LINEAR ESTIMATION OF BOUNDARY VALUE STOCHASTIC PROCESSES

by

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ABSTRACT

The linear minimum variance estimator for a broad class of linear boundary value stochastic processes is derived. This class includes boundary value processes governed by linear ordinary and partial differential and difference equations and contains initial value processes as a subclass. The estimator has been derived by an application of the method of complementary models introduced by Weinert and Desai [1], and notable is the fact that no Markov properties have been required. The product of the derivation is a representation of the estimate and estimation error in differential operator form. By taking this differential form as a starting point, efficient and numerically stable methods for implementing the estimator have been formulated for the cases of 1-D continuous parameter and 1-D and 2-D discrete parameter boundary value stochastic processes. These solutions are shown to be similar to two-filter methods for implementing the smoother for causal 1-D processes [19,20]. A Methodology for developing implementation schemes for the estimator of 2-D continuous parameter boundary value processes is presented. This methodology is based on operator transformations of the estimator dynamics to achieve either a diagonal or triangular form. It is shown that the existence of transformations leading to these decoupled forms is conditioned on the existence of solutions to certain operater Riccati equations.

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CHAPTER 1: **INTRODUCTION**

SECTION 1.1

PROBLEM MOTIVATION AND THESIS GOALS

Most of the natural laws of physics can be expressed as either ordinary or partial differential equations [5], e.g. Newton's law of cooling, Newton's laws of motion, Maxwell's equations governing the basic relations of classical electricity and magnetism and the Navier-Stokes equations of fluid motion. Although the differential equations associated with the above mentioned laws are generally well-defined, in any practical situation there may be uncertainties in the coefficients of these differential equations and/or in external processes which act as driving functions. In addition, it is often the case that there are uncertainties in the boundary values required for the complete specification of the underlying processes. Each of these uncertainties, in turn, results in uncertainties in our knowledge of the value of the process throughout the spatial region and/or time interval of interest.

There are many situations in which large errors in our knowledge of the value of a process are undesirable. As two examples, consider the control of the temperature of a steel slab during annealing and the control of the shape of a highly flexible orbiting antenna. In the first case, one will have little success in controlling the temperature distribution to some desired profile if the actual temperature of the slab is poorly known. In the second case, unknown solar pressures and temperature variations due to changes in the orientation of the antenna with respect to the sun will cause the antenna to deform from its desired shape. One's ability to correct for these deformations is dependent on how well the deformed shape itself can be estimated. Another example is provided by the fact that poor knowledge of the earth's gravitational field is a principal source of navigation error in highly accurate inertial navigation systems. In an effort to improve our knowledge of the value of a process (i.e., to reduce the uncertainties), measurements, which themselves will be in error due to imperfect measurement devices, are taken. A natural question is: given measures of the uncertainty in the boundary value, of the uncertainty in the external driving function and of the uncertainty in the measurements, how does one combine all of this along with the measured values to produce the best (in some sense) estimate of the underlying process? This brings us to the topic of this thesis: estimation of boundary value stochastic processes.

For those laws of physics which are stated in terms of partial differential equations, the most general form of each describes the transient (time-dependent) behavior of an underlying process at every point in some prescribed spatial region. Due to this space-time characteristic, a full specification of the value of the process throughout the region for future times requires both constraints on the value of the process at the boundary of the region for all time and knowledge of the value of the process over the entire region at an initial time (an initial condition). Processes with this type of boundary condition will be referred to as initial-boundary-value processes. In many cases, however, one is interested in the value of the process only after a temporal steady-state condition has been reached (i.e., the value of the process throughout the region of interest is time-independent). For those cases, the process can be completely specified in the region simply by defining constraints on the process at the boundary of the region. To differentiate between the time-independent processes and those with time variations, we will refer to the former as purely-boundary-value processes as opposed to initial-boundary-value processes. When it is not necessary to distinguish between the two, we will simply say boundary-value processes. Similarly, processes governed by ordinary differential equations with a specified initial condition will be referred to as initial value processes and those specified by a two-point boundary condition (one at each end of an interval) will be referred to as two-point boundary-value processes.

As advertised in the title, in this thesis we address <u>linear</u> estimation of boundary value stochastic processes. Linear is a key word, and it has been added to indicate that we have limited our study of the estimation problem in the following ways. First, we consider estimation of processes satisfying linear ordinary and partial differential and difference equations with uncertain inputs which are modelled formally as white noise. Second, the observations are limited to linear functions of the underlying process with an additive white measurement error. Finally, the estimate of the process is restricted to being a linear combination of the observations, and our objective is to find the linear combination which results in the minimum variance estimation error. In addition, it will be assumed that all random variables have a Gaussian probability density distribution. In this case, the linear minimum variance estimate is also the conditional mean. The following simple example of a noncausal 1-D process[1], which is considered in detail in Chapter 3, serves to illustrate all of this. Estimation of 1-D noncausal processes of this type was first suggested by Krener [17].

Example: A Cooling Rod

Consider the 1 ft long copper cooling rod depicted in Figure 1.1. At each end it is attached to a heat source at nominal temperature T_S, and along its length it is bathed in a coolant at nominal temperature T_C. For this example it is assumed that temporal variations in coolant temperature are

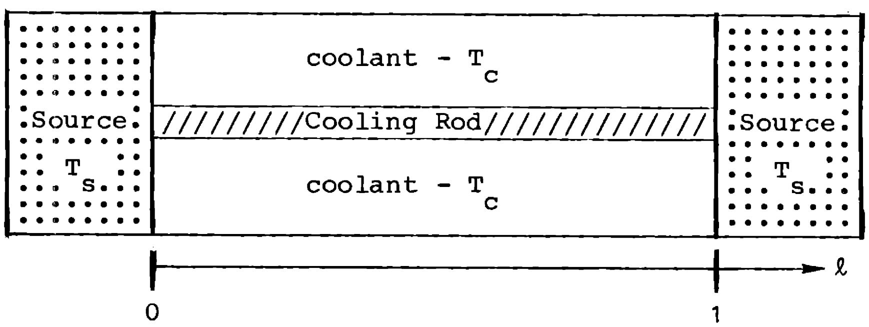


Figure 1.1.1
Cooling Rod

negligible. Let ℓ represent position along the rod and $x(\ell)$ represent the deviation of the temperature of the rod from T_C the nominal coolant temperature. It can be shown (cf. Section 3.6.1) that in temporal steady-state the temperature deviation $x(\ell)$ satisfies[2]

$$\frac{d^2x(\ell)}{d\ell^2} - m^2x(\ell) = u(\ell) \qquad (1.1.1)$$

---

[1] Here one-dimensional (1-D) refers to processes with a one dimensional independent variable. Multidimensional refers to a multidimensional independent variable.

[2] Equations and figures are numbered as follows: (Chapter# . Section# . Equation or Figure# within the Section).

where $u(\ell)$ is a white noise with variance $E[u(\ell)u(\ell')] = Q(\ell)\delta(\ell - \ell')$ which models the spatial variations in coolant termperature from the nominal T_c and m^2 is a model parameter which is a function of the diameter of the rod, the thermal conductivity of the rod and the rod-to-coolant heat transfer coefficient. In addition to the dynamics in (1.1.1), we also need a boundary condition in order to completely define the probability law for the stochastic process $x(\ell)$ on the interval [0,1]. The following two-point boundary condition is the appropriate one for the configuration depicted in Figure 1.1.1:

$$(T_s - T_c) + v_0 = x(0) \qquad (1.1.2a)$$

and

$$(T_s - T_c) + v_1 = x(1) \qquad (1.1.2b)$$

where v_0 and v_1 are zero mean independent Gaussian random variables both with variance σ^2 which represent constant variations of the source temperature from the nominal T_s.

The measurements will be modelled as

$$y(\ell) = x(\ell) + r(\ell) \qquad (1.1.3)$$

where $r(\ell)$ is a spatial white measurement error with variance $R(\ell)\delta(\ell - \ell')$. A final assumption is that u, r, v_0 and v_1 are pairwise uncorrelated. The estimation problem is: Given the probabilistic description of the underlying random processes and the values of the observations $y(\ell)$ on [0,1], what is the linear minimum variance estimate of $x(\ell)$ on [0,1]? $\underline{\equiv}$

Before discussing our approach to the solution of this estimation problem, we will briefly review some previous related work on linear estimation of stochastic processes. At the beginning of this section we made a distinction between what we called purely-boundary-value problems and initial-boundary-value problems. This distinction provides a framework for putting previous work on linear estimation in perspective with what we aim to accomplish in this thesis. In particular, the minimum variance linear estimation problem for causal stochastic processes in one and higher dimensions has been solved (see for instance, [6] and [10]). By causal, we mean 1-D initial value processes and, in higher dimensions, initial-boundary-value processes. Notable in the derivation of these estimators is the use of the Markov property for these causal processes (see [15] for a specific discussion of this for the 1-D case). In this thesis we formulate and solve the linear minimum variance estimation problem for a class of processes which includes both the causal stochastic processes mentioned above and noncausal processes specified by purely-boundary-value problems. As the class of processes we consider contains both causal and noncausal processes, the derivation of our estimator has necessarily not required any Markov properties. Because it is the estimation of purely-boundary-value processes that is completely new, our discussions will often be concentrated on these processes. In addition, we will often employ processes of this type in our examples, e.g. the cooling rod example above. Finally, we note that our differential operator solution yields a new estimator structure for multidimensional causal processes in comparison to those previously obtained [6].

By describing the process to be estimated, the boundary conditions and the measurements in a general operator form, we will be able to develop, within a single framework, the equations for the estimate and estimation error for not only continuous parameter processes[3] governed by linear ordinary and partial <u>differential</u> equations but also discrete parameter processes governed by linear ordinary and partial <u>difference</u> equations. The general form of the problem we address is given by:

Dynamics: $$Lx = Bu \tag{1.1.4}$$

Boundary Condition: $$v = Vx_b \tag{1.1.5}$$

and

Observations: $$y = Cx + r \tag{1.1.6a}$$

$$y_b = Wx_b + r_b \tag{1.1.6b}$$

where x is the process to be estimated, L is a linear differential or difference operator, u is a white input process, v is the random boundary value for x, V is a linear operator, x_b contains x and perhaps its

[3] A continuous parameter process is one whose independent variable, e.g. time, takes on values in a continuum. Similarly, a discrete parameter process is one for which the independent variable takes on discrete integer values.

derivatives on the boundary of the region of interest, y and y_b are observations and r and r_b are observation noise. Note that the dynamics, boundary condition and observations of our example can be written in this form if we let C and B be unity and define

$$L = \frac{d^2}{d\ell^2} - m^2 \quad , \tag{1.1.7}$$

x_b as the vector $\left((\dot{\ }) \text{ implies } d(\ )/d\ell \right)$:

$$x_b = \begin{bmatrix} x(0) \\ \dot{x}(0) \\ x(1) \\ \dot{x}(1) \end{bmatrix} \quad , \tag{1.1.8}$$

and V as the 2×4 matrix

$$V = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad . \tag{1.1.9}$$

Our solution to the estimation problem is based on a significant extension of the method of complementary models introduced by Weinert and Desai in [1] where they formulate the smoother for 1-D causal processes. In the context of our problem the complementary process $Z = \{z, z_b\}$, is defined to be that process which is orthogonal to the observations $Y = \{y, y_b\}$ and when combined with the observations, the two span the space spanned by all of the underlying processes which define the problem ,i.e. $\zeta = \{u, v, r, r_b\}$. The key to the derivation is the development of an internal differential realization for the complementary process which is in the same form as that for Y in (1.1.4) through (1.1.6). Critical to the development of this differential realization for Z has been the recognition of the role played by Green's identity. Augmenting the differential realizations for Y and Z into a single system and inverting that system gives a system with input $\{Y, Z\}$, output the underlying process ζ and an internal process which contains the process to be estimated x as a component. Projecting the solution of the inverted system onto the span of Y (i.e. setting Z to zero since Z and Y are orthogonal) yields the a differential realization of the estimator. Having the estimator expressed in differential form provides an excellent starting point for developing methods for its implementation.

The derivation of the differential operator governing the estimate is the first of three major goals of this thesis. The actual implementation of the solution (i.e., computing the estimate) poses different problems depending on whether the process to be estimated is continuous or discrete, 1-D or multidimensional. Thus, the second major goal of the thesis is to formulate efficient and numerically stable methods for implementing the solution to the estimate equation. To that end, we first consider 1-D discrete and 1-D continuous parameter boundary value processes and develop forward/backward two-filter implementations for each. These solutions are shown to be similar in many respects to two-filter smoothers developed for causal processes [19], [20], and most importantly, they are shown to have the same stability properties as the causal smoother solutions. Under fairly general conditions we are able to transform the estimator for 2-D discrete parameter processes (solutions of 2-D partial difference equations) into an equivalent, but high order, 1-D discrete parameter boundary value problem which can also be implemented in a two-filter form. In certain cases this high order 1-D process can be decomposed into a system of low order 1-D processes which, again lend themselves to a numerically stable two-filter implementation.

The last of the three major goals of this thesis is to unify the development of efficient and stable two-filter methods of implementation of the estimator into a single operator framework. Although some significant questions remain unanswered with respect to this unification problem, in this thesis we make a some major progress toward its resolution. In particular, we first show that the two-filter solutions for 1-D problems represent a diagonalization of the differential operator governing the estimator dynamics. Given this insight we are able to construct a methodology for developing (infinite dimensional) two-filter schemes for implementing the estimator for 2-D continuous parameter processes. In the course of our investigation of operator diagonalizations we also present a potentially more efficient method based on triangularization of the estimator dynamics. It is shown that a special case of this triangularized solution corresponds to smoothers for one and multidimensional causal processes developed by the innovations approach [6], [33].

SECTION 1.2

THESIS SUMMARY

In Chapter 2 linear estimation via the method of complementary models is introduced considering a simple static estimation example. Following the simple example, we apply the concept of complementary models to develop the differential operator form of the estimator for linear boundary value stochastic processes. A key to the derivation of this estimator is the use of Green's identity in developing an internal differential realization of the so-called complementary process. Having developed the general form of the estimator equations, we apply them to formulate explicit realizations of the estimators for two examples. The first example is for a 2-D continuous parameter process satisfying Poisson's equation on the unit disk, and we find in this case that the partial differential equation representing the estimator is in the form of a fourth order biharmonic equation. The second example is for a 1-D discrete process satisfying an n^{th} order difference equation with a two-point boundary condition. In this case, the estimator is shown to be given by a $2n^{th}$ order two-point boundary value process whose dynamics (but <u>not</u> its boundary condition) are identical to those of the smoother for discrete causal processes.

Having obtained an expression for the estimator, we turn to the question of implementation. In Chapter 3, we begin our investigation of implementation of the estimator solution by considering the estimator for an n^{th} order 1-D continuous parameter boundary value stochastic process. First we develop a special two-filter (one forward and one backward) form of the general solution for such processes. Then, applying the differential operator solution derived in Chapter 2, we show that the estimator for the n^{th} order boundary value process is a $2n^{th}$ order boundary value process whose dynamics are specified in terms of the same Hamiltonian matrix as that of the smoother for causal processes [30]. By diagonalizing the dynamics, we are able to apply the special two-filter form of the general solution developed earlier in the chapter. Establishing the stability of each of the filters in this two-filter solution, we attain our goal of formulating an efficient and numerically stable method for implementing the estimator. In the last part of the chapter we derive some matrix differential equations for computing the covariance of the estimation error and present numerical results for the covariance analysis of the cooling rod estimation problem introduced in Section 1.1.

In Chapter 4 we present an alternative method for developing the estimator for an n^{th} order 1-D boundary value process x(t). This alternative approach is based on establishing a Markov model for x(t), so that we can employ standard Kalman filtering and associated smoothing algorithms to provide the estimate we seek. The Markov model is constructed by the methodolgy introduced by Castanon et al in [37]. It is shown that a Markov model for the TPBVP x(t) is a linear system of order greater than n whose internal state process is a Markov process and whose output is the boundary value process x(t). A second topic of Chapter 4, somewhat unrelated with the first, is an alternative derivation of the two-filter implementation formulated in Chapter 3. This alternative derivation is carried out by viewing the estimation problem for x(t) as a linear scattering problem.

The estimator implementation problem for 1-D discrete boundary value processes is addressed in Chapter 5. The organization of this chapter is nearly identical to that of its continuous counterpart, Chapter 3. However, because the dynamics of the estimator are found to be in descriptor form [35], the problem of formulating a stable two-filter implementation by diagonalizing the estimator dynamics differs significantly from the diagonalization problem in the continuous case. This problem is resolved by considering the class of equivalent descriptor representations of the estimator and determining conditions for diagonal forms in that class. As in the continuous case, we also consider an alternative formulation of the two-filter solution from a scattering viewpoint.

Chapter 6 is a study of the estimator and its implementation for 2-D discrete boundary value stochastic processes. We begin by introducing a 2-D discrete dynamical model, referred to as the nearest neighbor model (NNM). By way of an example, we demonstrate how the NNM can be employed to describe general discrete 2-D partial difference equations. Next we investigate the general solution for the NNM and show that for a large class of processes the general solution can be expressed in a 1-D two-filter form. Under slightly more restrictive conditions we show that an FFT-based transformation, which we construct by extending the results of Jain and Angel [36], can be applied to decompose the 1-D two-filter solution into a system of low order 1-D two-filter problems. Having studied the basic properties of the NNM, we turn to the formulation of its estimator. First the Green's identity for the NNM is derived. This enables us to apply the operator solution from Chapter 2 and to write the estimator equations in the form of a 2-D difference equation. Transforming this difference equation to a NNM, we determine the conditions for the existence of 1-D two-filter forms discussed above. In the last section of the chapter we formulate the estimators for two 2-D discrete examples, one a purely-boundary-value process and the other an initial-boundary-value process. In each case we show that a 1-D form exists and that the FFT-based transformation can also be applied so that it is possible to implement the estimator for these processes by a system of low order 1-D two-filter problems. The estimator for 2-D discrete initial-boundary-value processes in the form of a system of decoupled low order two-filter forms differs from those previously obtained for 2-D discrete causal processes.

Chapter 7 is the last technical chapter of the thesis and our objective there is to unify the diagonalization procedures employed in earlier chapters to develop efficient methods for implementing the estimator. The chapter begins with a discussion of equivalent differential operator representations. It is shown that both the 1-D continuous and 1-D discrete two-filter solutions can be derived as diagonal differential operator representations of the estimator. A key step in these derivations is the determination of the operator transformations which ultimately lead to a diagonal form. We then extend the 1-D continuous results to a 2-D process governed by a parabolic partial differential equation and show that a similar operator diagonalization of the estimator dynamics is possible. The question of decoupling the boundary conditions for this example remains, in part, unanswered. Furthermore, when we consider the general class of 2-D continuous parameter boundary value processes, we find that the question of existence of the diagonalizing operator transformation (actually the existence of solutions to operator Riccati equations) is extremely difficult to answer. The resolution of this question for the general case is left to future research. In addition to diagonalization, we also study triangularization of the estimator dynamics. It is shown that in the 2-D continuous case triangularization can lead to an efficient form for implementation of the estimator which avoids some of the problems that are encountered for diagonalized forms.

The major contributions of the thesis are summarized in Chapter 8. Following the summary we present some suggestions for further research which include weakening of the assumptions under which our boundary value estimator has been derived and extensions of our efforts to obtain efficient methods for solving the estimator equations for other problems.

CHAPTER 2: **ESTIMATOR FORMULATION via THE METHOD OF COMPLEMENTARY MODELS**

SECTION 2.1

INTRODUCTION

In this chapter we present a substantial extension of Weinert and Desai's [1] method of complementary models for minimum variance linear estimation. Weinert and Desai showed that the fixed interval smoothing problem for causal one-dimensional[1] processes described by linear state equations driven by white noise could be solved by introducing the so-called complementary process. The complementary process has the property that it is orthogonal to the observations and that, when combined with the observations, containsinformation equivalent to the initial conditions, driving noise and measurement noise, i.e. all of the underlying variables which determine the system state. Here we generalize this idea and show how to employ the method of complementary models to solve estimation problems for both discrete and continuous parameter boundary value stochastic processes in one and higher dimensions. These processes include those defined through ordinary and partial linear differential equations and ordinary and partial linear difference equations and may be either causal or noncausal. By employing operator descriptions for these processes we are able to unify the development of the estimators for this wide variety of processes within a single framework. The major contribution of this chapter is a differential operator representation for the estimator which applies to all of the cases mentioned above. A key step in the derivation of the differential operator representation of the estimator is our formulation of a differential operator representation for the complementary process.

To help clarify our presentation we carry along two examples throughout. One example is a 2-D process governed by Poisson's equation with a white noise driving function. The other is a 1-D discrete two-point boundary value process. The emphasis in this chapter is on the development of the

[1] The terminology one-dimensional (1-D), two-dimensional (2-D) or multidimensional process is used here to indicate that the dimension of the independent variable for the process is one, two or multidimensional.

differential representation for the estimator. In later chapters we consider implementation of the estimator solution for various classes of processes. For instance, in Chapter 3 we study 1-D continuous pararmeter boundary value stochastic processes and address the details of the implementation and structure of their estimator and the computation of its error covariance.

In Section 2.2 we review the complementary models approach to linear estimation and motivate this approach by its application to a static problem of estimating a random vector. Following the example, the approach is generalized to second order stochastic processes by way of a restatement of the Projection theorem. Section 2.3 serves to introduce some notation which we employ for representing boundary value stochastic processes and their correlation functions. Utilizing this notation, we state the general form of the estimation problem for which we ultimately develop a solution. In Section 2.4 we present an operator representation for the complementary stochastic process associated with the general problem stated earlier. In that section we offer a proof that this operator representation satisfies the properties of a complementary process as defined in Section 2.2. In Section 2.5 a general form for the internal differential realization for the complementary process is derived. Given this realization, we are able to formulate an internal differential realization for the estimator. Using this recipe for the operator representation of the estimator, in Section 2.6 we present differential realizations for the estimators for the 1-D and 2-D examples. Finally, some observations and concluding remarks are offered in Section 2.7.

SECTION 2.2

LINEAR ESTIMATION AND COMPLEMENTARY STOCHASTIC PROCESSES

2.2.1 A Static Example

Before presenting the derivation of the estimator for a general class of noncausal stochastic processes, we illustrate the application of this approach for a familiar static problem of estimating a random vector. This example provides motivation for the complementary model approach in general, and in that there are many parallels between this static example and the more general problem we will ultimately address, the example also provides insight into the structure of the operator solution we obtain later.

