qquad (4.16)$$

The gains K and the conditional covariance matrix are given by:

$$K(k+1) = P(k+1|k+1)\,G'\,R^{-1} \qquad (4.17)$$

$$P(k+1|k+1) = P(k+1|k) - P(k+1|k)G'(GP(k+1|k)G'+R)^{-1}GP(k+1|k) \qquad (4.18)$$

The conditional third order moments Sijk(k+1|k) were set to zero based on the gaussian assumption.

In summary, the intent of this section has been to:

i. derive the central moment evolution equations to third order

ii. discuss approximations which result in a truncation of the infinite set of moment equations of the optimal filter

iii. discuss why the widely used EKF and second order filter approximations do not perform well as parameter estimation filters for the proposed model

iv. point out the equivalence of the gaussian technique using (4.13) and the fourth order cumulants method using (4.14).

4.2 An Alternate Approach to the Parameter Estimation Problem

Implementation of the third order filter described in section 4.1 involves a significant amount of computation due to the necessity of integrating the nonlinear moment evolution equations. The truncation of moments or cumulants also generates the possibility that the solution of (4.9), (4.11) and (4.15) can evolve in such a way that the moments no longer describe a valid joint probability density function. Variances, for example, may become negative or Schwartz's inequality may be violated. In view of these theoretical difficulties and the computational load of a full implementation, we describe here a possibly useful but as yet untried technique using the fixed correlation coefficient Singer type filter.

The variations in performance due to small variations in the correlation coefficient estimate do not appear critical in view of the results of Chapter III. It is expected, however, that very large variations would exists in actual filtering applications. A reasonable structure then might use a fixed correlation coefficient filter with the parameter estimation performed by a separate system using the acceleration estimates of the former filter to extract the desired parameter. Periodically, when some confidence was achieved in the correlation coefficient estimate, the previously fixed coefficient filter would be reintialized using the new coefficient estimate and then operated as a fixed coefficient filter until the next update. This is in contrast to maneuver detection schemes for which two or more predetermined values of the coefficient are chosen (only results for two values have been reported [4], [6], [11]) and the filters designed with these values are switched based on a filter residual test.

The structure of the proposed technique is shown in figure 6. The system is somewhat ad hoc and little experience with filters of this type has been gained. Several open questions about such a filtering system involve the choice of nonlinear parameter estimation system, the logic for deciding when to update the fixed coefficient filter with the new correlation coefficient estimate, and the manner in which the updated linear filter is initialized following an update. A useful nonlinear estimation system might be designed using the analysis of Chapter IV and considering only the last two states of the model of (2.1)-(2.4). We use y_3 and y_4 to refer to the acceleration and correlation coefficient states of the nonlinear parameter estimation filter and $\hat{a}$ to refer to the acceleration estimate of the Singer

