ANALYTICAL STUDIES OF THE GENERALIZED LIKELIHOOD RATIO TECHNIQUE FOR FAILURE DETECTION

by

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ABSTRACT

The generalized likelihood ratio (GLR) technique has been suggested for detecting failures in linear dynamical systems. This thesis reports a study of this technique in an effort to provide a framework in which one can systematically study the various tradeoffs involved in the design of GLR failure detection systems. Some performance indices are defined. Important questions related to the performance of the detection scheme such as the detectability and distinguishability of failures are examined. Possible modification of the original scheme for improved performance is also considered.

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Introduction

1.1 Motivation

Many aspects of recent developments of systems theory are concerned with the improvement of the performance of control systems. One such concern is the detection of abrupt changes in dynamical systems. Examples of abrupt changes are actuator or sensor failures in an aircraft and sudden changes in the rhythm of cardiac activities as measured on an electrocardiagram [6]. For simplicity, all such abrupt changes will be termed as "failures" even though a physical failure may not be the cause of the abrupt change. The detection of failures may be viewed as consisting of three tasks: set off an alarm when a failure develops, then isolate the failure type, and estimate the extent of the failure.

Most of the failure detection analysis has been performed in the context of a state space description of a linear dynamical system as follows:

State Equation:

$$x(k+1) = \Phi(k)x(k) + B(k)u(k) + w(k) \qquad (1\text{-}1)$$

Sensor Equation:

$$z(k) = H(k)x(k) + J(k)u(k) + v(k) \qquad (1\text{-}2)$$

where u is a known input, w and v are independent, zero mean, white gaussian random sequences with covariances:

$$E\{x(j)w'(k)\} = Q\delta_{jk} \; ; \; E\{v(j)v'(k)\} = R\delta_{jk} \qquad (1\text{-}3)$$

where δ_{jk} is the Kronecker delta.

Abrupt changes of the system may appear in (1-1) ("actuator failure") or in (1-2) ("sensor failures"). Actuator failures may take the form of a shift in the control gain matrix B, a bias on the right hand side (RHS) of (1-1) or a change in the process noise, corresponding, for example, to the failure of the actuator or control surfaces on an aircraft, a leak in the thruster of a space vehicle and a sudden shift in wind conditions (in an aircraft control situation) respectively. In these cases, an accurate knowledge of the failure is clearly vital (assuming there is a re-organizational procedure to compensate for the failure), as an undetected failure could easily lead to the loss of the vehicle. Changes in sensor noise and the H and J matrices are examples of sensor failures. These are the causes of erroneous state estimates and produce very undesirable effects in a feedback control system that utilizes these sensor outputs in the feedback loop.

Recent studies have provided different approaches to the problem of detecting failures. The "failure sensitive" filter developed by Beard [1] and Jones [2], the voting system studied by Broen [3] and the multiple hypotheses filters employed by Gustafson, Willsky and Wang in the classification of rhythms and detecting rhythm shifts in electrocardiagram [6] are examples of some failure detection schemes. Very often, the tradeoffs among the various approaches are detector complexity vs. detector sensitivity and detector sensitivity vs. detector false alarm rate. Hence the applicability of the different schemes depends on the particular situation and the performance criteria under consideration. With the decreasing cost of digital hardward and increasing availabiltiy of computers, many of these detection methods are becoming feasible for on-line implementation.

In [4], [5], Willsky and Jones have suggested the generalized likelihood ratio (GLR) approach to failure detection. As noted by Willsky [9], the GLR approach can be applied to a wide range of actuator and sensor failures. The method also provides an estimate of the failure size which is useful in system reorganization after the failure is determined to have occurred. The technique may be simplified in a number of ways making it more attractive from an implementation point of view. In addition, the tradeoff between complexity and performance may be studied analytically. In this thesis research, the GLR approach to failure detection is studied to obtain insights into its limitations and to develop some guidelines in the design of GLR failure detection systems.

1.2 Description of the GLR Technique

The GLR failure detection system assumes a linear system described by (1-1), (1-2) and a Kalman Bucy filter (that assumes no failure) characterized by the following:

$$\hat{x}(k+1|k) = \Phi(k)\hat{x}(k|k) + B(k)u(k) \qquad (1\text{-}4)$$

$$\hat{x}(k|k) = \hat{x}(k|k-1) + K(k)\gamma(k) \tag{1-5}$$

$$\gamma(k) = z(k) - H(k)\hat{x}(k|k-1) - J(k)u(k) \tag{1-6}$$

where $x(i|j)$ is the mean of $x(i)$ given $z(0), z(1), \ldots, z(j)$ and $P(i|j)$ is the associated error covariance. The quantity γ is the zero mean, white gaussian innovations process (or the residual) with covariance V. K is the Kalman Bucy filter gain computed as follows:

$$P(k+1|k) = \Phi(k)P(k|k)\Phi'(k) + Q \tag{1-7}$$

$$V(k) = H(k)P(k|k-1)H'(k) + R \tag{1-8}$$

$$K(k) = P(k|k-1)H'(k)V^{-1}(k) \tag{1-9}$$

$$P(k|k) = P(k|k-1) - K(k)H(k)P(k|k-1) \tag{1-10}$$

The four basic failure types (modes) under consideration are modeled as:

Type 1: state jump

$$x(k+1) = \Phi(k)x(k) + B(k)u(k) + w(k) + \nu\delta_{k+1;\theta} \tag{1-11}$$

Type 2: state step

$$x(k+1) = \Phi(k)x(k) + B(k)u(k) + w(k) + \nu\delta_{k+1;\theta} \tag{1-12}$$

Type 3: sensor jump

$$z(k) = H(k)x(k) + J(k)u(k) + v(k) + \nu\delta_{k;\theta} \tag{1-13}$$

Type 4: sensor step

$$z(k) = H(k)x(k) + J(k)u(k) + v(k) + \nu\delta_{k;\theta} \tag{1-14}$$

where

$$\delta_{k;\theta} = \begin{cases} 1 & \text{if } k = \theta \\ 0 & \text{otherwise} \end{cases} \tag{1-15}$$

$$\sigma_{k;\theta} = \begin{cases} 0 & k < \theta \\ 1 & k \geq \theta \end{cases} \quad (1\text{-}16)$$

Thus θ has the meaning as the "failure time" and ν is the failure vector of appropriate dimension. We note that the original GLR method devised by Willsky and Jones [4], [5] was developed for type 1 failures.

Due to the linearity of the system and filter, in the event of a failure, the residual can be expressed as:

$$\gamma(k) = \tilde{\gamma}(k) + G_i(k;\theta)\nu \quad (1\text{-}17)$$

where $\tilde{\gamma}$ is the residual in the absence of any failure. G_i is a matrix (i=1,2,3,4, denoting the failure type). The equations that one can use to compute the G_i are given in Section 2.1. Then $G_i(k;\theta)$ is the effect of the type i failure ν that occurred at time θ on the residual at time k. We can establish two hypotheses:

H_0 : no failure has occurred

H_i : a failure of type i (ν and θ unknown) has occurred.

Then the generalized likelihood ratio (GLR) is defined by

$$L_i(k) = \frac{p(\gamma(1),\ldots,\gamma(k)\,|\,H_i,\ \theta = \hat{\theta}(k),\ \nu = \hat{\nu}(k))}{p(\gamma(1),\ldots,\gamma(k)\,|\,H_0)} \quad (1\text{-}18)$$

where p denotes probability density function; $\hat{\theta}(k)$ and $\hat{\nu}(k)$ are the maximum likelihood estimates (MLE) of ν and θ assuming H_i to be true defined by:

$$\hat{\theta}(k),\ \hat{\nu}(k) = \arg\max_{\tilde{\theta},\tilde{\nu}}\ p(\gamma(1),\ \ldots\ \gamma(k)\,|\,H_i,\ \theta = \tilde{\theta},\ \nu = \tilde{\nu}) \quad (1\text{-}19)$$

Given H_i is true, the residual is

$$\gamma(k) = \tilde{\gamma}(k) + G_i(k;\theta)\nu \tag{1-20}$$

for some unknown θ and ν. When H_0 is true, the residual becomes

$$\gamma(k) = \tilde{\gamma}(k) \tag{1-21}$$

Using the fact that the γ's are gaussian independent variables and equations (1-20) and (1-21), the logarithm of (1-18) can be expressed as

$$\begin{aligned} \ell_i(k) &= 2 \ln L_i(k) \\ &= \sum_{j=1}^{k} \gamma'(j) V^{-1} \gamma(j) \\ &\quad - \sum_{j=1}^{k} [\gamma(j) - G_i(k;\hat{\theta}(k))\hat{\nu}(k)]' V^{-1}(j) [\gamma(j) - G_i(k;\hat{\theta}(k))\hat{\nu}(k)] \end{aligned} \tag{1-22}$$

To choose between H_0 and H_i we use the decision rule:

$$\ell_i(k) \underset{H_0}{\overset{H_i}{\gtrless}} \varepsilon \tag{1-23}$$

where ε is some predetermined threshold. Hence $\hat{\theta}(k)$ and $\hat{\nu}(k)$ also maximize $\ell_i(k)$. Also, $\hat{\nu}(k)$ can be solved as an explicit function of $\hat{\theta}(k)$:

$$\hat{\nu}(k) = C_i^{-1}(k;\ \hat{\theta}(k))\ d_i(k;\ \hat{\theta}(k)) \tag{1-24}$$

where $C_i(k;\ \hat{\theta})$ is the matrix

$$C_i(k;\ \hat{\theta}) = \sum_{j=1}^{k} G_i'(j;\ \hat{\theta})\ V^{-1}(j)G(j;\ \hat{\theta}) \tag{1-25}$$

and $d_i(k;\ \hat{\theta})$ is a linear combination of the residuals:

$$d_i(k;\ \hat{\theta}) = \sum_{j=1}^{k} G_i'(j;\ \hat{\theta})V^{-1}(j)\gamma(j) \tag{1-26}$$

Then $\hat{\theta}(k)$ is the value of $\theta \leq k$ that maximizes $\ell_i(k;\ \theta)$:

$$\ell_i(k;\ \theta) = d_i'(k;\ \theta)C_i^{-1}(k;\ \theta)\ d_i(k;\ \theta) \tag{1-27}$$

Therefore, the GLR system (also known, for reasons that will become clear, as full GLR) will declare a type i failure $\hat{\nu}$ occurring at $\hat{\theta}$ if $\ell_i(k;\ \hat{\theta}) > \varepsilon$ and $\ell_i(k;\ \hat{\theta}) > \ell_i(k;\ \theta)$ for $1 \leq \theta \leq k$. As time progresses, the number of possible values of $\hat{\theta}$ increases. Hence, the implementation of this scheme involves a growing bank of filters. (See Figure 1.)

When detectors for different failure types are implemented simultaneously, one is confronted with the additional problem of deciding among the failure types. A simple maximization of $\ell_i(k;\ \theta)$ over ν, θ and i may not provide satisfactory isolation of the failure type. In the following, the subscript i is dropped for the sake of simplifying the notation.

A number of simplifications of the approach have been suggested by Willsky and Jones [5] such as the finite window assumption where $\hat{\theta}$ is restricted to a range, $k\text{-}M \leq \hat{\theta} \leq k\text{-}N$. The physical assumptions made here are: 1) no decision can be made with less than N observations (an observability constraint), 2) failures before time k-M should have been

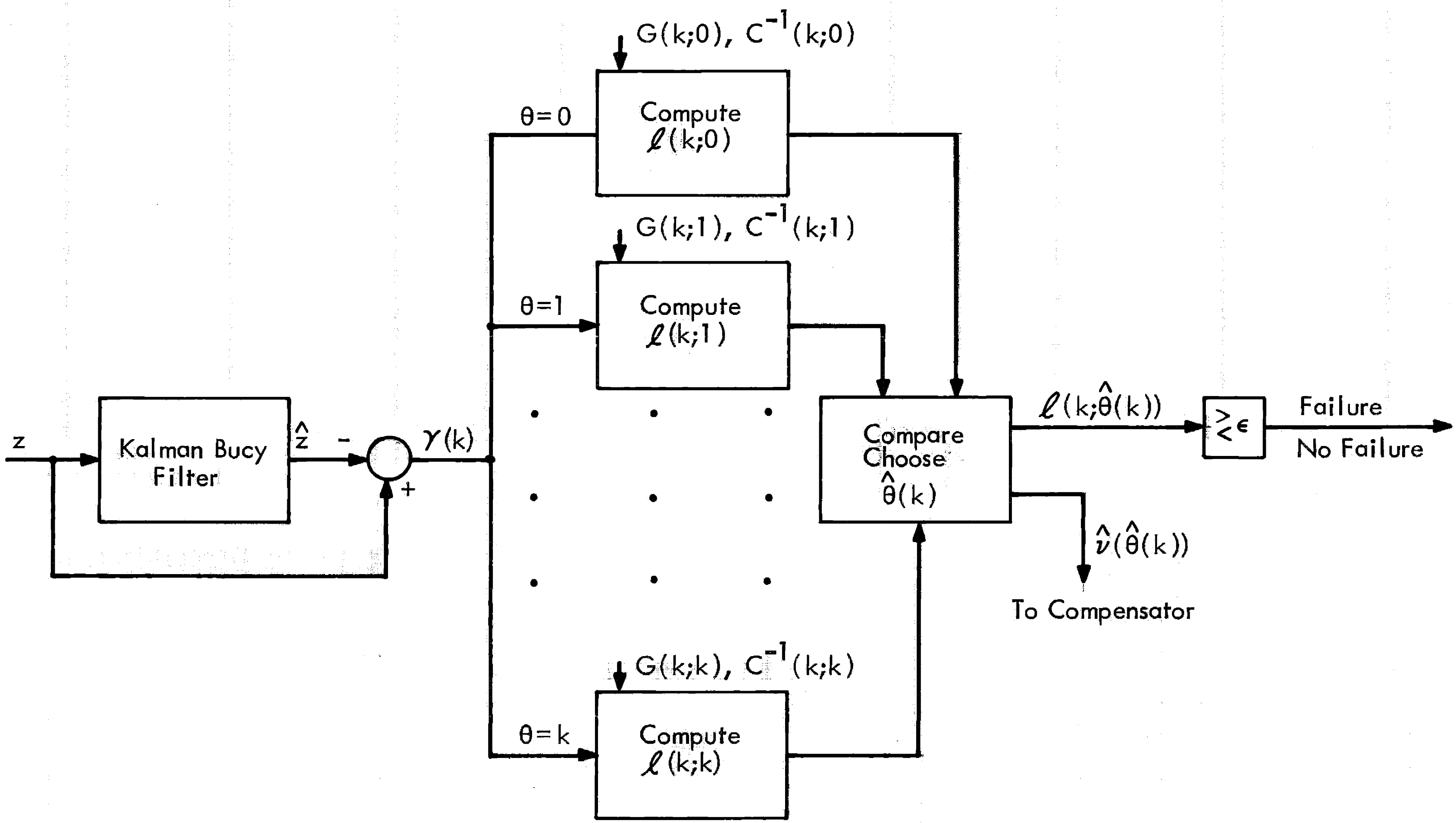

Fig. 1 Full GLR Detector Scheme with Growing Bank of Filters

detected at an earlier time and compensated for already (a limitation imposed by computational complexity). Both of these assumptions are reasonable, and the resulting "sliding window" reduces the computational burden imposed by the growing bank of filters described earlier (where calculation of $\ell(k;\theta)$ for $\theta = 1, \ldots, k$ is required). When the system under consideration is time invariant and the associated Kalman-Bucy filter (KBF) has reached a steady state, the G and C matrices become dependent on $k-\theta$ only. Thus, these matrices may be computed once and stored, greatly simplifying the required calculations. To reduce required calculations even further, one may wish to consider approximating G and C (by polynomials, for example). Of course this will degrade the quality of the estimate of ν.

Another simplification is the constrained GLR (CGLR) which involves the assumption that $\nu = \alpha f_j$ (where α is a scalar and f_j is one of a finite set of directions). Thus, in computing $\hat{\nu}(k)$, we require it to be along one of these directions and estimate its magnitude (α). The CGLR detector takes the form:

$$\hat{\alpha}(k) = \frac{b(k;\ \hat{\theta}(k),\ \hat{j}(k))}{a(k;\ \hat{\theta}(k),\ \hat{j}(k))} \tag{1-28}$$

where $\hat{\theta}(k)$ and $\hat{j}(k)$ are the quantities that maximize

$$\ell(k;\ \theta,\ j) = \frac{b^2(k;\ \theta,\ j)}{a(k;\ \theta,\ j)} \tag{1-29}$$

where

$$a(k;\ \theta,\ j) = f_j' C(k;\ \theta) f_j \tag{1-30}$$

$$b(k;\ \theta,\ j) = f_j'\ d(k;\ \theta) \qquad (1\text{-}31)$$

and $d(k;\ \theta)$ is defined by (1-26). The decision rule is:

$$\ell(k;\ \hat{\theta}(k),\ \hat{j}(k) \overset{H_1}{\underset{H_0}{\gtrless}} \varepsilon \qquad (1\text{-}32)$$

If ν is further restricted to be some constant ν_0, one has the simplified GLR (SGLR). We note that SGLR does not require maximization over ν and hence $\ell(k;\ \theta)$ becomes

$$\ell(k;\ \theta) = \sum_{j=1}^{k} [2\gamma(j) - G(j;\ \theta)\nu_0]'V^{-1}(j)G(j;\ \theta)\nu_0 \qquad (1\text{-}33)$$

Both CGLR and SGLR require less computation than full GLR. However, they are directionalized, i.e., most sensitive to certain directions only. This limits their ability to detect failure of other directions. Consequently, they may be suitable only for a certain class of failure detection problems.

From the above discussion, it is clear that the GLR method offers a range of implementations from the point of view of computational complexity. In order to develop a useful detector design methodology, one must study much more carefully the properties of the GLR method and the tradeoffs involved in the design. The purpose of this research is to study certain of these issues in order to provide some guidelines for the use of the GLR technique. Our aim is to develop an analytic framework in which one can systematically study the various tradeoffs involved in the design of a GLR failure detection system.

1.3 Overview of the Research

The two aspects of the performance of the GLR detector most closely examined in this research are the detector's sensitivity to failures and its ability to isolate the types of the failures. Effectively, the GLR detector concentrates failure information into the variable, $\ell(k;\ \theta)$ as the decision rule considers only these quantities. Hence, a starting point in the analysis of the GLR detection scheme is the study of this random variable.

In Chapter 2, we present the static analysis where we consider the ℓ's as static variables, i.e. the correlation among them is not considered. There, we derive expressions for probabilities such as the probability of correct detection and the probability of false alarm. We also consider the questions of the detectability of failures by a GLR detector and the ability of the detector to, in some way, distinguish among the different types of failure.

In Chapter 3, we study the correlation behavior of the ℓ's in an attempt to obtain more precise performance indices, such as the probability of time to detection and to derive more information about the failure from the temporal behavior of the likelihood ratios.

Finally, in Chapter 4, a summary of the study is presented along with a numerical example (failure detection for a simplified aircraft model) illustrating the performance of the GLR technique. We also outline several directions in which we feel further work should be done.

CHAPTER 2

Static Analysis

2.1 Summary of the GLR Equations

The implementation of GLR detectors requires the G matrices described in Section 1.2. In this section, we present the equations necessary for computing these matrices for the four basic failure types (state jump, state step, sensor jump and sensor step) as modelled by equations (1-11), (1-12), (1-13), and (1-14). The unfailed dynamical system is represented by equations (1-1) and (1-2) which are repeated here for easy reference.

$$x(k+1) = \Phi(k)x(k) + B(k)u(k) + w(k) \tag{2-1}$$

$$z(k) = H(k)x(k) + J(k)u(k) + v(k) \tag{2-2}$$

The associated KBF:

$$\hat{x}(k+1|k) = \Phi(k)x(k|k) + B(k)u(k) \tag{2-3}$$

$$\hat{x}(k|k) = \hat{x}(k|k-1) + K(k)\gamma(k) \tag{2-4}$$

$$\gamma(k) = Z(k) - H(k)\hat{x}(k|k-1) - J(k)u(k) \tag{2-5}$$

where K is the filter gain computed from equations (1-7), (1-8), (1-9) and (1-10).

Since the failures under consideration do not involve the known control u, the control terms in the above equations may be omitted to simplify the mathematics. However, the subsequent analysis is still valid for cases where the control is present due to the following reason. Since u is deterministic, its effects may be computed exactly;

by linearity of the system and filter, the control effects may be added directly to the uncontrolled system and filter to obtain the controlled situation. Consequently, the analysis throughout this report assumes the absence of controls without sacrificing the validity of the results for systems with deterministic controls.