Let ζ be an (n+p)-dimensional, zero mean random vector partitioned into $n\times 1$ and $p\times 1$ dimensional vectors x and r respectively as

$$\zeta = \begin{bmatrix} x \\ r \end{bmatrix} \begin{matrix} \leftarrow n\times 1 \\ \leftarrow p\times 1 \end{matrix} \tag{2.2.1a}$$

with invertible covariance matrix

$$\Sigma_\zeta = \begin{bmatrix} \Sigma_x & 0 \\ 0 & \Sigma_r \end{bmatrix} \quad . \tag{2.2.1b}$$

Let y be a p-dimensional observation ($p \leq n$)

$$y = M_y \zeta \tag{2.2.2a}$$

where M_y is the $p\times(n+p)$ matrix partitioned into a $p\times n$ matrix H and the $n\times n$ identity matrix I as

$$M_y = [\underset{\substack{\uparrow \\ p\times n}}{H} : \underset{\substack{\uparrow \\ p\times p}}{I}] \tag{2.2.2b}$$

with H full rank p. That is, we have the familiar linear observation:

$$y = Hx + r \quad . \tag{2.2.2c}$$

The vector to be estimated, x, can also be expressed as a linear function of ζ:

$$x = M_x \zeta \tag{2.2.3a}$$

where

$$M_x = [\, I \,\vdots\, 0 \,] \quad . \tag{2.2.3b}$$

Since both x and y are defined in terms of it, ζ will be referred to as the underlying random vector.

We show below that one way to calculate the linear minimum variance estimate of x given y is by establishing a complementary random vector z which has the following properties: 1) It is a linear function of the underlying random vector ζ

$$z = M_z \zeta \quad . \tag{2.2.4}$$

2) It is orthogonal to the observation y

$$E[yz'] = 0 \quad . \tag{2.2.5}$$

3) It is complementary with respect to y in that the augmented system

$$\begin{bmatrix} y \\ z \end{bmatrix} = M\zeta \quad ; \quad M = \begin{bmatrix} M_y \\ M_z \end{bmatrix} \tag{2.2.6}$$

is invertible.

Define the inverse of M as N and partition it compatibly with the dimensions of the vectors y and z as

$$M^{-1} \equiv N = [N_y \,\vdots\, N_z] \tag{2.2.7}$$

and denote the products of these partitions with y and z as

$$\zeta_y = N_y \, y \tag{2.2.8a}$$

and

$$\zeta_z = N_z \, z \quad . \tag{2.2.8b}$$

Then ζ can be written as the sum of orthogonal components

$$\zeta = \zeta_y + \zeta_z \quad . \tag{2.2.8c}$$

It follows from the orthogonality of ζ_y and ζ_z and from the Projection theorem that ζ_y is the linear minimum variance estimate of ζ given y (the projection of ζ onto span(y)), and that

$$\hat{x} = M_x \zeta_y \tag{2.2.9}$$

is the linear minimum variance estimate of x given y. In addition, from (2.2.3a), (2.2.8c) and (2.2.9) the estimation error can be written as a function of ζ_z:

$$\begin{aligned} \tilde{x} &= x - \hat{x} \\ &= M_x \zeta_z \end{aligned} \tag{2.2.10}$$

Thus the estimation error is the linear minimum variance estimate of x given z.

All of this is of little use if the matrix M_z is not known. It is shown in the Appendix that the three conditions stated above enable us to derive the following general expression for the n×(n+p) matrix M_z:

$$M_z = T[\ I \ \vdots \ -\Sigma_x H' \Sigma_r^{-1}] \tag{2.2.11}$$

where T is any invertible n×n matrix (indicating the obvious fact that the complementary process is only defined uniquely up to a choice of basis). We will see in the next section that the key to extending the method of complementary models to more general stochastic processes is the interpretation of the transpose of H in (2.2.11) as an adjoint mapping.

By performing the augmentation and inversion indicated in (2.2.6) through (2.2.8), the underlying process can be written as a linear function of y and z. If we define P as the matrix

$$P = [\Sigma_x^{-1} + H' \Sigma_r^{-1} H]^{-1} \quad , \tag{2.2.12a}$$

then inverting the matrix M defined by substituting (2.2.2b) and (2.2.11) into

(2.2.6) gives the following expression for ζ

$$\zeta = \begin{bmatrix} x \\ r \end{bmatrix} = \begin{bmatrix} P \\ - \; - \\ I-HP \end{bmatrix} H'\Sigma_r^{-1} y + \begin{bmatrix} P \\ - \; - \\ -HP \end{bmatrix} \Sigma_x^{-1} T^{-1} z \quad . \tag{2.2.12b}$$

From (2.2.3b) and (2.2.9) the estimate of x is given by

$$\hat{x} = PH'\Sigma_r^{-1} y \quad . \tag{2.2.12c}$$

From (2.2.10) the estimation error is

$$\tilde{x} = P\Sigma_x^{-1} T^{-1} z$$

or substituting for z from (2.2.4) and (2.2.11)

$$\tilde{x} = P[\Sigma_x^{-1} x - H'\Sigma_r^{-1} r] \quad . \tag{2.2.12d}$$

A direct calculation from (2.2.12d) and the defintion of P in (2.2.12a) gives the error covariance as

$$E\{\tilde{x}\tilde{x}'\} = P \quad . \tag{2.2.12e}$$

In summary, this static example illustrates the basic concepts of the method of complementary models. We have shown that two of the key elements in the development of the estimator are (1) the knowledge of the form of M_z and (2) the ability to easily invert the augmented system (2.2.6) to obtain the underlying variables ζ as a function of the observations y and the complementary process z. In the remainder of this section we formalize this approach to linear estimation as a restatement of the Projection theorem. Subsequently, we apply this theorem to establish an operator form for M_z which is appropriate for the general class of noncausal processes mentioned in the introduction.

2.2.2 The Projection Theorem Restated

Here we consider linear estimation for second order stochastic processes. Let $L_2(dP)$ denote the Hilbert space of finite variance random variables (on some given probability space). Let I denote an index set. A second order process over I is a set of elements in $L_2(dP)$ indexed by I:

$$\phi(\alpha) \in L_2(dP), \quad \alpha \in I. \tag{2.2.13}$$

The closed linear span in $L_2(dP)$ of ϕ (as α ranges over I) will be denoted by $Sp(\phi)$. The space of second order processes over I will be denoted by $L_2(I;dP)$. Linear mappings between two such spaces will be called second order operators.

With these definitions we can generalize the static example as follows. Define the underlying process as a second order process over a specified index set I_ζ

$$\zeta \in L_2(I_\zeta;\ dP) \equiv S_\zeta \quad . \tag{2.2.14}$$

The process to be estimated X, the observations Y and the complementary process Z are defined via second order linear operators acting on ζ:

$$X = M_x\zeta\ ; \quad M_x:\ S_\zeta \to L_2(I_x;dP) \equiv S_x \tag{2.2.15a}$$

$$Y = M_y\zeta\ ; \quad M_y:\ S_\zeta \to L_2(I_y;dP) \equiv S_y \tag{2.2.15b}$$

and

$$Z = M_z\zeta\ ; \quad M_z:\ S_\zeta \to L_2(I_z;dP) \equiv S_z \tag{2.2.15c}$$

where M_x and M_y are known and M_z must be chosen (if possible) so that the following conditions are satisfied:

Orthogonality:

$$E[Y(\alpha)Z(\beta)] = 0 \quad \text{for all } \alpha \in I_y,\ \beta \in I_z \tag{2.2.16}$$

Complementation:

With M defined as the augmented second order operator

$$M = \begin{bmatrix} M_y \\ M_z \end{bmatrix} \tag{2.2.17a}$$

the relation between ζ and $\{Y,Z\}$:

$$\begin{bmatrix} Y \\ Z \end{bmatrix} = M\zeta \tag{2.2.17b}$$

is invertible. This implies that $\mathbf{S}_\zeta$ and $\mathbf{S}_y \times \mathbf{S}_z$ are isomorphic.

Assume that M_z can be found so that these conditions are satisfied. Partitioning the inverse of the augmented system (2.2.17) as we did for the static example ($M^{-1} = N = [N_y : N_z]$), we will define the components of ζ as

$$\zeta_y = N_y Y \tag{2.2.18a}$$

and

$$\zeta_z = N_z Z \tag{2.2.18b}$$

so that

$$\zeta = \zeta_y + \zeta_z \quad . \tag{2.2.18c}$$

Projection Theorem: Given X, Y, and ζ as defined above and given an operator M_z and corresponding process Z which satisfies the stated orthogonality and complementation conditions, then

i) the linear minimum variance estimate of ζ given Y is

$$\hat{\zeta} = \zeta_y \tag{2.2.19}$$

ii) the linear minimum variance estimate of X given Y is

$$\hat{X} = M_x \zeta_y \tag{2.2.20}$$

$$= M_x N_y Y$$

iii) the estimation error is the linear minimum variance estimate of X given Z

$$\tilde{X} = M_x \zeta_z \tag{2.2.21}$$

$$= M_x N_z M_z \zeta \quad .$$

The proof follows from the same simple arguments used for the vector case in the static example. We remark that the theorem is applicable to processes defined on multidimensional index sets. In the next section we provide an example of such a processs.

The simple notation used to express the linear minimum variance estimate of X as a linear mapping of Y belies the complexity of the effort which may be required in (1) determining the form of the operator M_z, (2) augmenting and inverting to obtain M^{-1} and (3) implementing the solution. In the static example discussed in this section, a simple matrix form for M_z allowed for a direct inversion of the augmented system, yielding a simple form for the estimator and error equations. As one would expect, these three steps become substantially more difficult to accomplish in the more general case where we consider second order stochastic processes. For the estimation problems considered here, the mapping M_y in (2.2.15b) will be assumed to be of the same form as its matrix counterpart in the static example, i.e. $M_y = [H : I]$. Here I is the identity operator, and its action on the underlying process ζ produces the additive noise component to the observations. The operator H can be viewed as an input-output map describing the effect of the system dynamics. After introducing some notation and posing the general statement of the estimation problem in the next section, the first of the three issues stated above is addressed in Section 2.4 where a general operator representation for M_z is developed in the form of an input-output map. It will be shown that the form of the map M_z follows from that of the matrix M_z for the static example in (2.2.11). In particular, if one interprets the transpose of the matrix H in (2.2.11) as its Hilbert adjoint and appropriately interprets Σ_x and Σ_r as operators, then the operator form for M_z is of exactly the same form.

As we have just stated, a direct generalization of our static example is stated in terms of input-output representations for the observations and the complementary process. Unfortunately, working with these representations does not lead to a convenient or easily computed solution to the second step listed above, that is the augmentation of the observations with the complementary process and the inversion required to determine the estimator. However, by extending the approach taken by Levy et. al in [9], we will find that this second step is quite easily accomplished by working with internal differential realizations for the observations and complementary process. The internal realization for the observations is provided directly by the problem statement. We will see that an internal realization for the complementary process requires an internal realization for the Hilbert adjoint of H, H^*. A critical developement in our research has been the recognition that Green's identity (cf. Section 2.5.1) is the key to formulating an internal realization for the Hilbert adjoint map H^* in terms of the operators involved in the internal description of the observations. Given these internal realizations, we are able to perform the augmentation and inversion yielding an internal differential realization for the estimator. We feel that this representation for the estimator is an important one. In particular, if one directly applies the projection theorem to problems of the type which we consider here, the results are generally in the form of integral equations (e.g. Wiener-Hopf integral equations) which must be factored in some way in order to produce a realization for the estimator. In contrast, our solution, obtained via the method of complementary models, directly yields a differential realization of the estimator.

Much as in the case of causal processes described by finite-dimensional state equations, these realizations provide an excellent starting point for the construction of efficient algorithms for implementing the optimal estimator. This last step, determining estimator algorithms, is the subject of the remaining chapters of this thesis. As can be seen from those chapters, it is a decidedly nontrivial step to go from an internal realization to an efficient implementation of estimators for the variety of problems considered in this thesis.

SECTION 2.3

MATHEMATICAL BACKGROUND AND THE GENERAL PROBLEM STATEMENT

The noncausal stochastic processes for which we will be developing an estimator can be divided into two classes: 1) those with a continuous-valued independent variable and 2) those with a discrete-valued independent variable. More specifically, the processes in the first class are solutions of linear stochastic (partial) differential equations. Those in the second class are solutions of linear stochastic (partial) difference equations. In the first two parts of this section we introduce differential operator representations for each of the classes. By employing similar notation in the descriptions of each, we will be able to unify later discussions. An example will be provided for each of the two classes, and these examples will be carried along throughout the rest of the chapter.

The purpose of the development in this section is to describe general recipes for constructing complementary processes and, more importantly, for expressing the solutions to an extremely broad class of estimation problems involving processes with independent variable of one and higher dimensions. In order to highlight the basic concepts underlying these recipes, we will not discuss in detail the technical conditions that must be satisfied in order for our most general boundary value problems to be well-posed (i.e. for existence and uniqueness of solutions to the specified stochastic differential equations) but rather we will, in effect, assume that these conditions are met. Clearly, in any application one must verify the appropriate conditions, and we will illustrate this for our two examples.

2.3.1 Continuous Parameter Stochastic Processes

Differential Operators and Green's Identity

Our stochasitc differential equations are defined in terms of differential operators acting on Hilbert spaces of square-integrable functions as follows. Let Ω_N be a bounded convex region in $\mathbf{R}^N$ with smooth boundary [11]. The space of $n \times 1$ vector functions which are square-integrable on Ω_N is represented by $L_2^n(\Omega_N)$. Let L be a formal[1] differential operator defined on D(L), the subspace of sufficiently differentiable elements of $L_2^n(\Omega_N)$, so that

$$L: D(L) \rightarrow L_2^n(\Omega_N) \tag{2.3.1}$$

Note that D(L) is dense in $L_2^n(\Omega_N)$. With $\partial\Omega_N$ denoting the boundary of Ω_N, define a boundary condition associated with L through the mapping

$$V: D(L) \rightarrow L_2^{n_v}(\partial\Omega_N) \tag{2.3.2}$$

where the dimension n_v is briefly discussed below.

We will say that the pair (L,V) leads to a well-posed boundary value problem if the differential operator Λ formed by augmenting the formal differential operator L and boundary mapping V

$$\Lambda = \begin{bmatrix} L \\ V \end{bmatrix} \tag{2.3.3a}$$

has a unique continuous left inverse:

$$\Lambda^{\#}\Lambda = I \quad . \tag{2.3.3b}$$

We denote the components of the left inverse by

$$\Lambda^{\#} = [\ G_u : G_v] \tag{2.3.3c}$$

where

$$G_u: L_2^n(\Omega_N) \rightarrow D(L) \quad \text{and} \quad G_v: L_2^{n_v}(\partial\Omega_N) \rightarrow D(L) \quad . \tag{2.3.3d}$$

---

[1] The term formal differential operator will be used to denote operators which simply represent differentiation of a function. We will reserve the term differential operator to denote the combined action of a formal differential operator along with an appropriate boundary condition (see (3.3a)). That is, a differential operator implicitly defines an input-output map obtained by solving a well-posed boundary value problem.

In this case, the equation

$$\Lambda x = \begin{bmatrix} u \\ v \end{bmatrix} \tag{2.3.4a}$$

with u and v in the domains of G_u and G_v, respectively, has a unique solution which can be written as

$$x = G_u u + G_v v \quad . \tag{2.3.4b}$$

Thus for a given set of inputs u and v, x is unique and varies continuously with those inputs. The value of the vector dimension n_v in (2.3.2) which is required for a well-posed problem depends on the type and order of the operator L and the dimensions N and n. As mentioned earlier, we will assume that we have well-posed problems here. As discussed next for the examples, equation (2.3.4b) is the Green's function solution of (2.3.3).

<u>Example 1:</u>(Poisson's Equation) To illustrate the preceding development we consider Poisson's equation in $\mathbf{R}^2$ with Ω_2 the unit disk and with a Dirichlet boundary condition. In this case the formal differential operator is the Laplacian ∇^2 and its domain is the space $C_2(\Omega_2)$ of scalar (i.e. n=1) functions on Ω_2 with bounded continuous second partials.

$$L: C_2(\Omega_2) \rightarrow L_2(\Omega_2) \quad ; \quad Lx = \nabla^2 x \tag{2.3.5a}$$

The boundary operator V is the restriction of x to its values on the boundary of Ω_2:

$$V: C_2(\Omega_2) \rightarrow L_2(\partial\Omega_2) \quad ; \quad Vx = x\Big|_{\partial\Omega_2} \tag{2.3.5b}$$

(here $n_v = 1$). The Green's function solution for the pair Lx=u and Vx=v, where u and v are in the ranges of L and V as specified in (2.3.5), is shown in [3] to be given in polar coordinates as follows. Define the kernels g_v and g_u as

$$g_v(\rho,\theta;\beta) = \frac{1}{2\pi} \cdot \frac{(1-\rho^2)}{1 - 2\rho\cos(\theta-\beta) + \rho^2} \tag{2.3.6a}$$

and

$$g_u(\rho,\theta;\gamma,\beta) = \frac{1}{4\pi^2} \log\left\{ \frac{\rho^2 - 2\rho\gamma\cos(\theta-\beta) + \gamma^2}{1 - 2\rho\gamma\cos(\theta-\beta) + \rho^2\gamma^2} \right\} \quad . \tag{2.3.6b}$$

The solution x is given by

$$x(\rho,\theta) = \int_0^{2\pi} g_v(\rho,\theta;\beta)v(\beta)d\beta + \int_0^{2\pi}\int_0^1 g_u(\rho,\theta;\gamma,\beta)u(\gamma,\beta)d\gamma d\beta \quad . \tag{2.3.6c}$$

Note that the boundary value contribution to the solution in (2.3.6c) has been written as an integral over the interval $[0,2\pi]$. This is, of course, an integral over the boundary of the unit disk, i.e. the unit circle. $\underline{\equiv}$

As indicated in the previous section, Green's identity applied to the internal dynamics of the process to be estimated plays a key role in the construction of an internal realization of the complementary process and ultimately in the inversion required to solve the smoothing problem. In general, when it exists, Green's identity is obtained from integration by parts of the the N-fold integral specified by the following inner product:

$$\langle Lx,\lambda\rangle_{L_2^n(\Omega_N)} \quad .$$

This yields Green's Identity in the form

$$\langle Lx,\lambda\rangle_{L_2^n(\Omega_N)} = \langle x,L^\dagger\lambda\rangle_{L_2^n(\Omega_N)} + \text{boundary term} \quad . \tag{2.3.7}$$

Here $L^\dagger$ is also a formal differential operator of the same order as L and is referred to as the formal adjoint differential operator. The boundary term is in the form of an integral over the boundary $\partial\Omega_N$. This integral involves the processes x and λ and perhaps their derivatives evaluated on $\partial\Omega_N$. In general, the precise form of (2.3.7) is

$$\langle Lx,\lambda\rangle_{L_2^n(\Omega_N)} = \langle x,L^\dagger\lambda\rangle_{L_2^n(\Omega_N)} + \langle x_b,E\lambda_b\rangle_{H_b} \tag{2.3.8}$$

where x_b and λ_b are elements of a Hilbert space H_b of processes defined on $\partial\Omega_N$ and $E:H_b \to H_b$. In particular, these processes are defined through the action of an operator Δ_b:

$$\Delta_b : L_2^n(\Omega_N) \to H_b \tag{2.3.9a}$$

$$x_b = \Delta_b x \tag{2.3.9b}$$

$$\lambda_b = \Delta_b \lambda \quad . \tag{2.3.9c}$$

The nature of H_b, Δ_b, and E all depend upon L and Ω_N. Green's identity for ordinary differential operators can be found in [4] (also see Chapter 3 of this thesis); for elliptic, hyperbolic and parabolic second order partial differential operators see [3] and Chapter 7. In addition to the well-posedness assumption on (L,V), we will also restrict ourselves to operators L and regions Ω_N that admit a Green's identity.

<u>Example 1 Continued:</u>

Recall that in this example x is a scalar function (so that n=1), and Ω_2 is the unit disk. Performing the integration by parts on

$$\langle \nabla^2 x, \lambda \rangle_{L_2(\Omega_2)} = \iint_{\Omega_2} \big(\nabla^2 x(s,t) \big) \lambda(s,t) ds dt \tag{2.3.10a}$$

it can be shown that the Laplacian ∇^2 is formally self-adjoint [3], i.e.

$$\langle \nabla^2 x, \lambda \rangle = \langle x, \nabla^2 \lambda \rangle + \text{boundary term} \quad . \tag{2.3.10b}$$

The boundary term, for this example is expressed as follows. With x_n the normal derivative of x along $\partial\Omega_N$, define the function x_b as

$$x_b = \Delta_b x = \begin{bmatrix} x \big|_{\partial\Omega_N} \\ x_n \end{bmatrix}, \text{ or in polar coordinates } \quad x_b(\theta) = \begin{bmatrix} x(1,\theta) \\ x_n(1,\theta) \end{bmatrix} \quad . \tag{2.3.11}$$

Thus, in this case x_b is an element of the Hilbert space

$$H_b = L_2^2(\partial\Omega_2)$$

with inner product

$$\langle w,z\rangle_{H_b} = (1/2\pi)\int_0^{2\pi}\left[w_1(\theta)z_1(\theta) + w_2(\theta)z_2(\theta)\right]d\theta \quad . \tag{2.3.13}$$

The function λ_b is defined in terms of λ in the same fashion as x_b in (2.3.11). Furthermore, the action of the operator E in this case is simply multiplication of elements of $L_2^2(\partial\Omega_2)$ by a 2×2 matrix which we also denote by E. Specifically,

$$E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad . \tag{2.3.14}$$

Combining these, we have that the boundary term in Green's identity for this example is

$$\langle x_b, E\lambda_b\rangle_{L_2^2(\partial\Omega_2)} = \frac{1}{2\pi}\int_0^{2\pi}\left(\lambda(1,\theta)x_n(1,\theta) - x(1,\theta)\lambda_n(1,\theta)\right)d\theta \quad . \tag{2.3.15}$$

$\underline{\equiv}$

Operator Representations of Stochastic Processes

While we have so far been discussing deterministic functions, we can use the same formalism to describe stochastic processes. In particular, we will be concerned with stochastic processes defined as in (2.3.4b) where u and v are input stochastic processes. As is usually done, we will often represent such processes via a stochastic differential equation written as in (2.3.4a).

In general for the problems of interest here the operators G_u and G_v are integral operators over Ω_N and $\partial\Omega_N$ respectively[2] with piecewise continuous kernels. That is, they are integral operators of the form

$$G:\ L_2^n(\Gamma) \to L_2^m(\Gamma)$$

$$(Gf)(\underline{t}) = \int_\Gamma g(\underline{t},\underline{s})f(\underline{s})d\underline{s} \quad ; \quad \underline{t},\underline{s} \in \Gamma \tag{2.3.16a}$$

where Γ may be either Ω_N or $\partial\Omega_N$ and $d\underline{s}$ is the corresponding

[2] The only exception is in the case $N = 1$, in which Ω_N is an interval and $\partial\Omega_N$ consists of its two endpoints. In this case G_v reduces to a matrix operating on the vector v.

infinitesimal area element. Here we have underlined the independent variables $\underline{t}$ and $\underline{s}$ to emphasize the fact that they are multidimensional indices belonging to the region Γ. The adjoint of this operator is of the same form as (2.3.16a) but with kernel

$$g^*(\underline{t},\underline{s}) = g'(\underline{s},\underline{t}) \quad . \tag{2.3.16b}$$

By extending the class of kernels to include the multidimensional Dirac delta function, $\delta(\underline{t}-\underline{s})$, over Γ, we can express products of the form

$$g(\underline{t})f(\underline{t}) \quad ; \qquad \underline{t} \in \Gamma \tag{2.3.17a}$$

as

$$(Gf)(\underline{t}) = \int_\Gamma g(\underline{t})\delta(\underline{t}-\underline{s})f(\underline{s})d\underline{s} \quad . \tag{2.3.17b}$$

Note that in the case in which $\Gamma = \partial\Omega_N$, the Dirac delta function $\delta(\underline{t}-\underline{s})$ represents the measure on the manifold $\partial\Omega_N$ which is concentrated at the point $\underline{t}$ and has unit value.