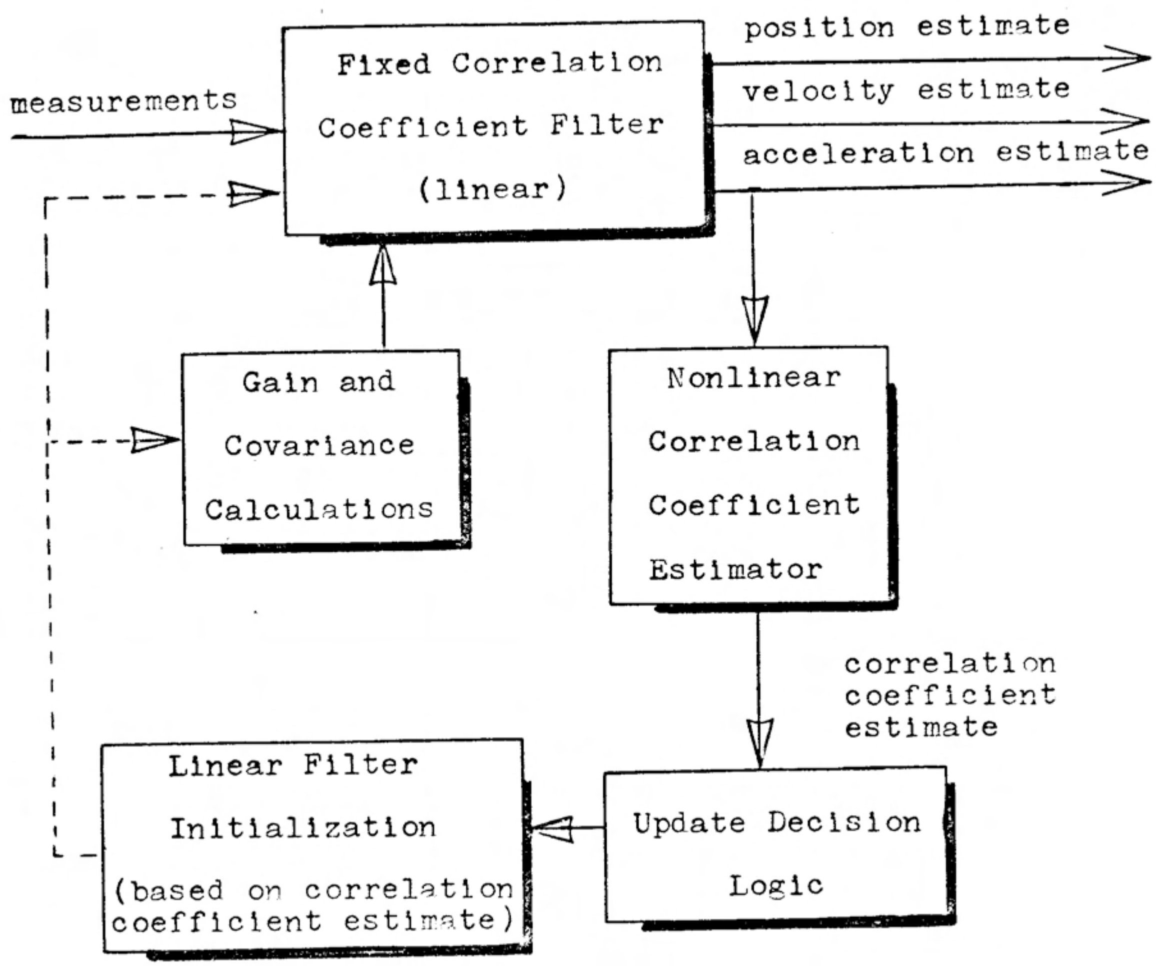

Figure 6. Components of the proposed alternate approach to the parameter estimation problem

type filter.

The idea is a follows: we have the dynamic model

$$dy_3 = (c+y_4)y_3dt + (2(c+y_4)q)^{\frac{1}{2}}dw_1$$

$$dy_4 = -by_4dt + dw_2$$

and we take as our observation the Singer filter acceleration estimate

$$z(kT) = \hat{a}(kT|kT)$$

We assume that the errors in these estimates are uncorrelated and that the covariance, r, of the estimation errors is known (this can be obtained from the Singer filter, P33, in Appendix A). In this case, we can write

$$z(kT)=y_3(kT) + u(k)$$

where u(k) is a white zero mean sequence with variance r. In this case, the same type of analysis as in the preceeding section yields the following equations for prediction between measurement updates.

$$d\hat{y}_3/dt = (c+\hat{y}_4)\hat{y}_3 + P_{34}$$

$$d\hat{y}_4/dt = -b\hat{y}_4$$

$$dP_{33}/dt = 2((c+\hat{y}_4)(P_{33}-q) + \hat{y}_3P_{34}+ S_{334})$$

$$dP_{34}/dt = (c+\hat{y}_4)P_{34} + \hat{y}_3P_{44} + S_{344}$$

$$dP_{44}/dt = -2bP_{44} + \sigma^2$$

$$dS_{334}/dt = [2(c+\hat{y}_4)-b]S_{334} + \hat{y}_3S_{344}+2P_{34}^2+2P_{44}[P_{33}-q]$$

$$dS_{344}/dt = (c+\hat{y}_4-2b)S_{344} + 2P_{34}P_{44} + \hat{y}_3S_{444}$$

$$dS_{444}/dt = -3bS_{444}$$

The estimates $\hat{y}_3$ and $\hat{y}_4$ would be conditioned on the acceleration estimate using:

$$\hat{y}_3(kT|kT)=\hat{v}_3(kT|kT-T)+K_3(\hat{a}(kT|kT)-y_3(kT|kT-T))$$

$$\hat{y}_4(kT|kT)=\hat{y}_4(kT|kT-T)+K_4(\hat{a}(kT|kT)-y_3(kT|kT-T))$$

with

$$K_3= P_{33}(kT|kT)/r$$

$$K_4= P_{34}(kT|kT)/r$$

$$P_{33}(kT|kT)=P_{33}(kT|kT-T)-P_{33}^2(kT|kT-T)/[P_{33}(kT|kT-T)+r]$$

$$P_{34}(kT|kT)=P_{34}(kT|kT-T)$$
$$-P_{33}(kT|kT-T)P_{34}(kT|kT-T)/ [P_{33}(kT|kT-T)+r]$$

$$P_{44}(kT|kT)=P_{44}(kT|kT-T)-P_{34}(kT|kT-T)/[P_{33}(kT|kT-T)+r]$$

$$S_{344}(kT|kT) = S_{334}(kT|kT) = S_{444}(kT|kT) = 0$$

Note that S_{444} remains at zero if initialized as such and need not be included.

CHAPTER V

MONTE CARLO PERFORMANCE WITH CONSTANT CORRELATION PLANT

Analytical error performance evaluation techniques similar to that performed in chapter III for nonlinear filters suffer from the same dimensionality problems encountered when attempting to derive optimal nonlinear filters. Since filters derived from the truncated set of moment equations of (4.9), (4.11) and (4.15) are themselves nonlinear systems, error performance predictions require the computation of the evolution of probability density functions through nonlinear dynamics. In view of these difficulties, Monte Carlo performance evaluation has been the standard approach. This chapter describes the results of extensive digital computer simulations using the moment evolution equations of (4.9), (4.11) and (4.15), and the conditional expectation equations of (4.16)-(4.19) in order to estimate the state and correlation coefficient of the colored acceleration noise system of (1.9) - (1.12) using noisy position measurements.

The position measurements were generated using the discrete time equivalent of the system described by (1.9) - (1.12). This requires initializing the state transition matrix (A.6) and a square root of the discrete plant noise covariance matrix (A.7) using the desired plant correlation coefficient h (instead of f in Appendix A.), maneuver variance q, and measurement interval T.

The initial acceleration a(0) is chosen using a pseudo-random gaussian number generator with zero mean and variance q. The initial position x(0) and velocity v(0) are set to zero. The values of actual position, velocity and acceleration at each measurement time are generated using the vector dynamic equation:

$$y(k+1) = F\, y(k) + Q^{1/2} w(k+1) \qquad (5.1)$$

In (5.1), y refers to the actual state vector $[x(k),v(k),a(k)]'$. The two position measurements required in (A.8) to initialize the first three filter states $(\hat{x}(1),\hat{v}(1),\hat{a}(1))$ are generated by corrupting x(0) and x(1) with zero mean additive gaussian noise of variance r. The parameter estimation tests were performed by choosing the filter correlation state to be some value mismatched from the actual plant. Note that p is negative and h is positive. The values of c,b,σ are inputs to the simulation program and for $c\neq 0$ were chosen to satisfy the stability condition of (B.19).