2.1.1 The General Case

The linearity of the system and the KBF enables us to mathematically decompose the residual γ and the state estimate $\hat{x}(k|k)$ into two parts:

$$\gamma(k) = \gamma_1(k) + \gamma_2(k) \tag{2-6}$$

$$\hat{x}(k|k) = \hat{x}_1(k|k) + \hat{x}_2(k|k) \tag{2-7}$$

when the variables with subscript 1 denote the residual and state estimate when no failure has occurred and the subscript 2 denote the "bias" developed in the KBF due to failures. (Note that γ_1 is the same as $\tilde{\gamma}$ defined in 1.2.) Similar decomposition is also applicable to x and z. In addition, we find that for the four failure types:

$$\gamma_2(k) = G(k;\ \theta)\nu \tag{2-8}$$

$$\hat{x}_2(k|k) = F(k;\ \theta)\nu \tag{2-9}$$

where G and F are matrices that are functions of the system and filter matrices, K, θ and failure type.

After some manipulation of equations (2-3), (2-4) and (2-5), we obtain a recursive expression for $\hat{x}(k|k)$:

$$\hat{x}(k|k) = \Theta(k-1)\hat{x}(k-1|k-1) + K(k)Z(k) \tag{2-10}$$

where

$$\Theta(k-1) = [I - K(k)H(k)]\Phi(k-1) \tag{2-11}$$

To simplify some of the notations, we define

$$\Theta(k, j) = \Theta(k-1)\Theta(k-2) \;\ldots\; \Theta(j) \tag{2-12}$$

$$\Phi(k, j) = \Phi(k-1)\Phi(k-2) \;\ldots\; \Phi(j) \tag{2-13}$$

Now, we are ready to consider the effects of the four types of failures.

State Step Failures

Consider a state step failure. Its effect on the system can be described by

$$x_2(k+1) = \Phi(k+1, k)\; x_2(k) + \sigma_{k+1,\theta}\nu \quad , \; x_2(0) = 0 \tag{2-14}$$

$$z_2(k) = H(k)x_2(k) \tag{2-15}$$

Thus

$$z_2(k) = x_2(k) = 0 \qquad k < \theta \tag{2-16}$$

$$x_2(k) = \sum_{i=\theta}^{k} \Phi(k, i)\nu \qquad k \geq \theta \tag{2-17}$$

$$z_2(k) = \sum_{i=\theta}^{k} H(k)\Phi(k, i)\nu \qquad k \geq \theta \tag{2-18}$$

The effect on the filter:

$$\hat{x}_2(k|k) = \Theta(k, k-1)\hat{x}_2(k-1|k-1) + K(k)Z_2(k), \quad \hat{x}_2(0|0) = 0 \tag{2-19}$$

We then calculate

$$\hat{x}_2(k|k) = 0 \qquad k < \theta \tag{2-20}$$

$$\hat{x}_2(k|k) = \sum_{j=\theta}^{k} \Theta(k,j)K(j)Z_2(j) \qquad k \geq \theta$$

$$= \sum_{j=\theta}^{k} \Theta(k,j)K(j) \sum_{i=\theta}^{j} H(j)\Phi(j,i)\nu$$

$$= \sum_{i=\theta}^{k} \sum_{j=i}^{k} \Theta(k,j)K(j)H(j)\Phi(j,i)\nu \tag{2-21}$$

Hence

$$\hat{x}_2(k|k) = F(k;\ \theta)\nu \tag{2-22}$$

$$F(k;\ \theta) = \begin{cases} 0 & k < \theta \\ \sum_{i=\theta}^{k} \sum_{j=i}^{k} \Theta(k,j)K(j)H(j)\Phi(j,\ \ell) & k \geq \theta \end{cases} \tag{2-23}$$

From the definition of the residual (2-5), we have

$$\gamma_2(k) = Z_2(k) - H(k)\Phi(k,\ k-1)\hat{x}_2(k-1|k-1) \tag{2-24}$$

$$\gamma_2(k) = G(k;\ \theta)\nu \tag{2-25}$$

$$G(k;\ \theta) = \begin{cases} 0 & k < \theta \\ H(k)\left[\sum_{j=\theta}^{k} \Phi(k,j) - \Phi(k,k-1)F(k-1;\theta)\right] & k \geq \theta \end{cases} \tag{2-26}$$

Following similar calculations, we obtain the expressions of the F and G matrices for the other failure types [7].

State Jump Failures

$$F(k;\ \theta) = \begin{cases} 0 & k < \theta \\ \sum_{j=\theta}^{k} \Theta(k,j)K(j)H(j)\Phi(j,\ \theta) & k \geq \theta \end{cases} \tag{2-27}$$

$$G(k;\ \theta) = \begin{cases} 0 & k < \theta \\ H(k)\ [\Phi(k,\theta) - \Phi(k,k-1)F(K-1;\ \theta) & k \geq \theta \end{cases} \tag{2-28}$$

Sensor Step Failures

$$F(k;\ \theta) = \begin{cases} 0 & k < \theta \\ \sum_{j=\theta}^{k} \Theta(k,j)K(j) & k \geq \theta \end{cases} \tag{2-29}$$

$$G(k;\ \theta) = \begin{cases} 0 & k > \theta \\ I & k = \theta \\ I-H(k)\Phi(k,k-1)F(k-1;\ \theta) & k > \theta \end{cases} \tag{2-30}$$

Sensor Jump Failures

$$F(k;\ \theta) = \begin{cases} 0 & k < \theta \\ \Theta(k,\ \theta)K(\theta) & k \geq \theta \end{cases} \tag{2-31}$$

$$G(k;\ \theta) = \begin{cases} 0 & k < \theta \\ I & k = \theta \\ -H(k)\Phi(k,\ k-1)F(k-1;\ \theta) & k > \theta \end{cases} \tag{2-32}$$

We note that the matrix G is essentially the only quantity that is needed in the implementation of the GLR detector. The matrix F is important in the implementation of a mechanism for compensation following detection. Examples of such a mechanism are discussed in [4] and are not pursued here.

Due to the fact that $G(K; \theta) = 0$ for $k < \theta$ and for all types of failures, the summation in the expression of $C(k; \theta)$ (1-25) need only be performed from $j = \theta$ to k instead of $j = 1$ to k.

$$C(k; \theta) = \sum_{j=\theta}^{k} G'(j; \theta) V^{-1} G(j; \theta) \tag{2-33}$$

2.1.2 Steady-State, Time-Invariance Simplification

When the system (2-1) and (2-2), under consideration is time invariant and the associated KBF has reached a steady state, $\Phi(k)$ and $\Theta(k)$ become constant matrices Φ and Θ respectively. Then

$$\Theta(k, j) = \Theta^{k-j} \tag{2-34}$$

$$\Phi(k, j) = \Phi^{k-j} \tag{2-35}$$

Substituting (2-33) and (2-34) in the expressions of F and G, one finds, after some simple manipulation, that these matrices become dependent on the value $k-\theta$ instead of k and θ explicitly. Letting $r = k-\theta$, we summarize the expressions under the steady-state time-invariance assumption in the following.

State Step Failures

$$F(r) = \begin{cases} 0 & k < \theta \\ \displaystyle\sum_{i=0}^{r} \sum_{j=i}^{r} \Theta^{r-j} K H \Phi^{j-i} & k \geq \theta \end{cases} \tag{2-36}$$

$$G(r) = \begin{cases} 0 & k < \theta \\ H \left[\displaystyle\sum_{j=0}^{r} \Phi^{j} - \Phi F(r-1) \right] & k \geq \theta \end{cases} \tag{2-37}$$

State Jump Failures

$$F(r) = \begin{cases} 0 & k < \theta \\ \displaystyle\sum_{j=0}^{r} \Theta^{r-j} K H \Phi^{j} & k \geq \theta \end{cases} \tag{2-38}$$

$$G(r) = \begin{cases} 0 & k < \theta \\ H[\Phi^{r} - \Phi F(r-1)] & k \geq \theta \end{cases} \tag{2-39}$$

Sensor Step Failures

$$F(r) = \begin{cases} 0 & k < \theta \\ \displaystyle\sum_{j=0}^{r} \Theta^{j} K & k \geq \theta \end{cases} \tag{2-40}$$

$$G(r) = \begin{cases} 0 & k < \theta \\ I & k = \\ I - H\Phi F(r-1) & k > \theta \end{cases} \tag{2-41}$$

Sensor Jump Failures

$$F(r) = \begin{cases} 0 & k < \theta \\ \Theta^r K & k \geq \theta \end{cases} \qquad (2\text{-}42)$$

$$G(r) = \begin{cases} 0 & k < \theta \\ I & k = \theta \\ -H\Phi F(r-1) & k > \theta \end{cases} \qquad (2\text{-}43)$$

We further note that under the same assumption the matrix $C(k;\ \theta)$ as defined in 1.2 becomes dependent on r $(r = k-\theta)$:

$$C(r) = \sum_{j=0}^{r} G;(j)V^{-1}G(j) \qquad (2\text{-}44)$$

where V is the steady state covariance of the residual under no failure.

2.2 Static Probabilities

As some measures of performance of a detection system, the probabilities of correct detection (P_D), false alarm (P_F), cross detection (P_{CD}), wrong time (P_{WT}) and time to detection (P_{TD}) are defined as follows:

$$P_D(k,\ \alpha,\ \theta,\ \nu) \triangleq \text{Prob } (\ell(k;\theta) > \varepsilon | \alpha,\ \theta,\ \nu) \qquad (2\text{-}45)$$

$$P_F(k,\ \alpha,\ \theta) \triangleq \text{Prob } (\ell(k;\theta) > \varepsilon | \alpha,\ \theta) \qquad (2\text{-}46)$$

$$P_{TD}(T,\ \alpha,\ \theta,\ \nu) \triangleq \text{Prob } (\ell(k;\theta) > \varepsilon \text{ for some } k \leq T | \alpha,\ \theta,\ \nu) \qquad (2\text{-}47)$$

$$P_{WT}(k,\ \alpha,\ \theta,\ \nu,\ \tilde{\theta}) \triangleq \text{Prob } (\ell(k;\tilde{\theta}) > \varepsilon | \alpha,\ \theta,\ \nu) \qquad (2\text{-}48)$$

where α is the actual type of the failure of size ν and is also the failure type the GLR detector hypothesizes, θ is the true time of failure and $\tilde{\theta}$ is the hypothesized time of failure. Also, we define:

$$P_{CD}\ (k,\ \alpha,\ \beta,\ \theta,\ \nu) \overset{\Delta}{=} \text{Prob}\ (\ell(k;\theta) > \varepsilon | \alpha,\ \beta,\ \theta,\ \nu) \tag{2-49}$$

where α is the failure mode the detector assumes, β is the actual failure mode, θ and ν are the time of failure and the failure vector respectively. We note that

$$P_D\ (k,\ \alpha,\ \theta,\ 0) = P_F\ (k,\ \alpha,\ \theta) \tag{2-50}$$

$$P_{CD}\ (k,\ \alpha,\ \alpha,\ \theta,\ \nu) = P_D\ (k,\ \alpha,\ \theta,\ \nu) \tag{2-51}$$

$$P_{WT}\ (k,\ \alpha,\ \theta,\ \nu,\ \theta) = P_D\ (k,\ \alpha,\ \theta,\ \nu) \tag{2-52}$$

There are many aspects to the evaluation of a detection scheme and a single index is not sufficient to indicate the quality of the scheme. The above defined probabilities are some convenient quantities defined to provide some insights into GLR detector performance. P_D is a measure of detector sensitivity, since it is the probability of detecting a failure when a failure actually occurred. P_F measures the negative quality of the detector, as it is the probability that a failure is signaled when none has developed. Both P_{CD} and P_{WT} are more subtle measures of performance, since they pertain to the ability of the detector to distinguish failures of different types and different failure times respectively. P_{TD} is the probability of the time delay until detection and therefore is a measure of the speed of detection. This quantity is of obvious importance in evaluating detector performance.

Excepting P_{TD}, these probabilities are defined at each point in time assuming no knowledge of the $\ell(k;\ \theta)$ at other times. It is evident that the variable $\ell(k;\ \theta)$ is correlated with the values of $\ell(k_1;\ \theta_1)$ for $k_1 \neq k$ and $\theta_1 \neq \theta$. Thus a better set of performance measures may take this temporal correlation into consideration. P_{TD} is an example of one such measure. The correlation behavior of the ℓ's will be investigated in Chapter 3.

In this chapter, we study the performance of the GLR detectors as measured by the above defined probabilities. The probability density of $\ell(k;\ \theta)$ is shown in sections 2.2.1 and 2.2.2 to be chi squared (χ^2) and gaussian for full GLR and SGLR respectively. As the density is determined, the required probabilities may be computed relatively easily.

2.2.1 Full GLR: χ^2 Probabilities

Consider a detector that hypothesizes a type i failure with failure time = θ while an actual failure ν of type j occurred at θ_t. The actual residuals and GLR outputs then are given by

$$\gamma(k) = \tilde{\gamma}(k) + G_j(k;\ \theta_t)\nu \tag{2-53}$$

$$\begin{aligned} d(k;\ \theta) &= \sum_{s=\theta}^{k} G_i'(s;\theta)V^{-1}(s)\gamma(s) \\ &= \sum_{s=\theta}^{k} G_i'(s;\theta)V^{-1}(s)\,[\gamma(s)+G_j(s;\theta_t)\nu] \end{aligned} \tag{2-54}$$

$$\ell(k;\ \theta) = d'(k;\theta)\ C_{i|i}^{-1}(k;\theta|\theta)d(k;\theta) \tag{2-55}$$

where

$G_j(k;\ \theta)$ is the G matrix corresponding to a type j failure, $\tilde{\gamma}(k)$ is the unbiased white part of the residual, and

$$C_{i|i}(k;\ \theta|\theta) \triangleq \sum_{m=\theta}^{k} G_i'(s;\ \theta)V^{-1}(s)G_i(s;\ \theta) \tag{2-56}$$

Note that $C_{i|i}(k;\ \theta|\theta) = C_i(k;\ \theta)$ of a type i detector.

Since the sensor noise covariance, R, is symmetric and positive definite, $V^{-1}(m)$ is a positive definite symmetric matrix. Therefore, $C_{i|i}(k;\ \theta|\theta)$ is positive semi-definite and symmetric. Then there exists an orthonormal matrix T such that

$$\Lambda_{i|i}(k;\ \theta|\theta) = T^{-1}C_{i|i}(k;\ \theta|\theta)T \tag{2-57}$$

where $\Lambda_{i|i}(k;\ \theta|\theta)$ is a diagonal matrix and the diagonal elements are the eigenvalues $\lambda_1, \lambda_2 \ldots \lambda_n$ of $C_{i|i}(k;\ \theta|\theta)$ (n is the dimension of $C_{i|i}(k;\ \theta|\theta)$). Assuming $C^{-1}_{i|i}(k;\ \theta|\theta)$ exists (we will consider this existence question later), define

$$\begin{aligned} \ell(k;\ \theta) &= \{d'(k;\ \theta)T\}\ \{T^{-1}C_{i|i}(k;\ \theta|\theta\ )T\}\ \{T^{-1}d(k;\ \theta)\} \\ &\triangleq\ v'(k;\ \theta)\Lambda^{-1}_{i|i}(k;\ \theta_1)v'(k;\ \theta_2) \end{aligned} \tag{2-58}$$

Then $v(k;\ \theta)$ is a gaussian random vector:

$$v(k;\ \theta) = T' \sum_{s=\theta}^{k} G_i'(s;\ \theta)V^{-1}(s)\ [\tilde{\gamma}(s) + G_j(s;\ \theta_t)\nu] \tag{2-59}$$

$$E\{v(k;\ \theta)\} = T' \sum_{s=\theta}^{k} G_i'(s;\ \theta)V^{-1}(s)G_j(s;\ \theta_t) \triangleq T'C_{i|j}(k;\ \theta|\theta_t)\nu \tag{2-60}$$

$$E\{v(k;\ \theta)v'(k;\ \theta)\}$$

$$= T'C_{i|i}(k;\ \theta|\theta)T + T'\nu C_{i|j}(k;\ \theta|\theta_t)C'_{i|j}(k;\theta|\theta_t)\nu'T$$

$$= \Lambda_{i|i}(k;\ \theta|\theta) + [E\{v(k;\theta)\}][E\{v(k;\theta)\}]' \qquad (2\text{-}61)$$

Hence $\Lambda_{i|i}(k;\ \theta|\theta)$ is the covariance of $v(k;\ \theta)$. Since $\Lambda_{i|i}(k;\ \theta|\theta)$ is diagonal, elements of $v(k;\ \theta)$ are independent of one another. Also, $\ell(k;\ \theta)$ can be expressed as the summation:

$$\ell(k;\ \theta) = \sum_{s=1}^{n} \frac{v_s^{\,2}(k;\ \theta)}{\lambda_s} \qquad (2\text{-}62)$$

where $v_s(k;\ \theta)$ is the s^{th} component of $v(k;\ \theta)$. Then each term in the above summation is the square of a gaussian random variable with unit variance and mean of $\frac{\bar{v}_s(k;\ \theta)}{\sqrt{\lambda_s}}$ ($\bar{v}_s(k;\theta)$ is the mean of $v_s(k;\ \theta)$). Therefore, $\ell(k;\ \theta)$ is a noncentral χ^2 random variable with n degrees of freedom [10]. The noncentrality parameter (δ^2) can be computed as follows.

$$\delta^2 \triangleq \sum_{s=1}^{n} \frac{\bar{v}_s^{\,2}(k;\ \theta)}{\lambda_s}$$

$$= [E\{v(k;\ \theta)\}]'\ \Lambda_{i|i}^{-1}(k;\ \theta|\theta)\ [E\{v(k;\ \theta)\}]$$

$$= \nu C'_{i|j}(k;\ \theta|\theta_t)\ C_{i|i}^{-1}(k;\ \theta|\theta)\ C_{i|j}(k;\ \theta|\theta_t)\nu \qquad (2\text{-}63)$$

The expected value of $\ell(k;\ \theta)$ is then simply $n + \delta^2$.

Note that no assumption is made on i, j, θ and θ_t. The derivation includes the conditions defining P_D, P_F, P_{CD}, and P_{WT} as special cases as well as others which are not considered presently. For instance, i may be different from j while θ is different from θ_t (but the G's are computed according to the same system and filter). This corresponds to a case of wrong time cross detection and it could be of interest. The associated probability (of wrong time cross detection) may be greater than the probability of correct time cross detection implying that a mismatched failure is more likely to be regarded as a matched failure but at a failure time different from the true one. In any event, $\ell(k;\ \theta)$ is a noncentral χ^2 random variable with n degrees of freedom and a noncentrality parameter δ^2 dependent on the conditions hypothesized.

Specializing to the four cases of current interest, we have,

Correct Detection

$\theta = \theta_t, \quad i = j$

$$\delta^2 = \nu' C_{i|i}(k;\ \theta|\theta)\nu \qquad (2\text{-}64)$$

False alarm

$i = j,\ \nu = 0$

$$\delta^2 = 0 \qquad (2\text{-}65)$$

$\ell(k;\ \theta)$ becomes a central χ^2 random variable

Cross Detection

$i \neq j,\ \theta = \theta_t$

$$\delta^2 = \nu' C'_{i|j}(k;\ \theta|\theta) C^{-1}_{i|i}(k;\ \theta) C_{i|j}(k;\theta|\theta\ )\nu \qquad (2\text{-}66)$$

Wrong time

$i = j \qquad \theta \neq \theta_t$

Note that the different relationships among θ, θ_t, k have different physical interpretations, for instance,

$$\left.\begin{array}{l} k < \theta_t \leq \theta \\ k < \theta \leq \theta_t \\ \theta_t < k < \theta \end{array}\right\} \quad \text{not meaningful}$$

$$\theta \leq k < \theta_t \qquad \text{false alarm}$$

$$\left.\begin{array}{l} \theta_t < \theta \leq k \\ \theta < \theta_t \leq k \end{array}\right\} \quad \text{wrong time} \tag{2-67}$$

then under the wrong time assumption and (2-66),

$$\delta^2 = \nu' C'_{i|i}(k;\ \theta|\theta_t) C_{i|i}(k;\ \theta|\theta) C_{i|i}(k;\ \theta|\theta_t)\nu \tag{2-68}$$

The probabilities, P_D, P_F, P_{CD} and P_{WT} can be computed by simply integrating the chi squared densities with the appropriate degrees of freedom and noncentrality parameters from $\ell = \varepsilon$ to $\ell = +\infty$. There are computer subroutines for computing central χ^2 probability [13]. An algorithm for computing noncentral χ^2 probabilities has been developed and is described in the appendix.