In addition to the operators described above we will also encounter integral operators which map $L_2^n(\Gamma)$ into R^m and are of the form

$$Gf = \int_\Gamma g(\underline{s})f(\underline{s})d\underline{s} \tag{2.3.18}$$

and those of the form of the adjoint of (2.3.18) which map R^m into $L_2^n(\Gamma)$:

$$(G^*f)(\underline{s}) = g'(\underline{s})f \quad . \tag{2.3.19}$$

Each of the integral operators in (2.3.16) through (2.3.18) can be applied to mean-square continuous processes with the output being either a mean-square continuous process or a finite variance random vector. In the case of the operator defined in (2.3.19), if the input is a finite variance random vector, then the output is a mean-square continuous process.

We can formally extend the domain of the operators in (2.3.16a) and (2.3.18) to include white noise as follows. Consider those operators with $\Gamma = \Omega_N$. In this case, as in [12], we can define Wiener integrals with respect to multidimensional Wiener processes. Similar to the one-dimensional case, these multidimensional Wiener integrals can be represented formally by multidimensional white noise integrals. Specifically, let $w(\underline{t})$ denote an N-dimensional white noise process with

$$E[w(\underline{t})w'(\underline{s})] = Q(\underline{t})\delta(\underline{t}-\underline{s}) \quad . \tag{2.3.20}$$

(Here δ is again the N-dimensional delta function.) We will use the following white noise integral representation

$$\int_{\Omega_N} g(\underline{t},\underline{s})w(\underline{s})d\underline{s} \tag{2.3.21}$$

to denote the Wiener integral over Ω_N with respect to the kernel g. With g continuous, (2.3.21) defines a mean-square continuous process. Note that the critical property of such white noise integrals is the following: If we define two processes x and y as

$$x(\underline{t}) = \int_{\Omega_N} g_1(\underline{t},\underline{s})w(\underline{s})d\underline{s} \quad \text{and} \quad y(\underline{t}) = \int_{\Omega_N} g_2(\underline{t},\underline{s})w(\underline{s})d\underline{s} \; ,$$

then their correlation function is given by

$$E[x(\underline{t})y'(\underline{\tau})] = \int_{\Omega_N} g_1(\underline{t},\underline{s})Q(\underline{s})g_2'(\underline{\tau},\underline{s})d\underline{s} \quad . \tag{2.3.22}$$

In an analogous fashion, we can define white noise integrals over the smooth, closed (N-1)-dimensional manifold $\partial\Omega_N$. Specifically, if we now let $w(\underline{t})$ denote a white noise process on $\partial\Omega_N$ and if we write

$$x(\underline{t}) = \int_{\partial\Omega_N} g_1(\underline{t},\underline{s})w(\underline{s})d\underline{s} \quad \text{and} \quad y(\underline{t}) = \int_{\partial\Omega_N} g_2(\underline{t},\underline{s})w(\underline{s})d\underline{s} \; , \tag{2.3.23a}$$

then

$$E[x(\underline{t})y'(\underline{\tau})] = \int_{\partial\Omega_N} g_1(\underline{t},\underline{s})Q(\underline{s})g_2'(\underline{s},\underline{\tau})d\underline{s} \tag{2.3.23b}$$

where in this case $d\underline{s}$ represents an infinitesimal area element on $\partial\Omega_N$. Note that if the support of $g_1(\underline{t},\cdot)$ and $g_2(\underline{t},\cdot)$ in (2.3.23) are on subsets of $\partial\Omega_N$ which are homeomorphic to a subset of an (N-1)-dimensional region then the integrals in (2.3.23) are precisely the same as those in (2.3.22) except on a space of dimension (N-1). For this reason, we will often represent the covariance function of a white noise process on $\partial\Omega_N$ exactly as in (2.3.20) where in this case the Dirac delta function on $\partial\Omega_N$ is to be interpreted as described previously.

Applying this formalism, we will consider the differential equation in (2.3.3a) with white noise inputs as a formal representation for the mean-square continuous process defined via the integral representations in (2.3.4b) where u is now an n×1 vector white noise over Ω_N with covariance

$$E[u(\underline{t})u'(\underline{s})] = Q(\underline{t}) \ \delta(\underline{t}-\underline{s}) \qquad , \qquad \underline{t},\underline{s} \in \Omega_N \qquad (2.3.24a)$$

and in problems for which N > 1, v is an n_v×1 vector white noise process over $\partial\Omega_N$ with covariance

$$E[v(\underline{\sigma})v'(\underline{\tau})] = \Pi_v(\underline{\tau}) \ \delta(\underline{\tau}-\underline{\sigma}) \qquad , \qquad \underline{\tau},\underline{\sigma} \in \partial\Omega_N \quad . \qquad (2.3.24b)$$

If N=1, for example when Ω_N is the interval [0,T] so that $\partial\Omega_N = \{0,T\}$, then v is simply an n_v×1 random vector with covariance matrix Π_v.

Throughout this thesis we will assume that both Π_v and Q are continuous in their arguments. Thus, given the continuity assumptions for the integration kernels in the Green's function solution and the continuity of Q and Π_v, x will be mean-square continuous. For the example of Poisson's equation on the unit disk, the dynamics in (2.3.5a) and the Dirichlet boundary condition in (2.3.5b) formally represent a 2-D random field x when the input u is a 2-D white noise over the unit disk and v is a 1-D white noise on the unit circle or, equivalently, on the interval [0,2π].

Correlation Operators

In this subsection we introduce some notation for representing the correlation functions of mean-square continuous stochastic processes as operators on Hilbert spaces of deterministic functions. In particular, the correlation functions of these stochastic processes can be viewed as kernels of such operators. In addition, we will also associate the correlation and cross-correlation functions of processes generated by second order mappings of the type discussed above with kernels of composite operators. Although we make use of this notation in stating the general estimation problem, its true worth will become evident later in Section 2.4.1 where we formulate an operator representation for complementary processes.

The correlation function of a mean-square continuous process can be viewed as the kernel of an operator as follows. Let $z(\underline{t})$ be a mean-square continuous process on Ω_N. Its correlation function

$$\Sigma_z(\underline{t},\underline{s}) = E[z(\underline{t})z'(\underline{s})] \tag{2.3.25}$$

is continuous on $\Omega_N \times \Omega_N$ and as such can be considered the kernel of an operator, which we will denote by Σ_z, of the same type as in (2.3.16a). Similarly, the correlation function for the white noise process w in (2.3.20),

$$\Sigma_w(\underline{t},\underline{s}) = Q(\underline{t})\delta(\underline{t}-\underline{s}) \quad , \tag{2.3.26}$$

can be viewed as the kernel of an operator Σ_w. In addition, the covariance matrix of a random vector can be viewed as an operator on a finite-dimensional space. Indeed, each of the operators defined in (2.3.16) through (2.3.19) can be associated with the cross-correlations of zero mean processes with each other or with random vectors.

More generally, the correlation functions of stochastic processes defined by a second order mapping of the type in (2.3.16) through (2.3.19) can be represented in terms of the composition of operators as follows. For example, let $r(\underline{s})$ be a mean-square continuous process with correlation operator Σ_r. Let the process $z(\underline{t})$ be defined by

$$z(\underline{t}) = (Gr)(\underline{t}) = \int_{\Omega_N} g(\underline{t},\underline{s})r(\underline{s})d\underline{s} \quad . \tag{2.3.27}$$

The correlation function of z and the cross-correlation function of z and r are the kernels of the operators

$$\Sigma_z = G\Sigma_r G^* \tag{2.3.28a}$$

and

$$\Sigma_{rz} = \Sigma_r G^* \quad . \tag{2.3.28b}$$

These are easily checked by directly computing the corresponding kernels from (2.3.27a). Note that these formulas are valid when r is either a second order process or a white noise process.

2.3.2 Discrete Parameter Stochastic Processes

In a parallel but much briefer fashion we define the class of discrete parameter stochastic processes to be considered. In order to unify later discussions to include both classes, we employ much of the same operator notation in describing the discrete process as was used to describe the continuous processes.

Let I_N denote the space of n-tuples $(i_1,\ldots,i_N)$ where each of the i_k is an integer. Let Ω_N be a bounded region in I_N and let $l_2^n(\Omega_N)$ be the space of square-summable n×1 vector sequences on Ω_N. For the discrete case, L represents a formal linear difference operator. As we will see below in the example, the support of the sequences in the range of a difference operator will be different from the support of those sequences in its domain. By properly defining the boundary set $\partial\Omega_N$, one can define the support of the sequences in the range of the formal difference operator as the union of Ω_N and $\partial\Omega_N$ (Again, the example will help clarify this point). Thus L is the mapping

$$L: \; l_2^n(\Omega_N \cup \partial\Omega_N) \rightarrow l_2^n(\Omega_N) \quad . \tag{2.3.29a}$$

Let V be an operator mapping[3]

$$V: \; l_2^n(\Omega_N \cup \partial\Omega_N) \rightarrow l_2^{n_v}(\partial\Omega_N) \tag{2.3.29b}$$

where, as in the continuous case, the value of the dimension n_v is such that the difference operator formed by combining L and V

$$\Lambda = \begin{bmatrix} L \\ V \end{bmatrix} \tag{2.3.29c}$$

has a unique left inverse, i.e. so that the problem is well-posed.

---

[3] As in the continuous paramenter case, when N = 1, the range of the boundary operator is simply R^{n_v}.

As in the continuous case, we will consider only those L and Ω_N which admit a Green's identity of the form

$$\langle Lx,\lambda\rangle_{l_2^n(\Omega_N)} = \langle x, L^{\dagger}\lambda\rangle_{l_2^n(\Omega_N)} + \langle x_b, E\lambda_b\rangle_{H_b} \tag{2.3.30}$$

where $L^{\dagger}$ is the formal adjoint difference operator and the nature of E, x_b, and the Hilbert space H_b all depend on L and Ω_N. Much as in the continuous case, x_b has support on $\partial\Omega_N$, and H_b is a Hilbert space of square-summable sequences with support on $\partial\Omega_N$. In contrast to the continuous case, Green's identity is not typically employed in the solution of boundary value difference equations, and therefore, it is not usually found in texts on difference equations. However, it can be derived in the same manner as its continuous counterpart, the difference being that integration by parts is replaced with summation by parts. The Green's identities for one- and two-dimensional difference operators are derived in Chapters 5 and 6, respectively. Below we present the Green's identity for a particular 1-D example.

<u>Example 2:</u>(1-D Discrete Boundary Value Problem) Let x be an $n\times 1$ vector 1-D discrete process, and let D denote the 1-D delay

$$(Dx)_k = x_{k-1} \quad , \tag{2.3.31}$$

Consider the 1-D difference operator

$$L = (D^{-1}I - A) \quad ; \; (Lx)_k = x_{k+1} - A_k x_k \quad . \tag{2.3.32}$$

If we define $\Omega_1 = [0,K-1]$ and its boundary as $\partial\Omega_1 = \{0,K\}$, then the range and domain of L are properly specified by

$$L\colon\; l_2^n[\Omega_1 \cup \partial\Omega_1] \rightarrow l_2^n[\Omega_1]$$

or

$$L\colon\; l_2^n[0,K] \rightarrow l_2^n[0,K-1] \quad .$$

This example already illustrates one important point. Due to sequencing issues for discrete dynamics, it will in general be the case that $\partial\Omega_N$ is neither disjoint from nor a subset of Ω_N. The Green's identity for this example is

$$\langle Lx,\lambda\rangle_{l_2^n[0,K-1]} = \langle x,L^\dagger\lambda\rangle_{l_2^n[0,K-1]} + \langle x_b,E\lambda_b\rangle_{R^{2n}} \tag{2.3.33}$$

where the formal adjoint difference operator is

$$L^\dagger = (I - A'D^{-1}) \quad ; \quad (L^\dagger x)_k = x_k - A_k' x_{k+1} \quad , \tag{2.3.34}$$

the boundary process is

$$x_b = \Delta_b x = \begin{bmatrix} x_0 \\ x_K \end{bmatrix} \tag{2.3.35a}$$

and E is a 2n×2n matrix partitioned into n×n blocks as

$$E = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} \quad . \tag{2.3.35b}$$

Combining these definitions, the Green's identity can be written as[4]

$$\sum_{k=0}^{K-1} (x_{k+1} - A_k x_k)'\lambda_{k+1} = \sum_{k=0}^{K-1} (\lambda_k - A_k'\lambda_{k+1})'x_k - x_0'\lambda_0 + x_K'\lambda_K \quad . \tag{2.3.35c}$$

We will assume a two-point boundary condition for this discrete example as follows. With V the n×2n matrix:

$$V = [\ V^0 : V^K\ ] \quad , \tag{2.3.36a}$$

the product

$$v = V\, x_b \tag{2.3.36b}$$

defines a two-point boundary condition (here $v \in R^n$, i.e. $n_v = n$). Let $\Phi[i,k]$ denote the transition matrix for A in (2.3.27). Then it can be shown

[4] Actually this is Green's identity written in terms of a shifted process. See Chapter 5 for details.

that the pair (L,V) is well-posed if the n×n matrix

$$F = V^0 + V^K \Phi[K,0] \qquad (2.3.36c)$$

is invertible [6]. $\equiv$

Returning to the general discrete case, a formal representation for a second order discrete stochastic process x is given by

$$\Lambda x = \begin{bmatrix} L \\ V \end{bmatrix} x = \begin{bmatrix} u \\ v \end{bmatrix} \quad ; \qquad \Lambda: l_2^n(\Omega_N \cup \partial\Omega_N) \to \left(l_2^n(\Omega_N) \times l_2^{n_v}(\partial\Omega_N)\right)$$

where u is now an n×1 vector discrete white noise process over Ω_N and v is a second order discrete process whose support is contained in $\partial\Omega_N$.

2.3.3 The Estimation Problem Statement

The process to be estimated is either defined on a continuous or discrete index set as described in the first two parts of this section. In stating the estimation problem, we unify the discussion to include both classes.

Let L be a formal linear differential or difference operator with range R(L) and domain D(L), where elements in each are n×1 vector functions with index set Ω_N. Let B(t) be an n×m matrix with $t \in \Omega_N$. For the continuous parameter case B(t) is assumed continuous in t. Let H_b be the Hilbert space of n_b×1 vector functions whose support is the boundary $\partial\Omega_N$. Recall that the dimension n_b is determined from Green's identity for L and Ω_N. Let V be a mapping from H_b to R(V) where the nature of the range space R(V) is determined by the well-posedness condition for the pair (L,V) (see, for example, (2.3.29b) and the discussion that follows this equation). As opposed to the more general form Vx = v for the boundary condition, we will restrict the boundary condition to be defined in terms of x_b as indicated in (2.3.37b) below. For example, in the discrete case we will consider only those operators V defined in (2.3.29b) which map sequences on $\partial\Omega_N$ and therefore can

be thought of as having $l_2^n(\partial\Omega_N)$ as their domain (see Example 2, and in particular (2.3.36), for a specific illustration of such a map V).

Let u be an m×1 vector white noise on Ω_N with an invertible correlation operator Q. Let v be an n_v×1 vector second order process over $\partial\Omega_N$, uncorrelated with u and with invertible correlation operator Π_v. Then the the process to be estimated is formally defined by

$$Lx = Bu \tag{2.3.37a}$$

with boundary condition

$$Vx_b = v \quad . \tag{2.3.37b}$$

The observations are defined as follows. Let C(t) be a p×n matrix, $t \in \Omega_N$. For the continuous parameter case it is assumed that C is continuous in t. Let W be an operator mapping elements of H_b into R(W), a space of $n_w \times 1$ vector functions defined over the index set $\partial\Omega_N$. Let r be a p×1 vector white noise over Ω_N with invertible correlation operator R, and let r_b be a $n_w \times 1$ vector process with invertible correlation operator Π_b. It will be assumed that u, v, r and r_b are mutually uncorrelated. The set of observations of x is given by:

$$y = Cx + r \qquad \text{on } \Omega_N \tag{2.3.38a}$$

and

$$y_b = Wx_b + r_b \qquad \text{on } \partial\Omega_N \quad . \tag{2.3.38b}$$

We will need to make some assumptions with respect to the relationship between the operators V and W. The importance of these assumptions will become apparent later in our development of Hilbert adjoint systems in Section 2.5.1. As explained in the 1-D continuous case studied in Chapter 3, one consequence of these assumptions is that no element of the boundary observation y_b can simply be absorbed into updating the boundary value v alone. That is, the boundary measurement contains information about the part of x_b not captured by Vx_b. In particular, we will assume that W and V are linearly independent, i.e. for any linear operators M_v and M_w whose range spaces are identical and whose domains are the range spaces of V and W respectively, we must have

$$M_v V + M_w W = 0 \quad \text{iff} \quad M_v = 0 \text{ and } M_w = 0 \quad . \tag{2.3.39a}$$

Furthermore, we will assume that if the operator obtained by augmenting V and W as

$$\begin{bmatrix} V \\ W \end{bmatrix} \tag{2.3.39b}$$

is not invertible, then there exists an operator W_c such that

$$\begin{bmatrix} V \\ W \\ W_c \end{bmatrix} \tag{2.3.39c}$$

is invertible.

Our estimation problem is to find the linear minimum variance estimate of x given the set

$$Y = \{ y, y_b \} \quad . \tag{2.3.40}$$

To transform this problem into notation similar to that used for the static example, let the inverse of (2.3.37) be denoted by

$$x = M_x \begin{bmatrix} u \\ v \end{bmatrix} \quad . \tag{2.3.41}$$

This is simply the Green's function solution from (2.3.4b) written in different notation. Recall from (2.3.9b) that $x_b = \Delta_b x$, so that the combined observation in (2.3.40) can be written as

$$\begin{bmatrix} y \\ y_b \end{bmatrix} = \begin{bmatrix} C \\ W\Delta_b \end{bmatrix} M_x \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} r \\ r_b \end{bmatrix} \quad . \tag{2.3.42}$$

If we define H as

$$H = \begin{bmatrix} C \\ W\Delta_b \end{bmatrix} M_x \tag{2.3.43}$$

and specify the underlying process as

$$\zeta = \begin{bmatrix} u \\ v \\ r \\ r_b \end{bmatrix} \quad , \tag{2.3.44a}$$

then the observations can be expressed in a form similar to (2.2):

$$\begin{bmatrix} y \\ y_b \end{bmatrix} = [\ H \ \vdots \ I\ ]\ \zeta \quad . \tag{2.3.44b}$$

Below we illustrate the problem statement for each of our two examples. In the succeeding sections, we formulate the solution to this class of problems in a differential operator form with y and y_b as the input and boundary condition respectively and the estimate of x as an element of the output.

Examples

Continuous case: (Example 1 continued) In this case Ω_2 is the unit disk and points within the disk will be represented by index variables $s,t \in \Omega_2$. Points on the unit circle $\partial\Omega_2$ will be denoted by an angle $\theta \in [0,2\pi]$. Let u be a scalar white noise over Ω_2 with continuous covariance parameter Q(s). Let v be a scalar white noise over $\partial\Omega_2$ with continuous covariance parameter $\Pi_v(\theta)$ (which, of course, is periodic with period 2π). Let B(s) be a continuous function on Ω_2 and $V(\theta)$ be a nonzero continuous function on $\partial\Omega_2$. The process to be estimated is formally defined by

$$\nabla^2 x(s) = B(s)u(s) \tag{2.3.45a}$$

with boundary condition (in polar coordinates)

$$V(\theta)x(1,\theta) = v(\theta) \quad . \tag{2.3.45b}$$

Let r be a scalar white noise over Ω_2 and r_b be a scalar white noise over $\partial\Omega_2$ with continuous covariance parameters R(s) and $\Pi_b(\theta)$ respectively. Let C(s) be a continuous function on Ω_2 and $W(\theta)$ be a nonzero continuous function on $\partial\Omega_2$. The observations are defined by

$$y(s) = C(s)\ x(s) + r(s) \qquad \text{on } \Omega_2 \tag{2.3.46a}$$

and

$$y_b(\theta) = W(\theta)x_n(\rho,\theta) + r_b(\theta) \quad ; \quad \rho = 1 \quad . \tag{2.3.46b}$$

The estimation problem is to find the least squares estimate of x given y on Ω_2 and y_b on its boundary. Note that since the augmented operator

$$\begin{bmatrix} V \\ W \end{bmatrix} x_b = \begin{bmatrix} V(\theta) & \vdots & 0 \\ - - & \vdots & - - \\ 0 & \vdots & W(\theta) \end{bmatrix} \begin{bmatrix} x(1,\theta) \\ x_n(1,\theta) \end{bmatrix}$$

is invertible, there will be no complementing operator W_c as in (2.3.39c).

$\equiv$

Discrete case:(Example 2 continued) Recall that Ω_1 is the set of integers [0,K-1], and $\partial\Omega_1$ is the set {0,K}. Let u be a m×1 vector white noise over Ω_1 with nonsingular covariance matrix Q_k, $k \in \Omega_1$. Let v be a n×1 random vector with nonsingular covariance matrix Π_v. Let B_k be an n×m matrix and A_k be a n×n matrix both on Ω_1, and let V be a full rank n×2n matrix with n×n partitions $[V^0 \vdots V^K]$. The process to be estimated is defined by the difference equation

$$x_{k+1} = A_k x_k + B_k u_k \tag{2.3.47a}$$

with a two-point boundary condition

$$v = V^0 x_0 + V^K x_K \quad . \tag{2.3.47 }$$

To define the observations, let r be a p×1 white noise over Ω_1 whose covariance matrix R_k is nonsingular on Ω_1, and let r_b be a q×1 random vector with nonsingular covariance matrix Π_b. Let C_k be a p×n matrix on Ω_1 and let W be a full rank q×2n matrix with $q \leq n$, with the rows of W linearly independent of the rows of V and with q×n partitions: $[W^0 : W^K]$. Then the observations are given by

$$y_k = C_k x_k + r_k \qquad \text{on } \Omega_1 \tag{2.3.48a}$$

along with the random vector

$$y_b = W^0 x_0 + W^K x_K + r_b \quad . \tag{2.3.48b}$$

For both examples the input processes u and v and the observation noises r and r_b are all assumed to be mutually uncorrelated. For this example, when

$q < n$, the augmented $(n+q)\times 2n$ matrix in the following equation

$$\begin{bmatrix} V \\ \\ W \end{bmatrix} x_b = \left[\begin{array}{c:c} V^0 & V^K \\ \hdashline W^0 & W^K \end{array}\right] \begin{bmatrix} x_0 \\ \\ x_K \end{bmatrix} \tag{2.3.49}$$

will not be invertible. Thus, to attain an invertible matrix we must choose W_c as an $(n-q)\times 2n$ matrix whose rows are linearly independent of the rows of V and W so that

$$\begin{bmatrix} V \\ W \\ W_c \end{bmatrix}$$

is invertible. As we will see in Section 2.5.4, we only need to actually construct such a matrix in those cases for which Π_V is singular.