The filter covariance parameters P_{ij} for $i\leq 3$, $j\leq 3$, were initialized using (A.9) with $f = -(c+\hat{p}(1))$. P_{i4} for $i\neq 4$ and S_{ijk} were initialized to zero and P_{44} was inputted to the program since its initial value was found to have a significant effect on filter performance. The desired number (typically about 50) of estimation stages were then computed by repeating the following steps:

1. integrating (4.9), (4.11), and (4.15) using a fourth order Runge Kutta integration scheme with automatic step size correction (initial step size was T/10)

2. using (3.1) and (3.2) for measurement generation

3. using (4.16)-(4.18) for conditioning $\hat{x}, \hat{v}, \hat{a}, \hat{p}$ and the state covariance matrix based on the new measurement

4. conditioning third order moments by setting them to zero.

At each stage the information necessary for computing the sample mean and variance of the estimation error for each state was stored so that these sample statistics could be updated when the next trial was performed.

Following completion of the desired number of stages the filter and plant are reinitialized and a new trial begun. The sample statistics are outputted every ten trials in order to observe performance variations due to averaging. In most cases satisfactory results were obtained after only 21 trials. The required number of trials was primarily determined by comparing the sample results (transient and steady state) with the exact results of Chapter III by setting $b=\sigma=P44=0$ and observing only small changes in these results when 10 more trials were performed.

Prior to investigating the performance of the third order filter, the two lower order approximations discussed in Chapter IV were tested. The EKF algorithm was implemented using predictions based on the 4x4 state transition matrix and plant noise covariance matrix obtained by linearizing about the previous estimate. The second order filter was implemented by integrating the mean and covariance equations with third order moments set to zero. Both of these filter implementations provided no adaptation, i.e. the correlation coefficient estimate $\hat{p}$ remained at its initial value. Filter performance was observed to match the results predicted using the analysis of chapter III over a large variety of correlation mismatch cases. The inability of these filters to change the correlation coefficient is caused by the P14 covariance matrix element remaining at zero in the EKF filter and very near zero in the truncated third order moment implementation. Since the filter gain for the p state is directly proportional to this element (see Appendix A, equation (A.11)) no changes in this state estimate occur.

Considerable difficulty was also encountered in making the third order filter adapt effectively when the correlation coefficient estimate was greater than 1/T, the reciprocal of the measurement interval, and in low signal to noise ratio situations. Since these conditions show the greatest variation in performance due to mismatch ($c+p \neq h$), the largest potential improvements(based on the results of Chapter III) in the position, velocity and acceleration estimates were not achieved using the third order filter. Figures 7,8,9,10 illustrate one case where performance improvements were achieved. These figures illustrate the sample estimation error statistics computed from the results of 21 Monte Carlo trials of 60 simulated measurements(taken at 1 second intervals) generated using the procedure described earlier in this chapter. The initial correlation coefficient assumed by the filter was chosen to be 1 and the measurements were generated using a system with a correlation coefficient of h=0.01. This corresponds to the actual acceleration noise being highly correlated and the third order filter being initialized assuming little acceleration correlation exists between measurements. A value of q/r of 100, corresponding to the large signal to noise ratio results in Table 3, was used.

The parameters b,σ an P44 were chosen on the basis of experience gained on earlier runs where many variations were tried somewhat unsuccessfully. A value of 0.01 for b was chosen to provide an assumption of