2.2.2 SGLR: Gaussian Probabilities

Consider a simplified GLR detector set to detect a failure ν_0 of type i with failing time θ while a true failure ν of type j actually occured at θ_t. The actual residuals and log likelihood ratios are given by

$$\gamma(k) = \tilde{\gamma}(k) + G_j(k;\ \theta_t)\nu \qquad (2\text{-}69)$$

$$\ell(k;\theta) = \sum_{s=\theta}^{k} [2\gamma(s) - G_i(s;\theta)\nu_0]' V^{-1}(s) G_i(s;\theta)\nu_0$$

$$= \sum_{s=\theta}^{k} 2\tilde{\gamma}'(s) V^{-1}(s) G_i(s;\theta)\nu_0$$

$$+ 2 \sum_{s=\theta}^{k} \nu' G_j'(s;\theta_t) V^{-1}(s) G_i(s;\theta)\nu_0$$

$$- \nu_0' \sum_{s=\theta}^{k} G_i'(s;\theta) V^{-1}(s) G_i(s;\theta)\nu_0$$

$$= \sum_{s=\theta}^{k} 2\ \nu_0'\ G_i'(s;\theta) V^{-1}(s)\tilde{\gamma}(s)$$

$$+ 2\ \nu_0'\ C_{i|j}(k;\ \theta|\theta_t)\nu - \nu_o'\ C_{i|i}(k;\ \theta|\theta)\nu_0 \qquad (2\text{-}70)$$

Since $\tilde{\gamma}(s)$ are zero mean, independent gaussian random vectors, $\ell(k;\theta)$ is a gaussian random variable with mean (m) and variance (σ^2):

$$m = E\{\ell(k;\theta)\} = 2\nu_0' C_{i|j}(k;\ \theta|\theta_t)\nu - \nu_0' C_{i|i}(k;\ \theta|\theta)\nu_0 \qquad (2\text{-}71)$$

$$\sigma^2 = E\{[\ell(k;\theta) - m]^2\}$$

$$= 4\ \nu_0' \left[\sum_{m=\theta}^{k} G_i'(m;\theta) V^{-1}(m) G_i(m;\theta) \right] \nu_0$$

$$= 4\ \nu_0'\ C_{i|i}(k;\theta)\nu_0 \qquad (2\text{-}72)$$

Note that the variance is the same for all cases whereas the mean varies. For the four cases of interest:

Correct Detection

$i = j,\ \theta = \theta_t,\quad \nu = \nu_0$

$$m = \nu_0{}'\ C_{i|i}(k;\ \theta|\theta)\ \nu_0 \qquad (2\text{-}73)$$

False Alarm

$i = j,\ \nu = 0$

$$m = -\nu_0{}'\ C_{i|i}(k;\ \theta|\theta)\ \nu_0 \qquad (2\text{-}74)$$

Cross Detection

$i \neq j,\ \nu \neq \nu_0,\ \theta = \theta_t$

$$m = 2\ \nu_0{}'\ C_{i|j}(k;\ \theta|\theta)\nu - \nu_0{}'\ C_{i|i}(k;\ \theta|\theta)\ \nu_0 \qquad (2\text{-}75)$$

Wrong Time

$i = j,\ \nu = \nu_0,$

$\theta < \theta_t < k$ or

$\theta_t < \theta < k$

$$m = 2\ \nu_0{}'\ C_{i|i}(k;\ \theta|\theta_t)\nu - \nu_0{}'\ C_{i|i}(k;\ \theta|\theta)\nu_0 \qquad (2\text{-}76)$$

Then the desired probabilities can be obtained by integrating the appropriate gaussian densities. This involves the evaluation of error functions.

2.2.3 Discussion

For chi squared densities, the probabilities P_D, P_{CD}, P_{WT}, are increasing functions of the noncentrality parameter δ^2 for a fixed threshold ε (see Figure 15). For SGLR, a similar relation between the probabilities and the mean of ℓ holds. The variance of the likelihood ratio of a SGLR set to detect a particular failure is a constant for all different time failures. Only the mean m varies with the time failures. Hence, both δ^2 and m are (alternative measures of GLR performance.

When considering P_D, δ^2 takes the form $\nu' C(k;\theta)\nu$, where ν is the true failure. Then for any threshold, the probability (P_D) of detecting this failure ν is directly determined by the effect of ν on δ^2. The failure becomes more "detectable" as ν results in a larger δ^2 and consequently a higher P_D. A zero δ^2 will make P_D equal P_F signifying that the detector is unable to tell between the failure and the noise in the system. Hence, a failure ν that occurred at θ is viewed as "undetectable" at time k if $\nu' C(k;\theta)\nu$ (the δ^2 for computing P_D) is zero.

Simplified GLR behaves in a similar manner. In considering P_D for SGLR, σ is $2\sqrt{m}$, and thus $\frac{\sigma}{m} = \frac{\sqrt{m}}{2}$ (which represents effective SGLR signal to noise ratio). Hence, an increase in m will give a larger P_D for any threshold. The only difference is as follows. Here, the mean value (m) of the likelihood ratio takes the same form as δ^2 for full GLR. A failure ν that causes a zero m will make σ zero and consequently $\ell(k;\theta)$ zero deterministically and independent of the residuals. In this case, failure detection is clearly meaningless.

As the C matrix is closely related to the detectability of failures, it is studied in section 2.3 to explore its significance and behavior as functions of k and θ. In section 2.4, the C matrix is examined to determine undetectable failure directions.

After consideration of P_{CD} and P_{WT} similar to that of P_D in the above, we note that the δ^2 associated with these probabilities and consequently the $C_{i|j}$ matrix are crucial factors of the detector's ability to "distinguish" between different failures and failure times. This issue is further explored in section 2.5.

2.3 The Information C Matrix

The C matrix is called the failure information matrix for reasons that will become apparent. In addition to its relation to the probability of correct detection, the C matrix has one other important property. In section 2.3.1, we will show that $C^{-1}(k;\ \theta)$ is the error covariance of the MLE of the true failure assuming that the full GLR detector has determined the true failure type and failure time.

The general time varying situation is too complex for initial analysis. To obtain some basic understanding of the behavior of $C(k;\theta)$ as a function of k and θ, we have turned to the time invariant, steady state situation. In 2.3.2 we will discuss the asymtotic behavior of $C(k-\theta)$.

2.3.1 $C^{-1}(k;\theta)$: the Error Covariance of $\hat{\nu}$

Consider the situation where a full GLR detector has determined the type of a true failure ν and the true failure time θ. Then the MLE of ν is $\hat{\nu}$ as described by equation (1-24).

$$\hat{\nu} = C^{-1}(k;\ \theta)\ d(k;\ \theta) \qquad (2\text{-}77)$$

For easy reference, we repeat the definition of C and d here.

$$C(k;\ \theta) = \sum_{j=\theta}^{k} G'(j;\ \theta)\ V^{-1}(j)G(j;\ \theta) \tag{2-78}$$

$$d(k;\ \theta) = \sum_{j=\theta}^{k} G'(j;\ \theta)V^{-1}(j)\gamma(j) \tag{2-79}$$

where

$$\gamma(j) = \tilde{\gamma}(j) + G(j;\ \theta)\nu \tag{2-80}$$

The actual residual γ contain a zero mean white independent component $\tilde{\gamma}$ with covariance V and a bias $G(j;\ \theta)\nu$ due to the actual failure. We also define

$$\tilde{d}(k;\ \theta) = \sum_{j=\theta}^{k} G'(j;\ \theta)V^{-1}(j)\tilde{\gamma}(j) \tag{2-81}$$

The $\tilde{d}$ is also zero mean, white independent with covariance:

$$E\{\tilde{d}(k;\ \theta)\tilde{d}'(k;\ \theta)\} = C(k;\ \theta) \tag{2-82}$$

Furthermore, we have

$$d(k;\ \theta) = \tilde{d}(k;\ \theta) + C(k;\ \theta)\nu \tag{2-83}$$

We can compute the error covariance of the MLE $\hat{\nu}$ as follows:

$$E\{(\hat{\nu} - \nu)(\hat{\nu} - \nu)'\} = \nu\nu' - E\{\hat{\nu}\nu'\} - E\{\nu\hat{\nu}'\} + E\{\hat{\nu}\hat{\nu}'\} \tag{2-84}$$

Second term on the RHS of (2-82):

$$\begin{aligned} E\{\hat{\nu}\nu'\} &= E\{C^{-1}(k;\theta)d(k;\theta)\nu'\} \\ &= E\{C^{-1}(k;\theta)\ \tilde{d}(k;\theta)\nu' + C^{-1}(k;\theta)C(k;\theta)\nu\nu'\} \\ &= \nu\nu' \end{aligned} \tag{2-85}$$

Third term:

$$E\{\nu\hat{\nu}'\} = E\{\hat{\nu}\nu'\}' = \nu\nu' \qquad (2\text{-}86)$$

Fourth term:

$$E\{\hat{\nu}\hat{\nu}'\} = E\{C^{-1}(k;\theta)d(k;\theta)d'(k;\theta)C^{-1}(k;\theta)\}$$
$$= C^{-1}(k;\theta)E\{[\tilde{d}(k;\theta) + C(k;\theta)\nu][\tilde{d}(k;\theta) + C(k;\theta)\nu]'\}C^{-1}(k;\theta)$$
$$= C^{-1}(k;\theta)\ E\{\tilde{d}(k;\theta)\tilde{d}'(k;\theta) + \tilde{d}(k;\theta)\nu'C(k;\theta)$$
$$+ C(k;\theta)\nu\tilde{d}'(k;\theta) + C(k;\theta)\nu\nu'\ C(k;\theta)\}\ C^{-1}(k;\theta)$$
$$= C^{-1}(k;\theta)\ [C(k;\theta) + C(k;\theta)\nu\nu'\ C(k;\theta)]\ C^{-1}(k;\theta)$$
$$= C^{-1}(k;\theta) + \nu\nu' \qquad (2\text{-}87)$$

Summing up the terms,

$$E\{(\hat{\nu} - \nu)(\hat{\nu} - \nu)'\} = C^{-1}(k;\ \theta) \qquad (2\text{-}88)$$

Under the assumption that the full GLR detector has decided on the true failure type and failure time (θ), we have shown that $C^{-1}(k;\theta)$ is the error covariance of the MLE of the true failure. From the theory of linear algebra [16] we know that both C and C^{-1} have the same eigenvectors and that these eigenvectors are orthogonal to one another since the matrices C and C^{-1} are symmetric. Suppose a failure ν lies in the direction of an eigenvector x_1 of C corresponding to a large eigenvalue λ_1 ($\lambda >> 1$). Then δ^2 ($= \nu'C\nu$) and P_D for this failure is large. The error covariance of the MLE of ν is λ_1^{-1} ($= x_1'C^{-1}x_1$) which is small. The failure direction x_1 is a detectable direction as it can be detected easily as well as estimated accurately. If ν lies in the direction of an eigenvector x_2 with small eigenvalues λ_2 ($\lambda_2 << 1$), the associated δ^2 is small and consequently P_D is close to P_F. In addition, the error covariance of the MLE of ν, λ_2^{-1}, is large. Hence, the failure direction x_2 is "less" detectable. Indeed, the matrix C describes the directional sensitivity of GLR.

We will describe a failure direction as undetectable if it results in a zero δ^2. Thus a direction is detectable if it produces a nonzero δ^2. In order to gain insight into the detection problem, it is appropriate to study the subspace of undetectable directions (which is similar to the concept of unobservable subspace in linear systems). By understanding it one can provide an analytical foundation for questions such as the distinguishability of various failure modes. Note that the set of detectable directions is not a subspace and two directions are totally indistinguishable if they differ by an undetectable direction. With this idea as a foundation, one may be in a position to define a concept of distance between failure directions -- i.e., a measure of the degree of distinguishabiliy (see section 2.5).

For full GLR, we have assumed the existence of C^{-1}. (Otherwise, a pseudo inverse may be used [5].) When C is noninvertible, it must have some zero eigenvalues. If we allow some eigenvalues of an invertible C to approach zero, the noninvertible situation is reached. By a limit argument, the above discussion may be extended to a noninvertible C matrix and hence leads to the notion of undetectable direction mentioned earlier. The relationship between invertibility of C and detectable failure directions is further examined in 2.4.

2.3.2 Asymptotic Behavior of C(r)

In a time invariant system, $C(k;\ \theta)$ becomes dependent on the difference between the true failure time and observation time $(k-\theta)$. For convenience, we let $r = k-\theta$. Furthermore, we assume the associated Kalman filter has reached a steady state. The four different types of detectors are considered separately.

State Jump Detector

$$F(r) = \sum_{j=0}^{r} \Theta^{r-j}\ KH\Phi^{j} \tag{2-89}$$

where $\Theta = [I - KH]\Phi$; K is the steady state Kalman gain, Φ is the system matrix and H is the observation matrix.

Both the system and the filter are assumed to be stable. Then the magnitude of the eigenvalues of Φ and Θ is strictly less than 1, i.e.

$$|\lambda_i(\Phi)| < 1 \qquad i = 1, 2, \ldots n \tag{2-90}$$

$$|\lambda_i(\Theta)| < 1 \qquad i = 1, 2, \ldots n \tag{2-91}$$

where $\lambda_i(\Phi)$ and $\lambda_i(\Theta)$ denote the i^{th} eigenvalues of Φ and Θ respectively and n is the dimension of Φ and Θ. Consider the norm $||\cdot||$ of an nxm matrix A,

$$||A|| = \max_{||x||=1} (x'A'Ax)^{\frac{1}{2}} \tag{2-92}$$

where x is an m-vector.

For a square matrix A with all eigenvalues of magnitudes less than 1, it can be shown that $||A|| < 1$.

For a jump in the state,

$$F(r) = \sum_{j=0}^{r} \Theta^{r-j} KH \Phi^{j} \tag{2-93}$$

$$||F(r)|| \leq \sum_{j=0}^{r} ||\Theta^{r-j} KH \Phi^{j}|| \leq \sum_{j=0}^{r} ||KH|| \rho^{r}$$

$$= ||KH|| \ (r+1) \rho^{r} \tag{2-94}$$

where $\rho = \max \{||\Phi||, \ ||\Theta||\}$. Since $\rho < 1$, there exist an $\alpha > 0$ such that $\rho = e^{-\alpha}$. Then

$$||F(r)|| \leq ||KH|| \ (r+1) e^{-\alpha r} \tag{2-95}$$

The RHS goes to zero as $r \to \infty$. Therefore

$$\lim_{r \to \infty} F(r) = 0 \tag{2-96}$$

Similarly, for G(r),

$$\begin{aligned} ||G(r)|| &= ||H[\Phi^{r} - \Phi F(r-1)]|| \\ &\leq ||H|| \ [||\Phi||^{r} + ||\Phi|| \ ||F(r-1)||] \\ &\leq ||H|| \ [\rho^{r} + ||KH|| \ r\rho^{r}] \end{aligned} \tag{2-97}$$

Hence G(r) also approaches zero as $r \to \infty$. Now consider C(r). Define

$$\Delta C(r,s) \triangleq C(r) - C(s) \qquad , r < s$$

$$= \sum_{j=r+1}^{s} G'(j) V^{-1} G(j) \tag{2-98}$$

$$||\Delta C(r,s)|| \leq \sum_{j=r+1}^{s} ||V^{-1}|| \; ||H||^2 \; [\rho^j + ||KH|| \; j\rho^j]^2$$

$$\leq ||V^{-1}|| \; ||H||^2 \; [(s-r)\rho^r + ||KH|| \; (s-r) \; r\rho^r]^2 \qquad (2\text{-}99)$$

As $r \to \infty$, the terms in the bracket approach 0. Hence

$$\lim_{r \to \infty} ||\Delta C(r, s)|| = 0 \qquad r < s \qquad (2\text{-}100)$$

This shows that $\{C(1), C(2), \ldots C(r) \ldots\}$ is a Cauchy sequence and hence converges to a finite constant matrix. This has the implication that δ^2 in the state jump case will approach a finite limit; as this limit is reached, we are getting no more information about the jump from γ. Thus, waiting further will not improve P_D nor the error in the failure estimate. Therefore, the rate of convergence of $C(r)$ to its limit may be used in determining the length of the detector window (i.e., the value of M for the window: $k-M \leq \theta \leq k-N$).

<u>Step in the state</u>

$$F(r) = \sum_{i=0}^{r} \sum_{j=i}^{r} \Theta^{r-j} \, KH\Phi^{j-i} = \sum_{j=0}^{r} \Theta^{r-j} \sum_{i=0}^{j} KH\Phi^{i}$$

$$= \sum_{j=0}^{r} \Theta^{r-j} \, KH[I-\Phi^{j+1}][I-\Phi]^{-1}$$

$$= \sum_{j=0}^{r} \Theta^{r-j} \, KH[I-\Phi]^{-1} - \sum_{j=0}^{r} \Theta^{r-j} \, KH\Phi^{j+1} \, [I-\Phi]^{-1}$$

$$= [I-\Theta^{r+1}]\ [I-\Theta]^{-1}KH[I-\Phi]^{-1} - [\sum_{j=0}^{\ell} \Theta^{r-j}KH\Phi^{j}]\Phi[I-\Phi]^{-1} \qquad (2\text{-}101)$$

As $r \to \infty$, the first term becomes $[I-\Theta]^{-1}KH[I-\Phi]^{-1}$ and the second goes to 0 following the reasoning for the state jump case. $[I-\Theta]^{-1}$ and $[I-\Phi]^{-1}$ exist because $|\lambda_i(\Theta)| < 1$, $|\lambda_i(\Phi)| < 1$ for $i = 1, 2, \ldots n$.

$$G(r) = H\ [\sum_{j=0}^{r} \Phi^{r-j} - \Phi F(r-1)]$$

$$= H[I-\Phi^{r+1}]\ [I-\Phi]^{-1} - H\Phi F(r-1) \qquad (2\text{-}102)$$

As $r \to \infty$, the first term becomes $H[I-\Phi]^{-1}$ and the second, $H\Phi[I-\Theta]^{-1}KH[I-\Phi]^{-1}$. Hence $G(r)$ reaches a constant, $H\{I-\Phi[I-\Theta]^{-1}\}[I-\Phi]^{-1}$ as $r \to \infty$. $G'(j)V^{-1}G(j)$ is positive semi-definite and attains a steady state value $G'(\infty)V^{-1}G(\infty)$. Thus at least some eigenvalues of $C(r)$ grow as r increases indicating that some failure vectors will cause a growing δ^2. Therefore, an actual failure lying in the direction of an eigenvector of $G'(\infty)V^{-1}G(\infty)$ with a nonzero eigenvalue will cause P_D to approach 1 as the waiting time (r) increases.

Jump in Sensor

$$\lim_{r \to \infty} F(r) = \lim_{r \to \infty} \Phi^{r}R = 0 \qquad (2\text{-}103)$$

$$\lim_{r \to \infty} G(r) = \lim_{r \to \infty} -H\Phi F(r-1) = 0 \qquad (2\text{-}104)$$

Hence sensor jump C(r) behaves much like that of state jump failures: C(r) approaches constant as r approaches ∞.

Step in Sensor

$$\lim_{r\to\infty} F(r) = \lim_{r\to\infty} \sum_{j=0}^{r} \Phi^j K$$
$$= [I-\Phi]^{-1}K \qquad (2\text{-}105)$$

$$\lim_{r\to\infty} G(r) = \lim_{r\to\infty} [I-H\Phi F(r-1)]$$
$$= I-H\Phi[F\Theta]^{-1}K \qquad (2\text{-}106)$$

Therefore, the sensor step C(r) behaves in a manner similar to that of the state step C(r) as r approaches ∞.

2.4 Undetectable Failure Directions

One moment's reflection upon the physical meaning of an undetectable failure leads us to believe that a failure is not detectable if it cannot be observed by the system's sensors. Therefore, state failures that lie in the unobservable subspace are not detectable and all sensor failures are detectable as they have direct effects on the sensor outputs. From the discussion in 2.3.1, all failure directions are detectable if C is positive definite (i.e., invertible). Now we will determine the condition for C to be invertible under the time invariant steady state assumption.

The positive semi-definite synmetric matrix C(r) may be rewritten as

$$G(r) = [G'(0) \vdots G'(1) \vdots \cdots \vdots G'(r)] \begin{bmatrix} V^{-1} & & & 0 \\ & V^{-1} & & \\ & & \ddots & \\ 0 & & & V^{-1} \end{bmatrix} \begin{bmatrix} G(0) \\ \cdots \\ G(1) \\ \cdots \\ \vdots \\ \cdots \\ G(r) \end{bmatrix}$$

$$\triangleq \mathcal{G}(r)\, \mathcal{V}\mathcal{G}(r) \qquad (2\text{-}107)$$

Since V^{-1} is positive definite, $\mathcal{V}$ is also positive definite. It is clear that $C(r)$ is positive definite if and only if the null space of $\mathcal{G}(r)$ is $\{0\}$. We will examine $\mathcal{G}(r)$ for the four different failure types separately.