SECTION 2.4

OPERATOR FORM FOR M_z

Based on the matrix form for M_z established for the static example in Section 2.2.1, in this section we present an expression for a mapping of the underlying process defining our estimation problem, and we then prove that the resulting process satisfies the orthogonality and complementation conditions required of the complementary process. Only the continuous parameter case is addressed here; however, with a few obvious changes the same arguments can be adapted to the discrete parameter case. Indeed, since all discrete stochastic processes can be represented by a (possibly very large) random vector, the matrix representation for M_z from the static example in Section 2.2.1 is itself applicable for the discrete case.

2.4.1 An Operator Representation for the Complementary Process

It will be convenient to partition the underlying process ζ into two parts denoted by ζ_1 and ζ_2. The first part ζ_1 corresponds to the boundary value and input process, and the second part ζ_2 represents the additive noise on the observations:

$$\zeta = \begin{bmatrix} \zeta_1 \\ -- \\ \zeta_2 \end{bmatrix} = \begin{bmatrix} u \\ v \\ -- \\ r \\ r_b \end{bmatrix} \quad . \tag{2.4.1}$$

The covariance parameters of the elements of ζ are assumed to be continuous and the covariance parameters and covariance matrices are all assumed invertible. As discussed previously, the second order statistics of ζ can be defined by way of a correlation operator. The range and domain of this operator are identical and are defined by way of the following spaces:

$$S_1 = L_2^n(\Omega_N) \times L_2^{n_v}(\partial\Omega_N) \quad , \tag{2.4.2a}$$

$$S_2 = L_2^p(\Omega_N) \times L_2^q(\partial\Omega_N) \tag{2.4.2b}$$

and

$$S = S_1 \times S_2 \quad . \tag{2.4.2c}$$

As discussed earlier in Section 2.3.1, when $\partial\Omega_N$ is finite (i.e. when N = 1), the L_2 spaces of functions over $\partial\Omega_N$ should be replaced by the Euclidian spaces R^{n_v} and R^q. The correlation operator Σ_ζ is the self-adjoint invertible mapping

$$\Sigma_\zeta : S \rightarrow S \tag{2.4.3a}$$

which we will express in partitioned form as

$$\Sigma_\zeta = \begin{bmatrix} \Sigma_{\zeta_1} & 0 \\ 0 & \Sigma_{\zeta_2} \end{bmatrix} \quad . \tag{2.4.3b}$$

The observations are defined via the operator M_y

$$M_y : S \rightarrow S_2 \tag{2.4.4a}$$

where from (3.40b)

$$M_y = [\ H : I\ ] \quad . \tag{2.4.4b}$$

Thus the observation Y is given by the second order mapping

$$\begin{aligned} Y &= M_y \zeta \\ &= H\zeta_1 + \zeta_2 \quad . \end{aligned} \tag{2.4.5}$$

Theorem: (Complementary Process) Let M_z be the mapping

$$M_z = [\ -I \ \vdots \ H^* \ ] \ \Sigma_\zeta^{-1} \quad ; \quad M_z : \mathbf{S} \rightarrow \mathbf{S}_1 \tag{2.4.6a}$$

where

H^* is the Hilbert adjoint of H in (2.3.40),

I is the identity on $\mathbf{S}_1$,

and

Σ_ζ^{-1} is the inverse of Σ_ζ in (2.4.2).

Then the stochastic process given by the second order mapping

$$Z = M_z \ \zeta \tag{2.4.6b}$$

is the complementary process for the observations Y in (2.4.4), i.e. Z in (2.4.6b) satisfies both the orthogonality and complementation conditions as prescribed in our restatement of the projection theorem.

Proof:

Orthogonality: This condition requires that the correlation between elements of Y and Z are zero, or equivalently that the kernel of the correlation operator

$$\Sigma_{yz} = M_y \ \Sigma_\zeta \ M_z^* \tag{2.4.7a}$$

is identically zero. Substituting from (2.4.4) and (2.4.6), Σ_{yz} can be written as

$$\begin{aligned} \Sigma_{yz} &= [\ H \ \vdots \ I \ ] \ \Sigma_\zeta \ \Sigma_\zeta^{-1} \begin{bmatrix} -I \\ H \end{bmatrix} \\ &= 0 \quad , \end{aligned} \tag{2.4.7b}$$

verifying the orthogonality condition.

Complementation: In order to prove the complementation condition, we will need the following lemma [7] which is the multidimensional version of a basic result from the theory of Fredholm integral equations.

> Lemma: Let Ω_N be a multidimensional index set. Let $\underline{s}$ and $\underline{t}$ be index variables in Ω_N. Let $g(\underline{t},\underline{s})$ be a symmetric $\left(g(\underline{t},\underline{s}) = g(\underline{s},\underline{t}) \right)$ continuous kernel defining an integral operator
>
> $$G: L_2(\Omega_N) \rightarrow L_2(\Omega_N)$$
>
> with G having no negative eigenvalues. Then the operator $(I + G)$ has a unique inverse of the same form: $(I + G)^{-1} = (I + K)$, where the kernel of the integral operator K is also symmetric and continuous.

To establish the complementation condition it is sufficient to show that the augmented map M given by (see (2.2.17a))

$$M = \begin{bmatrix} M_y \\ \\ M_z \end{bmatrix} \quad ; \qquad M: \mathbf{S} \rightarrow \mathbf{S} \tag{2.4.8a}$$

is invertible. Substituting for M_y and M_z we have the explicit representation for M

$$M = \left[\begin{array}{c:c} H & I \\ \hdashline -\Sigma_{\zeta_1}^{-1} & H^* \Sigma_{\zeta_2}^{-1} \end{array}\right] \quad . \tag{2.4.8b}$$

Assuming the existence of the inverses in its partitions, the following operator can be shown by direct calculation to be the inverse of M

$$M^{-1} = \left[\begin{array}{c|c} \Sigma_{\zeta_1} H^* (\Sigma_{\zeta_2} + H \Sigma_{\zeta_1} H^*)^{-1} & -(\Sigma_{\zeta_1}^{-1} + H^* \Sigma_{\zeta_2}^{-1} H)^{-1} \\ \hdashline \Sigma_{\zeta_2} (\Sigma_{\zeta_2} + H \Sigma_{\zeta_1} H^*)^{-1} & H(\Sigma_{\zeta_1}^{-1} + H^* \Sigma_{\zeta_2}^{-1} H)^{-1} \end{array}\right] \tag{2.4.9}$$

i.e. $\quad M M^{-1} = M^{-1} M = I \quad .$

The lemma is invoked to establish the invertiblity of the operators

$$\left(\Sigma_{\zeta_2} + H\Sigma_{\zeta_1} H^*\right) \quad \text{and} \quad \left(\Sigma_{\zeta_1}^{-1} + H^* \Sigma_{\zeta_2}^{-1} H\right)$$

by the following argument. Rewrite these two operators as

$$\Sigma_{\zeta_2}^{1/2}\left(I + \Sigma_{\zeta_2}^{-1/2} H\Sigma_{\zeta_1} H^* \Sigma_{\zeta_2}^{-1/2}\right)\Sigma_{\zeta_2}^{1/2} \tag{2.4.10a}$$

and

$$\Sigma_{\zeta_1}^{-1/2}\left(I + \Sigma_{\zeta_1}^{1/2} H^* \Sigma_{\zeta_2}^{-1} H \Sigma_{\zeta_1}^{1/2}\right)\Sigma_{\zeta_1}^{-1/2} \quad . \tag{2.4.10b}$$

If we choose symmetric continuous kernels for the square roots in (2.4.10a,b), then the kernels of the operators in the parentheses will be symmetric, positive and continuous, and the lemma ensures the aforementioned invertibility.

2.4.2 An Operator Representation for the Estimator

Recall from the Projection theorem that the estimator (2.2.20) for x is given by

$$\hat{x} = M_x \, M^{-1} \begin{bmatrix} Y \\ 0 \end{bmatrix} \quad . \tag{2.4.11}$$

By substituting for M^{-1} from (2.4.9), we obtain an explicit operator representation for the estimator:

$$\hat{x} = M_x \begin{bmatrix} \Sigma_{\zeta_1} H^* \\ - - \\ \Sigma_{\zeta_2} \end{bmatrix} \left(\Sigma_{\zeta_2} + H\Sigma_{\zeta_1} H^*\right)^{-1} Y \quad . \tag{2.4.12}$$

Similarly, it can be shown that the estimation error (2.2.21) can be expressed as a linear function of the underlying process ζ

$$\tilde{x} = M_x \begin{bmatrix} \Sigma_{\zeta_1} H^* \\ - - \\ \Sigma_{\zeta_2} \end{bmatrix} \left(\Sigma_{\zeta_2} + H\Sigma_{\zeta_1} H^*\right)^{-1} \left[\, -I \,:\, H \,\right] \Sigma_{\zeta}^{-1} \zeta \quad . \tag{2.4.13}$$

A direct implementation of the estimator in (2.4.12) requires a realization of the indicated inverse. As an alternative, in Section 5 we obtain a realization for the estimator without explicitly performing this inversion. As we indicated in Section 2.2, Green's identity plays a critical role in this formulation. In particular, by invoking Green's identity we will be able to formulate a differential realization of the Hilbert adjoint H^*, yielding a differential realization of the complementary process Z. We will find that augmenting the differential operator representation for Y with that for Z results in a system which is easily inverted to give a differential operator representation of the estimator and estimation error.

2.4.3 M_z For Σ_{ζ_1} Singular

There are cases of interest for which there may be a singular correlation operator for either the input process or boundary condition. For example, in Section 2.6.3 we consider the example of a 1-D periodic process for which the boundary condition is, by the nature of the process, known without error. In this section we state a form for the operator M_z which does not require the invertibility of Σ_{ζ_1}. An outline of the proof of the orthogonality and complementation conditions is given. We remark that although the form for M_z presented earlier in (2.4.6a) could be derived directly from the one given below in this section, we have deliberately separated the two. As we will see later, the estimator for the case when Σ_{ζ_1} is singular is somewhat more complex than that for the case when it is nonsingular.

The action of the following operator on the underlying process (cf. (2.4.6b)) defines the complementary process and does not require the invertibility of Σ_{ζ_1}:

$$M_z = [\ -I \ : \ \Sigma_{\zeta_1} H^* \Sigma_{\zeta_2}^{-1}] \tag{2.4.14}$$

The orthogonality condition for the complementary process formed in this way is easily established by the same approach taken in (2.4.7a,b). The complementation condition is proved by showing that the inverse of

$$M = \begin{bmatrix} M_y \\ M_z \end{bmatrix} = \begin{bmatrix} H & I \\ -I & \Sigma_{\zeta_1} H^* \Sigma_{\zeta_2}^{-1} \end{bmatrix} \tag{2.4.15}$$

is given by

$$M^{-1} = \begin{bmatrix} \Sigma_{\zeta_1} H^* (\Sigma_{\zeta_2} + H\Sigma_{\zeta_1} H^*)^{-1} & -(I + \Sigma_{\zeta_1} H^* \Sigma_{\zeta_2}^{-1} H)^{-1} \\ \Sigma_{\zeta_2} (\Sigma_{\zeta_2} + H\Sigma_{\zeta_1} H^*)^{-1} & H(I + \Sigma_{\zeta_1} H^* \Sigma_{\zeta_2}^{-1} H)^{-1} \end{bmatrix} \tag{2.4.16}$$

The existence of the inverse of the operators in the left hand column of (2.4.16) is established in (2.4.10a). The existence of the inverse required in the right hand column is proved by invoking the operator version of the matrix inversion lemma [13] to write:

$$(I + \Sigma_{\zeta_1} H^* \Sigma_{\zeta_2}^{-1} H)^{-1} = I - \Sigma_{\zeta_1} H^* (\Sigma_{\zeta_2} + H\Sigma_{\zeta_1} H^*)^{-1} H \quad . \tag{2.4.17}$$

Note that the form of M_z in (2.4.6a) is obtained by operating on the left of (2.4.14) by the inverse of Σ_{ζ_1}.

SECTION 2.5

A DIFFERENTIAL OPERATOR GOVERNING THE ESTIMATE

In this section we derive a differential operator representation for the estimator. The key to its derivation is the formulation of a differential operator representation for the complementary process whose I/O map is given in (2.4.6). It is in the formulation of this differential representation for the complementary process that the Green's Identity introduced in Section 2.2 plays an important role. With differential representations for both the process to be estimated and the corresponding complementary process, we will find that the augmentation and inversion steps (cf. Section 2.2.3) required in the formulation of the estimator become trivial.

2.5.1 The Hilbert Adjoint System

In the previous section we proved that the complementary process is given by

$$z = \left[\; -I \;\vdots\; H^* \right] \Sigma_\zeta^{-1} \zeta \qquad . \tag{2.5.1a}$$

Substituting from (2.4.1) and (2.4.3) into (2.5.1a), we can view the complementary process as an output signal plus noise:

$$z = H^* \Sigma_{\zeta_2}^{-1} \zeta_2 - \Sigma_{\zeta_1}^{-1} \zeta_1 \qquad . \tag{2.5.1b}$$

Our objective in this section is to formulate an internal realization for the input-output map H^*. The internal process in this realization is defined by a differential operator whose input process and boundary condition are the inputs to H^*.

To determine an internal differential realization for H^*, we temporarily leave the stochastic setting. That is, throughout the rest of this subsection all processes should be considered as elements of Hilbert spaces of deterministic functions rather than stochastic processes.

The internal realization for the input-output map H is

$$Lx = Bu \tag{2.5.2a}$$

$$Vx_b = v \tag{2.5.2b}$$

$$\Phi = \begin{bmatrix} \phi \\ \phi_b \end{bmatrix} = \begin{bmatrix} Cx \\ Wx_b \end{bmatrix} \quad \text{i.e.} \quad \Phi = H\begin{bmatrix} u \\ v \end{bmatrix} \quad . \tag{2.5.3}$$

Each of the maps L, B, V, C and W has been defined in Section 2.3.3. It will be convienient to define the spaces containing u and v as D_u and D_v respectively so that the domain of H can be written as $D(H) = D_u \times D_v$. Similarly, define the range spaces containing the output elements ϕ and ϕ_b as R_ϕ and R_{ϕ_b} so that the range of H is $R(H) = R_\phi \times R_{\phi_b}$.

The Hilbert adjoint of H is defined to be that operator which maps from the range of H into the domain of H

$$H^*: R(H) \rightarrow D(H) \tag{2.5.4a}$$

and for which the inner product identity

$$\langle H\xi, \eta\rangle_{R(H)} = \langle \xi, H^*\eta\rangle_{D(H)} \tag{2.5.4b}$$

is satisfied for arbitrary ξ and η in D(H) and R(H) respectively [8].

The first step in determining an internal realization for H^* is to rewrite (2.5.4b) in a more convenient form. Since the input u in (2.5.2a) enters only through the action of B, we can decompose H as

$$H = \tilde{H}\begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix} \tag{2.5.5a}$$

If we denote the range of B by R_B, then $\tilde{H}: (R_B \times D_v) \rightarrow R(H)$. Given this decomposition of H, its adjoint H^* can be decomposed as

$$H^* = \begin{bmatrix} B^* & 0 \\ 0 & I \end{bmatrix} \tilde{H}^* \quad . \tag{2.5.5b}$$

If we denote the input process by

$$\xi = \begin{bmatrix} u \\ v \end{bmatrix} \tag{2.5.6}$$

and if we partition η, which is an element of $R(H) = R_\phi \times R_{\phi_b}$ and denote its partitions as

$$\begin{bmatrix} u_\lambda \\ v_\lambda \end{bmatrix} \equiv \eta \quad ; \quad \begin{matrix} u_\lambda \in R_\phi \\ v_\lambda \in R_{\phi_b} \end{matrix} \quad , \tag{2.5.7}$$

then substituting (2.5.5b) and (2.5.7) into the right hand side of (2.5.4b) gives

$$\langle H\xi, \eta\rangle = \langle \xi, \begin{bmatrix} B^* & 0 \\ 0 & I \end{bmatrix}, \tilde{H}^* \begin{bmatrix} u_\lambda \\ v_\lambda \end{bmatrix} \rangle$$

$$= \langle \begin{bmatrix} Bu \\ v \end{bmatrix}, \tilde{H}^* \begin{bmatrix} u_\lambda \\ v_\lambda \end{bmatrix} \rangle \quad . \tag{2.5.8}$$

Similarly, defining the partitions of $\tilde{H}^*\eta$ which is an element of $D(\tilde{H})$, as

$$\begin{bmatrix} \lambda \\ \psi_b \end{bmatrix} \equiv \tilde{H}^* \begin{bmatrix} u_\lambda \\ v_\lambda \end{bmatrix} \quad ; \quad \begin{matrix} \lambda \in R_B \\ \psi_b \in D_v \end{matrix} \tag{2.5.9}$$

and substituting for Bu and v from (2.5.2a) and (2.5.2b), (2.5.8) becomes

$$\begin{aligned} \langle H\xi, \eta\rangle &= \langle Lx, \lambda\rangle + \langle Vx_b, \psi_b\rangle \\ &= \langle Lx, \lambda\rangle + \langle x_b, V^*\psi_b\rangle \quad . \end{aligned} \tag{2.5.10}$$

Finally, by noting from (2.5.3) that

$$H\xi = \begin{bmatrix} \phi \\ \phi_b \end{bmatrix} = \begin{bmatrix} Cx \\ Wx_b \end{bmatrix} \quad ,$$

we can rewrite the left hand side of (2.5.10) so that the inner product identity in (2.5.4b) can be expressed as

$$\langle x, C^* u_\lambda \rangle + \langle x_b, W^* v_\lambda \rangle = \langle Lx, \lambda \rangle + \langle x_b, V^* \psi_b \rangle \quad . \tag{2.5.11}$$

Up to this point we have simply combined some new notation along with that for the internal representation for H to re-express the inner product identity (2.5.4b). The next step is more substantial and is a key one in the development of the internal realization for H^*. In particular, we employ Green's identity from (3.18) to replace $\langle Lx, \lambda \rangle$ in (2.5.11). Then (2.5.4b) can be written in terms of the formal adjoint differential (difference) operator $L^\dagger$:

$$\langle x, [C^* u_\lambda - L^\dagger \lambda] \rangle = \langle x, [E\lambda_b + V^* \psi_b - W^* v_\lambda] \rangle \quad . \tag{2.5.12}$$

Although the Hilbert adjoint H^* is a unique map, there exists a family of equivalent internal differential realizations. Using the notation introduced above, we will verify one internal realization for H^* with input η and output $\Psi = \{\psi, \psi_b\}$ by showing that it satisfies (2.5.12).

Let W_c be one of the family of operators which complements V and W (see equation (2.3.35c)), in that

$$\Gamma \equiv \begin{bmatrix} -V \\ W_c \\ W \end{bmatrix} \tag{2.5.13}$$

is invertible (as will become clear shortly, we have included a minus sign in defining Γ for convenience). Employing the inverse of Γ and the operator E in the boundary term of Green's identity (2.3.13), define the partitioned operator

$$\begin{bmatrix} W_\lambda \\ V_{\lambda c} \\ V_\lambda \end{bmatrix} \equiv (\Gamma^*)^{-1} E \quad . \tag{2.5.14}$$

This leads to an expression for E that will be useful later:

$$E = \left[-V^* : W_c^* : W^*\right]\begin{bmatrix} W\lambda \\ V_{\lambda c} \\ V_\lambda \end{bmatrix} = -V^* W_\lambda + W_c^* V_{\lambda c} + W^* V_\lambda \quad . \tag{2.5.15}$$

The following theorem establishes an internal differential realization for H^*.

Theorem: (Hilbert Adjoint System) An internal differential realization for the input-output map

$$\Psi = \begin{bmatrix} \psi \\ \psi_b \end{bmatrix} = H^* \begin{bmatrix} u_\lambda \\ v_\lambda \end{bmatrix} \tag{2.5.16a}$$

is given by an internal process λ satisfying

$$L^\dagger \lambda = C^* u_\lambda \tag{2.5.16b}$$

with boundary condition

$$\begin{bmatrix} V_\lambda \\ V_{\lambda c} \end{bmatrix} \lambda_b = \begin{bmatrix} v_\lambda \\ 0 \end{bmatrix} \tag{2.5.16c}$$

and output map

$$\Psi = \begin{bmatrix} \psi \\ \psi_b \end{bmatrix} = \begin{bmatrix} B^* \lambda \\ W_\lambda \lambda_b \end{bmatrix} \quad . \tag{2.5.16d}$$

Proof: With the dynamics of λ given by (2.5.16b), the left hand side of (2.5.12) is zero. To show that the right hand side is also zero, we employ (2.5.16d) and the first row of (2.5.16c) to rewrite the right hand side of (2.5.12) as

$$<x_b, [E\lambda_b + V^* \psi_b - W^* v_\lambda]> = <x_b, [E + V^* W_\lambda - W^* V_\lambda]\lambda_b> \quad . \tag{2.5.17a}$$

Substituting for E from (2.5.15) gives

$$\langle x_b, [E\lambda_b + V^*\psi_b - W^*v_\lambda]\rangle = \langle x_b, W_c^* V_{\lambda c}\lambda_b\rangle \quad , \tag{2.5.17b}$$

A further substitution from the second row of (2.5.16c) completes the proof:

$$\begin{aligned}\langle x_b, [E\lambda_b + V^*\psi_b - W^*v_\lambda]\rangle &= \langle x_b, 0\rangle \\ &= 0 \quad .\end{aligned} \tag{2.5.17c}$$

Thus (2.5.12) is satisfied with both the left and right hand sides identically zero. Although this differential realization is not unique due to the degrees of freedom in choosing W_c, we will show that the estimator itself is invariant with respect to the choice of W_c, as it must be.

2.5.2 Augmentation and Inversion

The internal differential realization for H^* in (2.5.16) defines a representation for the complementary stochastic process in (2.5.1a) as:

$$\begin{bmatrix} z \\ z_b \end{bmatrix} = [\ -I \ \vdots \ H^*]\Sigma_\zeta^{-1}\zeta \tag{2.5.18}$$

In this subsection we augment the internal realization for (2.5.18) with that for the observations to get an internal differential realization for the combined system:

$$\begin{bmatrix} Y \\ Z \end{bmatrix} = M\zeta \quad ; \quad M = \begin{bmatrix} M_Y \\ M_Z \end{bmatrix} \quad . \tag{2.5.19}$$

We then invert this realization to obtain an internal differential realization for the estimator.