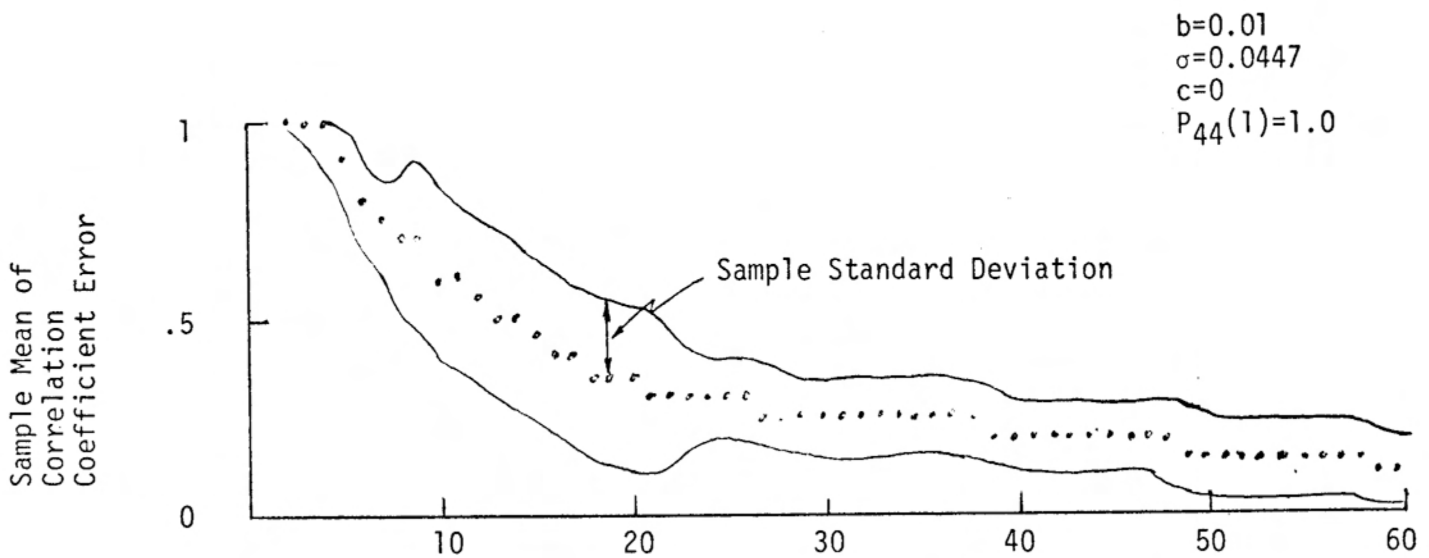

Figure 7. Sample Mean Error and Standard Deviation in the Correlation Coefficient Estimate (21 trials) with h=0.01, q=100, r=1.

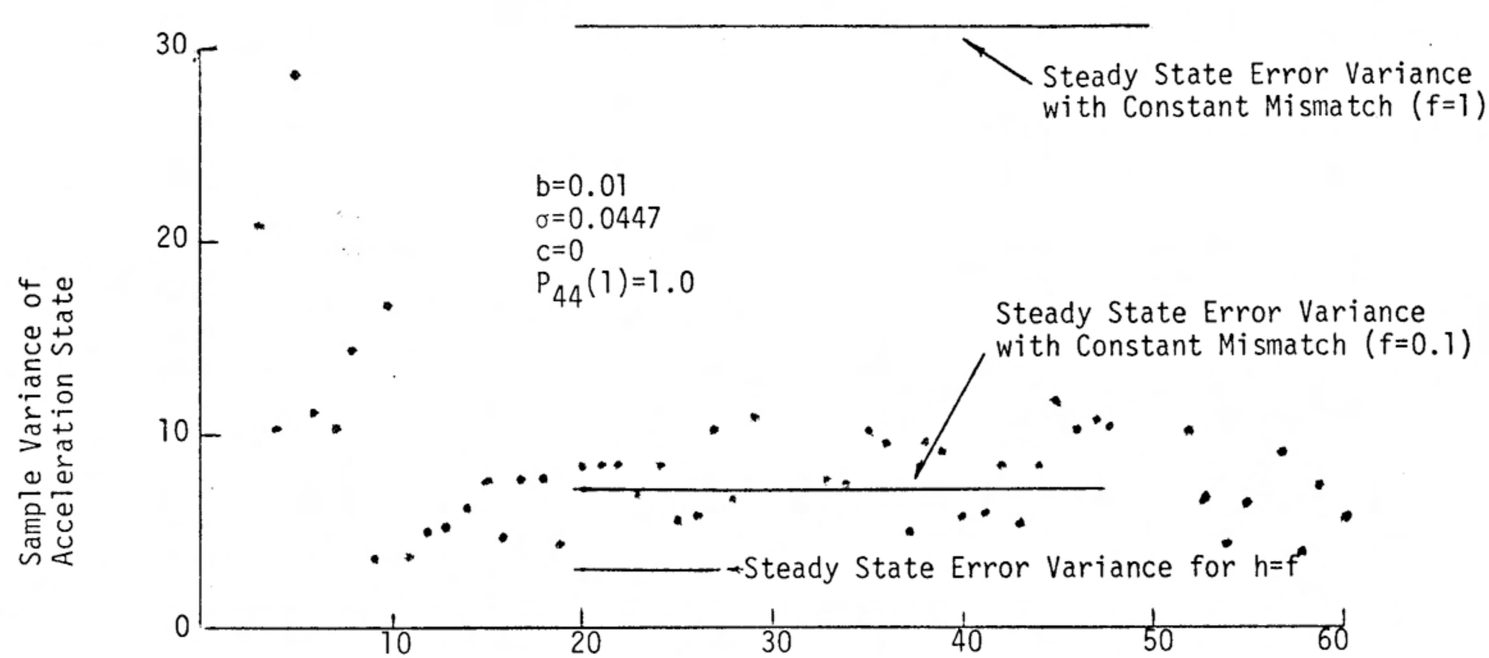

Figure 8. Sample Acceleration Estimate Error Variance with h=0.01, q=100, r=1. (21 trials)