<u>State Jumps</u>

After some manipulation of the expression of $G(r)$ (2-39), we can write $\mathcal{G}(r)$ as a product of two matrices:

$$\mathcal{G}(r) = \begin{bmatrix} I & & & 0 \\ & \ddots & & \\ -H\Phi K & & I & \\ \vdots & & \vdots & \ddots \\ -H\Phi\Theta^{r-1}K & & -H\Phi\Theta^{r-2}K & \cdots & I \end{bmatrix} \begin{bmatrix} H \\ \cdots \\ H\Phi \\ \cdots \\ \vdots \\ \cdots \\ H\Phi^{r} \end{bmatrix}$$

$$\triangleq A_1(r)\, B(r) \qquad (2\text{-}108)$$

$A_1(r)$ is a lower triangular matrix with identity blocks on the diagonal and is thus of full rank. On the other hand, the null space of B(r) is the null space of $\mathcal{G}(r)$. We note that B'(r) B(r) is the observability gaussian. Therefore, C(r) is positive definite if the system is observable in r steps. Since the unobservable subspace in r steps is the null space of B(r) and $\mathcal{G}(r)$, the observable subspace coincides with the "undetectable subspace". One other property of system observability maybe applied here, i.e., if an n dimensional constant system is not completely observable in n-1 steps, it will never be completely observable. Then if C(r) of an n dimensional system is not invertible for r = n-1, it will not be invertible for r > n-1. Hence, after waiting n-1 steps (r = n-1), the undetectable subspace becomes constant. However, C(r) may be noninvertible for $r < r_0$ but invertible for $r > r_0$ and $r_0 \leq n-1$. These last two properties of state jump detection may be used to determine the value of N in the window $k-M \leq \theta \leq k-N$.

That is, if a jump ν is in the nullspace of C(r-1) but $C(r)\nu \neq 0$, we will certainly take $N \geq r$ when looking for a jump of this type. In SGLR and CGLR (where we prespecify failure direction), we will set different window constraints for different jump directions, depending upon their observability. In full GLR, the failure direction (as well as magnitude) is determined on-line. One obvious design concept for full GLR is to choose N large enough so that C(N) is invertible (assuming the system is observable). If, because of a desire to detect certain failures more quickly, we choose N such that det $C(N) = 0$, we must take care in utilizing the full GLR equations given earlier. That is, we must replace C^{-1} by a pseudo-inverse and should take care in making sure that $\hat{\nu}$ lies in the orthogonal complement of the nullspace of C (as it will if we use the standard Penrose inverse).

State Steps

$$G(r) = \begin{bmatrix} I & & & 0 \\ & \ddots & & \\ I-H\Phi K & & I & \\ \vdots & & \vdots & \ddots \\ I-H\Phi\sum_{j=0}^{r-1}\Theta^j K & & I-H\Phi\sum_{j=0}^{r-2}\Theta^2 K & \cdots \quad I \end{bmatrix} \begin{bmatrix} H \\ \cdots \\ H\Phi \\ \cdots \\ \vdots \\ \cdots \\ H\Phi^r \end{bmatrix}$$

$$\triangleq A_2(r)\ B(r) \qquad (2\text{-}109)$$

$A_r(r)$ is of full rank. $B(r)$ is identical with that of the state jump cases. Therefore, similar comments applies here also.

Sensor Jumps and Steps

Upon examining the expressions of $G(r)$ (2-40) and (2-41) of these two failure types, we find that $G(0) = I$ for both types. Consequently, the null space of $G(r)$ is $\{0\}$ implying that $C(r)$ is always invertible for sensor failures.

The above discussion on state jump, senior jump and sensor step failures maybe extended easily to the general time varying cases. It is clear that in the state jump case, observability has to be considered for a time varying system and the value of r for which $C(k; k-r)$ is invertible will generally vary with time k.

For state steps in an n dimensional time varying systems, the invertibility of $C(k; \theta)$ is related to the observability of an augmented system. An n dimensional system with state step failures (2-110) (2-111) can be represented by a 2n dimensional system with state jump failures (2-112) (2-113).

<u>An n dimensional system with state step failure</u>

$$x(k+1) = \Phi(k)\ x(k) + \nu\sigma_{k+1,\theta} \tag{2-110}$$

$$z(k) = H(k)\ x(k) \tag{2-111}$$

<u>An augmented system with state jump failure</u>

$$\begin{bmatrix} x(k+1) \\ b \end{bmatrix} = \begin{bmatrix} \Phi(k) & I \\ 0 & I \end{bmatrix} \cdot \begin{bmatrix} x(k+1) \\ b \end{bmatrix} + \begin{bmatrix} 0 \\ \nu \end{bmatrix} \delta_{k+1,\ \Theta-1} \tag{2-112}$$

$$z(k) = [H(k) \quad 0] \begin{bmatrix} x(k) \\ b \end{bmatrix} \tag{2-113}$$

Recall that in the state jump case, a failure direction is undetectable if it lies in the null space of the corresponding C matrix. Then, jump failures of the form as described in (2-112) i.e., $[0, \nu]'$, are detectable if they do not lie in the null space of the state jump C matrix of the augmented system. Therefore, we deduce that the state step C matrix of the original system is invertible if the null space of the state jump C matrix and hence the unobservable subspace of the augmented system does not contain directions of the form $[0, \nu]'$.

Furthermore, we have made the following observation. Suppose all state jumps in the original system are detectable. Then if some state jumps in the augmented system are not detectable, we know either or both of the following are true:

1. At least some state steps in the original system are undetectable.

2. Certain state steps cannot be distinguished from jumps in the original system, e.g., consider the 2 dimensional system

$$x(k+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(k) \tag{2-114}$$

$$x(k) = [1 \quad 0] \; x(k) \tag{2-115}$$

This system is observable hence all state jumps are detectable. But a jump in the second state is indistinguishable from a step in the first. A check of the observability grammian of the augmented system shows that the augmented system is not completely observable i.e., some state jumps here are undetectable.

2.5 Distinguishability of Failures

Recall that when the likelihood ratio crosses the threshold, a failure is declared: in full GLR, a failure (size and direction unknown) of the type hypothesized is declared while in SGLR, a failure of the hypothesized type and direction (size unknown) is declared. However, there are many possible causes of the likelihood ratio's exceeding the threshold e.g., a noise spike, an actual failure of the hypothesized type or another type of failure. Therefore, failures other than the ones hypothesized by the detector can be mistaken as the hypothesized failures. The sensitivity of a particular detector to other types of failures makes it difficult to distinguish between various failure modes. In this section, we will make a first attempt to consider this problem analytically in order to provide basis for detector design and reliability analysis.

Consider a SGLR detector that is set to detect a failure in the direction f_1 of type i. Then the probability of detecting f_1 when it occurs is P_D. In the event that another failure in direction f_2 of type j occurs, the probability that the SGLR declares a failure in the direction f_1 of type i is P_{CD}. Clearly the quantity P_{CD} provides a measure of how distinguishable various failure modes and directions are. If $P_{CD} \leq P_F$, the modes are distinguishable. If $P_{CD} \geq P_D$ the modes are correlated and the size of P_{CD} is a measure of the degree of the indistinguishability between the failure modes.

Since full GLR is sensitive to failures in all directions of the specified type, we should only consider the distinguishability of different failure types. Suppose we have a type i full GLR and a type j failure of size f occurs. Since full GLR will choose the most likely failure direction Let us find the direction ν of type i failure for which SGLR has the highest P_{CD} under the failure f. Then this highest P_{CD} is a measure of how distinguishable the type of failure of size f is to full GLR of type i. To obtain this quantity, we need to maximize P_{CD} over all directions ν. Thus it is clear that full GLR will have more complicated distinguishability problems than SGLR. The analytical study of these problems is beyond the scope of this thesis. But we will consider SGLR as it should provide insights into the problems in full GLR.

Although the foregoing discussion concerns the cross detection situation, the same reasoning is applicable to wrong time detection where P_{WT} is the measure of distinguishability between true and hypothesized failure times. A more general situation is the wrong time cross detection. This is also an important case. The physical interpretation of the situation is that a particular failure mode may not look much like another one occurring at the same time, but it may be highly correlated with the other mode started at a different time (e.g. ,$\sin(\omega_i t+\theta)$ is uncorrelated with $\cos \omega_i t$ for $\theta=0$ but they are highly correlated for $\theta=\pi/2$). In the following, this general case is considered and the wrong time and cross detection cases may be regarded as specialized results of this general one.

Now consider the situation where a SGLR (D1) is set to detect a type i failure ν_1. From equations (2-71) and (2-72), the mean $(m_{1|1})$ and variance (σ_1^2) of $\ell(k;\theta_1)$ of D1 when ν_1 occurs at θ_1 are given by

$$m_{1/1} = \nu_1{}'C_{i/i}(k;\theta_1/\theta_1)\nu_1 \tag{2-116}$$

$$\sigma_1^2 = 4\ m_{1/1} \tag{2-117}$$

In the event that another failure ν_2 of type j occurring at θ_2, the variance of $\ell(k;\theta_1)$ of D1 is still σ_1^2 while the mean is $m_{1|2}$.

$$m_{1/2} = 2\nu_1{}'\ C_{i/j}(k;\theta_1/\theta_2)\nu_2 - m_{1/1} \tag{2-118}$$

We define

$$m_{1,2} \overset{\Delta}{=} \nu_1{}'C_{i/j}(k;\theta_1/\theta_2)\ \nu_2 \tag{2-119}$$

Then it follows that

$$m_{1,2} = m_{2,1} \tag{2-120}$$

$$m_{1/1}' = m_{1,1} \tag{2-121}$$

For a fixed threshold, P_{CD} is an increasing function of $m_{1/2}$, or equivalently, $m_{1,2}$ (as $m_{1/1}$ is fixed for D1). Therefore, $m_{1,2}$ is a key (wrong time) cross detection parameter and a measure of distinguishability.

From (2-60), we have

$$C_{i/j}(k;\theta_1/\theta_2) = \sum_{s=\theta_1}^{k} G_i'(s;\theta_1)\ V^{-1}(s)\ G_j(s;\theta_2) \tag{2-122}$$

Since for $k < \theta$ and $i = 1,2,3,4$,

$$G_i(k;\theta) = 0 \tag{2-123}$$

equation (2-122) may be expressed as

$$C_{i/j}(k;\theta_1/\theta_2) = \sum_{s=\theta}^{k} G_i(s;\theta_1)\ V^{-1}(s)\ G_j(s;\theta_2) \tag{2-124}$$

where

$$\theta = \max\{\theta_1, \theta_2\} \tag{2-125}$$

Then we can express $m_{1,2}$ as

$$m_{1,2} = \nu_1'[G_i'(\theta;\theta_1) \vdots G_i'(\theta+;\theta_1) \vdots \cdots \vdots G_i'(k;\theta_1)] \begin{bmatrix} V^{-1}(\theta) & & 0 \\ & V^{-1}(\theta+1) & \\ & & \ddots \\ 0 & & V^{-1}(k) \end{bmatrix} \text{x}$$

$$\begin{bmatrix} G_j(\theta;\theta_2) \\ \cdots\cdots\cdots \\ G_j(\theta+1;\theta_2) \\ \cdots\cdots\cdots \\ \vdots \\ G_j(k;\theta_2) \end{bmatrix} \nu_2$$

$$\triangleq \nu_1' \mathcal{G}_i'(k;\theta;\theta_1) \mathcal{V}(k;\theta)\ \mathcal{G}_j(k;\theta;\theta_2)\ \nu_2 \tag{2-126}$$

Since $V(k;\theta)$ is a positive definite matrix, $m_{1,2}$ is in fact an inner product:

$$m_{1,2} \triangleq \langle G_i(k;\theta;\theta_1)\nu_1, G_j(k;\theta;\theta_2)\nu_2 \rangle_{V(k;\theta)} \quad (2\text{-}127)$$

To simplify notations, the explicit arguments of G_i and G_j are suppressed in the following. Hence

$$m_{1,2} \triangleq \langle G_i\nu_1, G_j\nu_2 \rangle_{V(k;\theta)} \quad (2\text{-}128)$$

Now suppose that a second SGLR (D2) is set up to detect the failure ν_2. When ν_2 occurs at θ_2, the mean ($m_{2/2}$) and variance (σ_2^2) of $\ell(k;\theta_2)$ of D2 are:

$$m_{2/2} = \nu_2' C_{j/j}(k;\theta_2/\theta_2)\ \nu_2 \quad (2\text{-}129)$$

$$\sigma_2^2 = 4\ m_{2/2} \quad (2\text{-}130)$$

If in reality ν_1 occurs at θ, instead of ν_2 at θ_2, the variance of $\ell(k;\theta_2)$ of D2 is still σ_2^2 but the mean is $m_{2/1}$:

$$m_{2/1} = 2\nu_2' C_{j/i}(k;\theta_2|\theta_1)\nu_1 - m_{2/2} \quad (2\text{-}131)$$

If we assume that we design the magnitude of the assumed failures ν_1 and ν_2 so that

$$m_{1/1} = m_{2/2} \quad (2\text{-}132)$$

we than have equivalently set the P_0's of both SGLR (D1 and D2) to be equal. Physically, this means that at time k, D1 is as sensitive to ν_1 that occurred at θ_1 as D2 is sensitive to ν_2 that occurred at θ_2. From the Schwartz inequality, we have

$$<G_i\nu_i, G_j\nu_2>^2_{V(k;\theta)} \leq <G_i\nu_1, G_i\nu_1>_{V(k;\theta)} <G_j\nu_2, G_j\nu_2>_{V(k;\theta)} \tag{2-133}$$

A closer examination of $m_{1/1}$ is appropriate here. By equations (2-121) and (2-127), we have

$$m_{1/1} = <G_i\nu_1, G_i\nu_1>_{V(k;\theta_1)} \tag{2-134}$$

If $\theta_1 = \theta$, then

$$m_{1/1} = <G_i\nu_1, G_i\nu_i>_{V(k;\theta)} \tag{2-135}$$

Suppose $\theta_1 < \theta$, then

$$m_{1/1} = \nu_1 [\sum_{s=\theta}^{\theta-1} G_i'(s;\theta_1)V(s)^{-1}G_i(s;\theta_1) + \sum_{s=\theta}^{k} G_i'(s;\theta_1)V(s)^{-1}G_i(s;\theta_1)]\nu_1 \tag{2.136}$$

The second term in the bracket is actually $<G_i\nu_1, G_i\nu_1>_{V(k;\theta)}$. Since the first term in the bracket is a positive semi-definite matrix,

$$m_{1/1} \geq <G_i\nu_1, G_i\nu_1>_{V(k;\theta)} \tag{2-137}$$

Both (2-135) and (2-137) are summarized in (2-137).

Then we have

$$m_{1,2} \le m_{1/1} \tag{2-138}$$

$$m_{1,2} \le m_{2/2} \tag{2-139}$$

Now we have arrived at the following result. For a fixed threshold ε that is common to both D1 and D2, the failures ν_1 at θ_1 and ν_2 at θ_2 resulting in the same mean value for the ℓ of their matched detectors (d1 and D2 respectively) will give the same P_D. Furthermore, the wrong-time, cross-detection probabilities of both detectors (with respect to the above failures) are not greater than P_D. From another point of view, this result -- i.e., the cross correlation (2-127) provides a measure of the degree of indistinguishability between failures (and failure times) under the assumption (2-134). This result may be extended to include cases of more than two detectors and provides a first step for the development of method for designing "mutually distinguishable" detectors.

It is desirable for the mutually distinguishable detectors to have the same P_D and P_F for all times. This means that we can detect smaller failures in the "more sensitive" directions as well as we can detect larger failures of "less sensitive" types and directions. We note that the sensitivity of SGLR to different type and size failures varies with time (depending upon the shape of the $G\nu$'s). In order to maintain the constancy of P_D and P_F over a period of time, we may have to scale the ν's of the SGLR detectors in a time-varying manner. This can be done by normalizing the G's appropriately. This problem is an interesting one for future research. In addition, once mutually distinguishable directions are dtermined, one will probably find it more useful to use constrained GLR tan full GLR. This question is also of interest for the future.

CHAPTER 3

Correlation Studies

In the previous chapter, we have derived some properties of the GLR failure detection system by studying the statistical properties of $\ell(k; \theta)$. The likelihood ratio $\ell(k; \theta)$ was treated as a static variable, i.e. it was regarded as a function of the fixed time parameters k and θ. The ℓ's at different k's and θ's are clearly correlated. Hence the study of the correlation is important in the understanding of the behavior of the ℓ's with respect to k and θ. This will clearly be important in the development of a reliable detection rule. In full GLR, $\ell(k; \theta)$ is a χ^2 random variable and the correlation study of such variables is difficult. However, SGLR provides a very manageable situation as the ℓ's are gaussian random variables. Thus we will focus mainly on the correlation of the ℓ's in SGLR. In section 3.1, we will derive the covariance function. Some additional probabilities such as P_{TD} and their computations are discussed in 3.2. As the behavior of the ℓ's as functions of k and θ are known, possible modified decision rules for GLR failure detection systems are discussed in section 3.3.

3.1 The Covariance of $\ell(k; \theta)$

Consider two likelihood ratios of a SGLR detector at different k's and θ's under a certain failure condition. By equation (2-69), these ℓ's have the following expressions:

$$\ell_1(k;\ \theta_1) = 2\nu_0' \sum_{s=\theta_1}^{k_1} G_i'(s;\ \theta_1)V^{-1}(s)\tilde{\gamma}(s) + m_1(k_1,\theta_1,\theta_t,\nu,j) \tag{3-1}$$

$$\ell_2(k_2;\theta_2) = 2\nu_0' \sum_{t=\theta_2}^{k_2} G_i'(t;\theta_2)V^{-1}(t)\tilde{\gamma}(t) + m_2(k_2,\theta_2,\theta_t,\nu,j) \tag{3-2}$$

where $\tilde{\gamma}$ is the zero mean, independent, white gaussian sequence, θ_t is the true failure time of the type j failure ν, and the m's are the expected values of the ℓ's as given by (2-71).

The cross covariance (R_{12}) of ℓ_1 and ℓ_2 can be simply computed as:

$$R_{12} = E\{(\ell_1 - m_1)(\ell_2 - m_2)'\}$$

$$= 4\nu_0' E\left\{\sum_{s=\theta_1}^{k_1} \sum_{t=\theta_2}^{k_2} G_i'(s;\theta_1)V^{-1}(s)\tilde{\gamma}(s)\tilde{\gamma}'(t)V^{-1}(t)G_i(t;\theta_2)\right\}\nu_0$$

$$= 4\nu_0' \left[\sum_{s=\theta}^{k} G_i'(s;\theta)V^{-1}(s)G_i(s;\theta)\right]\nu_0$$

$$= 4\nu_0' C_{i|i}(k;\ \theta|\theta)\nu_0 \tag{3-3}$$

where

$$k = \min\ \{k_1,\ k_2\} \tag{3-4}$$

$$\theta = \max\ \{\theta_1,\ \theta_2\} \tag{3-5}$$

The third equality in (3-3) is a result of the whiteness of $\tilde{\gamma}$ and the fact that $G(s;\ \theta) = 0$ for $s < \theta$. Note that $R_{12} = R_{21}$ and $R_{12} \geq 0$. R_{12} is zero if ν_0 lies in the null space of $C_{i|i}(k;\ \theta|\theta)$, i.e. if ν_0

is not observable on the interval $[k;\ \theta]$. Therefore, $\ell_1(k_1;\ \theta_1)$ and $\ell_2(k_2;\ \theta_2)$ are not correlated if ν_0 is not observable on the common interval $[k;\ \theta]$ with k and θ defined by (3-4) and (3-5).

Since the density function of a gaussian random vector is determined by its mean and covariance, the joint density function of the likelihood ratios at different k's and θ's under the same failure condition may be constructed as in the following. We define

$$L \triangleq [\ell_1(k_1;\ \theta_1),\ \ell_2(k_2;\ \theta_2),\ \ldots,\ \ell_n(k_n;\ \theta_n)]' \quad (3\text{-}6)$$

$$m \triangleq E\{L\} \quad (3\text{-}7)$$

$$R \triangleq E\ (L-m)(L-m)' \quad (3\text{-}8)$$

The elements of m can be computed using (2-71). The matrix R is symmetric with the diagonal elements as the variance (σ^2) of the ℓ's and the off diagonal elements as the cross covariances of the ℓ's. The variances and cross covariances can be determined using (2-72) and (3-3). Then the probability density function of L is

$$p(L) = (2\pi)^{-\frac{n}{2}}(\det R)^{-\frac{1}{2}} \exp[-\tfrac{1}{2}\ (L-m)'R^{-1}(L-m)] \quad (3\text{-}9)$$

An assumption on the failure determines the expected values of the ℓ's. Therefore, under such an assumption, the statistical properties of the likelihood ratios of a SGLR detector are determined by their joint density (3-9).