The differential form for the augmented system in (2.5.19) is

$$\begin{bmatrix} L & 0 \\ 0 & L^\dagger \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} B & 0 \\ 0 & C^*R^{-1} \end{bmatrix} \begin{bmatrix} u \\ r \end{bmatrix} \tag{2.5.20a}$$

with boundary condition

$$\begin{bmatrix} v \\ \hline 0 \\ \Pi_b^{-1} r_b \end{bmatrix} = \begin{bmatrix} V & \vdots & 0 \\ \hline 0 & \vdots & V_{\lambda c} \\ 0 & \vdots & V_\lambda \end{bmatrix} \begin{bmatrix} x_b \\ \lambda_b \end{bmatrix} \tag{2.5.20b}$$

and outputs

$$\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} C & 0 \\ 0 & B^* \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} + \begin{bmatrix} 0 & I \\ -Q^{-1} & 0 \end{bmatrix} \begin{bmatrix} u \\ r \end{bmatrix} \quad \text{on } \Omega_N \tag{2.5.20c}$$

$$\begin{bmatrix} y_b \\ z_b \end{bmatrix} = \begin{bmatrix} W & 0 \\ 0 & W_\lambda \end{bmatrix} \begin{bmatrix} x_b \\ \lambda_b \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & -\Pi_v^{-1} \end{bmatrix} \begin{bmatrix} v \\ r_b \end{bmatrix} \quad \text{on } \partial\Omega_N . \tag{2.5.20d}$$

As indicated by (2.5.19), the inverse system we seek is one with $\{Y,Z\} = \{y,y_b,z,z_b\}$ as input and $\zeta = \{u,v,r,r_b\}$ as output. To this end, following the approach taken by Levy et al for the 1-D causal case in [9], we first solve for the elements of ζ by inverting the output equations (2.5.20c) and (2.5.20d):

$$\begin{bmatrix} u \\ r \end{bmatrix} = \begin{bmatrix} -Q & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} z \\ y \end{bmatrix} + \begin{bmatrix} 0 & QB^* \\ -C & 0 \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} \tag{2.5.21a}$$

$$\begin{bmatrix} v \\ r_b \end{bmatrix} = \begin{bmatrix} -\Pi_v & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} z_b \\ y_b \end{bmatrix} + \begin{bmatrix} 0 & \Pi_v W_\lambda \\ -W & 0 \end{bmatrix} \begin{bmatrix} x_b \\ \lambda_b \end{bmatrix} . \tag{2.5.21b}$$

Substituting these expressions into the dynamics and boundary conditions in (2.5.20a) and (2.5.20b) yields an internal differential realization of the inverse system with dynamics:

$$\begin{bmatrix} L & \vdots & -BQB^* \\ \hline C^*R^{-1}C & \vdots & L^\dagger \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} -BQ & \vdots & 0 \\ \hline 0 & \vdots & C^*R^{-1} \end{bmatrix} \begin{bmatrix} z \\ y \end{bmatrix} \tag{2.5.22}$$

and with boundary condition:

$$\begin{bmatrix} z_b \\ 0 \\ \Pi_b^{-1} y_b \end{bmatrix} = \begin{bmatrix} -\Pi_v^{-1} V & : & W_\lambda \\ - - - & : & - - \\ 0 & : & V_{\lambda c} \\ - - - & : & - - \\ \Pi_b^{-1} W & : & V_\lambda \end{bmatrix} \begin{bmatrix} x_b \\ \lambda_b \end{bmatrix} \quad . \tag{2.5.23}$$

This boundary condition can be simplified so that its dependence on W_c, $V\lambda$, and $V\lambda_c$ is eliminated. Recalling the relation between these operators and E in Green's identity from (2.5.15), it can be shown that operating on the left of (2.5.23) by $[-V^* : W_c{}^* : W^*]$ gives the boundary condition as

$$[\, W^* \Pi_b^{-1} y_b - V^* z_b \,] = [\, W^* \Pi_b^{-1} W + V^* \Pi_v^{-1} V \; \vdots \; E \,] \begin{bmatrix} x_b \\ \lambda_b \end{bmatrix} \quad . \tag{2.5.24}$$

The estimator is the solution of (2.5.22) and (2.5.24) projected onto Sp(Y), i.e. the solution with $Z = \{z, z_b\} = 0$:

$$\begin{bmatrix} L & : & -BQB^* \\ - - - - & : & - - \\ C^* R^{-1} C & : & L^\dagger \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{\lambda} \end{bmatrix} = \begin{bmatrix} 0 \\ - - - \\ C^* R^{-1} y \end{bmatrix} \tag{2.5.25a}$$

$$W^* \Pi_b^{-1} y_b = [\, W^* \Pi_b^{-1} W + V^* \Pi_v^{-1} V \; \vdots \; E \,] \begin{bmatrix} \hat{x}_b \\ \hat{\lambda}_b \end{bmatrix} \tag{2.5.25b}$$

The estimates of the elements of the underlying process ζ, if desired, can be computed form the output equations (2.5.21a) and (2.5.21b) evaluated at the solution of (2.5.25) and with z and z_b equal to zero. Note that since L and $L^\dagger$ are of the same order, the order of the estimator is twice that of L. Also note the remarkable fact that in addition to the original problem statement, we only need to know E and $L^\dagger$ from Green's identity in (2.3.13) to completely define the differential realization for the estimator. In Appendix 2B we show how the estimator boundary condition (2.5.25b) changes when the process boundary condition v is nonzero mean.

2.5.3 The Estimation Error

The estimation error

$$\tilde{x} = x - \hat{x} \qquad (2.5.26)$$

is obtained as the solution of (2.5.22) and (2.5.24) projected onto Sp(Z) rather than Sp(Y). Here we formulate a differential realization of the estimation error which is driven by ζ whose probability law is known. The second order statistics of the estimation error can be computed from those of ζ using this relation.

Recall from the restatement of the Projection Theorem in Section 2.2.3 that

$$\tilde{x} = M_x M^{-1} \begin{bmatrix} 0 \\ Z \end{bmatrix} = M_x M^{-1} \begin{bmatrix} 0 \\ M_z \end{bmatrix} \zeta \quad . \qquad (2.5.27)$$

That is, Z has been replaced by its representation given in terms of ζ. Consider the boundary condition (2.5.24) projected onto Sp(Z), i.e. (2.5.24) evaluated with y_b equal to zero:

$$-V^* z_b = [\, W^* \Pi_b^{-1} W + V^* \Pi_v^{-1} V \,\vdots\, E \,] \begin{bmatrix} \tilde{x}_b \\ \tilde{\lambda}_b \end{bmatrix} \qquad (2.5.28)$$

Substituting for z_b from (2.5.23b)

$$z_b = W_\lambda \lambda_b - \Pi_v^{-1} v \qquad (2.5.29a)$$

and using the basic definition:

$$\tilde{\lambda}_b = \lambda_b - \hat{\lambda}_b \quad , \qquad (2.5.29b)$$

(5.28) becomes

$$V^* \Pi_v^{-1} v = [\, W^* \Pi_b^{-1} W + V^* \Pi_v^{-1} V \,\vdots\, E + V^* W_\lambda \,\vdots\, V^* W_\lambda \,] \begin{bmatrix} \tilde{x}_b \\ \tilde{\lambda}_b \\ \hat{\lambda}_b \end{bmatrix} \quad . \qquad (2.5.30)$$

To eliminate the dependence on W_λ, V_λ and $V_{\lambda c}$ as we had done for the estimator, recall from (2.5.15) and (2.5.16b) that

$$E + V^* W_\lambda = W^* V_\lambda + W_c^* V_{\lambda c} \quad , \tag{2.5.31a}$$

$$V_\lambda \lambda_b = V_\lambda(\tilde{\lambda}_b + \hat{\lambda}_b) = \Pi_b^{-1} r_b \tag{2.5.31b}$$

and

$$W_c^* V_{\lambda c} \lambda_b = W_c^* V_{\lambda c}(\tilde{\lambda}_b + \hat{\lambda}_b) = 0 \quad . \tag{2.5.31c}$$

From these three equations we can write

$$[\, E + V^* W_\lambda]\tilde{\lambda}_b = [\, W^* V_\lambda + W_c^* V_\lambda] \, [\lambda_b - \hat{\lambda}_b]$$

$$= W^* \Pi_b^{-1} r_b - [W^* V_\lambda + W_c^* V_{\lambda c}]\hat{\lambda}_b \quad , \tag{2.5.32}$$

and substituting into (2.5.28), the boundary condition becomes

$$[\, V^* \Pi_v^{-1} v - W^* \Pi_b^{-1} r_b] = [\, W^* \Pi_b^{-1} W + V^* \Pi_v^{-1} V \;\vdots\; E \,] \begin{bmatrix} \tilde{x}_b \\ -\hat{\lambda}_b \end{bmatrix} \quad . \tag{2.5.33}$$

We have chosen $-\hat{\lambda}_b$ instead of $\hat{\lambda}_b$ to highlight the similarity between the structure of the boundary condition for the estimation error in (2.5.33) and that of the estimator in (2.5.25).

The projection of (2.5.22) onto Sp(Z) gives the error dynamics as

$$\begin{bmatrix} L & \vdots & -BQB^* \\ - & \vdots & - \\ C^* R^{-1} C & \vdots & L^* \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{\lambda} \end{bmatrix} = \begin{bmatrix} -BQz \\ - \\ 0 \end{bmatrix} \quad . \tag{2.5.34}$$

Replacing z from (2.5.20c)

$$z = B^* \lambda - Q^{-1} u \quad , \tag{2.5.35a}$$

employing

$$\tilde{\lambda} = \lambda - \hat{\lambda} \tag{2.5.35b}$$

and recalling from (2.5.20a) that the dynamics of λ are given by

$$L^{\dagger}\lambda = C^{*}R^{-1}r \quad , \tag{2.5.35c}$$

(2.5.34) can be rewritten as

$$\begin{bmatrix} L & \vdots & -BQB^{*} \\ \hline C^{*}R^{-1}C & \vdots & L^{\dagger} \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \hline -\hat{\lambda} \end{bmatrix} = \begin{bmatrix} Bu \\ \hline -C^{*}R^{-1}r \end{bmatrix} \quad . \tag{2.5.36}$$

Thus (2.5.33) and (2.5.36) completely define the estimation error in terms of $\zeta = \{u,v,r,r_b\}$ whose probability law is known. In addition, the dynamics and boundary conditions of the estimation error have been shown to be similar to those of the estimator. One should be able to take advantage of these similarities when computing the estimate and its error covariance. For example, see the discussion of the implementation of the estimator and the computation of the error covariance for the 1-D noncausal process in Chapter 3.

2.5.4 Special Case: Π_v Singular

In Section 2.4.3 we presented a model for the complementary process which did not require the invertiblity of the covariance parameters Q and Π_v (i.e. the invertibility of Σ_{ζ_1}). In this section we define the estimator for the case when Π_v is singular.

By augmenting with the complementary process defined through (2.4.14) and inverting, it can be shown that we arrive at the same dynamics for the estimator as obtained previously in (2.5.22). However, the boundary condition for the inverted system is slightly different than that in (2.5.23). In particular, the boundary condition in this case is:

$$\begin{bmatrix} \Pi_v z_b \\ \hline 0 \\ \hline \Pi_b^{-1} y_b \end{bmatrix} = \begin{bmatrix} -V & \vdots & \Pi_v W_\lambda \\ \hline 0 & \vdots & V_{\lambda c} \\ \hline \Pi_b^{-1} W & \vdots & V_\lambda \end{bmatrix} \begin{bmatrix} x_b \\ \lambda_b \end{bmatrix} \quad . \tag{2.5.37}$$

As we had done for (2.5.23), we will rewrite this boundary condition in terms of the operator E found in Greeen's Identity. Let Ψ be the partitioned operator:

$$\Psi = \begin{bmatrix} \Pi_v & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \tag{2.5.38a}$$

with the partitions compatible with those of Γ in (2.5.13). Let Θ be the operator

$$\Theta = \Psi \, \Gamma^{*-1} \quad . \tag{2.5.38b}$$

Then recalling (2.5.15), we can write the boundary condition (2.5.37) as

$$\begin{bmatrix} \Pi_v z_b \\ \hline 0 \\ \hline \Pi_b^{-1} y_b \end{bmatrix} = \left[\begin{array}{c:c} -V & \\ \hline 0 & \Theta E \\ \hline \Pi_b^{-1} W & \end{array}\right] \begin{bmatrix} x_b \\ \lambda_b \end{bmatrix} \tag{2.5.39}$$

Projecting (2.5.39) onto Sp(Y) gives the boundary condition for the estimator as

$$\begin{bmatrix} 0 \\ \hline 0 \\ \hline \Pi_b^{-1} y_b \end{bmatrix} = \left[\begin{array}{c:c} -V & \\ \hline 0 & \Theta E \\ \hline \Pi_b^{-1} W & \end{array}\right] \begin{bmatrix} \hat{x}_b \\ \hat{\lambda}_b \end{bmatrix} \tag{2.5.40}$$

Following arguments similar to those in Section 2.5.3 and applying (2.5.38), it can be shown that the estimation error dynamics for this case are the same as those in (2.5.36), while the boundary condition is

$$\begin{bmatrix} -v \\ \hline 0 \\ \hline -\Pi_b^{-1} r_b \end{bmatrix} = \left[\begin{array}{c:c} -V & \\ \hline 0 & \Theta E \\ \hline \Pi_b^{-1} W & \end{array}\right] \begin{bmatrix} \tilde{x}_b \\ -\hat{\lambda}_b \end{bmatrix} \quad . \tag{2.5.41}$$

The boundary conditions in (2.5.40) and (2.5.41) may appear to be in as simple a form as their counterparts for the case when Π_V is invertible. However, the requirement to invert Γ^* in order to compute Θ in (2.5.38b) makes the boundary conditions derived in this subsection considerably more complex. Furthermore, in this formulation of the boundary condition, the operator W_C, one of the partitions of Γ, has not been eliminated. Since the estimator and error equations must be independent of the choice of W_C it should be possible to reduce our formulation to one which is independent of this choice. This remains an open question at this time. Nevertheless, we will see in an example of a periodic process in Section 2.6.3 that the form of the boundary condition presented here is useful in deriving the estimator for a particular case when Π_V is singular.

SECTION 2.6

THE ESTIMATE EQUATIONS FOR THE TWO EXAMPLES

By considering the two examples introduced earlier, we demonstrate the ease with which one can apply (2.5.25) to obtain an internal differential representation for the estimator of a noncausal stochastic process. We show that the estimator for the process governed by Poisson's equation takes the form of a fourth order biharmonic equation. In the case of the 1-D discrete boundary value process, it will be shown that a special case of the solution we obtain from (2.5.25) is a well-known form of the solution for the fixed-interval smoother for 1-D discrete causal processes [10]. In addition, in Section 2.6.3 we apply the solution for the estimator when Σ_{ζ_1} is singular to a discrete 1-D periodic process.

2.6.1 2-D Continuous Case: Poisson's Equation (Example 1)

The problem statement has been given in Section 2.3.3 by equations (2.3.45) and (2.3.46). As we have done previously for this example, s denotes an index variable representing elements of the unit disk, and θ represents elements of $[0,2\pi]$ which we have identified with $\partial\Omega_2$. Substituting for $L^{\dagger}$ in the estimator solution (2.5.25) from (2.3.10a), we obtain the estimator dynamics as

$$\begin{bmatrix} \nabla^2 & -B^2(s)Q(s) \\ -C^2(s)R^{-1}(s) & -\nabla^2 \end{bmatrix} \begin{bmatrix} \hat{x}(s) \\ \hat{\lambda}(s) \end{bmatrix} = \begin{bmatrix} 0 \\ -C(s)R^{-1}(s) \end{bmatrix} y(s) \quad . \tag{2.6.1}$$

Note from (2.3.45b) and (2.3.46b) that the boundary condition and boundary observation can be expressed by functions on $[0,2\pi]$ as

$$(Vx_b)(\theta) = [\ V(\theta) \ \vdots \ 0\ ]\, x_b(\theta) \tag{2.6.2a}$$

and

$$(Wx_b)(\theta) = [\ 0 \ \vdots \ W(\theta)\ ]\, x_b(\theta) \tag{2.6.2b}$$

where we recall that $x_b'(\theta) = [x(1,\theta), x_n(1,\theta)]$. Using this expression and substituting for E from (2.3.14), it can be shown that the boundary condition for (2.6.1) is (in polar coordinates evaluated at $\rho = 1$)

$$\begin{bmatrix} 0 \\ - - - - \\ W(\theta)\Pi_b^{-1}(\theta)y_b(\theta) \end{bmatrix} = \begin{bmatrix} V^2(\theta)\Pi_v^{-1}(\theta) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{x}(\rho,\theta) \\ \hat{\lambda}(\rho,\theta) \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ W^2(\theta)\Pi_b^{-1}(\theta) & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_n(\rho,\theta) \\ \hat{\lambda}_n(\rho,\theta) \end{bmatrix} . \quad (2.6.3)$$

When $B^2(s)Q(s) > 0$ for all s, we can solve for $\hat{\lambda}$ in (2.6.1) as

$$\hat{\lambda}(s) = [B^2(s)Q(s)]^{-1}\nabla^2 \hat{x}(s) \quad . \qquad (2.6.4)$$

Substituting (2.6.4) back into (2.6.1), we find that the estimator dynamics are given by the biharmonic equation:

$$\{\nabla^2[B^2(s)Q(s)]^{-1} + C^2(s)R^{-1}(s)\}\nabla^2 \hat{x}(s) = C(s)R^{-1}(s)y(s) \quad . \qquad (2.6.5)$$

With $\partial/\partial n$ denoting the normal derivative and substituting from (2.6.4), the boundary condition in (2.6.3) can be rewritten as

$$0 = \Pi_v^{-1}(\theta)\hat{x}(\rho,\theta) - (\partial/\partial n)\{[B^2(\rho,\theta)Q(\rho,\theta)]^{-1}\nabla^2 \hat{x}(\rho,\theta)\} \qquad (2.6.6a)$$

and

$$W(\theta)y_b(\theta) = W(\theta)^2\hat{x}_n(\rho,\theta) + \Pi_b(\theta)[B^2(\rho,\theta)Q(\rho,\theta)]^{-1}\nabla^2 \hat{x}(\rho,\theta) \qquad (2.6.6b)$$

evaluated at $\rho = 1$. We investigate methods for implementing this solution in Chapter 7. Indeed, one could employ one of the many available numerical techniques such as finite difference approximations [14].

2.6.2 1-D Discrete Case: Two-Point Boundary Value Process (Example 2)

Again we simply substitute from the problem statement in equations (2.3.47) and (2.3.48) and from (2.3.34) and (2.3.35b) for $L^\dagger$ and E into (2.5.25) to obtain the estimator dynamics

$$\begin{bmatrix} \hat{x}_{k+1} \\ \hat{\lambda}_k \end{bmatrix} = \begin{bmatrix} A_k & -B_kQ_kB_k' \\ -C_k'R_k^{-1}C_k & A_k' \end{bmatrix} \begin{bmatrix} \hat{x}_k \\ \hat{\lambda}_{k+1} \end{bmatrix} + \begin{bmatrix} 0 \\ C_k'R_k^{-1} \end{bmatrix} y_k \qquad (2.6.7a)$$

and boundary condition

$$\begin{bmatrix} W^{0'} \\ W^{K'} \end{bmatrix} \Pi_b^{-1} y_b = \begin{bmatrix} V^{0'}\Pi_v^{-1}V^0 + W^{0'}\Pi_b^{-1}W^0 & -I \\ V^{K'}\Pi_v^{-1}V^0 + W^{K'}\Pi_b^{-1}W^0 & 0 \end{bmatrix} \begin{bmatrix}\hat{x}_0 \\ \hat{\lambda}_0\end{bmatrix} + \begin{bmatrix} V^{0'}\Pi_v^{-1}V^K + W^{0'}\Pi_b^{-1}W^K & 0 \\ V^{K'}\Pi_v^{-1}V^K + W^{K'}\Pi_b^{-1}W^K & I\end{bmatrix}\begin{bmatrix}\hat{x}_K\\ \hat{\lambda}_K\end{bmatrix} . \tag{2.6.7b}$$

If we consider the special case of <u>no</u> boundary observation y_b (i.e. $W^0 = W^K = 0$) and an <u>initial</u> condition for x (i.e. $V^0 = I$, $V^K = 0$), then the boundary condition in (2.6.7b) becomes

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} I & -\Pi_v \\ 0 & 0 \end{bmatrix} \begin{bmatrix}\hat{x}_0 \\ \hat{\lambda}_0\end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix}\hat{x}_K \\ \hat{\lambda}_K\end{bmatrix} . \tag{2.6.8}$$

This boundary condition along with the dynamics in (2.6.7a) is recognized as the well-known solution for the fixed-interval smoother for causal discrete 1-D stochastic processes [10].

2.6.3 <u>1-D Discrete Case: A Periodic Process</u>

To illustrate the case when Π_v is singular, we consider a 1-D discrete periodic process. The dynamics of this process are the same as those introduced earlier except for the following periodicity constraints:

$$A_{k+K} = A_k \quad , \tag{2.6.9a}$$

$$B_{k+K} = B_k \tag{2.6.9b}$$

and

$$u_{k+K} = u_k \quad . \tag{2.6.9c}$$

Along with (2.6.9), the following boundary condition guarantees that x repeats itself with a period of K:

$$0 = x_0 - x_K \tag{2.6.10a}$$

i.e.

$$V = [\ I \ : \ -I\ ] \tag{2.6.10b}$$

and

$$\Pi_v = 0 \quad . \tag{2.6.10c}$$

We also assume for the moment that there is no boundary observation.

As discussed earlier in Section 2.5.4, the estimator dynamics for the case when Π_v is singular are the same as those in (2.6.7a) above. From (2.5.40) it can be shown that in this case the boundary conditions for the estimator can always be put in the following form (i.e. for any admissable choice of W_c)

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \hat{x}_0 \\ \hat{\lambda}_0 \end{bmatrix} + \begin{bmatrix} -I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} \hat{x}_K \\ \hat{\lambda}_K \end{bmatrix} . \tag{2.6.11}$$

If there is some additional a priori information about the zero mean random variable x_0, then it can be included as a boundary measurement as follows. Let Π_0 be the a priori variance of x_0. Then the boundary measurement

$$y_b = 0 = [\ I : 0\ ]x_b - r_b \quad , \tag{2.6.13a}$$

$$\text{i.e.} \quad W = [\ I : 0\ ] , \tag{2.6.13b}$$

where the variance of r_b is Π_0, will account for this information in the estimator. When y_b in (2.6.13) is included, the boundary condition for the estimator of the periodic process becomes (from (2.5.40)):

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ -\Pi_0^{-1} & I \end{bmatrix} \begin{bmatrix} \hat{x}_0 \\ \hat{\lambda}_0 \end{bmatrix} + \begin{bmatrix} -I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} \hat{x}_K \\ \hat{\lambda}_K \end{bmatrix} . \tag{2.6.14}$$

As shown in Chapter 5, solutions for discrete two-point boundary value problems representing the estimators in these last two sections can be implemented via a two-filter form similar to two-filter forms of the smoother for causal processes.