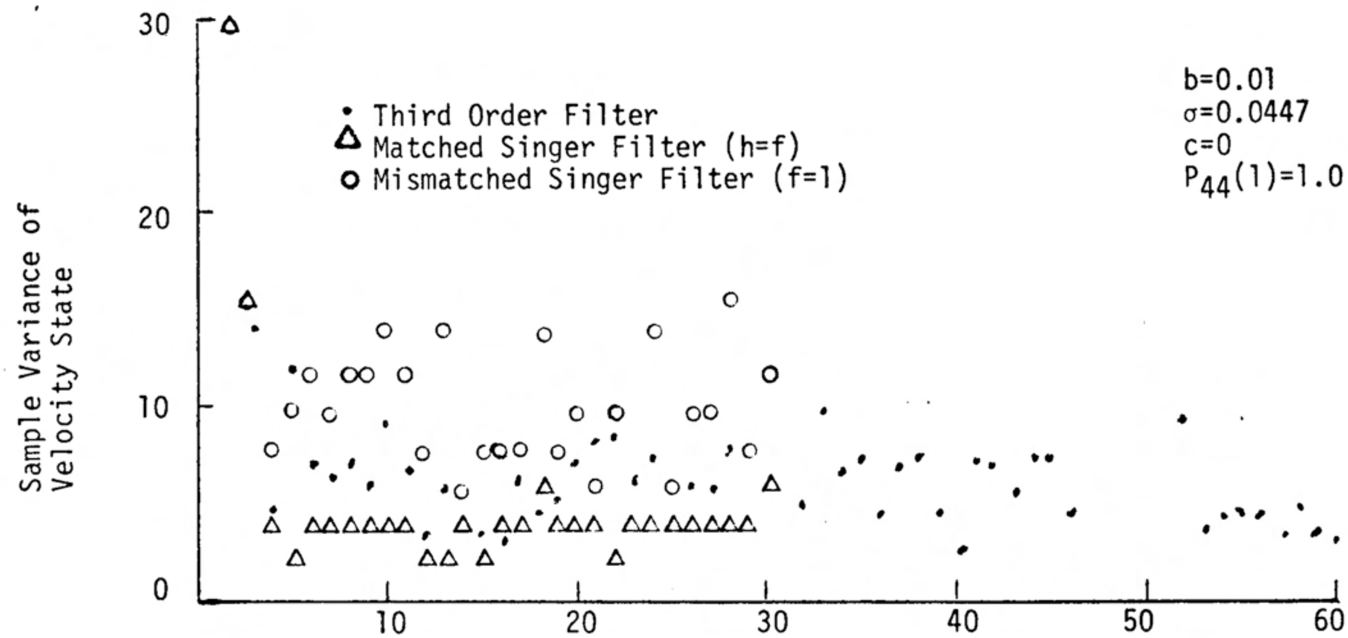

Figure 9. Sample Velocity Estimate Error Variance with h=0.01, q=100, r=1. (21 trials)

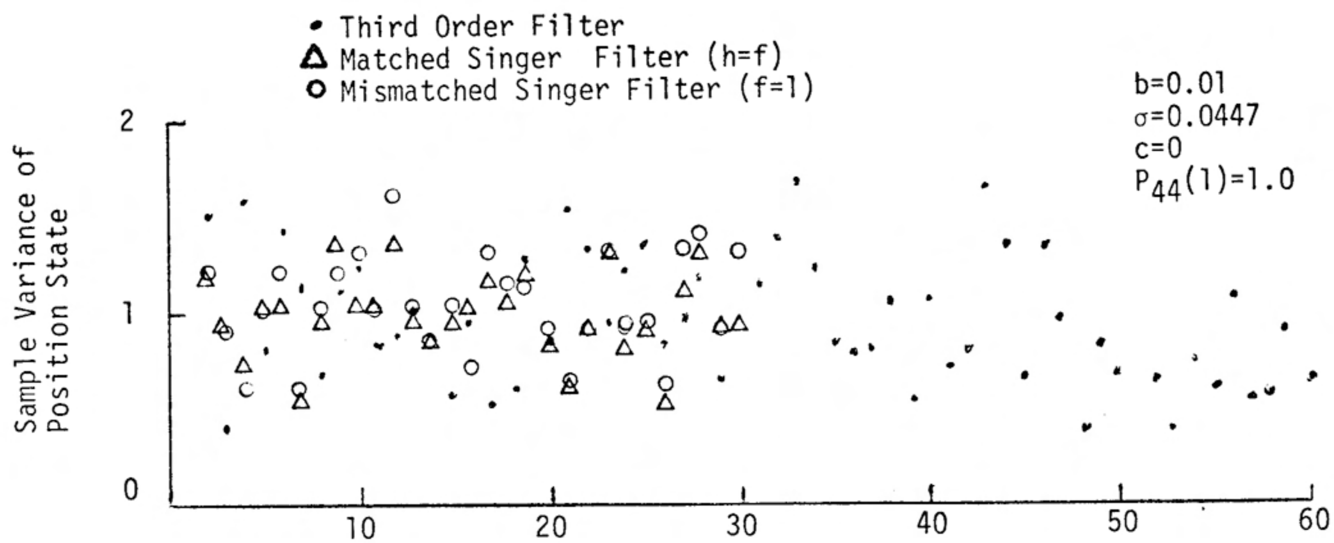

Figure 10. Sample Position Estimate Error Variance with h=0.01, q=100, r=1. (21 trials)

nearly constant correlation coefficient over relatively long periods. A steady state P44 prediction was chosen so as to maintain some uncertainty in the p estimate. Having defined b and P44(∞) in this way, the use of (4.11j) requires: σ^2=2bP44(∞) or σ=0.0447. The initial value of P44 used for these results was 1 but several other values were attempted. When set at 0.1, some adaptation occured but $\hat{p}$ convergence was not as rapid and larger sample variances were observed for the correlation coefficient estimate. A value of 10 caused significant problems when integrating the prediction equations due to a great many Schwartz's inequality violations occuring (i.e. $Pij^2 > Pii \cdot Pjj$). This resulted in poor overall performance. Similar problems having been encountered for other mismatch cases, we have found that a useful initial value for P44 appears to be approximately the square of the initial correlation coefficient estimate (c+$\hat{p}$(1)). For values of initial P44 an order of magnitiude less than $(c+\hat{p}(1))^2$, little adaptation occurs and for values an order of magnitude larger, significant problems are encountered integrating the prediction equations (in terms of the size of the integration step size required to achieve the same relative error), and in many cases negative variances and Schwartz's inequality violations are prevalent, causing extremely large errors for all the estimates.

These observations are summarized in Table 4.