3.2 Some Additional Probabilities

With the joint density function of the likelihood ratios, we are now able to compute probabilities such as P_{TD}. Recall the definitions of P_D and P_{TD}.

$$P_D(k, \alpha, \theta, \nu) \triangleq \text{Prob } (\ell(k;\ \theta) > \varepsilon | \alpha, \theta, \nu) \tag{3-10}$$

$$P_{TD}(T, \alpha, \theta, \nu) \triangleq \text{Prob } (\ell(k;\theta) > \varepsilon \text{ for some } k \leq T | \alpha, \theta, \nu) \tag{3-11}$$

where α denotes the failure type the GLR detector hypothesizes (which is the actual failure type here where we are considering delay in correct detection), ν is the true failure, and θ is the true failure time. To simplify notations, we will suppress the dependence of the probabilities on α and ν. Thus we have

$$P_D(k, \theta) = P_D(k, \alpha, \theta, \nu) \tag{3-12}$$

$$P_{TD}(T,\theta) = P_{TD}(T, \alpha, \theta, \nu) \tag{3-13}$$

In the case where the parameter N of the detector window, $k-M \leq \theta \leq k-N$, is nonzero, decision concerning whether a failure has occurred at the hypothesized failure time θ cannot be made until time $\theta + N$ has been reached. With this observation and definitions (3-10) and (3-11), we find that

$$P_{TD}(\theta+N, \theta) = P_D(\theta+N, \theta) \tag{3-14}$$

$$P_{TD}(T+1, \theta) = P_{TD}(T,\theta) + \Delta P_{TD}(T,\theta) \qquad T \geq \theta+N \tag{3-15}$$

where

$$\Delta P_{TD}(T,\theta) = \text{Prob } (\ell(k;\theta) < \varepsilon \text{ for } \theta \leq k \leq T \text{ and } \ell(T+1;\ \theta) > \varepsilon) \tag{3-16}$$

The computation of P_D has already been discussed in section 2.2. The quantity $\Delta P_{TD}(T,\theta)$ is the integral of the joint density of $\ell_0(\theta+N,\theta)$, $\ell(\theta+N+1,\ \theta)$, ..., $\ell_r(T,\ \theta)$, $\ell_{r+1}(T+1,\theta)$ as follows ($r = T-N$):

$$\Delta P_{TD}(T,\theta) = \int_{\varepsilon}^{\infty}\int_{-\infty}^{\varepsilon}\cdots\int_{-\infty}^{\varepsilon}\int_{-\infty}^{\varepsilon} p(\ell_0,\ell_1,\ldots,\ell_r,\ell_{r+1})\,d\ell_0 d\ell_1\ldots d\ell_r d\ell_{r+1} \tag{3-17}$$

The joint density p for SGLR can be derived using the method described in the last section. Hence the probability of time to detection (P_{TD}) for SGLR can be computed using equations (3-14), (3-15), and (3-17).

The expression (3-17) is generally very difficult to evaluate even for the simplest nontrivial case:

$$\Delta P_{TD}(\theta+N,\ \theta) = \int_{\varepsilon}^{\infty}\int_{-\infty}^{\varepsilon} p(\ell_0,\ \ell_1)\,d\ell_0 d\ell_1 \tag{3-18}$$

However, intuition may be developed by examining the behavior of the integral. In this case, we define

$$P(\ell_1 > \varepsilon|\ell_0) \triangleq \int_{\varepsilon}^{\infty} p(\ell_1|\ell_0)\,d\ell_1 \tag{3-19}$$

Then (3-18) can be written as

$$\Delta P_{TD}(\theta+N,\theta) = \int_{-\infty}^{\varepsilon} p(\ell_0)\int_{\varepsilon}^{\infty} p(\ell_1|\ell_0)\,d\ell_1 d\ell_0$$

$$= \int_{-\infty}^{\varepsilon} p(\ell_0)P(\ell_1 > \varepsilon|\ell_0)\,d\ell_0 \tag{3-20}$$

Note that the conditional density $p(\ell_1|\ell_0)$ has a constant variance and ℓ_0 and ℓ_1 have fixed means. Therefore, for a fixed ε, the probability $P(\ell_1 > \varepsilon|\ell_0)$ is an increasing function of the conditional expectation of ℓ_1, i.e. $E(\ell_1|\ell_0)$. Since ℓ_0 and ℓ_1 are always nonnegatively correlated ($R_{01} \geq 0$), $E(\ell_1|\ell_0)$ increases with ℓ_0 and we have

$$\Delta P_{TD}(\theta+N,\theta) \leq P(\ell_1 > \varepsilon|\ell_0 = \varepsilon)P(\ell_0 < \varepsilon) \tag{3-21}$$

In the case of the correct detection of the same failure, we would indeed expect a ℓ_1 that has a larger ℓ_0 to be greater than the ℓ_1 that has a smaller ℓ_0.

Thus, in order to use this correlation analysis for detailed system analysis, one will need to develop approximation methods. Arguments such as those above should be useful in obtaining bounds. Specifically, the shape of $E(\ell_1|\ell_0)$ as a function of ℓ_0 and the more complex version $E(\ell_i|\ell_0,\ldots, \ell_{i-1})$ should be the crucial factor in this analysis.

From the definition (3-11), $P_{TD}(T,\theta)$ is the probability of declaring a failure before or at time T when a failure ν of the hypothesized type occurred at θ. This probability becomes a measure of false alarm rate if we allow $\nu = 0$. Under this assumption, the quantity P_{TD} is the probability of a false alarm being signalled before or at T. We call this the probability of false alarm in an interval (P_{FI}) and it is defined by

$$P_{FI}(k, \alpha, \theta) = P_{TD}(k, \alpha, \theta, 0) \tag{3-22}$$

Note that for SGLR, the covariance of the ℓ's under no failure is the same as under a failure. Only their means are different and these can be computed via (2-70).

From (3-15), it is clear that the probability of declaring a failure when one actually occurred (P_{TD}) increases as the observation is continued. By the same token, in the case of no failure, the probability of declaring a false alarm (P_{FI}) also increases as more observations are made. Hence, P_{TD} and P_{FI} represent a pair of tradeoff factors in GLR design, especially in choosing the threshold ε and the window size.

Similar to P_{TD}, another useful probability can be defined - the probability of detection over the window P_{DW}:

$$P_{DW}(k,\ M,\ N,\ \alpha,\ \theta,\ \nu) \triangleq \text{Prob}(\ell(k;\tilde{\theta}) > \varepsilon \text{ for some } \tilde{\theta} \text{ s.t. } k-M \leq \tilde{\theta} \leq k-N \mid \alpha,\ \theta < k,\ \nu) \qquad (3\text{-}23)$$

where α, θ and ν denote the same quantities as in the definition of P_{TD} (3-11). Where P_{TD} dealt with fixed θ and variable k, P_{DW} deals with fixed k and variable θ, reflecting the fact that signals at times other than θ may be important. Here M and N are the parameters defining the data window: $k-M \leq \tilde{\theta} \leq k-N$ of the GLR detector. Recall the definition of P_{WT}:

$$P_{WT}(k,\ \alpha,\ \theta,\ \nu,\ \tilde{\theta}) = \text{Prob}\ (\ell(k;\tilde{\theta}) > \varepsilon \mid \alpha,\ \theta,\ \nu) \qquad (3\text{-}24)$$

We will suppress the arguments α, θ, ν and N of the probabilities.

Hence,

$$P_{DW}(k, M) = P_{DW}(k, M, N, \alpha, \theta, \nu) \qquad (3\text{-}25)$$

$$P_{WT}(k, \tilde{\theta}) = P_{WT}(k, \alpha, \theta, \nu, \tilde{\theta}) \qquad (3\text{-}26)$$

Then P_{DW} can be computed in a manner similar to that of P_{TD} as follows.

$$P_{DW}(k, k-N) = P_{WT}(k, k-N) \qquad (3\text{-}27)$$

$$P_{DW}(k, T-1) = P_{DW}(k, T) + \Delta P_{DW}(k, T) \qquad k-M < T \leq k-N \qquad (3\text{-}28)$$

$P_{WT}(k, k-N)$ can be calculated using methods described in section 2.2. The quantity $\Delta P_{DW}(k, T)$ is the integral of the joint density of $\ell_0(k, k-N), \ell_1(k, k-N-1), \ldots, \ell_r(k, T), \ell_{r+1}(k, T-1)$

$$\Delta P_{DW}(k,T) = \int_{\varepsilon}^{\infty}\int_{-\infty}^{\varepsilon} \cdots \int_{-\infty}^{\varepsilon}\int_{-\infty}^{\varepsilon} p(\ell_0,\ell_1,\ldots\ell_r,\ell_{r+1})\, d\ell_0 d\ell_1 \ldots d\ell_r d\ell_{r+1} \qquad (3\text{-}29)$$

where $r = k - T - N$.

For a nonzero failure ν, P_{DW} is the probability that the detector will declare a failure in the window and therefore, is a measure of the detector's sensitivity to failures. A more interesting situation is when no failure has occurred and, in this case, we define the probability of false alarm over the window (P_{FW}):

$$P_{FW}(k, M, N, \alpha) \stackrel{\Delta}{=} P_{DW}(k, M, N, \alpha, \theta, 0) \qquad (3\text{-}30)$$

Then P_{FW} is a measure of false alarm rate over the window. Since (3-26) implies that both P_{DW} and P_{FW} increase with larger windows, the size of P_{FW} is an additional consideration in setting a detector window size.

3.3 Other Possible Decision Rules

The decision rule used so far in GLR design is the comparison of a single likelihood ratio with a constant threshold. As we have studied the ℓ's and have a better understanding of their behavior, we are able to exploit this knowledge to construct other decision rules that improve certain performance criteria such as the reduction of false alarm rate. Depending on the system and failures under consideration, many different decision rules are possible. To illustrate the idea, one such possibility is discussed here.

Consider a GLR detector and the associated ℓ's with the same θ. Suppose a matched detectable failure occurred at θ. These ℓ's (i.e., $\ell(k;\theta)$ for $k \geq \theta$) are now expected to be larger than in the case of no failure as the mean values of these ℓ's are larger. But in the case of a noise disturbance at θ without a failure, $\ell(\theta;\theta)$ may be large while subsequent ℓ's will become small again as their mean values are indeed small, i.e. the effect of a burst of noise will be localized in time, while the effect of a failure will persist in time. With this qualitative insight, we can devise the following "interval decision" rule. A failure is declared if K_1 ($K_1 \leq T$) or more of the ℓ's of the set $\{\ell(k_0;\theta) \mid \theta \text{ is fixed}, k-T \leq k_0 \leq k\}$ exceed the threshold. This decision rule will specify a set of possible failure times (θ's). The failure time is determined by the $\hat{\theta}$ in the set of θ's that has the largest value of $\ell(k_0;\hat{\theta})$ ($k-T \leq k_0 \leq k$).

By implementing the interval decision rule, we hopefully have reduced the false alarm rate. However, the speed of detection is decreased as the detector must wait at least K_1 observations after the occurrence of the failure before it can detect it.

Associated with this decision rule, the probability of correct detection (P_D^I) and false alarm (P_F^I) can be defined as measures of the performance of the modified GLR detection scheme.

$$P_D^I(k,\ T,\ K_1,\ \theta,\ \alpha,\ \theta_t,\ \nu)$$

$$\triangleq \text{Prob } (K_1 \text{ of the } \ell\text{'s in } \{\ell(k_0;\theta)\,|\,\theta \text{ fixed, } k-T \leq k_0 \leq k\} \text{ exceeds } \varepsilon\,|\,K_1 \leq T,\ \alpha,\ \theta_t,\ \nu) \qquad (3\text{-}31)$$

$$P_F^I(k,\ T,\ K_1,\ \theta,\ \alpha) \triangleq P_D^I(k,\ T,\ K_1,\ \theta,\ \alpha,\ \theta_t,\ 0) \qquad (3\text{-}32)$$

where θ_t is the true failing time, and α and ν denote quantities as defined in section 3.2.

It is clear that the calculation of these probabilities require integrating the joint density of the ℓ's. The actual computations involved are very complex. A general formula for such computations is impractical and is not pursued here. Intuitively, one would expect P_F^I to fall off rather rapidly as a function of K_1, while the effect of a persistent failure ($G(k;\theta)$ not decaying too rapidly) will probably only make P_D^I a modestly decreasing function of K_1.

Similar to the interval decision, a window decision rule could be devised - here we consider the ℓ's in a window at one given time k in the same manner as we consider the ℓ's in an interval ($k-T \leq k_0 \leq k$) for fixed θ. Therefore, the window decision will declare a failure if

K_2 of the ℓ's in the window at time k exceed the threshold. The failure time might then be chosen as that value of θ in the window with the largest ℓ. The performance of the detection system using this rule can also be evaluated by computing the corresponding probabilities of correct detection and false alarm.

It is clear that many modifications of the decision rule are now possible. The resulting performance may be evaluated with the corresponding probabilities of correct detection, false alarm, cross detection, etc. as in the original detection system. The probability computations generally require the joint density of the ℓ's. Such a density is difficult to obtain for full GLR. But in SGLR, the joint density can be obtained via the method discussed in section 3.1.

CHAPTER 4

A Numerical Example and Conclusions

4.1 A Simplified Aircraft Model

In order to gain some practical insights into the nature of the GLR detection scheme, we proceed to examine GLR failure detections for a second order dynamical system. The simplified longitudinal dynamics of an aircraft as examined in [8] is the subject of the numerical studies. The pitch rate (q, in radians/sec) and angle of attack (α, in radians) are considered to be the two states constituting the linearized longitudinal dynamics. Furthermore, we assume a sensor for each state and the dynamics and sensor outputs are assumed to be affected by additive white noise. After appropriate discretization (sampling period of 1/32 second), we have obtained the following model.

$$\begin{bmatrix} q(k+1) \\ \alpha(k+1) \end{bmatrix} = \begin{bmatrix} 0.9826 & -0.1465 \\ 0.0306 & 0.9179 \end{bmatrix} \begin{bmatrix} q(k) \\ \alpha(k) \end{bmatrix} + \begin{bmatrix} 0.0226 & 0 \\ 0.0043 & 0.0002 \end{bmatrix} w(k) \tag{4-1}$$

$$\begin{bmatrix} Z_1(k) \\ Z_2(k) \end{bmatrix} = \begin{bmatrix} 1.000 & 0 \\ 0 & 16.15 \end{bmatrix} \begin{bmatrix} q(k) \\ \alpha(k) \end{bmatrix} + \begin{bmatrix} 0.0087 & 0 \\ 0 & 0.06 \end{bmatrix} v(k) \tag{4-2}$$

where w and v are zero mean, independent, white gaussian sequences with unit covariance. The steady state gain (K) and the inverse of the residual covariance (V^{-1}) of the associated KBF have been computed to be:

$$K = \begin{bmatrix} 0.7535 & 0.0463 \\ 0.1353 & 0.0128 \end{bmatrix} \tag{4-3}$$

$$V^{-1} = \begin{bmatrix} 3234.6 & -607.0 \\ -607.0 & 220.6 \end{bmatrix} \tag{4-4}$$

The system (4-1) has eigenvalues of 0.977 ± 0.0667 and hence is stable.

To facilitate the numerical studies, we have developed a Fortran computer package consisting of the Multiple Detector Simulation Program (MDSP) and routines for computing different types of probabilities (e.g., P_D, P_F, etc.). The MDSP is used to compute all the detector matrices (G's, C's, etc.) and to simulate the full GLR detection mechanism for all four basic types of failures in a time-invariant system with KBF at steady state. (MDSP is fully documented in [8]. The documentation of the remaining routines will appear in an Electronic Systems Laboratory research report in the near future.) This computer package is, therefore, capable of providing analytical data (e.g., the probabilities) as well as simulation results.

To illustrate some of the issues brought forth in this thesis, we will discuss the analytical data generated for the simplified aircraft model. This data is presented here in graphical form in Figures 2 to 14.

In Figures 2 to 9, the elements of the G and C^{-1} matrices of each type of failure are plotted against the elapsed time r ($r = k-\theta$). It is

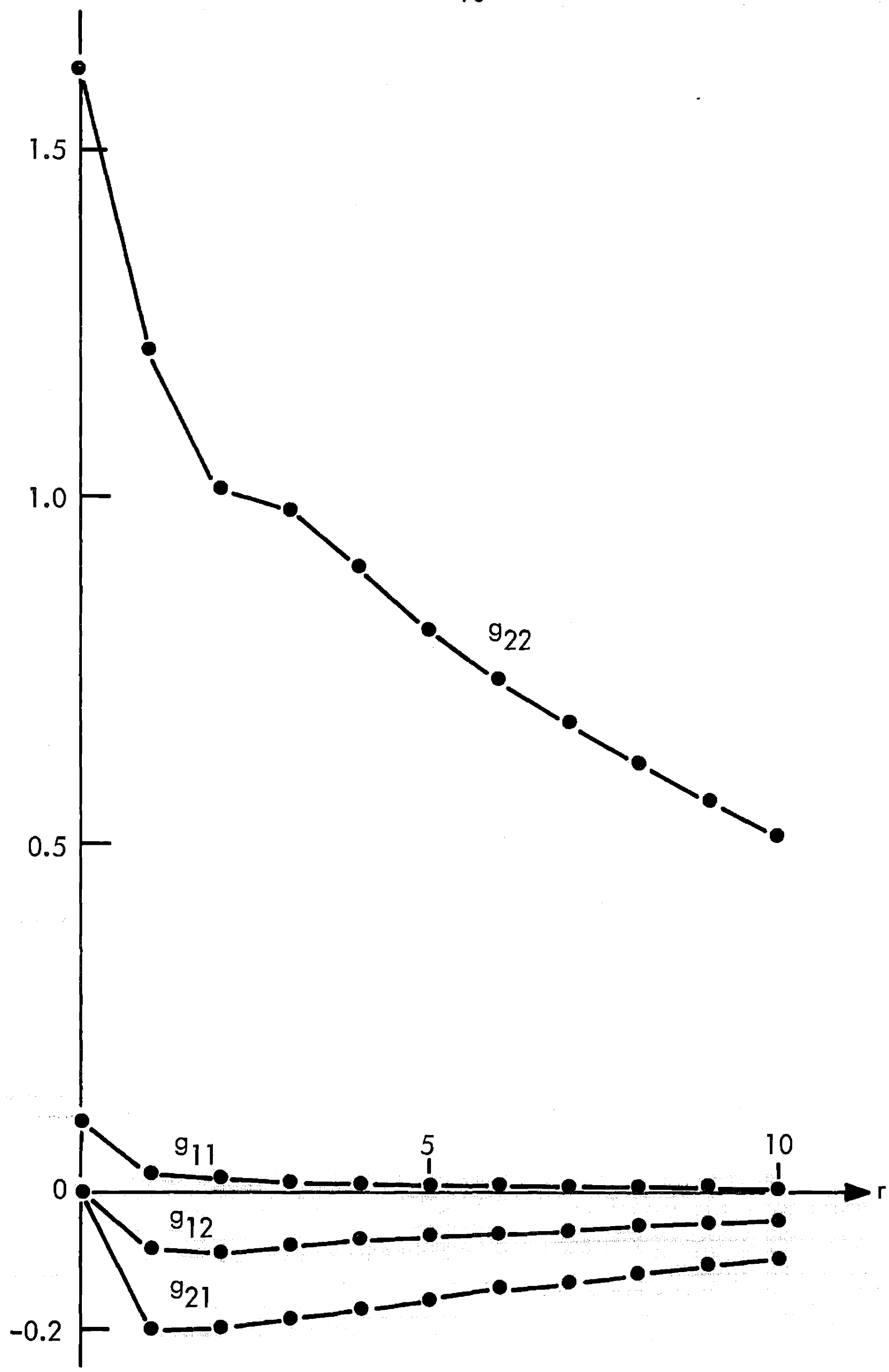

Fig. 2 State Jump G(r)