SECTION 2.7

CONCLUSIONS

Through an extension of the method of complementary models [1], we have developed a procedure for writing the estimator for both discrete and continuous parameter linear boundary value stochastic processes in a differential operator form. The two major steps in the development of the estimator have been (1) the formulation of an input-output operator representation for the complementary process in Section 2.4 and (2) the use of Green's identity in Section 2.5 in the derivation of an internal differential realization for this input-output map. We emphasize that at no point in our derivations have we required a Markov representation for the process to be estimated. The variety of problems for which our estimator solution is applicable has been illustrated through two examples: a 1-D discrete parameter process and a 2-D continuous parameter process.

The major advantage in specifying the estimate as the solution of a differential equation is that this form of representation provides an excellent starting point for the development of methods for obtaining the estimate. This is in contrast to estimators derived by a direct application of the projection theorem, which usually leads to integral equations (e.g. Wiener-Hopf) requiring factorization in order to obtain an implementation. Furthermore, we have also derived an internal differential realization for the estimation errors in a form which is nearly identical to that for the estimator. In Chapter 3 we apply the estimator solution formulated in this chapter to a continuous 1-D two-point boundary value stochastic process and develop a stable, recursive implementation for the resulting differentential form of the estimator. In addition, by following the same procedures as used to obtain the recursive estimator implementation, we develop recursions for the computation of the smoothing error covariance. Implementation of the estimator for discrete 1-D, discrete 2-D and continuous 2-D processes is investigated in Chapters 5, 6 and 7, respectively.

In addition to questions of implementation, there are also unanswered questions which relate to the boundary conditions for multi-D problems. For example, recall from (2.5.25b) that the boundary condition for our estimator is defined in terms of the operator adjoints V^* and W^* and the inverses of the correlation operators Π_v and Π_b. In our 2-D example we have tacitly avoided any complications which might arise in determining these adjoints and inverses by choosing v and r_b as white noise and by choosing V and W as a simple scaling of the process on the boundary (see (2.6.2a,b)). It would be of interest to investigate the estimator's boundary condition for this 2-D example when the boundary value v is, for instance, a 1-D periodic stochastic process on the unit circle. Another open issue concerning to the estimator boundary condition has already been raised in Section 2.5.4. That is, it should be possible to further simplify the expression in (2.5.41) for the estimator's boundary condition in the case when Π_v is singular.

In summary, this chapter presents what we feel is an extremely useful and broadly applicable method for deriving optimal estimators for noncausal processes in several dimensions. Given this valuable tool, one is then in a position to focus one's attention on the problem of implementing the optimal estimator in an efficient fashion. As mentioned previously, this is precisely the subject of Chapters 3, 5, 6 and 7.

APPENDIX 2A

DERIVATION OF M_z FOR THE STATIC EXAMPLE

In Section 2.2.1 it was stated that the complementary process associated with the observation vector y in (2.2.2a) is the n-dimensional random vector

$$z = M_z \zeta \qquad (2.A.1)$$

where, with T any invertible n×n matrix, M_z is given by

$$M_z = T[\, I \,\vdots\, -\Sigma_x H' \Sigma_r^{-1}\,] \quad . \qquad (2.A.2)$$

In this appendix we derive this general form for the matrix M_z.

Let M_z be denoted by

$$M_z = [Z_x \,\vdots\, Z_r] \qquad (2.A.3)$$

where the partitions are compatible with the partitions of ζ in (2.2.1a). With M_z given by (2.A.3), M in (2.2.6) is

$$M = \begin{bmatrix} M_y \\ M_z \end{bmatrix} = \begin{bmatrix} H & \vdots & I \\ - & - & - \\ Z_x & \vdots & Z_r \end{bmatrix} \quad . \qquad (2.A.4)$$

Now use the orthogonality and complementation conditions stated in Section 2.2.1 to find expressions for Z_x and Z_r.

Complementation

This condition implies that the (n+p)×(n+p) matrix M is invertible or that for k an (n+p)×1 vector

$$Mk = 0 \implies k = 0 \quad . \qquad (2.A.5)$$

If we denote partitions of k by

$$k = \begin{bmatrix} k_x \\ k_r \end{bmatrix} \begin{matrix} \leftarrow n\times 1 \\ \leftarrow p\times 1 \end{matrix} \quad , \qquad (2.A.6)$$

then Mk = 0 implies that

$$Hk_x = -k_r \qquad (2.A.7a)$$

and

$$Z_x k_x = -Z_r k_r \quad . \qquad (2.A.7b)$$

Substituting (2.A.7a) into (2.A.7b) we get the following condition which is equivalent to (2.A.5)

$$(Z_x - Z_r H)k_x = 0 \implies k_x = 0 \quad . \qquad (2.A.7c)$$

<u>Orthogonality</u>

This condition requires that the elements of y and z are uncorrelated

$$E[yz'] = 0$$

$$= [H \; : \; I] \begin{bmatrix} \Sigma_x & 0 \\ 0 & \Sigma_r \end{bmatrix} \begin{bmatrix} Z'_x \\ Z'_r \end{bmatrix} \qquad (2.A.8)$$

Thus

$$H\Sigma_x Z'_x = -\Sigma_r Z'_r \qquad (2.A.9a)$$

or

$$Z_r = -Z_x \Sigma_x H' \Sigma_r^{-1} \quad . \qquad (2.A.9b)$$

Combining (2.A.7c) and (2.A.9b) we have

$$Z_x(I + \Sigma_x H' \Sigma_r^{-1} H)k_x = 0 \implies k_x = 0 \quad . \qquad (2.A.9c)$$

Since

$$(I + \Sigma_x H' \Sigma_r^{-1} H)$$

is invertible, the statement (2.A.9c) is true if and only if Z_x is invertible. Therefore, let Z_x equal some invertible matrix T:

$$Z_x = T \quad . \tag{2.A.10a}$$

Then from (2.A.9b)

$$Z_r = -T\Sigma_x H'\Sigma_r^{-1} \quad , \tag{2.A.10b}$$

and from (2.A.3) we obtain the expression we seek

$$M_z = T[I \ \vdots \ -\Sigma_x H'\Sigma_r^{-1}] \quad . \tag{2.A.10c}$$

As mentioned in Section 2.2.1, different choices for T simply represent different bases for the complementary process. Below we consider two values for T which suggest the operator forms for M_z used in Sections 4.1 and 4.3

Two Special Cases

(1) If we let (Here we assume Σ_x is nonsingular)

$$T = -\Sigma_x^{-1} \quad ,$$

then

$$M_z = [-I \ \vdots \ H']\Sigma_\zeta^{-1} \quad ,$$

which leads to the operator form hypothesized in Section 2.4.1

(2) Another simple form is obtained if we set

$$T = -I \quad ,$$

then

$$M_z = [-I \ \vdots \ \Sigma_x H'\Sigma_r^{-1}] \quad .$$

This suggests the operator form employed in (2.4.14) which is applicable when Σ_x is nonsingular.

APPENDIX 2B

THE ESTIMATOR WHEN THE BOUNDARY VALUE v IS NONZERO MEAN

In deriving the estimator dynamics and boundary condition in Section 2.5.2, it was assumed that the a priori boundary condition v for the process to be estimated was zero mean. By applying the superposition principle for linear systems, in this appendix we determine how the estimator boundary condition should be changed to account for a nonzero mean boundary value v. First define this mean value as

$$E[v] = v_0 \tag{2.B.1a}$$

and the difference between v and its mean as

$$\delta v = v - v_0 \quad . \tag{2.B.1b}$$

Applying the superposition principle, we can write the solution of our original boundary value problem

$$Lx = Bu \tag{2.B.2a}$$

$$v = Vx_b \tag{2.B.2b}$$

as the sum of the solutions of the following two problems

$$Lx_0 = 0 \tag{2.B.3a}$$

$$v_0 = Vx_{b_0} \tag{2.B.3b}$$

and

$$L\delta x = Bu \tag{2.B.4a}$$

$$\delta v = V\ \delta x_b \quad . \tag{2.B.4b}$$

That is,

$$x = x_0 + \delta x \quad \text{and} \quad x_b = x_{b_0} + \delta x_b \quad . \tag{2.B.5}$$

The boundary value δv for the process δx is zero mean and as such we can apply the estimator derived in Section 2.5 to this process. First, the measurements

$$y = Cx + r \tag{2.B.6a}$$

and

$$y_b = Wx_b + r_b \tag{2.B.6b}$$

as defined in Section 2.3.4 must be rewritten in terms of δx. Define these mean values of the measuerments as

$$y_0 = Cx_0 \tag{2.B.7a}$$

and

$$y_{b_0} = Wx_{b_0} \quad . \tag{2.B.7b}$$

Removing these mean values from the original measurements gives measurements which are linear functions of δx:

$$\delta y = y - y_0$$

$$= C\delta x + r \tag{2.B.8a}$$

and

$$\delta y_b = y_b - y_{b_0}$$

$$= W\delta x_b + r_b \quad . \tag{2.B.8b}$$

With measurements given in this form we are ready to apply the estimator solution in (2.5.25) for the process δx. The estimator dynamics are

$$\begin{bmatrix} L & -BQB^* \\ C^*R^{-1}C & L^\dagger \end{bmatrix} \begin{bmatrix} \hat{\delta x} \\ \hat{\lambda} \end{bmatrix} = \begin{bmatrix} 0 \\ C^*R^{-1} \end{bmatrix} \delta y \tag{2.B.9a}$$

with boundary condition

$$W^*\Pi_b^{-1}\delta y_b = \left(W^*\Pi_b^{-1}W + V^*\Pi_v^{-1}V\right) \hat{\delta x}_b + E\hat{\lambda}_b \quad . \tag{2.B.9b}$$

Next we use the estimate of δx to construct an estimate of x.

The estimator for the complete process x is obtained by adding the dynamics for x_0 (2.B.3a) to the first row of (2.B.9a) and by adding both of the following to the boundary condition (2.B.9b):

$$V^* \Pi_v^{-1} v_0 = V^* \Pi_v^{-1} V x_{b_0} \tag{2.B.10a}$$

and

$$W^* \Pi_b^{-1} y_{b_0} = W^* \Pi_b^{-1} W x_{b_0} \quad . \tag{2.B.10b}$$

Carrying out the addition and noting from (2.B.5) that

$$\hat{x} = x_0 + \hat{\delta x} \tag{2.B.11a}$$

and

$$\hat{x}_b = x_{b_0} + (\hat{\delta x})_b \quad , \tag{2.B.11b}$$

gives the estimator for x as:

$$\begin{bmatrix} L & \vdots & -BQB^* \\ - - - - & \vdots & - - - \\ C^* R^{-1} C & \vdots & L^\dagger \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{\lambda} \end{bmatrix} = \begin{bmatrix} 0 \\ - - - \\ C^* R^{-1} \end{bmatrix} y \tag{2.B.12a}$$

with boundary condition

$$W^* \Pi_b^{-1} y_b + V^* \Pi_v^{-1} v_0 = \left(W^* \Pi_b^{-1} W + V^* \Pi_v^{-1} V \right) \hat{x}_b + E \hat{\lambda}_b \quad . \tag{2.B.12b}$$

Note that the estimator dynamics remain unchanged from those originally derived in (2.5.25a) and that the only change in the boundary condition (2.5.25b) is the addition of the term on the left hand side containing the nonzero mean value v_0. Of course, when this mean value is zero the two estimators are identical, as they should be.

CHAPTER 3: 1-D CONTINUOUS PARAMTER BOUNDARY VALUE PROCESSES

SECTION 3.1
INTRODUCTION

Both linear filtering and linear smoothing for one-dimensional (1-D), nonstationary, causal processes have been extensively studied. Many of the classical solutions to these problems are discussed in the review paper by Kailath [1]. The derivations of these solutions have relied heavily on the Markovian nature of the models for these 1-D processes [2]. However, inasmuch as stochastic processes in higher dimensions (random fields) are typically noncausal and consequently are not Markovian in the usual sense, their estimators cannot be derived through a direct extension of these 1-D derivations. Thus linear estimation problems for noncausal processes require new approaches. One such new approach has been developed in Chapter 2 where we extended Weinert and Desai's [1] method of complementary models. This extension allows us to write equations governing estimates for a broad class of noncausal processes in one and higher dimensions. In this chapter,we build upon this solution procedure in order to perform a detailed investigation of the smoothing problem for 1-D noncausal processes.

The processes that we consider are governed by the linear noncausal 1-D dynamic models introduced by Krener in [16]. In his study of these models, he has developed results on controllability, observability and minimality and has solved a deterministic linear control problem. In addition, he has posed the fixed-interval smoothing problem for these systems [17] and has derived integral equations for both the weighting pattern and error covariance of the optimal smoother. Working directly with these equations he has had success in obtaining a dynamic realization of the smoother for a special "stationary-cyclic" class of these models [18]. In this paper we begin by applying the solution for linear estimation of boundary value processes developed in Chapter 2, and we obtain a differential realization for the optimal smoother and the smoothing error for the complete class of 1-D noncausal processes considered by Krener. For a noncausal process defined by an n^{th} order model, this solution takes the form of a $2n^{th}$ order two-point boundary value problem. Typically, solutions for this type of boundary value problem are given in the Green's function form [3], and the smoother implementation implied by this form is such that the estimate at each point in the interval of interest is obtained by numerical quadrature over the entire interval. As an alternative, in this chapter we develop a two-filter implementation for our smoother which is remarkably similar to, and of nearly the same complexity as two-filter implementations developed for the fixed-interval smoother for causal processes [19,20]. As we will show, the advantage of such a two-filter form is that the estimate at each point in the interval is obtained through a linear combination of stable forward and stable backward recursions rather than numerical quadrature.

This chapter is organized as follows. In Section 3.2 the linear stochastic differential equation and boundary conditions which define the noncausal 1-D process that we study are presented.Along with the model for this process, two forms of the general solution are outlined and the matrix differential equation governing the evolution of the process covariance is given. The fixed-interval smoothing problem for this model is described in Section 3.3. In Section 3.4 we formulate a two-filter implementation of the smoother by applying a decoupling transformation to the smoother dynamics which are specified by the complementary models solution. Transformations of this type have previously been applied to the smoother for causal processes by Kailath and Ljung [21] and Desai [22]. A discussion of the properties of the smoother for some special cases including causal processes and a class of systems related to Krener's [23] "separable" systems is given in Section 3.5. In Section 3.6 we apply our smoother solution to a noncausal model representing a cooling fin. Finally, Section 3.7 contains some concluding remarks.

SECTION 3.2

LINEAR STOCHASTIC TWO-POINT BOUNDARY VALUE PROCESS (TPBVP)

3.2.1 The General Solution

The model for the one-dimensional boundary value stochastic process we consider here was introduced by Krener in [16]. The process is governed by an n^{th} order linear stochastic differential equation together with a specified two-point boundary condition. Accordingly, the process will be referred to as a linear stochastic two-point boundary value process or TPBVP. This linear boundary value process has been used to model a variety of space-time processes in temporal steady-state including the deflection of a beam under loading [3], the deflection of a rotating shaft [24] and the temperature distribution in a cooling fin [25]. (See the example in Section 3.6.)

As we have shown in Chapter 2, the formal structure of the linear stochastic differential equation governing the complementary process is defined by way of the structure of a related deterministic differential equation. For this reason, in Chapter 2 and here in Chapter 3 we find it convenient to employ the white noise formalism for representing linear stochastic differential equations. Let u(t) be a $m\times 1$ white noise process with covariance parameter Q(t). Let v be a $n\times 1$ random vector, independent of u(t), with covariance matrix Π_v. The $n\times 1$ boundary value process x(t) is governed on the interval [0,T] by

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \qquad (3.2.1a)$$

with boundary condition

$$v = V^0x(0) + V^Tx(T) \quad . \qquad (3.2.1b)$$

It will be assumed that A and B are continuous on [0,T] and that all random variables are zero-mean since the contribution of any nonzero mean can be added separately by invoking superposition. See Appendix 3B for a further discussion of nonzero mean boundary values.

It is instructive to derive one form of the general solution for (3.2.1) as the approach we take in this derivation will be used later. The form of the solution which we obtain differs from the usual Green's function solution (e.g. see [16]). Specifically, this derivation which is posed in the terminology of linear systems theory highlights the role of a process which we will denote below by x^0. Let $\Phi(t,s)$ be the transition matrix associated with $A(t)$. If $x(0)$ were known, then $x(t)$ could be represented in the variation-of-constants form

$$x(t) = \Phi(t,0)x(0) + x^0(t) \tag{3.2.2a}$$

where $x^0(t)$ is the solution of (3.2.1a) with $x^0(0) = 0$:

$$x^0(t) = \int_0^t \Phi(t,s)B(s)u(s)ds \quad . \tag{3.2.2b}$$

Substituting from (3.2.2a) at t = T into the boundary condition (3.2.1b), we can write

$$v - V^T x^0(t) = \left[V^0 + V^T\Phi(T,0)\right]x(0) \quad . \tag{3.2.3a}$$

For a well-posed problem, there will be a unique $x(0)$ for a given v and u on $[0,T]$. Thus well-posedness requires that the $n\times n$ matrix

$$F = V^0 + V^T\Phi(T,0) \tag{3.2.3b}$$

be nonsingular. With F invertible, we can solve for $x(0)$ as

$$x(0) = F^{-1}\left(v - V^T x^0(T)\right) \quad . \tag{3.2.3c}$$

Substituting $x(0)$ into (3.2.2a) gives the general solution for (3.2.1a,b) as

$$x(t) = \Phi(t,0)F^{-1}\left(v - V^T x^0(T)\right) + x^0(t) \quad . \tag{3.2.4}$$

The Green's function form of the general solution can be obtained from (3.2.4) by combining the two integrals representing $\Phi(t,0)F^{-1}V^T x^0(T)$ and $x^0(t)$ into a single integral over $[0,T]$.

The noncausal nature of the TPBVP x(t) is clearly displayed if we correlate the value of x at t = 0 with future values of the input u:

$$E\{x(0)u'(t)\} = -F^{-1}V^T\Phi(T,t)B(t)Q(t) \qquad t \in [0,T] \quad . \tag{3.2.5}$$

Thus, the n^{th} order model in (3.2.1) is not Markovian, and consequently Kalman filtering and associated smoothing techniques are not directly applicable.

It is often the case for a TPBVP that the system dynamics matrix A in (3.2.1a) will have both positive and negative eigenvalues (see the example in Section 3.6). In these cases, when implementing a solution for $x^0(t)$ in (3.2.2b) as an initial value problem, the positive eigenvalues may cause numerical instabilities. Below, as an alternative, we present a second form for the general solution of (3.2.1) which leads to a numerically stable implementation. Consider the equivalent process obtained by transforming x as

$$\begin{bmatrix} x_f(t) \\ x_b(t) \end{bmatrix} = T(t)x(t) \tag{3.2.6a}$$

where the transformation matrix T(t) is chosen so that 1) the dynamics of the system model in (3.2.1) become decoupled[1]:

$$\begin{bmatrix} \dot{x}_f \\ \dot{x}_b \end{bmatrix} = \begin{bmatrix} A_f & 0 \\ 0 & A_b \end{bmatrix} \begin{bmatrix} x_f \\ x_b \end{bmatrix} + \begin{bmatrix} B_f \\ B_b \end{bmatrix} u \tag{3.2.6b}$$

and 2) A_f is exponentially stable in the forward direction and A_b is exponentially stable in the backward direction. For "time"-invariant systems this is always possible by assigning those modes associated with eigenvalues greater than or equal to zero to A_f and those less than zero to A_b. For time-varying dynamics, it may be difficult to determine the dynamics and boundary conditions for a transformation T(t) which transforms the system dynamics into this form. However, we will find that by invoking results obtained previously for smoothing solutions for causal processes we can overcome this difficulty for the systems of interest to us later in this

[1] When there is no risk of confusion we will often omit explicit reference to the independent variable, i.e. $A(t) \to A$.

The boundary condition for the transformed process will be written in the following partitioned form:

$$v = \left[V_f^0 \;\vdots\; V_b^0 \right] \begin{bmatrix} x_f(0) \\ x_b(0) \end{bmatrix} + \left[V_f^T \;\vdots\; V_b^T \right] \begin{bmatrix} x_f(T) \\ x_b(T) \end{bmatrix} \quad . \tag{3.2.6c}$$

The reason for our choice of subscripts f and b, denoting forward and backward respectively, will become apparent below.

If $x_f(0)$ and $x_b(T)$ were known, then we could solve for $x_f(t)$ and $x_b(t)$ as

$$x_f(t) = \Phi_f(t,0)x_f(0) + x_f^0(t) \tag{3.2.7a}$$

and

$$x_b(t) = \Phi_b(t,T)x_b(T) + x_b^0(t) \tag{3.2.7b}$$

where $x_f^0(t)$ is governed by (3.2.6b) with $x_f^0(0) = 0$ and $x_b^0(t)$ is governed by (3.2.7b) with $x_b^0(T) = 0$. Following a derivation similar to that used to obtain the general solution in (3.2.4), it can be shown that

$$\begin{bmatrix} x_f(t) \\ x_b(t) \end{bmatrix} = \begin{bmatrix} \Phi_f(t,0) & \vdots & 0 \\ - & \vdots & - \\ 0 & \vdots & \Phi_b(t,T) \end{bmatrix} F_{fb}^{-1}\left(v - V_f^T x_f^0(T) - V_b^0 x_b^0(0)\right) + \begin{bmatrix} x_f^0(t) \\ x_b^0(t) \end{bmatrix} \tag{3.2.8}$$

where

$$F_{fb} = \left[V_f^0 + V_f^T \Phi_f(T,0) \;\vdots\; V_b^T + V_b^0 \Phi_b(0,T) \right] \quad . \tag{3.2.9}$$

The TPBVP x is recovered from (3.2.8) by inverting (3.2.6a):

$$x(t) = T^{-1}(t) \begin{bmatrix} x_f(t) \\ x_b(t) \end{bmatrix} \quad . \tag{3.2.10}$$

As we will see, the <u>two-filter</u> form of the general solution in (3.2.8) is the foundation for the implementation of the estimator that we develop later in Section 3.4. The term two-filter is used to signify that the numerical solution of (3.2.8) requires the integration of a forward process x_f^0 and a backward process x_b^0.

3.2.2 Covariance of the TPBVP x(t)

By a direct calculation, it can be shown that the covariance of x(t)

$$P_x(t) = E\{x(t)x'(t)\} \qquad (3.2.11a)$$

satisfies the differential equation

$$\dot{P}_x = AP_x + P_xA' + BQB' - BQB'\Phi'(T,t)V^{T'}F^{-1'}\Phi'(t,0) - \Phi(t,0)F^{-1}V^T\Phi(T,t)BQB' \;; \qquad (3.2.11b)$$

$$P_x(0) = F^{-1}(\Pi_V + V^T\Pi^0(T)V^{T'})F^{-1} \qquad (3.2.11c)$$

where Π^0 is governed by

$$\dot{\Pi}^0 = A\Pi^0 + \Pi^0A' + BQB' \quad ; \quad \Pi^0(0) = 0 \quad . \qquad (3.2.11d)$$

An alternative expression for P_x which requires the solution of only one matrix differential equation can be derived from (3.2.4) as

$$P_x(t) = P_x^0(t) + \Phi(t,0)F^{-1}[\Pi_V + V^TP_x^0(T)V^{T'}]F^{-1'}\Phi'(t,0) - \Phi(t,0)F^{-1}V^TP_x^0(t) - P_x^0(t)V^{T'}F^{-1'}\Phi'(t,0) \qquad (3.2.12a)$$

where $P_x^0(t)$ is the covariance of $x^0(t)$ satisfying

$$\dot{P}_x^0 = AP_x^0 + P_x^0A' + BQB' \quad ; \qquad P_x^0(0) = 0 \quad . \qquad (3.2.12b)$$

An additional expression for P_x can be derived from the two-filter form of the general solution (equation (3.2.8)). However, because this expression is somewhat complex, we will wait until later in Section 3.4 to present it in the context of our examination of the estimation error covariance.