Twenty one trials was deemed to be a large enough sample size in order to evaluate the performance of the third order filter for each set of parameter values, since this sample size provided adequate sample statistics and much larger sample sizes would have been needed to reduce the sample variances further. Identical measurement sets were filtered using the Singer filter with both constant mismatch and matched correlation parameters. Figures 9 and 10 show the results of these runs for $t \leq 30$ (steady state performance is achieved by the Singer filter at approximately t=10) as well as the results for the third order filter. The theoretical predictions of Chapter III and the nonlinear filter velocity estimate error variances clearly lie between these limits. Since the nonlinear filter has on the average reduced the p error from .99 to approximately 0.1 the, the velocity error variance was reduced to approximately 7, which corresponds to the value for f=.1 in Table 3d. The improvement in the acceleration esimates are even more significant. Using Table 3f, the steady state acceleration estimate variance using a constant value of f=1 for the correlation coefficient is 31. Figure 8 again shows the third order filter performing such that the experimental variance values for t>20 cluster quite

Table 4. Comparison of Initial P44 Value and Third Order Filter Performance for $b=0.1, \sigma=0.0447, q=100, r=1$.

$P44(1) = 0.1[c+\hat{p}(1)]^2$	Little adaptation in $\hat{p}$ occurs and performance compares with a Singer filter with $f=c+p(1)$
$P44(1) = [c+\hat{p}(1)]^2$	Significant filter adaptation occurs, causing overall filter performance improvement
$P44(1) = 10\ [c+\hat{p}(1)]^2$	Violations of Schwartz's inequality occur frequently resulting in poor overall filter performance

closely to the f=.1 value of 7.16. No improvement in the position error variance is observed since only about a 5 per cent reduction is predicted (0.931 versus 0.979--see Table 3a). For large signal to noise ratios the position error variance approaches the measurement variance r and all three cases in Figure 10 are observed to cluster about this value.

Table 5 summarizes the improvements obtained with the third order filter for the case illustrated by Figures 7-10. Note that the most marked improvement was obtained in the acceleration estimate (see also Figure 8), while the least improvement occurs in the position estimate. This can be explained by the observation that at high signal to noise ratios, the position estimation error is dominated by position measurement errors (i.e. filtering is insignificant); however, filtering does become important when we consider the estimation of derivatives of position--i.e. velocity and acceleration. This case serves to illustrate that by using a nonlinear filter, marked improvements can be achieved (e.g. if position-velocity predictions must be made beyond the latest measurement update time, the improvement in acceleration and velocity estimates could prove to be extremely significant). Note also that in this particular case in which the third order filter adapts to a model mismatch, the transient behavior is

Table 5. Comparison of Third Order Filter Performance with Matched and Mismatched Singer Filters with an Actual Correlation Coefficient of h=0.01, and with q=100, r=1, b=0.01, σ =0.0447.

	Matched Singer Filter with f=0.01	Mismatched Singer Filter with f=1.0	Third Order Filter with initial $\hat{p}$ set at 1.0
Position Estimate Variance	0.931	0.979	0.95
Velocity Estimate Variance	4.42	13	7
Acceleration Estimate Variance	3.43	31	7

sufficiently short that the adaptive filter rapidly takes advantage of the improved estimate of the correlation coefficient.

CHAPTER VI

CONCLUSIONS AND RECOMMENDATIONS

In this Chapter we summarize our major conclusions and discuss several recommendations for future study.

1. We have demonstrated how the Singer (constant correlation coefficient) filter behaves under mismatch conditions. This information is useful in itself in providing guidelines for the choice of a value of the coefficient for applications in which implementation restrictions require the use of the simpler constant correlation coefficient filter. That is, if we have a priori information about the range of the coefficient, we should choose the filter coefficient to minimize the maximum performance degradation as the actual coefficient ranges over the given range. This sensitivity analysis was not found in the earlier literature.

2. Based on the several nonlinear filters considered, we observed that the state product nonlinearity caused obviously poor performance for approximations which ignored third order moments. The third order approximation used here also encountered difficulties in low signal to noise ratio situations and when the actual correlation coefficient of the system from which the measurements were derived was constant and greater than the reciprocal of the measurement interval (note that

this last case is not terribly significant, as we would not expect to be able to detect correlation times less that the measurement interval).

The major obstacle to be overcome appears to be finding a way to treat effectively the large dynamic range (0 to $-\infty$) of the correlation coefficient estimate in order to take advantage of the potentially large filtering error reductions predicted in Chapter III. In many of the cases simulated, a set of initial conditions on the moment prediction system at some stage of the estimation process was encountered which caused violations of Schwartz's inequality on some of the covariance terms ($P32^2 > P33 \cdot P22$ was the most common). Clearly, the Schwartz's inequality problems will have to be overcome. The development of techniques for generating truncated moment filters which do not evolve such that the moments no longer describe a valid probability density function appears to be central to the development of useful filters of the type studied here, since extremely poor performance resulted when such violations occurred.

Some success has been obtained by choosing P44 to be a function of the correlation coefficient estimate, and further investigation of this concept is recommended. In view of the difficulties described above, a significant amount of effort would still be required to achieve a useful real time filter implementation based on a continuously adaptive parameter estimation model.

3. The simpler two step estimation technique described in Section 4.2 provides a structure for which a correlation coefficient estimator based on a model using direct measurements of the noise process can be used. In view of the success Licht[10] had using a third order filter for this simpler problem, the alternate technique appears promising as a useful adaptive filter. Simulations of this technique are needed to determine its usefulness. This method can also be generalized to other filtering problems in which estimation of plant noise parameters is desired.