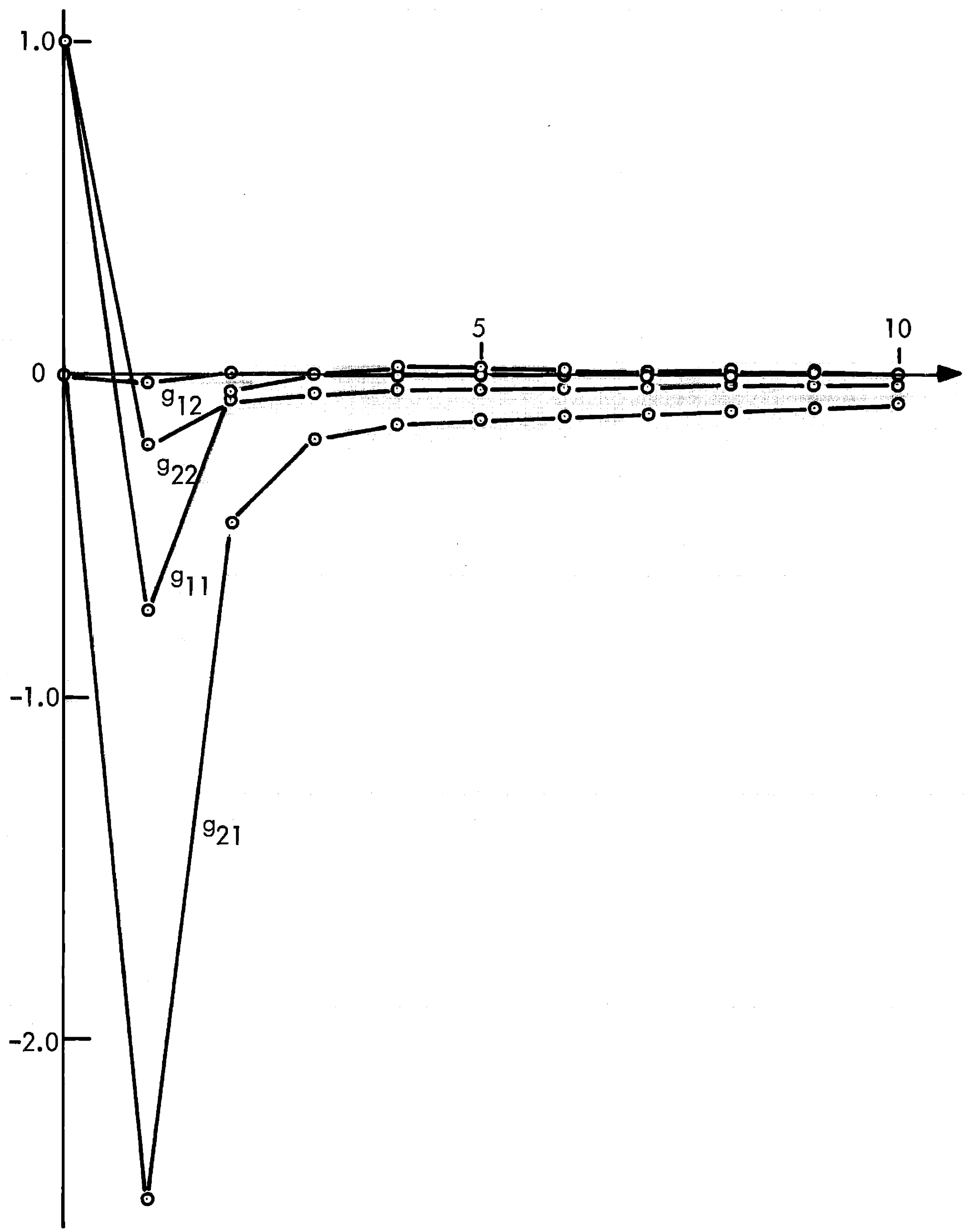

Fig. 3 Sensor Jump G(r)

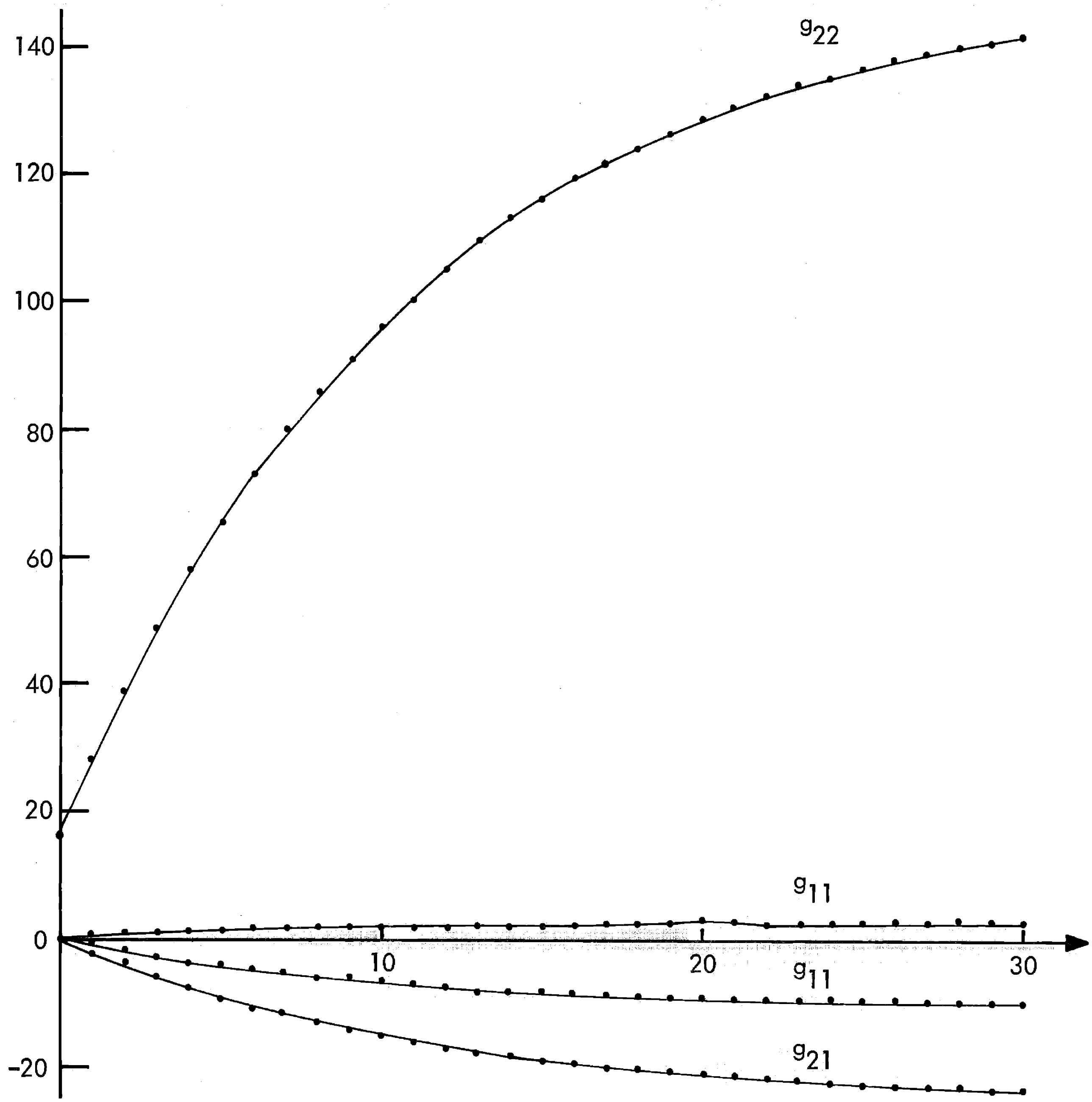

Fig. 4 State Step G(r)

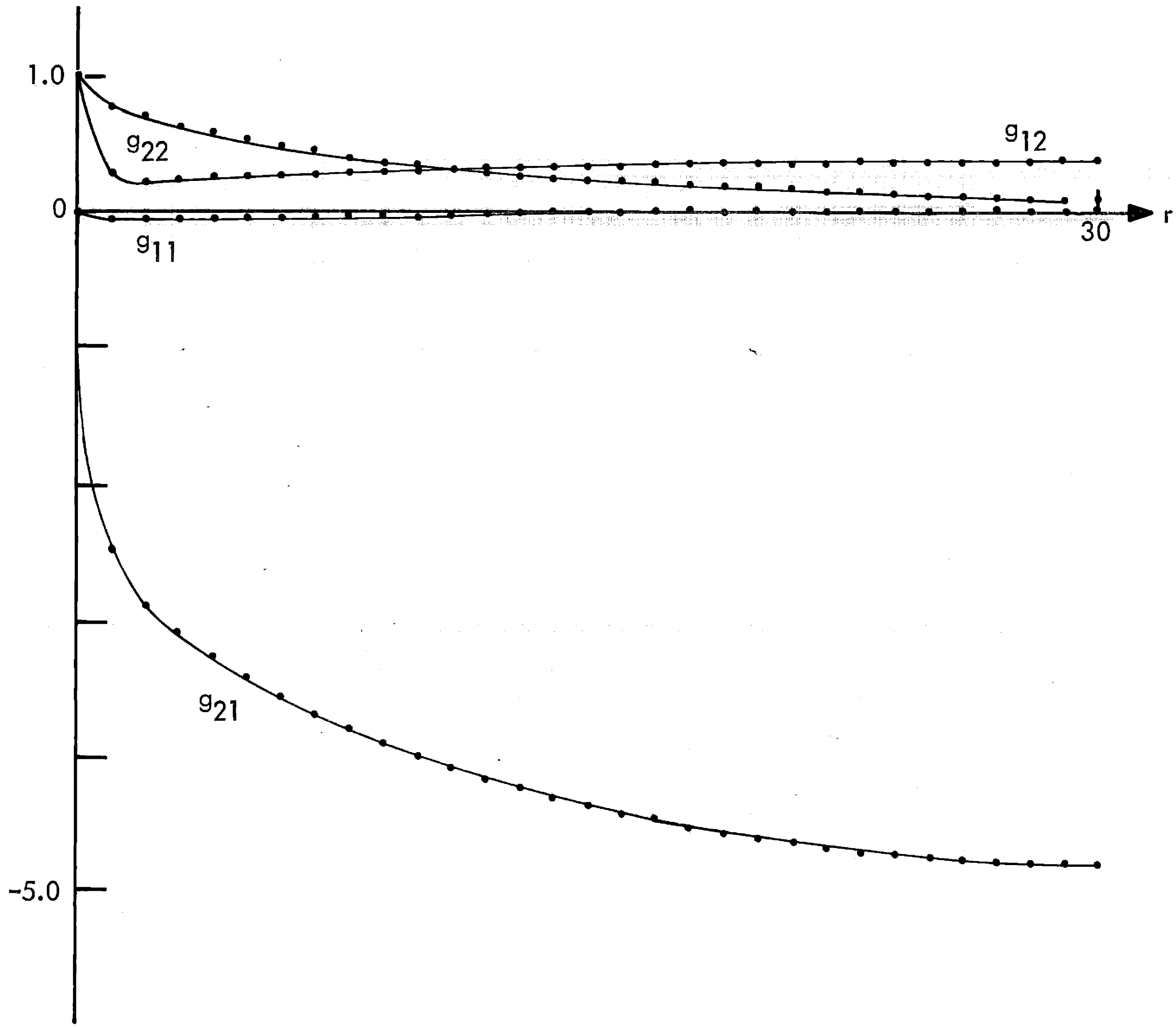

Fig. 5 Sensor Step G(r)