3.2.3 Green's Identity

It was shown in Chapter 2 that the differential realization for the estimator is written in terms of the operators which define the Green's Identity for the differential operator governing the dynamics of the process to be estimated. In terms of the notation introduced in Chapter 2, the differential operator representing the dynamics in (3.2.1a) is

$$L:D(L) \rightarrow R(L) \quad ; \; (Lx)(t) = \dot{x}(t) - A(t)x(t) \tag{3.2.13}$$

where D(L) is the space of once continuously differentiable $n\times1$ vector functions on [0,T] and R(L) is the Hilbert space of square integrable $n\times1$ vector functions on [0,T]. Let E be the $2n\times2n$ matrix partitioned into $n\times n$ blocks with:

$$E = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} \tag{3.2.14a}$$

and define the $2n\times1$ vector

$$x_b = \begin{bmatrix} x(0) \\ x(T) \end{bmatrix} \quad . \tag{3.2.14b}$$

The formal adjoint of the operator L is [26]

$$(L^{\dagger}\lambda)(t) = -\dot{\lambda}(t) - A'(t)\lambda(t) \quad . \tag{3.2.14c}$$

Given these definitions, the Green's Identity for L on the interval [0,T] is obtained directly by integration by parts, yielding

$$\langle Lx, \lambda\rangle_{L_2^n[0,T]} = \langle x, L^{\dagger}\lambda\rangle_{L_2^n[0,T]} + \langle x_b, E\lambda_b\rangle_{R^{2n}} \quad . \tag{3.2.15}$$

SECTION 3.3

PROBLEM STATEMENT

The fixed-interval smoothing problem for the noncausal process $x(t)$ defined earlier in Section 3.2.1 is stated as follows. Let $r(t)$ be a $p \times 1$ white noise process uncorrelated with v and $u(t)$ and with continuous covariance parameter $R(t)$. Let $C(t)$ be a $p \times n$ matrix whose elements are continuous on $[0,T]$. The observations of $x(t)$ are given by the $p \times 1$ vector stochastic process:

$$y(t) = C(t)x(t) + r(t) \quad . \tag{3.3.1}$$

In addition to the observation $y(t)$, we assume that there may be available a boundary observation y_b defined as follows. Let r_b be a $q \times 1$ random vector uncorrelated with $r(t)$, $u(t)$ and v with covariance matrix Π_b. Define a $q \times 2n$ matrix W partitioned into $q \times n$ blocks as

$$W = [W^0 \; \vdots \; W^T] \quad . \tag{3.3.2a}$$

The boundary observation is the $q \times 1$ random vector:

$$y_b = Wx_b + r_b \quad . \tag{3.3.2b}$$

Define an $n \times 2n$ matrix V as

$$V = [V^0 \; \vdots \; V^T] \tag{3.3.3a}$$

so that the boundary condition in (3.2.1b) can be written as

$$v = Vx_b \tag{3.3.3b}$$

A condition imposed in Part I is the assumption that the rows of W and the rows of V are linearly independent. The significance of this assumption is explained as follows. If, say, the i^{th} row of W were a linear combination of the rows of V:

$$W_i = M_i V \quad , \tag{3.3.4a}$$

then the i^{th} element of y_b could be written as

$$y_{b_i} = M_i V x_b + r_{b_i}$$
$$= M_i v + r_{b_i} \qquad . \qquad (3.3.4b)$$

Thus, y_{b_i} in (3.3.4b) can be viewed as a measurement of the boundary condition v. Without loss of generality we can assume that y_b has been transformed so that the elements of r_b are mutually orthogonal. As such, y_{b_i} could be eliminated from the boundary observation to be used to update our knowledge of v. This relationship between y_b and v implies that the dimension of y_b is less than or equal to n, the dimension of v.

The concept of the boundary measurement has been introduced previously in a simpler form ($W^0 = 0$, $W^T = I$) into a smoothing problem for causal processes by Bryson and Hall [27]. They included a "post-flight" measurement and showed that this additional measurement results in a nonzero initial condition for the backward filter in the two-filter implementation of the causal smoother solution. Thus, the boundary measurement introduces additional symmetry into the structure of the two-filter solution. This type of boundary measurement has a natural analog in higher dimensions where measurements of a random field may often be made along the boundary of the region over which it is defined. For example, one might have observations of temperature on the surface of an object whose internal temperature distribution is of interest. Measurements of gravity at the surface of the earth or some other body provides another example.

Returning to the 1-D problem of interest here, the fixed-interval smoothing problem is to find the linear minimum variance estimate of the noncausal TPBVP x(t), t ε [0,T], given the complete observation set Y:

$$Y = \{y_b, \ y(t) \ \vdots \ t \in [0,T]\} \quad . \qquad (3.3.5)$$

In Appendix 3A we develop the estimator equations for a 1-D continuous parameter process with the following integral form boundary condition

$$v = \int_0^T V(s)x(s)ds \qquad . \qquad (3.3.6)$$

In that appendix it is shown that for this integral condition the estimator takes the form of an integro-differential equation.

SECTION 3.4

THE TPBVP SMOOTHER

A direct application of the differential operator representation for the estimator developed in Chapter 2 immediately yields the TPBVP smoother as a $2n^{th}$ order boundary value process. Given this two-point boundary value process, we show how it can be transformed into a two-filter form as discussed in Section 3.2.1. In a similar manner, we also apply the results of Chapter 2 to write a $2n^{th}$ order boundary value representation of the smoothing error and use the same transformation to develop expressions for the error covariance.

3.4.1 A Differential Realization for the Smoother

Let the $2n\times 2n$ matrix H be given by

$$H = \begin{bmatrix} A & \vdots & BQB' \\ \hdashline C'R^{-1}C & \vdots & -A' \end{bmatrix} \quad . \tag{3.4.1a}$$

Let the $2n\times p$ matrix G be given by

$$G = \begin{bmatrix} 0 \\ \hdashline -C'R^{-1} \end{bmatrix} \quad . \tag{3.4.1b}$$

Then substituting into (2.5.25a), it can be shown that the smoother dynamics are given by the $2n^{th}$ order differential equation

$$\begin{bmatrix} \dot{\hat{x}} \\ \dot{\hat{\lambda}} \end{bmatrix} = H \begin{bmatrix} \hat{x} \\ \hat{\lambda} \end{bmatrix} + Gy \quad . \tag{3.4.2}$$

To obtain an expression for the boundary condition for this differential equation, first define two $2n \times 2n$ matrices

$$V^{0}_{x\lambda} = \left[\begin{array}{c:c} V^{0'}\Pi_v^{-1}V^{0} + W^{0'}\Pi_b^{-1}W^{0} & -I \\ \hdashline V^{T'}\Pi_v^{-1}V^{0} + W^{T'}\Pi_b^{-1}W^{0} & 0 \end{array}\right] \tag{3.4.3a}$$

and

$$V^{T}_{x\lambda} = \left[\begin{array}{c:c} V^{0'}\Pi_v^{-1}V^{T} + W^{0'}\Pi_b^{-1}W^{T} & 0 \\ \hdashline V^{T'}\Pi_v^{-1}V^{T} + W^{T'}\Pi_b^{-1}W^{T} & I \end{array}\right] . \tag{3.4.3b}$$

Then from (2.5.25b), with the transpose of the matrices V and W identified as the operator adjoints V^* and W^*, the boundary condition for the smoother can be shown to be given by

$$\left[\begin{array}{c} W^{0'}\Pi_b^{-1}y_b \\ \hdashline W^{T'}\Pi_b^{-1}y_b \end{array}\right] = V^{0}_{x\lambda}\left[\begin{array}{c} \hat{x}(0) \\ \hat{\lambda}(0) \end{array}\right] + V^{T}_{x\lambda}\left[\begin{array}{c} \hat{x}(T) \\ \hat{\lambda}(T) \end{array}\right] . \tag{3.4.3c}$$

3.4.2 Hamiltonian Diagonalization

The solution of the $2n^{th}$ order boundary value process in (3.4.2) and (3.4.3) could be implemented by either of the two forms of the general solution derived in Section 3.2.1. However, by considering the "time"-invariant case we can anticipate, as discussed in that section, that there may be numerical instabilities associated with the first of those methods. In the time-invariant case the eigenvalues of the $2n \times 2n$ Hamiltonian[1] matrix H defined in (3.4.1a) are symmetric about the imaginary axis [29], i.e. there are n eigenvalues in each of the left and right half planes. Thus, for the time-invariant case, the right half plane eigenvalues will result in numerical instabilities for the unidirectional implementation suggested by (3.2.4). Recall that these stability problems can be avoided in general by transforming the smoother dynamics into the stable forward/backward form in (3.2.8). To achieve this second form, we need a transformation which diagonalizes the dynamics of H into two $n \times n$ blocks, one stable in the forward

[1] The terminology Hamiltonian is employed for historical reasons [28].

direction and the other backwards stable. As discussed below, this transformation is readily obtained by adapting results from previous studies of the smoother for causal processes.

Since the dynamics of our smoother as represented by H are identical to those of the smoother for causal processes as originally derived by Bryson and Frazier [30], any transformation which results in a two-filter smoother for causal processes will also diagonalize our smoother. As mentioned earlier, these diagonalizing transformations have been studied in [21] and [22], see also Appendix 3B. However, choosing a diagonalizing transformation for our problem requires special considerations not encountered in the causal case. First, because the two-point boundary condition provides incomplete information for both the initial and final values of the process, we will choose a transformation which corresponds to a two-filter solution for causal processes with both filters in information form. Second, as we will see, it is important to choose the boundary conditions properly for the Riccati differential equations which govern the time-varying elements of the diagonalizing transformation. In particular, the choice that we make here leads to an explicit representation for both the smoother and smoothing error covariance in terms of a single critical variable. With the smoother in this form we will be able to interpret some special cases in the next section. Finally, as discussed later, our choice of diagonalizing transformation and corresponding boundary conditions makes it possible to formulate a numerically stable two-filter form for our smoother which is remarkably similar to two-filter smoothers for causal processes.

Define the time-varying transformation T(t) as the $2n\times 2n$ matrix partitioned in $n\times n$ blocks as

$$T(t) = \left[\begin{array}{c:c} \theta_f(t) & -I \\ \hdashline \theta_b(t) & I \end{array}\right] . \tag{3.4.4a}$$

Let the transformed process be denoted by

$$q(t) = \begin{bmatrix} q_f(t) \\ q_b(t) \end{bmatrix} = T(t) \begin{bmatrix} \hat{x}(t) \\ \hat{\lambda}(t) \end{bmatrix} . \tag{3.4.4b}$$

Also define

$$\dot{H}_q = \dot{T}T^{-1} + THT^{-1} \tag{3.4.5a}$$

and

$$G_q = TG \tag{3.4.5b}$$

so that the dynamics of the transformed process can be written as

$$\begin{bmatrix} \dot{q}_f \\ \dot{q}_b \end{bmatrix} = H_q \begin{bmatrix} q_f \\ q_b \end{bmatrix} + G_q y \quad . \tag{3.4.5c}$$

If we use the following form for the inverse of T:

$$T^{-1}(t) = \left[\begin{array}{c:c} I & I \\ \hdashline -\theta_b(t) & \theta_f(t) \end{array}\right] \left[\begin{array}{c:c} P_s(t) & 0 \\ \hdashline 0 & P_s(t) \end{array}\right] \tag{3.4.6a}$$

where

$$P_s(t) = \left[\theta_f(t) + \theta_b(t)\right]^{-1} \quad , \tag{3.4.6b}$$

and if we choose the dynamics for θ_f and θ_b as

$$-\dot{\theta}_f = \theta_f A + A'\theta_f + \theta_f BQB'\theta_f - C'R^{-1}C \tag{3.4.6c}$$

and

$$-\dot{\theta}_b = \theta_b A + A'\theta_b - \theta_b BQB'\theta_b + C'R^{-1}C \quad , \tag{3.4.6d}$$

then carrying out the calculation in (3.4.5a), it can be shown that H_q is diagonalized with diagonal blocks

$$H_f = -[A' + \theta_f BQB'] \tag{3.4.6e}$$

and

$$H_b = -[A' - \theta_b BQB'] \quad . \tag{3.4.6f}$$

Thus the dynamics of q_f and q_b are decoupled and are given by

$$\dot{q}_f = H_f q_f + C'R^{-1}y \tag{3.4.7a}$$

and

$$\dot{q}_b = H_b q_b - C'R^{-1}y \quad . \tag{3.4.7b}$$

If we assume for time-invariant dynamics that $\{A,B\}$ is stabilizable and that $\{A,C\}$ is detectible and for time-varying dynamics that $\{A,B\}$ is uniformly completely controllable and $\{A,C\}$ is uniformly completely reconstructable, then the invertibility of P_s in (3.4.6b) is guaranteed if both $\theta_f(0)$ and $\theta_b(T)$ are nonnegative definite [29]. Furthermore, these conditions guarantee that θ_f and θ_b and their derivatives are bounded and that H_f and H_b are forward and backward stable respectively.

Under the transformation (3.4.4a), the boundary condition (3.4.3c) becomes

$$\begin{bmatrix} W^{0'}\Pi_b^{-1}y_b \\ W^{T'}\Pi_b^{-1}y_b \end{bmatrix} = V_q^0 \begin{bmatrix} q_f(0) \\ q_b(0) \end{bmatrix} + V_q^T \begin{bmatrix} q_f(T) \\ q_b(T) \end{bmatrix} \tag{3.4.8a}$$

where

$$V_q^0 = V_{x\lambda}^0 T^{-1}(0) \tag{3.4.8b}$$

and

$$V_q^T = V_{x\lambda}^T T^{-1}(T) \quad . \tag{3.4.8c}$$

To simplify the expressions for the boundary value coefficient matrices in (3.4.8b) and (3.4.8c), choose the following nonnegative definite initial and final conditions for the Riccati equations (3.4.6c) and (3.4.6d):

$$\theta_f(0) = V^{0'}\Pi_v^{-1}V^0 + W^{0'}\Pi_b^{-1}W^0 \tag{3.4.9a}$$

and

$$\theta_b(T) = V^{T'}\Pi_v^{-1}V^T + W^{T'}\Pi_b^{-1}W^T \quad . \tag{3.4.9b}$$

Then defining θ_c as the following $n \times n$ matrix:

$$\theta_c = V^{T'}\Pi_v^{-1}V^0 + W^{T'}\Pi_b^{-1}W^0 \quad , \tag{3.4.10}$$

it can be shown that the boundary value coefficient matrices can be written as

$$V_q^0 = \begin{bmatrix} I & \vdots & 0 \\ - & \vdots & - \\ \theta_c P_s(0) & \vdots & \theta_c P_s(0) \end{bmatrix} \equiv \left[\; V_f^0 \;\vdots\; V_b^0 \;\right] \tag{3.4.10a}$$

and

$$V_q^T = \begin{bmatrix} \theta_c' P_s(T) & \vdots & \theta_c' P_s(T) \\ - & \vdots & - \\ 0 & \vdots & I \end{bmatrix} \equiv \left[\; V_f^T \;\vdots\; V_b^T \;\right] \quad . \tag{3.4.10b}$$

Since the dynamics of q_f and q_b are decoupled, the only coupling between the two enters through the boundary condition. By our choice of initial and final conditions for the Riccati equations, we have been able to display this coupling solely as a function of the matrix θ_c.

The smoothed estimate of x is recovered by inverting T(t) in (3.4.4b) so that we obtain

$$\hat{x}(t) = P_s(t)\left[q_f(t) + q_b(t)\right] \quad . \tag{3.4.11}$$

Following (3.2.8), an explicit expression for the two-filter solution for q_f and q_b is formulated as follows. Let q_f^0 and q_b^0 be governed by

(3.4.7a) and (3.4.7b) respectively with boundary conditions: $q_f^0(0) = 0$ and $q_b^0(T) = 0$. Define F_{fb} and Φ_{fb} as the $2n \times 2n$ matrices

$$F_{fb} = \left[V_f^0 + V_f^T \Phi_f(T,0) \;\vdots\; V_b^T + V_b^0 \Phi_b(0,T)\right]$$

$$= \begin{bmatrix} I + \theta_c' P_s(T)\Phi_f(T,0) & \vdots & \theta_c' P_s(T) \\ \theta_c P_s(0) & \vdots & I + \theta_c P_s(0)\Phi_b(0,T) \end{bmatrix} \tag{3.4.12}$$

and

$$\Phi_{fb}(t) = \begin{bmatrix} \Phi_f(t,0) & \vdots & 0 \\ 0 & \vdots & \Phi_b(t,T) \end{bmatrix} . \tag{3.4.13}$$

Then the two-filter solution for q(t) is given by

$$\begin{bmatrix} q_f(t) \\ q_b(t) \end{bmatrix} = \Phi_{fb}(t) F_{fb}^{-1} \left\{ \begin{bmatrix} W^{0'} \\ W^{T'} \end{bmatrix} \Pi_b^{-1} y_b - \begin{bmatrix} \theta_c' P_s(T) q_f^0(T) \\ \theta_c P_s(0) q_b^0(0) \end{bmatrix} \right\} + \begin{bmatrix} q_f^0(t) \\ q_b^0(t) \end{bmatrix} . \tag{3.4.14}$$

The computational complexity of the noncausal smoother implementation suggested by (3.4.11) and (3.4.14) is nearly the same as that of the two-filter smoothers for causal processes such as the Mayne-Fraser form [19,20]. We note, however, that before q_f and q_b can be evaluated for any $t \in [0,T]$, both q_f^0 and q_b^0 must be computed and stored along with P_s and Φ_{fb} for the entire interval [0,T]. Thus, the required storage exceeds that of the smoother for causal processes. Indeed, the Mayne-Fraser solution and ours differ significantly in one aspect. That is, for our smoother the contribution of the forward filter to the smoothed estimate at some point t depends not only on past observations, as does the Mayne-Fraser solution, but also on future observations through the term $\theta_c' P_s(T) q_f^0(T)$ in (3.4.14). A similar statement applies for the backward process.

3.4.3 Smoothing Error

From (2.5.36), the differential realization of the smoothing error is

$$\begin{bmatrix} \dot{\tilde{x}} \\ \dot{\hat{\lambda}} \\ -\lambda \end{bmatrix} = H \begin{bmatrix} \tilde{x} \\ \hat{\ } \\ -\lambda \end{bmatrix} + \begin{bmatrix} Bu \\ - - - \\ C'R^{-1}r \end{bmatrix} \qquad (3.4.15b)$$

with boundary condition (from (2.5.33))

$$v_e = \begin{bmatrix} V^{0'} & W^{0'} \\ V^{T'} & W^{T'} \end{bmatrix} \begin{bmatrix} \Pi_v^{-1} & 0 \\ 0 & \Pi_b^{-1} \end{bmatrix} \begin{bmatrix} v \\ -r_b \end{bmatrix} = V_x^0 \begin{bmatrix} \tilde{x}(0) \\ \hat{\ } \\ -\lambda(0) \end{bmatrix} + V_x^T \begin{bmatrix} \tilde{x}(T) \\ \hat{\ } \\ -\lambda(T) \end{bmatrix} \quad . \qquad (3.4.15b)$$

The same diagonalizing transformation in (3.4.4a) can be applied to the error dynamics with the result that, as we will see, the error covariance can be computed from many of the same quantities required for computing the smoothed estimate.

In a manner similar to (3.4.4b) let

$$e(t) = \begin{bmatrix} e_f(t) \\ e_b(t) \end{bmatrix} = T(t) \begin{bmatrix} \tilde{x}(t) \\ \hat{\ } \\ -\lambda(t) \end{bmatrix} \quad . \qquad (3.4.16)$$

Then the smoothing error is

$$\tilde{x}(t) = P_s(t)\left[e_f(t) + e_b(t)\right] \qquad (3.4.17)$$

where e_f and e_b satisfy the decoupled dynamics

$$\dot{e}_f = H_f e_f + \left[\theta_f B \ \vdots\ -C'R^{-1}\right] \begin{bmatrix} u \\ r \end{bmatrix} \qquad (3.4.18a)$$

and

$$\dot{e}_b = H_b e_b + \left[\theta_b B \ \vdots\ C'R^{-1}\right] \begin{bmatrix} u \\ r \end{bmatrix} \quad . \qquad (3.4.18b)$$

Under this transformation the boundary condition takes the form (see (3.4.10a,b))

$$v_e = [V_f^0 \vdots V_b^0] \begin{bmatrix} e_f(0) \\ e_b(0) \end{bmatrix} + [V_f^T \vdots V_b^T] \begin{bmatrix} e_f(T) \\ e_b(T) \end{bmatrix} \quad . \tag{3.4.19}$$

Below we develop an expression for the error covariance. Let

$$\Sigma_f(t) = E\{e_f(t)e_f'(t)\} , \tag{3.4.20a}$$

$$\Sigma_b(t) = E\{e_b(t)e_b'(t)\} \tag{3.4.20b}$$

and

$$\Sigma_{fb}(t) = E\{e_f(t)e_b'(t)\} \quad . \tag{3.4.20c}$$

The covariance of the smoothing error can be written directly from (3.4.17) as

$$\begin{aligned} P(t) &= E\{\tilde{x}(t)\tilde{x}'(t)\} \\ &= P_s(t)\left[\Sigma_f(t) + \Sigma_b(t) + \Sigma_{fb}(t) + \Sigma_{fb}'(t)\right]P_s(t) \quad . \end{aligned} \tag{3.4.21}$$

We derive expressions for each of the individual covariances in (3.4.20) by expressing e(t) in the two-filter form of (3.2.8). Accordingly, let $e^0{}_f$ and $e^0{}_b$ be governed by (3.4.18a) and (3.4.18b) respectively with boundary conditions: $e^0{}_f(0) = 0$ and $e^0{}_b(T) = 0$. Then e_f and e_b can be written as

$$\begin{bmatrix} e_f(t) \\ e_b(t) \end{bmatrix} = \Phi_{fb}(t)F_{fb}^{-1}\left\{ v_e - \begin{bmatrix} \theta_c' P_s(T)e_f^0(T) \\ \theta_c P_s(0)e_b^0(0) \end{bmatrix} \right\} + \begin{bmatrix} e_f^0(t) \\ e_b^0(t) \end{bmatrix} \quad . \tag{3.4.22}$$

Thus the covariances in (3.4.20) can be expressed in terms of the covariance of v_e and the covariances and cross-covariance of $e_f{}^0$ and $e_b{}^0$.