4. Although no attempt was made to use higher than the third order moment equations derived by setting fourth order cumulants to zero, this research has led us to conjecture the following:

a) Higher order cumulant approximations are just the gaussian approximation at a higher level (i.e. assuming the m th moment is a function of the mean and variance as in the gaussian case). These higher order filters, therefore, will run into problems similar to those encountered using the second and third order filters used in this study.

b) These difficulties may be caused by the unimodal, unskewed nature of these approximations. Thus, we conjecture that a multimodal assumed density approximation would work much better (e.g. if we approximate our density functions as a weighted sum of two gaussians[18], we can obtain an expression for higher order moments in terms of lower order ones.

5. A comparison of continuous parameter estimation systems and techniques based on switching between two or more filters with different fixed parameters would be very useful since the latter is at present the most feasible to implement for real time applications where filters using a single fixed parameter are not deemed accurate enough.

6. Actual trajectories in more than one dimension (for example a vehicle traveling at constant speed along some course and abruptly changing heading) were not used as tests in this study due to the difficulties encountered with the parameter estimation filter. The results of such tests would be highly desirable once a large dynamic range correlation coefficient estimator has been developed.

APPENDIX A

COLORED ACCELERATION NOISE FILTER EQUATIONS

This appendix lists the discrete time vector equations and matrix definitions necessary for implementation of the colored acceleration noise model described in Singer [1].

The filter equations in vector form are given by (3.6) - (3.9) and are repeated here for convenience:

$$\hat{x}(k+1) = F\hat{x}(k) + K(k+1)\{m(k+1) - GF\hat{x}(k)\} \qquad (A.1)$$

$$K(k+1) = P(k+1|k+1)\, G'\, R^{-1} \qquad (A.2)$$

$$P(k+1|k+1) = P(k+1|k)$$
$$- P(k+1|k)G'\{GP(k+1|k)G' + R\}^{-1} GP(k+1|k) \qquad (A.3)$$

$$P(k+1|k) = FPF' + Q \qquad (A.4)$$

The measurement matrix G in (2.6) is:

$$G = [1 \;\; 0 \;\; 0 \;\; 0] \qquad (A.5)$$

with measurement error variance $E\{uu'\}=r$.

The transition matrix F is derived from the linear continuous system of (2.1) - (2.4) with $f=-(c+p)$ and $b=\sigma=0$:

$$F[T,f] = \begin{bmatrix} 1 & T & [-1+fT+\exp(-fT)]/f^2 \\ 0 & 1 & [1-\exp(-fT)]/f \\ 0 & 0 & \exp(-fT) \end{bmatrix} \qquad (A.6)$$

The maneuver excitation covariance matrix Q has the form:

$$Q = q \begin{bmatrix} Q11 & Q12 & Q13 \\ Q12 & Q22 & Q23 \\ Q13 & Q23 & Q33 \end{bmatrix} \qquad (A.7)$$

where

$$Q11 = [1-\exp(-2fT)+2fT+2f^3T^3/3 - 2f^2T^2-4fT\exp(-fT)]/f^4$$

$$Q12=[\exp(-2fT)+1-2\exp(-fT)+2fT\exp(-fT)-2fT+f^2T^2]/f^3$$

$$Q13=[1-\exp(-2fT)-2fT\ \exp(-fT)]/f^2$$

$$Q22=[4\exp(-fT)-3-\exp(-2fT)+2fT]/f^2$$

$$Q23=[\exp(-2fT)+1-2\exp(-fT)]/f$$

$$Q33=[1 - \exp(-2fT)]$$

The Kalman filter equations are initialized using two measurements, m(0) and m(1), such that:

$$\hat{x}(1|1) = m(1) \qquad (A.8a)$$

$$\hat{v}(1|1) = [m(1)-m(0)]/T \qquad (A.8b)$$

$$\hat{a}(1|1) = 0 \qquad (A.8c)$$

The elements of the initial covariance matrix P(1|1) are derived using (A.8) and are given by:

$$P11(1|1)=r \qquad (A.9a)$$

$$P12(1|1)=r/T \qquad (A.9b)$$

$$P13(1|1)=0 \qquad (A.9c)$$

$$P22(1|1)=2r/T^2 \qquad (A.9d)$$
$$+q[2-f^2T^2+2f^3T^3/3-2\exp(-fT)-2fT\exp(-fT)]/(f^4T^4)$$

$$P23(1|1)=q[\exp(-fT)+fT-1]/(f^2T) \qquad (A.9e)$$

$$P33(1|1)=q \qquad (A.9f)$$

Equation (A.3) computes the conditional state covariance matrix based on a new measurement m(k+1). Expanding (A.3) for this problem results in the individual elements of P(k+1|k) being given by:

$$Pij(k+1|k+1)=Pij(k+1|k) - [Pi1(k+1|k)\ P1j(k+1|k)]/(P11(k+1|k)+r) \quad (A.10)$$

Using (A.4) the individual filter gains are given by:

$$Kj(k+1) = [\ 1 - P11(k+1|k)/\{P11(k+1|k)+r\}]\ P1j/r \quad (A.11)$$

The gain for changing the j th state at measurement updates is observed to be directly proportional to the covariance of the j th state and the measured state.

APPENDIX B

DERIVATION OF THE STABILITY REQUIREMENTS FOR BOUNDED ACCELERATION VARIANCE

The introduction of the acceleration correlation coefficient as a state in Chapter II allows the possibility of unbounded acceleration variance predictions for certain values of the stabilizing constant c in equation (2.4).

An expression for the evolution of the acceleration variance in time under the condition that the initial acceleration (a) and coefficient (p) are zero is derived and the boundedness condition presented.