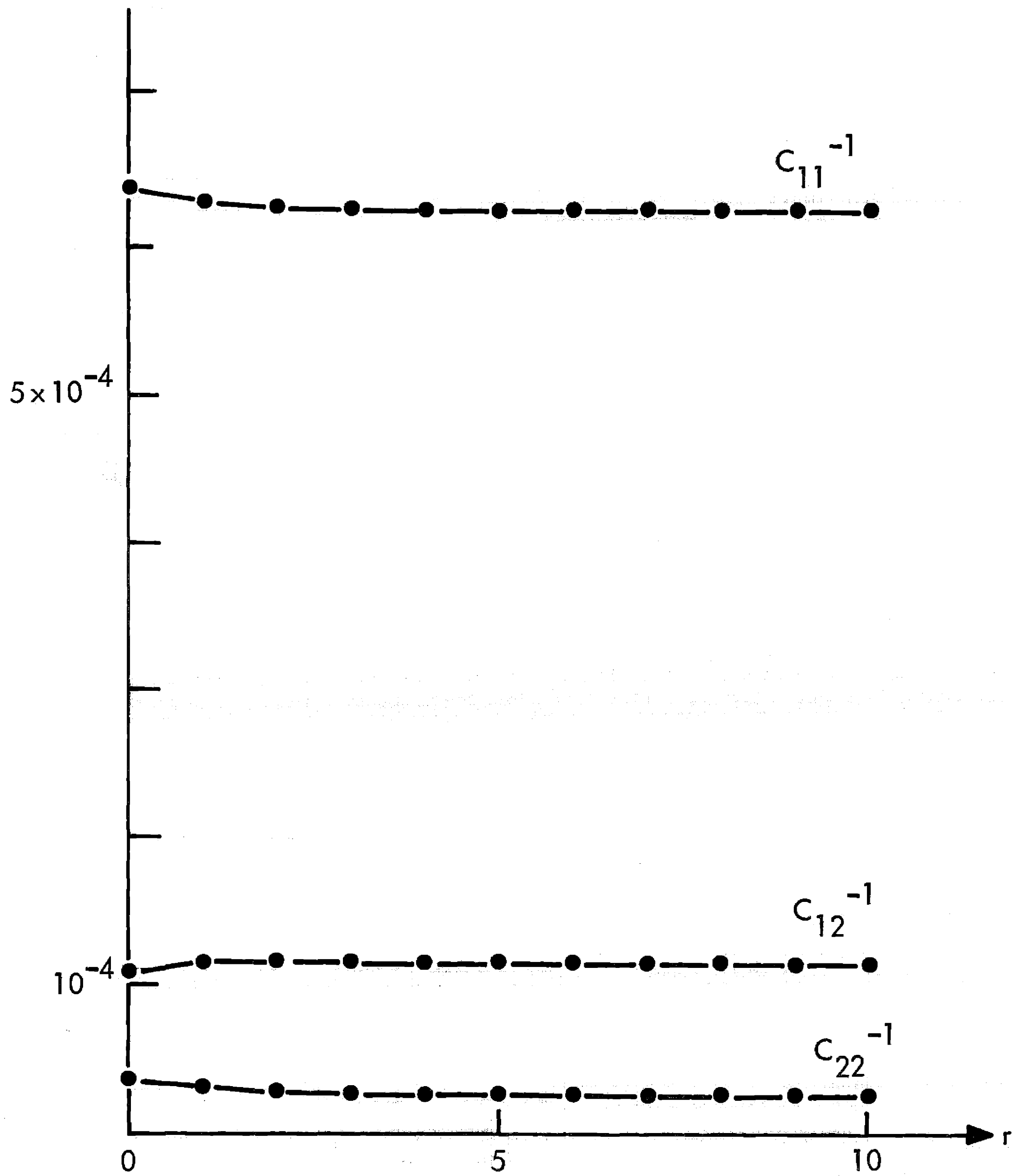

Fig. 6 State Jump $C^{-1}(r)$

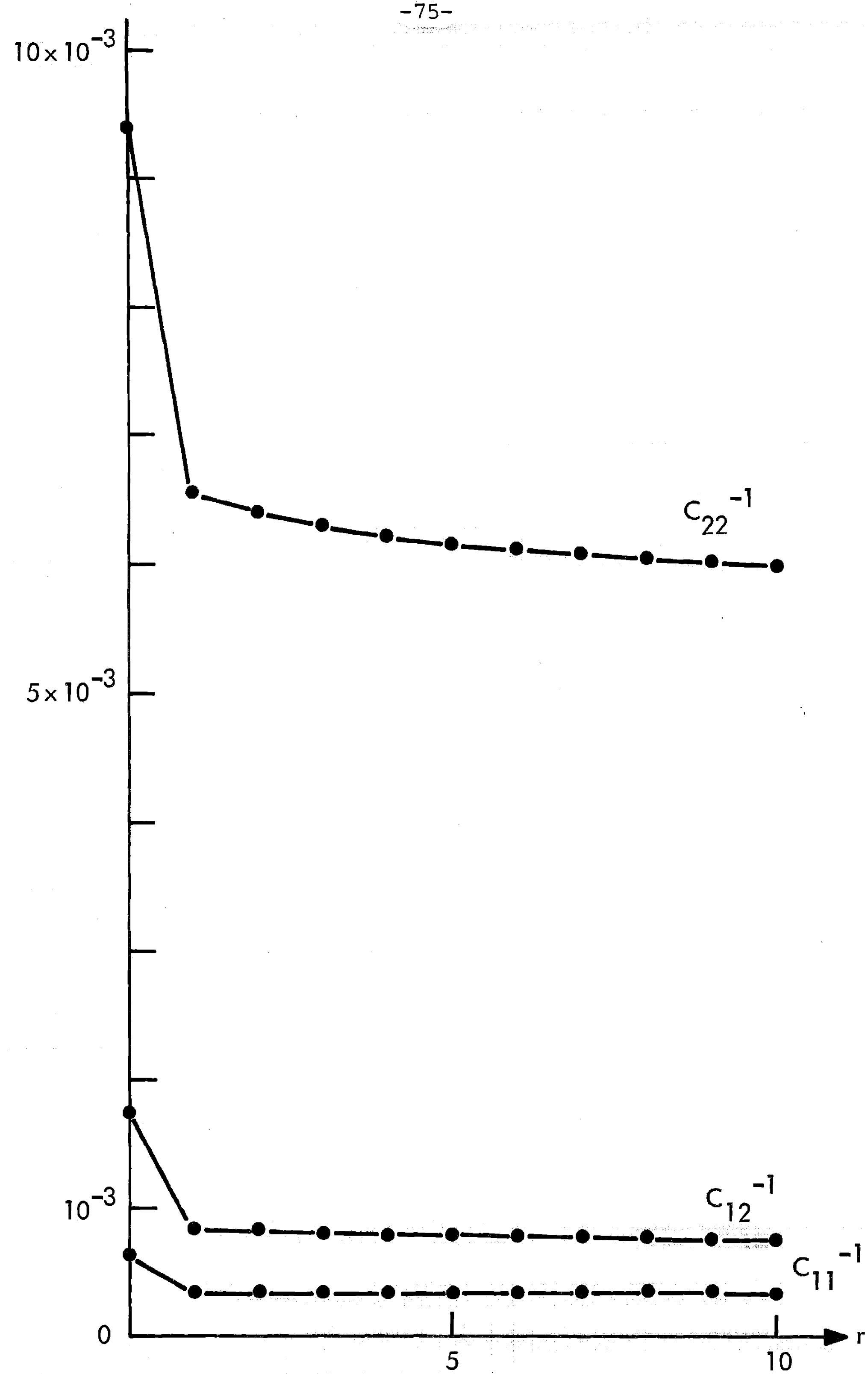

Fig. 7 Sensor Jump $C^{-1}(r)$

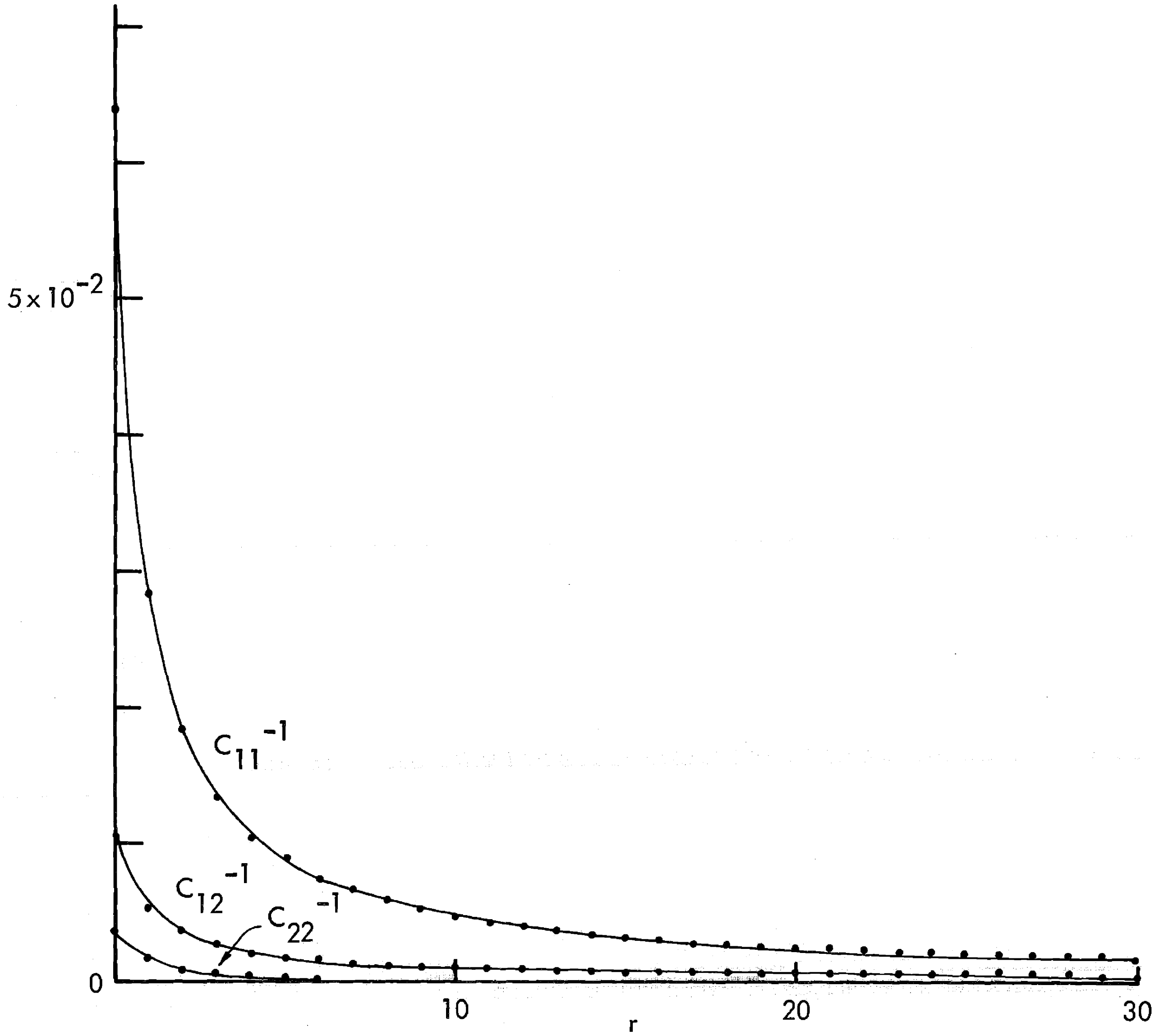

Fig. 8 State Step $C^{-1}(r)$

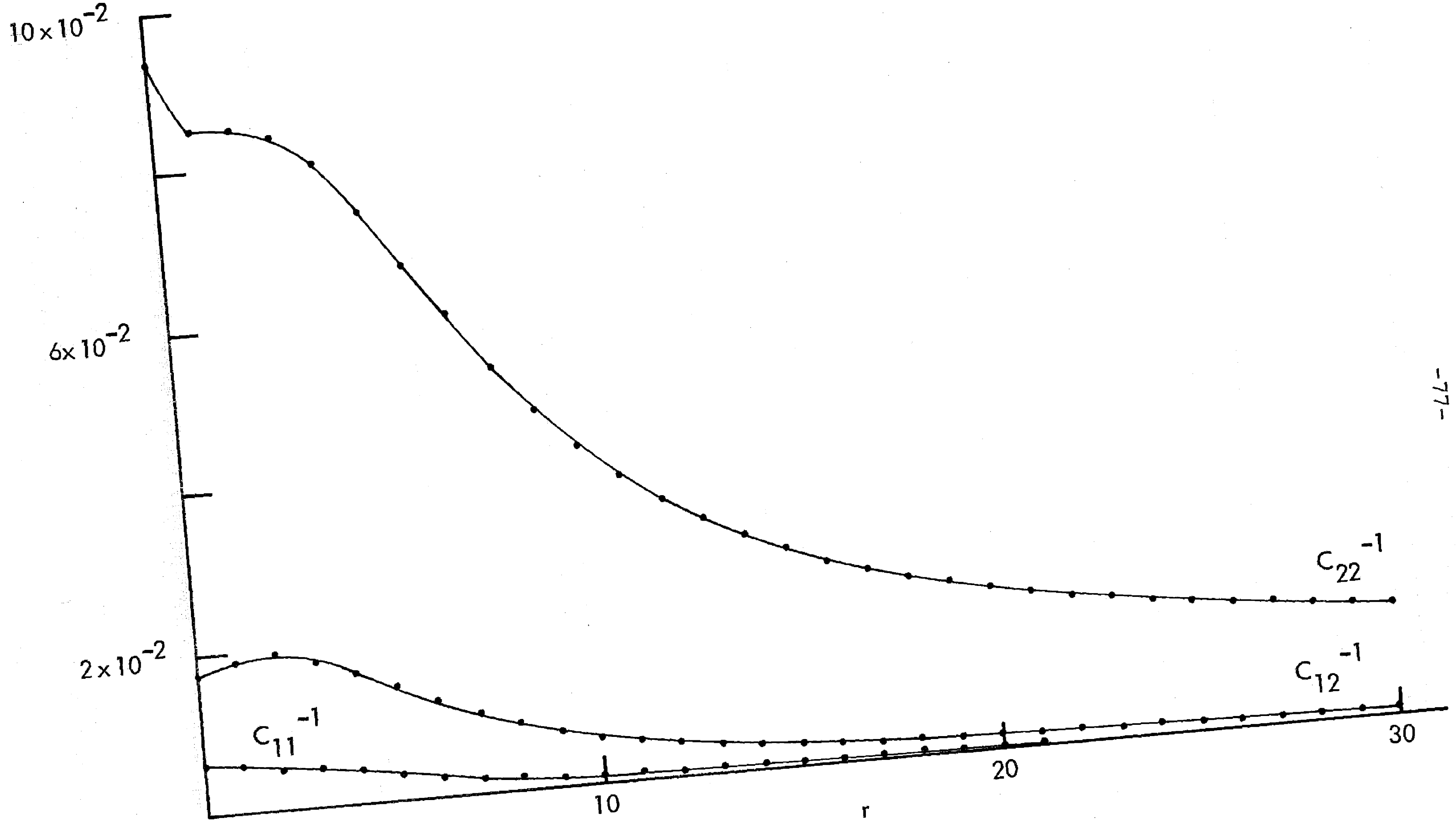

Fig. 9 Sensor Step $C^{-1}(r)$

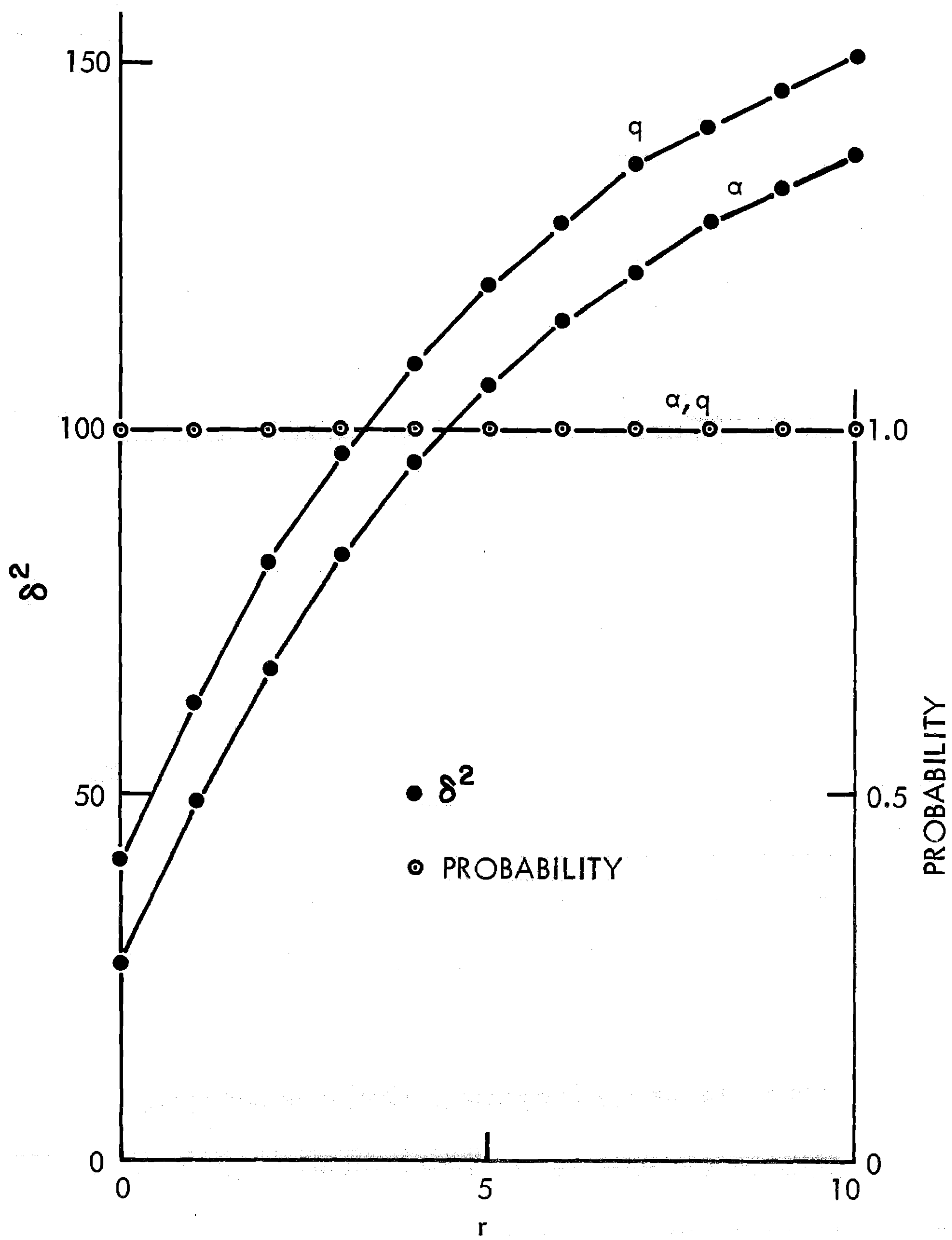

Fig. 10 5σ State Jump δ^2 and P_D with $\epsilon = 5$

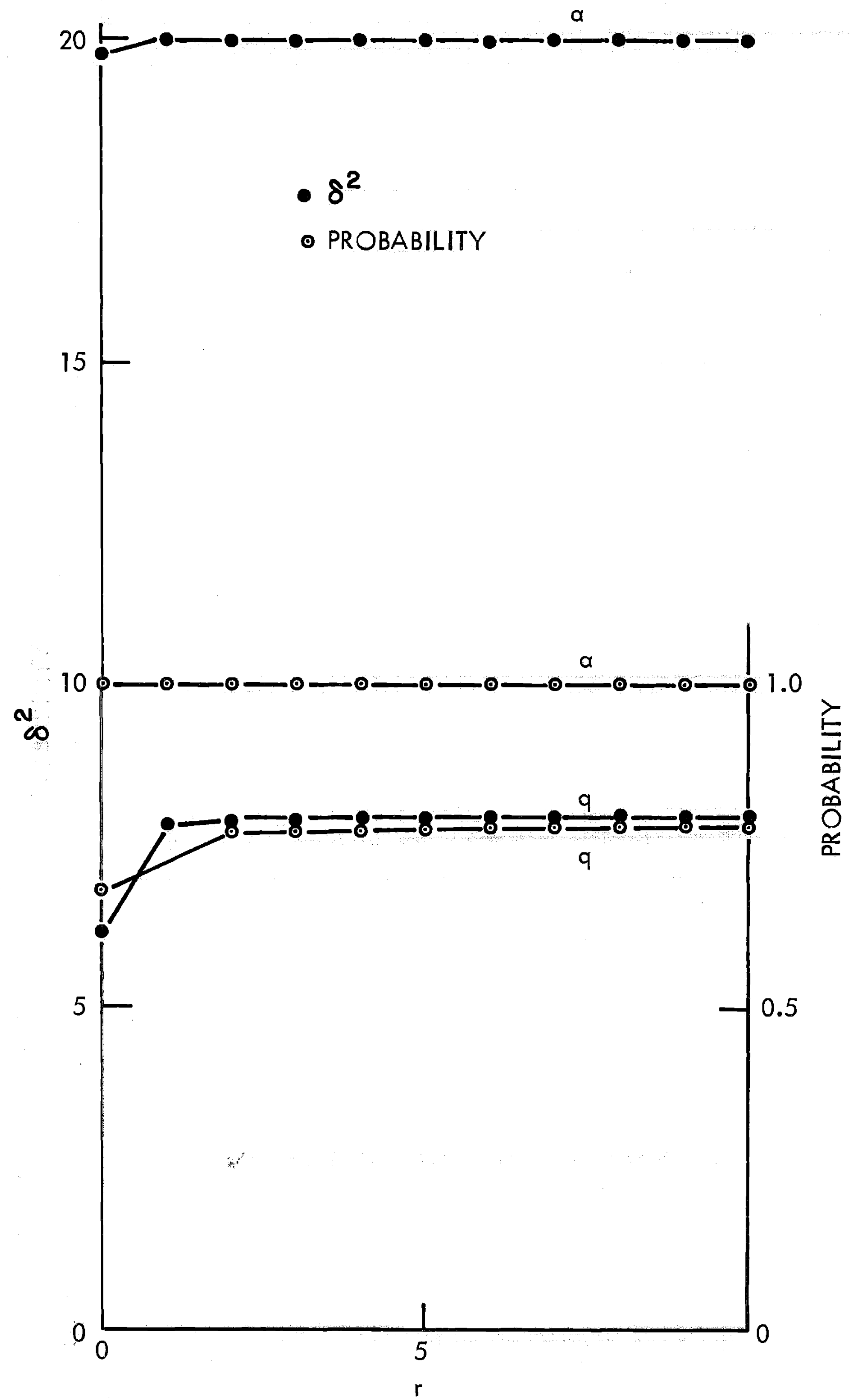

Fig. 11 5σ Sensor Jump δ^2 and P_D with $\epsilon = 5$

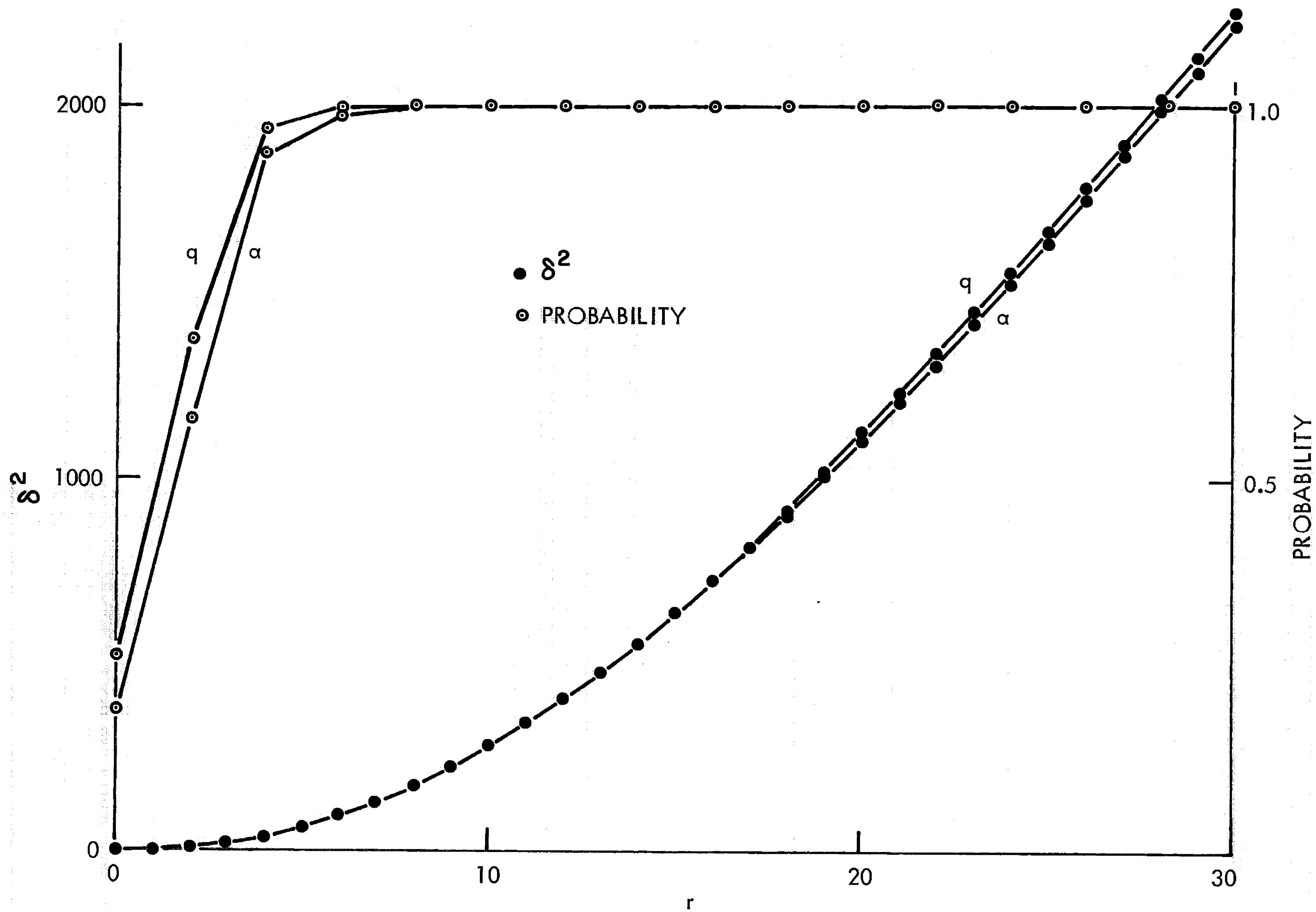

Fig. 12 1σ State Step δ^2 and P_D with $\epsilon = 5$

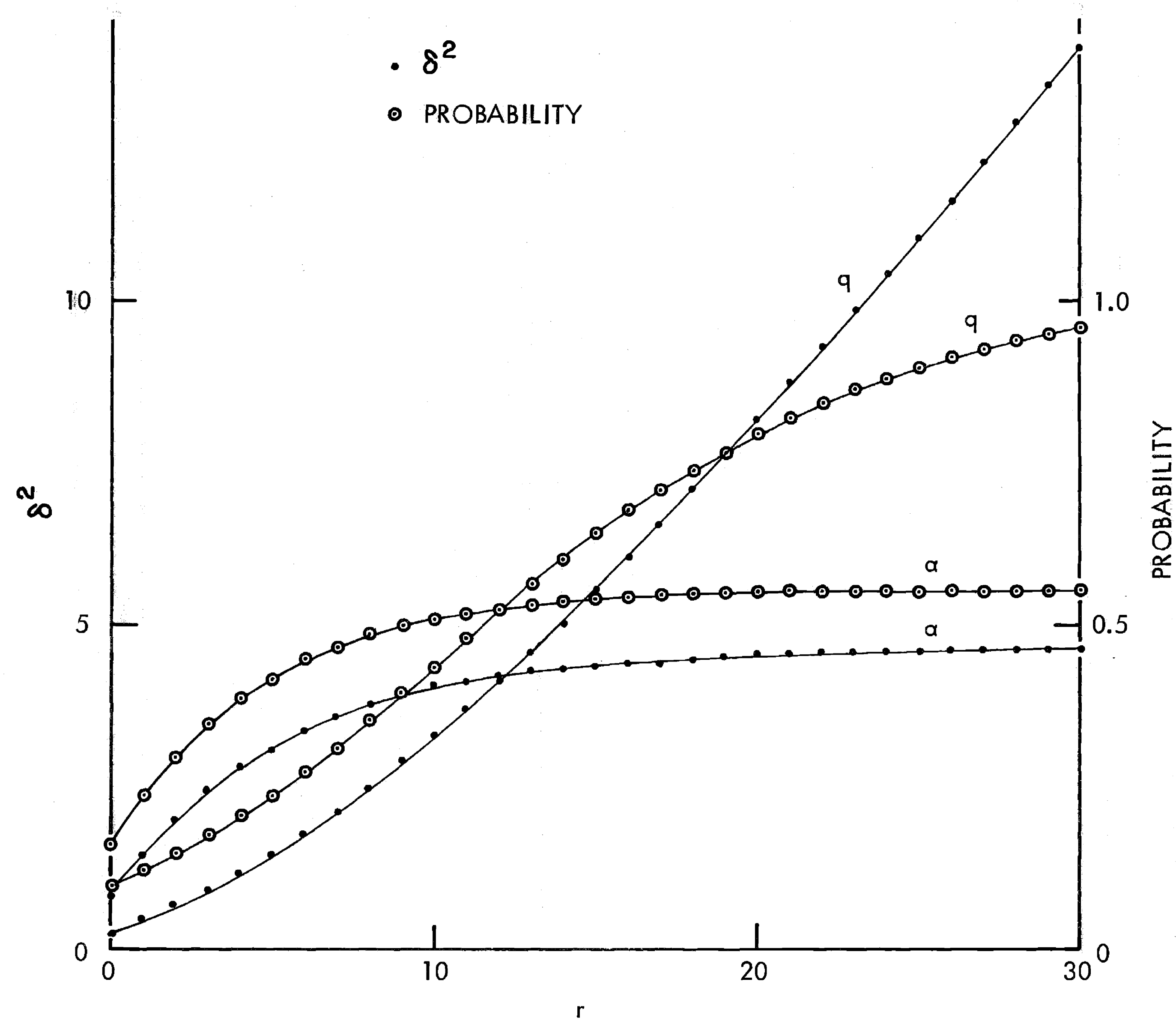

Fig. 13 1σ Sensor Step δ^2 and P_D with $\epsilon = 5$

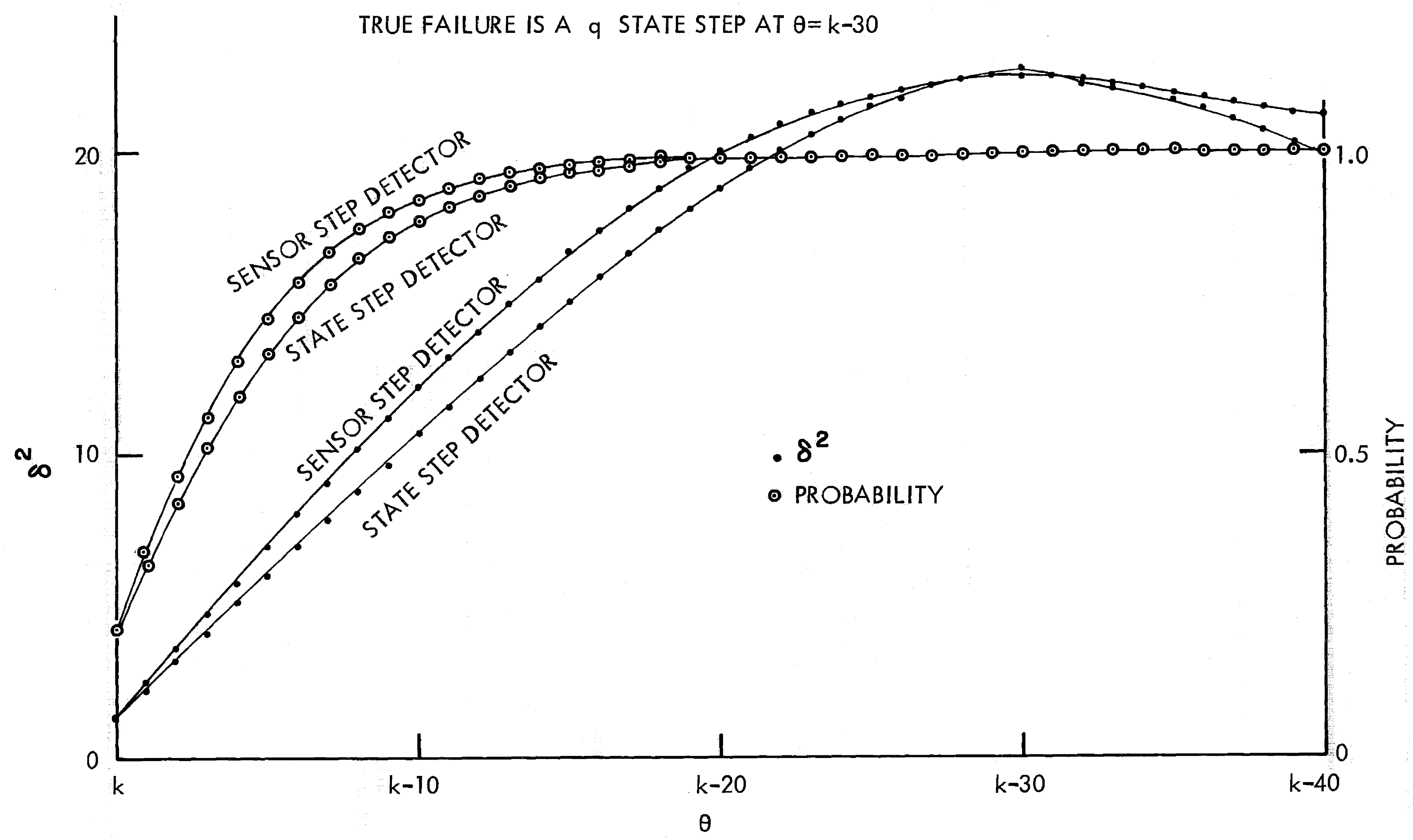

Fig. 14 Wrong Time Cross Detection Probability

evident that all elements of the G's of jump failures tend to zero while at least some elements of the G's of step failures are non-diminishing. This is in agreement with intuition and the arguments in Section 2.3.2. Since both the system and the associated KBF are stable, the effect of a jump (either in state or in sensor) on the residual is expected to decay to zero as the elapsed time increases. The fact that the system and KBF behave like low pass filters enables them to track certain types of steps. Consequently, the effect of steps (in state or sensor) of at least some directions on the residual is non-diminishing. The problem of finding out precisely which directions lead to nonvanishing G's can be solved by examining the figures or by evaluating the steady-state G as described in Section 2.3.2.

As the effect of jump failures on the residuals decreases with the elapsed time, less information about the failure is available to the detector. As a result, one would expect the estimate of the failure to improve very little after some initial period. In fact, the accuracy of the estimate reaches a limit as shown in Section 2.3.2. Figures 6 and 7 display such behavior for the C^{-1} matrices (error covariance matrix of the failure estimate).

Since some step failures have non-diminishing effect on the residual, the detector is provided with more information about these failures as time progresses. Therefore, an improvement of the failure estimates as the elapsed time increases is possible. Indeed, the error covariances of the estimates of failures in these directions do go to zero as time progresses. The fact that some elements of C^{-1} of both sensor and state step failures in Figures 8 and 9 tend to zero is evidence of this behavior.

Figures 10 to 13 are plots of δ^2 (of the correct detection cases) and P_D of full GLR versus r for the four failure types. The P_D's are computed for a threshold ε of 5. With this threshold, we have a P_F of 0.082. For each failure type, a failure in α alone, $(0, \nu_2)'$ and a failure in q alone, $(\nu_1, 0)'$ are separately considered in the computation of δ^2 and P_D. For jumps, we have set the failure to take on the size of 5 σ's (5 standard deviations) of the noise affecting that component of state or output vector and the step failure sizes are of 1σ. The following is a summary of the failures considered.

a jump in q state $\nu = (.1129, 0)'$

a jump in α state $\nu = (0, 0.0217)'$

a jump in q sensor $\nu = (0.0437, 0)'$

a jump in α sensor $\nu = (0, 0.3)'$

a step in q state $\nu = (0.0226, 0)'$

a step in α state $\nu = (0, 0.0043)'$

a step in q sensor $\nu = (0.0087, 0)'$

a step in α sensor $\nu = (0, 0.06)'$

With a threshold of 5, all jump failures considered yield a high P_D immediately after the failure has occurred. If we only consider the P_D's, the different detectors seem to be equally "sensitive" to their matched failures. However, an examination of the δ^2 reveals a different fact. Sensor jump δ^2's reach steady state values almost immediately and have smaller values than the δ^2 of state jumps of the same "size" (i.e., 5σ of the corresponding noise). State jump δ^2's actually grow with r, at least in the range of r considered, reflecting the different nature of state jumps and sensor jumps and hence the difference in sensitivity between state jump and sensor jump detectors. Due to the growing nature of δ^2 within the range of elapsed time (0, 10), a state jump failure that causes a small initial δ^2 (and P_D) will probably have a larger P_D as r increases implying that if this failure is not detected immediately, the probability of its being detected increases as one waits. But the same does not apply in the sensor jump situation because δ^2 practically stays at a constant value.

All step failures considered here have growing δ^2's except the step failure in the α sensor. Let us consider this exception. As a first order approximation, the angle of attack (α) is the integral of the pitch rate (q). Hence, a jump in the q state and a step in the α sensor produce similar effects on the residual. Therefore, the information about the failure in the residual diminishes with time for both failures, and we should anticipate the δ^2 due to a step in the α sensor to reach a steady state value (Figure 13) in a way similar to the jump in the q state situation.

Finally, in Figure 14, we present a plot of the wrong time cross detection δ^2 and probabilities of two full GLR detectors having the same window and threshold ($\varepsilon = 5$). The detectors considered are a state step detector and a sensor step detector. In this figure, the ordinate represents the noncentrality parameter and probability. The abscissa represents the hypothesized failure times (θ) of the detector window which is taken to be $k-40 \leq \theta \leq k$ (here k is fixed). A 0.1σ q state step failure is assumed to have occurred at $\theta = k-30$.

The δ^2 profile of both detectors over the window are very much alike, signifying a strong correlation between sensor step failure and the q state step failure and, therefore, the indistinguishability between them. We note that full GLR may often have severe cross-detection problems. Suppose an actual failure ν of type i occurs. If <u>any</u> failure $\tilde{\nu}$ of type j is correlated with the true failure, we will have cross-detection problems since full GLR is designed to choose the most likely direction. Thus, CGLR may be useful in avoiding the cross detection problem and this idea should be investigated in the future.

We also note that the δ^2 of the state step detector across the window peaked at the true failure time while the sensor step detector does not have a distinct peak in δ^2. This indicates that the sensor step detector will have more difficulty in determining the true failure time (of the q state failure).

In Figure 14, k is fixed. As k varies, the δ^2 for both detectors will vary accordingly (and the matched (state step detector) δ^2 profile will probably remain very peaked). Thus, examining the δ^2 behavior as a function of k and θ may lead to some rules for distinguishability (essentially hypothesis testing on the ℓ's). For instance, the δ^2 profiles mentioned above are indications of correlation of the failures over a window and in turn may be used to determine the type of the true failure by distinguishing between the shapes of the likelihood ratio profiles.