The system being considered is that of equations (2.4) and (2.5):

$$da = (c+p)a\, dt + [-2(c+p)q]^{1/2}\, dw_1(t) \qquad (B.1)$$

$$dp = -bp\, dt + \sigma\, dw_2(t) \qquad (B.2)$$

with dw_1 and dw_2 uncorrelated unit variance Brownian motion processes.

For each experimental outcome defined on the joint sample space of a and p:

$$a(t) = \phi(t,0)a(0) + \int_0^t \phi(t,s)[2(c+p(s))q]^{1/2} dw_1(s) \qquad (B.3)$$

where

$$\phi(t,s) = \exp(\int_s^t [c+p(\tau)]d\tau) = \exp[\, c(t-s)+\int_s^t p(\tau)d\tau] \qquad (B.4)$$

Letting $a(0)=0$ and using $E\{dw_1(t)\ dw_1(t)\}=dt$,

$$E\{a^2(t)\}=E\{2q\int_0^t [c+p(s)]\ \exp[\ 2c(t-s)+2\int_s^t p(\tau)d\tau]ds \qquad (B.5)$$

Defining $\theta(s) = 2\int_s^t p(\tau)\ d\tau$, B.5 can be written as:

$$E\{a(t)\} = 2q[c\int_0^t \exp(2c(t-s))\ E\{\exp(\theta(s))\}\ ds$$
$$+ \int_0^t \exp(2c(t-s))\ E\{p(s)\exp(\theta(s))\}\ ds\ ] \qquad (B.6)$$

Similarly, the correlation function of $p(s)$ is:

$$E\{p(s)p(t)\} = (\sigma^2/2b)\exp(-b|t-s|) + E\{p^2(0)-\sigma^2/2b\} \qquad (B.7)$$

Since (B.2) is a linear system driven by a gaussian process, $p(s)$ is gaussian and therefore $\theta(s)$ is gaussian. The expectations on the right side of B.6 can be conveniently derived using the known form of the characteristic function of gaussian random variables. Since p and θ are jointly gaussian their joint characteristic function is of the form:

$$\psi_{p\theta}(u_p,u_\theta) = E\{\ \exp(iu_p p(s)+iu_\theta\theta(s)\ )\ \}$$
$$=\exp(iu_p m_p + iu_\theta m_\theta -(1/2)[P_p u_p^2 + 2u_p u_\theta P_{p\theta} + P_\theta u_\theta^2]) \qquad (B.8)$$

with m and P defined as the appropriate means and covariances, respectively.

Using $E\{p(s)\} = E\{\theta(s)\} = 0$

note that

$$E\{\exp(\theta(s))\} = \psi(u_p=0,u_\theta=-i) = \exp[P_\theta/2] \qquad (B.9)$$

with

$$P_\theta(s) = 4\int_s^t\int_s^t E\{p(\xi)p(\gamma)\}\ d\xi\, d\gamma \qquad (B.10)$$

and using (B.7)

$$P_\theta(s) = (4\sigma^2/b^2)[\exp(-b(t-s)) + b(t-s) - 1\ ] \qquad (B.11)$$

Combining (B.9) and (B.11) gives:

$$E\{\exp(\theta(s))\}=\exp\{(2\sigma^2/b^3)[\exp(-b(t-s))+b(t-s)-1]\} \quad (B.12)$$

The second expectation is obtained similarly by observing that:

$$E\{p(s)\ \exp(\theta(s))\} = (1/i)\frac{\partial\psi}{\partial u_p}(u_p = 0, u_\theta = -i)$$
$$= P_{p\theta}\ \exp[P_\theta/2] \quad (B.13)$$

The covariance of p and θ can also be found using (B.7).

$$P_{p\theta} = E\{p(s)\theta(s)\} = 2\int_s^t E\{p(s)p(\tau)\}\ d\tau$$
$$= 2\ (\sigma^2/2b)\int \exp[-b|s-\tau|\ ]\ d\tau$$
$$= (\sigma^2/b^2)\ [1-\exp(-b(t-s))] \quad (B.14)$$

Substituting (B.11) and (B.14) into (B.13) results in:

$$E\{p(s)\ \exp(\theta(s))\} = (\sigma^2/b^2)[1-\exp(-b(t-s))]$$
$$\cdot\exp\{(2\sigma^2/b^3)[\exp(-b(t-s))+b(t-s)-1]\} \quad (B.15)$$

Combining (B.12) (B.15), and (B.6), letting ξ=t-s, and performing some simplification produces the desired result:

$$E\{a\ (t)\}=2\alpha e^{-2\frac{\sigma^2}{b^3}}\{[1+\frac{\sigma^2}{b^2}]\int_0^t e^{2(C+\frac{\sigma^2}{b^2})\xi}\ e^{2\frac{\sigma^2}{b^3}e^{-b\xi}}\ d\xi$$
$$+\frac{\sigma^2}{b^2}\int e^{-b\xi}\ e^{2(C+\frac{\sigma^2}{b^2})\xi}\ e^{2\frac{\sigma^2}{b^3}e^{-b\xi}}\ d\xi\} \quad (B.16)$$

Now if b>0(stability requirement on B.2)

$$\lim_{\xi\to\infty} \exp[(2\sigma^2/b^3)\ \exp(-b\xi)\ ] = 1 \qquad (B.17)$$

and convergence of (B.16) requires

$$c+\sigma^2/b^2 < 0$$

or

$$c < -\sigma^2/b^2 \qquad (B.18)$$

Thus for a given c<0 and b>0 the range of permissible values of σ is given by:

$$0<\sigma^2<-cb^2 \qquad (B.19)$$

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