4.2 Conclusions and Suggestions for Future Research

Having recognized that the likelihood ratio $\ell(k;\ \theta)$ is a crucial quantity in the GLR detection scheme, we have studied its properties in this research. In Chapter 2 where the ℓ's were treated as static random variables, we have considered probabilities such as P_D as performance measures of the GLR system. As a result, we were able to determine some guidelines for setting detector window sizes and threshold. We were also able to gain some insights into two important questions related to the performance of the detection system, namely, the detectability and distinguishability of failures. A relationship between the detectability and observability of failures has been determined. But we were only able to make an initial study of the distinguishability issue and further investigation of this subject is necessary to achieve an "optimal" performance GLR detection system.

In Chapter 3 where the ℓ's were regarded as a random sequence, we have studied the correlation behavior of the ℓ's at different observation time (k) and hypothesized failure time (θ) in SGLR. As a result of this study, we were able to consider more "precise" performance measures such as the probability of time to detection (P_{TD}). We have also developed a framework in which we can consider modified and improved decision rules that involves the temporal characteristics of the ℓ's. But we have only laid out some groundwork and much remains to be done (in particular we need algorithms for approximating certain integrals of Gaussian random variables).

As we have gained some understanding of the GLR technique, we are also confronted with questions that require additional studies. In concluding this report, we now outline the areas that we feel require further investigation.

The first major area of future research is the indistinguishability problem. A better understanding of the nature of this problem is desirable to fully utilize the GLR technique. Insights may be obtained by following research directions similar to the approach discussed in Section 2.5 and by the study of the correlation behavior of the ℓ's and δ^2 for the wrong time cross detection situation. Such a study will provide information about the degree of correlation between different failures. This information may be used to distinguish different failure modes, for instance, in an interval decision rule detector scheme.

As pointed out in Section 2.5, once the mutually distinguishable failure directions are determined, it is more appropriate to use CGLR. As this thesis research did not consider this scheme, an analysis of CGLR similar to the one performed for full GLR and SGLR is desirable (and the present work provides all the tools needed for such a study). There, we can develop similar performance measures and modified decision rules. As CGLR has an indistinguishability problem of its own, we will have to consider it also, and, in fact, the choice of detector assumed failure directions will be governed by distinguishability considerations.

Lastly, the joint density functions of the ℓ's in full GLR is a valuable piece of information to obtain. As the ℓ's here are χ^2, we expect the joint densities to be complicated and, therefore, approximations are necessary. After these densities have been determined, modified decision rules as discussed in Section 3.3 may be developed for full GLR performance.

APPENDIX

The Chi Squared (χ^2) Random Variable

The central χ^2 random variable u with n degrees of freedom is the sum of squares of n independent, zero mean, unit variance gaussian random variables or more precisely,

$$u = \sum_{i=1}^{n} x_i^{\,2}$$

where $x_i \sim N(0, 1)$ and $E\{x_i x_j\} = 0$ for $i \neq j$. Then the density function of u is [10]:

$$f_u^n(u) = \begin{cases} \dfrac{1}{2^{n/2}\,\Gamma(\frac{1}{2}n)} \; e^{-\frac{u}{2}} \, u^{\frac{n}{2}-1} & u > 0 \\ 0 & u \leq 0 \end{cases}$$

where $\Gamma(\cdot)$ is the gamma function.

There is a FORTRAN subroutine (CDTR) in the IBM Scientific Subroutine Package that can be readily used to compute the integral of the above density, i.e. the quantity

$$P_u^n(u \leq \varepsilon) = \int_{-\infty}^{\varepsilon} f_u^n(u)\,du = \int_{0}^{\varepsilon} f_u^n(u)\,du$$

Then the false alarm probability (P_F) of a detector set to detect a n dimensional failure is

$$P_F = 1 - P_u^n(u \leq \varepsilon).$$

The noncentral χ^2 random variable ω with n degrees of freedom is the sum of squares of n independent, nonzero mean, unit variance gaussian random variables with the noncentrality parameter defined as

$$\delta^2 = \sum_{i=1}^{n} [E(x_i)]^2$$

With $\delta^2 = 0$, ω is central χ^2. The density for ω is [10]:

$$f^n_{\omega,\delta^2}(\omega) = \begin{cases} \dfrac{1}{\sqrt{\pi}\; 2^{n/2}} \; e^{-\frac{1}{2}(\delta^2+\omega)} \; \omega^{\frac{n-2}{2}} \displaystyle\sum_{j=0}^{\infty} \dfrac{(\delta^2\omega)^j \Gamma(j+\frac{1}{2})}{(2j)!\,\Gamma(j+\frac{1}{2}n)} & \omega > 0 \\ 0 & \omega \leq 0 \end{cases}$$

Recall

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

Then

$$f^n_{\omega,\delta^2}(\omega) = e^{-\frac{1}{2}\delta^2} \sum_{j=0}^{\infty} (\delta^2)^j \omega^{\frac{n}{2}+j-1} e^{-\frac{1}{2}\omega} \times$$

$$\frac{2^{-j}(2j-1)(2j-3)\ldots(3)(1)\Gamma(\frac{1}{2})}{2^{\frac{n}{2}}(2j)(2j-1)(2j-2)\ldots(2)(1)\Gamma(\frac{1}{2})\Gamma(j+\frac{1}{2}n)}$$

$$= e^{-\frac{1}{2}\delta^2} \sum_{j=0}^{\infty} (\delta^2)^j \frac{1}{j!\; 2^{\frac{n}{2}+2j}\,\Gamma(j+\frac{1}{2}n)} \omega^{\frac{n}{2}+j-1} e^{-\frac{1}{2}\omega}$$

$$= e^{-\frac{1}{2}\delta^2} \sum_{j=0}^{\infty} \frac{(\delta^2)^j}{2^j j!} f_u^{n+2j}(\omega)$$

Hence

$$P^n_{\omega,\delta^2}(\omega \leq \varepsilon) = e^{-\frac{1}{2}\delta^2} \sum_{j=0}^{\infty} \frac{(\frac{\delta^2}{2})^j}{j!} P^{n+2j}_u (u \leq \varepsilon)$$

Therefore, $P^n_{\omega,\delta^2}(\omega \leq \varepsilon)$ can be easily calculated by summing the above series. We note that $P^n_u (u \leq \varepsilon) = P^n_{\omega,\delta^2} = 0$ $(\omega \leq \varepsilon)$.

In Figure 15, we have plotted P_x versus ε for various values of δ^2 and P_x is defined as

$$P_x \triangleq 1 - P^n_{\omega,\delta^2}(\omega \leq \varepsilon)$$

Note that

$$\lim_{\delta^2 \to \infty} P_x = 1 \qquad \text{fixed } \varepsilon$$

$$\lim_{\varepsilon \to \infty} P_x = 0 \qquad \text{fixed } \delta^2$$

Figure 15 may be used to determine the various probabilities defined in section 2.2 once the noncentrality parameters are determined.

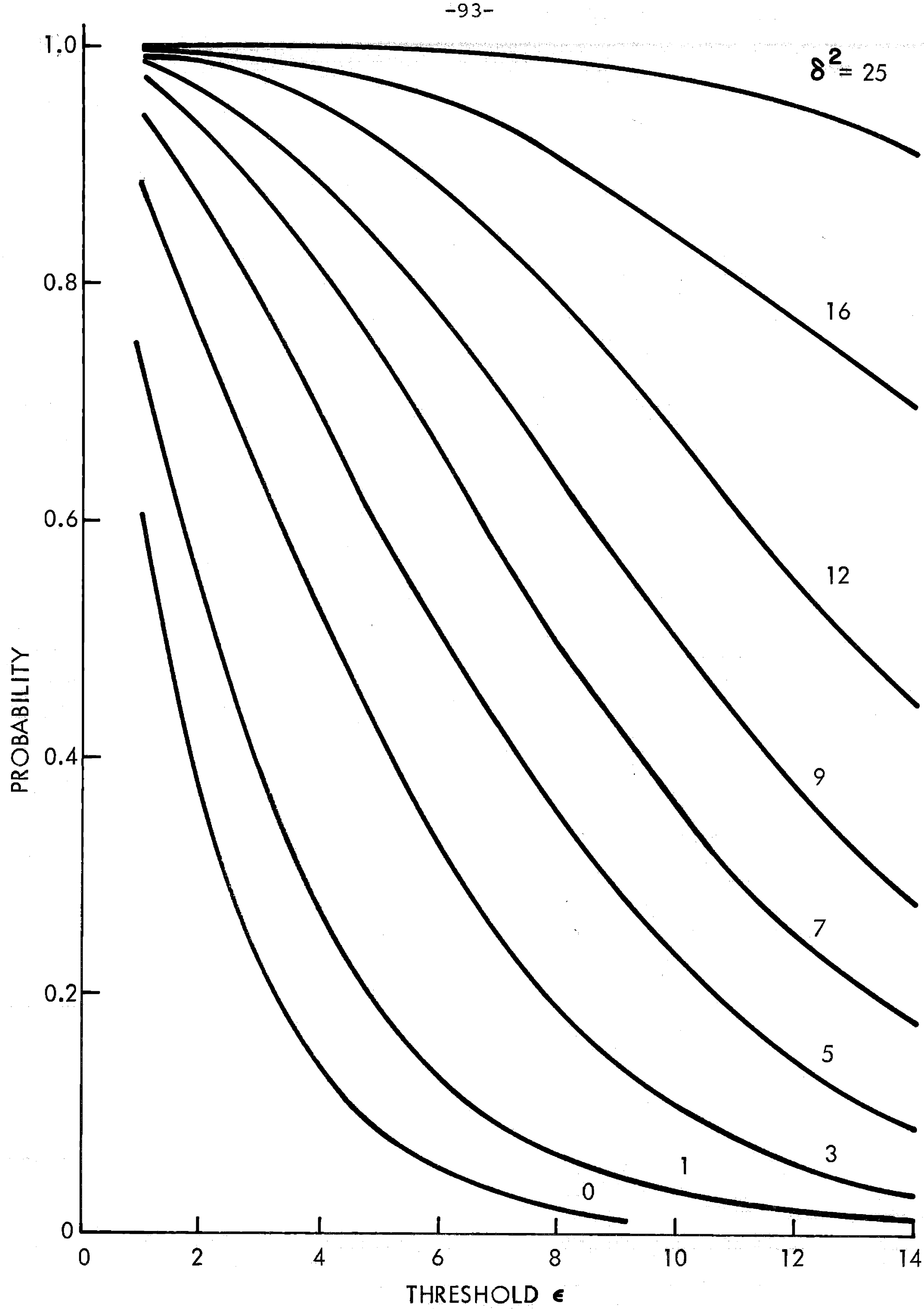

Fig. 15 P_x of a Chi Squared Random Variable with 2 Degrees of Freedom

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