First note from (3.4.15b) that the covariance of v_e is given by

$$\Pi_{v_e} = E\{v_e v_e'\} = \begin{bmatrix} \theta_f(0) & \vdots & \theta_c' \\ \theta_c & \vdots & \theta_b(T) \end{bmatrix} \quad . \tag{3.4.23}$$

The covariance of e^0_f

$$\Sigma_f^0(t) = E\{e_f^0(t)e_f^{0'}(t)\} \tag{3.4.24a}$$

satisfies

$$\dot{\Sigma}_f^0 = H_f\Sigma_f^0 + \Sigma_f^0 H_f' + \theta_f BQB'\theta_f + C'R^{-1}C \quad ; \quad \Sigma_f^0(0) = 0 \quad . \tag{3.4.24b}$$

Similarly, the covariance matrix for $e_b{}^0$ satisfies

$$\dot{\Sigma}_b^0 = H_b\Sigma_b^0 + \Sigma_b^0 H_b' - \theta_b BQB'\theta_b - C'R^{-1}C \quad ; \quad \Sigma_b^0(T) = 0 \quad . \tag{3.4.25}$$

To obtain an expression for the cross-correlation:

$$E\{e_f^0(t)e_b^{0'}(\tau)\} = \begin{cases} \Sigma_{fb}^0(t,\tau), & t > \tau \\ 0 \quad , & t \le \tau \end{cases} \quad , \tag{3.4.26}$$

first define

$$\dot{\Pi}_{fb}^0 = H_f\Pi_{fb}^0 + \Pi_{fb}^0 H_b' + \theta_f BQB'\theta_b - C'R^{-1}C \quad ; \quad \Pi_{fb}^0(0) = 0 \quad . \tag{3.4.27}$$

Substituting the variation of constants integral expressions for the processes in the expectation in (3.4.26), it can be shown that for $t > \tau$:

$$\Sigma_{fb}^0(t,\tau) = \Phi_f(t,\tau)\Pi_{fb}^{0'}(\tau) - \Pi_{fb}^0(t)\Phi_b'(\tau,t) \tag{3.4.28a}$$

and that

$$\Sigma_{bf}^0(\tau,t) = \Sigma_{fb}^{0'}(t,\tau) \quad . \tag{3.4.28b}$$

Finally, combining these identities we can express

$$\Sigma_e(t) = E\{e(t)e'(t)\} = \begin{bmatrix} \Sigma_f(t) & \Sigma_{fb}(t) \\ \Sigma_{fb}'(t) & \Sigma_b(t) \end{bmatrix} \tag{3.4.29}$$

as

$$\Sigma_e(t) = \Phi_{fb}(t)F_{fb}^{-1}\left[\begin{array}{c:c} \theta_f(0) + \theta_c' P_s(T)\Sigma_f^0(T)P_s(T)\theta_c & \theta_c' P_s(T)\Sigma_{fb}^0(T,0)P_s(0)\theta_c' + \theta_c' \\ \hdashline \theta_c P_s(0)\Sigma_{bf}^0(0,T)P_s(T)\theta_c + \theta_c & \theta_b(T) + \theta_c P_s(0)\Sigma_b^0(0)P_s(0)\theta_c' \end{array}\right]\Phi_{fb}(t)F_{fb}^{-1}$$

$$+ \begin{bmatrix} \Sigma_f^0(t) & 0 \\ 0 & \Sigma_b^0(t) \end{bmatrix} - \Phi_{fb}(t)F_{fb}^{-1}\left[\begin{array}{c:c} \theta_c' P_s(T)\Phi_f(T,t)\Sigma_f^0(t) & \theta_c' P_s(T)\Sigma_{fb}^0(T,t) \\ \hdashline \theta_c P_s(0)\Sigma_{bf}^0(0,t) & \theta_c P_s(0)\Phi_b(0,t)\Sigma_b^0(t) \end{array}\right]$$

$$- \left[\begin{array}{c:c} \Sigma_f^0(t)\Phi_f'(T,t)P_s(T)\theta_c & \Sigma_{fb}^0(t,0)P_s(0)\theta_c' \\ \hdashline \Sigma_{bf}^0(t,T)P_s(T)\theta_c & \Sigma_b^0(t)\Phi_b'(0,t)P_s(0)\theta_c' \end{array}\right] F_{fb}^{-1'}\Phi_{fb}'(t) \quad . \qquad (3.4.30)$$

Next, note that it can be shown that the solutions of (3.4.24) and (3.4.25) are related to θ_f and θ_b in (3.4.6c) and (3.4.6d) by

$$\Sigma_f^0(t) = \theta_f(t) - \Phi_f(t,0)\theta_f(0)\Phi_f'(t,0) \qquad (3.4.31a)$$

and

$$\Sigma_b^0(t) = \theta_b(t) - \Phi_b(t,T)\theta_b(T)\Phi_b'(t,T) \quad . \qquad (3.4.31b)$$

That is,

$$\Sigma_f^0(t) = \theta_f^0(t) \quad \text{and} \quad \Sigma_b^0(t) = \theta_b^0(t) \quad . \qquad (3.4.31c)$$

When $\Sigma_f{}^0$ and $\Sigma_b{}^0$ are replaced in (3.4.30) by the expressions in (3.4.31a) and (3.4.31b), it can be seen that the only computation required in excess of that already performed for the smoother solution is the integration of $\Pi^0{}_{fb}$ in (3.4.27).

Although the expression for the covariance in (3.4.30) may seem forbidding, it does explicitly display the dependence of Σ_e on θ_c. In the next section we discuss a special class of problems for which θ_c is zero. As a preview to that discussion, we note that when $\theta_c = 0$,

i) $F_{fb} = I$

and

$$\text{ii)}\quad \Sigma_e(t) = \Phi_{fb}(t)\begin{bmatrix}\theta_f(0) & 0\\ 0 & \theta_b(T)\end{bmatrix}\Phi_{fb}'(t) + \begin{bmatrix}\Sigma_f^0(t) & 0\\ 0 & \Sigma_b^0(t)\end{bmatrix} .$$

Substituting from (3.4.29), $\Sigma_e(t)$ for this case becomes simply

$$\Sigma_e(t) = \begin{bmatrix}\theta_f(t) & 0\\ 0 & \theta_b(t)\end{bmatrix}$$

which implies that the forward and backward error processes e_f and e_b are orthogonal and that the smoothing error covariance in (3.4.21) is

$$P(t) = P_s(t) = \left[\theta_f(t) + \theta_b(t)\right]^{-1} .$$

Also, when θ_c is zero, the noncausal contributions of the forward and backward processes $q_f{}^0$ and $q_b{}^0$ to the smoothed estimate are eliminated (see (3.4.14)). Note that all of these are also properties of the two-filter smoothers for causal processes [15]. In the next section we will show that for the case when θ_c is zero, q_f and q_b can be interpreted as the forward and backward information vectors for a causal process smoother with special nonzero boundary values for θ_f and θ_b.

SECTION 3.5

SPECIAL CASES

In the first part of this section we discuss some properties of the smoother for a class of noncausal processes with special boundary conditions and boundary observations. A subset of this class was first studied by Krener [23]. Here we show for this class that the smoother described in the previous section is equivalent to a previously derived smoother for causal processes. The last topic of the section is alternative transformations which lead to two of the popular forms of the smoother for causal processes, namely the Mayne-Fraser and the Rauch-Tung-Striebel. The former belongs to the class of diagonalizing transformations studied by Kailath and Ljung [21] and Desai [22] and the latter is a triangularizing transformation [2].

3.5.1 Separable Systems

In the context of 1-D linear stochastic TPBVPs, Krener first introduced the terminology separable to describe a class of n^{th} order noncausal stationary processes which are, in fact, n^{th} order Markov, i.e. their evolution can be described by an n^{th} order linear stochastic differential equation with a prescribed initial condition which is orthogonal to future inputs. Recall that, in general, the boundary value representation for noncausal processes which we presented in Section 3.2.1 is not a Markov model. Along with stationarity, Krener's criteria for separability includes a block-diagonal form for Π_v and the orthogonality condition: $V^{T'}V^0 = 0$. In fact, the slightly less restrictive condition

$$V^{T'}\Pi_v^{-1}V^0 = 0 \tag{3.5.1}$$

could have been imposed. In [2], the stationarity condition was shown to be unnecessary so that (3.5.1) is both necessary and sufficient for the existence of an n^{th} order Markov model. With respect to the smoothing problem, the existence of such a model implies that when there is no boundary measurement, any of the smoothers for causal processes can be applied directly to the Markov model. Here we will extend the notion of separability to include cases for which there is a boundary measurement and say that a system is separable if

$$\theta_c = V^{T'} \Pi_v^{-1} V^0 + W^{T'} \Pi_b^{-1} W^0 \tag{3.5.2}$$

is zero. Note that this condition is compatible with Krener's original condition when there is no boundary measurement ($W^0 = W^T = 0$).

When θ_c is zero, the boundary condition in (3.4.8) becomes decoupled (see (3.4.10)) and F_{fb} in (3.4.12) becomes the identity so that q_f and q_b are completely decoupled with boundary conditions

$$q_f(0) = W^{0'} \Pi_b^{-1} y_b \tag{3.5.3a}$$

and

$$q_b(T) = W^{T'} \Pi_b^{-1} y_b \quad . \tag{3.5.3b}$$

Based on this observation, we can interpret the smoother for the separable case as being equivalent to Bryson and Hall's [27] problem with a "post-flight" measurement as follows.

Here we consider the information in the boundary condition v and observation y_b when <u>combined</u> into a single measurement:

$$\begin{bmatrix} 0 \\ y_b \end{bmatrix} = \begin{bmatrix} V^0 & \vdots & V^T \\ W^0 & \vdots & W^T \end{bmatrix} \begin{bmatrix} x(0) \\ x(T) \end{bmatrix} + \begin{bmatrix} -v \\ r_b \end{bmatrix} \quad . \tag{3.5.4}$$

This information will be viewed in the form of an information vector [31]. An information vector is used to store information about a random vector when the apriori uncertainty for that random vector (or at least some of its components) is infinite, i.e. it is totally unkown. When sufficient measurement information has been gathered so that the error-covariance matrix for the random vector becomes finite, the stored information in the form of the information vector can be transformed by the inverse of the covariance matrix (the information matrix) to produce a finite error-variance estiatmte of the random vector. In (3.5.4) above we have posed the boundary condition for $\{x(0),x(T)\}$ as a measurement. In this way we can consider the aproiri information as totally uncertain. Since v and r_b are orthogonal random variables, it can be shown that the information matrix ψ_x and information vector i_x associated with (3.5.4) are

$$\psi_x = \begin{bmatrix} \theta_f(0) & \vdots & \theta_c' \\ \theta_c & \vdots & \theta_b(T) \end{bmatrix} \tag{3.5.5a}$$

and

$$i_x = \begin{bmatrix} V^{0'} & \vdots & W^{0'} \\ V^{T'} & \vdots & W^{T'} \end{bmatrix} \begin{bmatrix} \Pi_v^{-1} & \vdots & 0 \\ 0 & \vdots & \Pi_b^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ y_b \end{bmatrix} = \begin{bmatrix} W^{0'} \\ W^{T'} \end{bmatrix} \Pi_b^{-1} y_b \tag{3.5.5b}$$

where $\theta_f(0)$ and $\theta_b(T)$ are given by (3.4.9a,b). Separability is thus the case when the information about x(0) contained in the combined boundary measurement (3.5.4) is orthogonal to that for x(T), i.e. ψ_x is block-diagonal. By considering (3.5.3a) as the initial value for an information form Kalman filter for x(t) with associated information matrix $\theta_f(0)$ and by considering (3.5.3b) as the information vector corresponding to a "post-flight" measurement with associated information matrix $\theta_b(T)$, we find that separability is equivalent to a causal process with (possibly) incomplete information about its initial value plus a post-flight measurement. Finally, we remark that from (3.5.2) we see that even when (3.5.1) is not satisfied it is still possible to achieve separability if the boundary measurement is designed so that $W^{T'}\Pi_b{}^{-1}W^0$ cancels $V^{T'}\Pi_v{}^{-1}V^0$.

3.5.2 Alternative Transformations

As Kailath and Ljung [21] have noted, there exists a family of transformations which diagonalize the Hamiltonian H (also see Appendix 3B). In addition to diagonalization, there are other special structures for the smoother dynamics which lead to smoother implementations which may also be of interest. For example, here we present both a diagonalizing and a triangularizing transformation each with appropriate boundary conditions so that their application results in the Mayne-Fraser and Rauch-Tung-Striebel smoothers respectively for causal processes.

I) Mayne-Fraser

The Mayne-Fraser two-filter smoother is obtained by choosing the transformation

$$T(t) = \begin{bmatrix} I & \vdots & -P(t) \\ \hline \theta_b(t) & \vdots & I \end{bmatrix} \tag{3.5.6a}$$

where P satisfies

$$\dot{P} = AP + PA' + BQB' - PC'R^{-1}CP \quad ; \quad P(0) = \Pi_v \tag{3.5.6b}$$

and θ_b satisfies (3.4.9b) with boundary condition $\theta_b(T) = 0$.

II) Rauch-Tung-Striebel

As an alternative to diagonalization, the smoother dynamics are triangularized by applying the transformation

$$T(t) = \begin{bmatrix} 0 & \vdots & I \\ \hline I & \vdots & -P(t) \end{bmatrix} \tag{3.5.7}$$

with the dynamics and boundary condition of P given by (3.5.6b). With this transformation, the Hamiltonian dynamics become block-triangular yielding the Rauch-Tung-Striebel smoother for causal processes. Later in Chapter 7 we expand upon this discussion of triangular forms.

SECTION 3.6

EXAMPLE: THIN ROD HEAT EXCHANGER

Thin rods or fins are commonly used as the medium for dissipating heat from some primary source to a coolant fluid which passes over the rods [25]. We will consider the temporal steady-state heat transfer for the two configurations depicted in Figures 3.6.1a and 3.6.1b[1]. That is, we will be looking at the heat distribution for some snapshot in time when temporal variations have settled out.

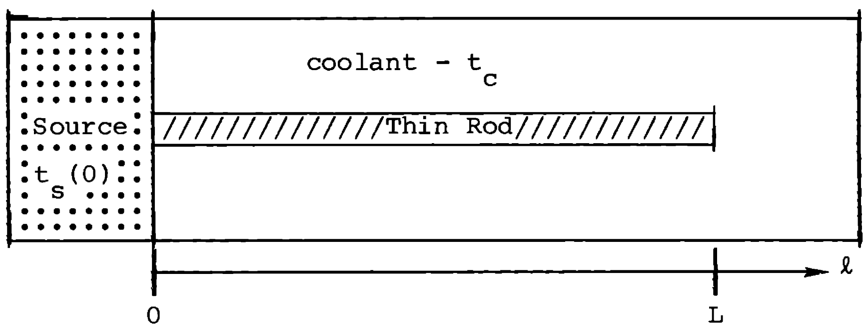


Figure 3.6.1a)
Thin Rod Case

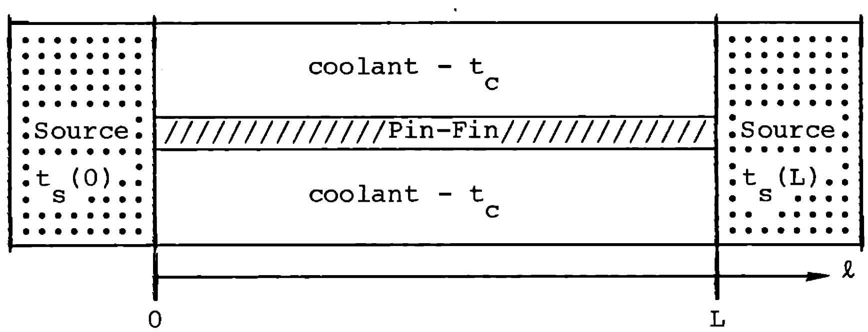


Figure 3.6.1b)
Pin-Fin Case

In this section we present a probabilistic two-point boundary value representation for the steady-state temperature distribution and heat flow along the rod for these two cases. The corresponding deterministic TPBVP models for these configurations in temporal steady-state can be found in most

[1] Temperatures are denoted by lower case t and the independent variable, length along the rod, by ℓ.

introductory texts on heat transfer such as [25] or [32]. Following the discussion of these models, some numerical results for a covariance analysis of the TPBVP smoother as applied to these cases are presented.

3.6.1 The Dynamics

As is typically done [32], it will be assumed that the rod is sufficiently thin so that in temporal steady-state the temperature of the rod can be considered constant throughout any cross-section. Given this assumption, the spatial dynamics of the temporal steady-state temperature and heat flow are derived by balancing the rod-to-coolant heat energy exchange with the along-rod heat energy conduction.

For our probabalistic approach, the coolant temperature along the rod, $t_c(\ell)$, will be modelled as a constant ambient value plus a white noise fluctuation:

$$t_c(\ell) = t_{amb} + \eta(\ell) \tag{3.6.1}$$

$$E\{\eta(\ell)\eta(s)\} = Q\delta(\ell - s)$$

The fluctuation is meant to account for spatial variations in coolant temperature. Note that $\eta(\ell)$ might be a second order process which could be modelled as the output of a shaping filter and incorporated into our state model below via state augmentation. We have used white noise here for simplicity in presentation.

One state variable, $t(\ell)$, is defined as the difference between the rod temperature and the coolant ambient:

$$t(\ell) = t_{rod}(\ell) - t_{amb} \tag{3.6.2}$$

The other state variable is the derivative of $t(\ell)$:

$$\dot{t}(\ell) = \frac{dt(\ell)}{d\ell} \quad . \tag{3.6.3}$$

Defining

k = thermal conductivity of the rod (Btu/(hr ft F))
A = cross-sectional area of the rod (sq ft)
p = rod perimeter (ft)
h = rod-coolant heat transfer coefficient (Btu/(sq ft hr F))

and

$$m^2 = hp/kA \quad ,$$

the state dynamics with t in degrees F are given by

$$\begin{bmatrix} \dot{t} \\ \ddot{t} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ m^2 & 0 \end{bmatrix} \begin{bmatrix} t \\ \dot{t} \end{bmatrix} + \begin{bmatrix} 0 \\ m^2 \end{bmatrix} \eta \quad . \tag{3.6.4}$$

The heat flow at any point along the rod is given by [32]

$$q(\ell) = -kA\dot{t}(\ell) \quad \text{(Btu/hr)} \quad . \tag{3.6.5}$$

3.6.2 Measurement Model

The dynamics in (3.6.4) are common to both the thin rod and pin-fin configurations. Before discussing their boundary conditions, we describe the measurement which is assumed to be available for both cases. Let

$$y(\ell) = [1:0] \begin{bmatrix} t(\ell) \\ \dot{t}(\ell) \end{bmatrix} + r(\ell) \tag{3.6.6}$$

$$E\{r(\ell)r(s)\} = R\delta(\ell - s)$$

represent a noisy measurement of temperature along the rod. One could conceive of these measurements as being obtained optically by infra-red techniques. Here we have modelled the measurement noise as white, while in practice optical measurements might also contain some noncausal blurring which could be accounted for via state augmentation.

3.6.3 Boundary Conditions

The two cases depicted in Figure 3.6.1 are distinguishable through their boundary conditions. The boundary condition for the thin rod case in Figure

3.6.1a is determined by a) the temperature of the rod at the source:

$$t_{rod}(0) = t_s$$
$$= t_m + v_t(0) \qquad (3.6.7a)$$

where t_m is an a priori mean, and $v_t(0)$ is a zero mean variation about t_m with variance $\sigma_t^2(0)$; and by b) equating conduction and convection at the end of the rod:

$$v_q(L) = h'A[t_{rod}(L) - t_{amb}] + kA\dot{t}(L) \qquad (3.6.7b)$$

where h' is the coefficient of heat transfer through the end of the rod and $v_q(L)$ is a zero mean random variable with variance σ_q^2 used to compensate for errors in determining k and h' and the effect of deviations of the coolant temperature at the end of the rod from ambient $(t_c(L) - t_{amb})$.

Thus, we have the following boundary condition for the thin rod case:

$$\begin{bmatrix} (t_m - t_{amb}) + v_t(0) \\ v_q(L) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} t(0) \\ \dot{t}(0) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ Ah' & Ak \end{bmatrix} \begin{bmatrix} t(L) \\ \dot{t}(L) \end{bmatrix} . \qquad (3.6.7c)$$

Note that when $v_t(0)$ and $v_q(L)$ are uncorrelated, (3.6.7c) satisfies the separability condition (3.5.2).

The boundary condition for the pin-fin case in Figure 3.6.1b is obtained from (3.6.7a) at both $\ell = 0$ and $\ell = L$:

$$\begin{bmatrix} (t_m - t_{amb}) + v_t(0) \\ (t_m - t_{amb}) + v_t(L) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} t(0) \\ \dot{t}(0) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} t(L) \\ \dot{t}(L) \end{bmatrix} . \qquad (3.6.8)$$

Similar to the thin rod case, if $v_t(0)$ and $v_t(L)$ are uncorrelated, then (3.6.8) would represent a separable case. However, in many pin-fin configurations, the physical proximity of the two ends of the fin will result in the variations $v_t(0)$ and $v_t(L)$ being correlated. For example, consider the correlated case represented by

$$\Pi_v = E\left\{ \begin{bmatrix} v_t(0) \\ v_t(L) \end{bmatrix} [\, v_t(0) \vdots v_t(L) \,] \right\} = \begin{bmatrix} \sigma_t^2 & \rho\sigma_t^2 \\ \rho\sigma_t^2 & \sigma_t^2 \end{bmatrix} . \qquad (3.6.9a)$$

In this case due to the nonzero correlation ρ, θ_c is nonzero:

$$\theta_c = V^{T'} \Pi_v^{-1} V^0 = \begin{bmatrix} -\rho\sigma_t^{-2}(1-\rho^2)^{-1} & \vdots & 0 \\ - \; - \; - \; - & \vdots & - \\ 0 & \vdots & 0 \end{bmatrix} \tag{3.6.9b}$$

resulting in a nonseparable case.

3.6.4 Numerical Results

Error covariance results are presented for the three examples. The first is a thin rod case and the last two are pin-fin cases. For one pin-fin case the correlation ρ in (3.6.9) is assumed to be zero and for the other ρ is assumed nonzero. For all three examples we assume a 0.25 ft long copper rod with outer diameter 0.1 ft: $L = 0.25$ ft, $D_0 = 0.1$ ft and $k = 280$ Btu/(hr ft F). The coolant is water at 100 degrees F passing over the rod at a velocity of 5 ft/sec. These conditions correspond to a Reynolds number $R_e \simeq 6.75 \times 10^5$, a Prandtl number $P_r \simeq 4.52$ and a coefficient of heat transfer for the water of $k_w = 0.364$ Btu/(hr ft F). Applying an approximation from [32], the water-to-rod convective heat transfer coefficient is

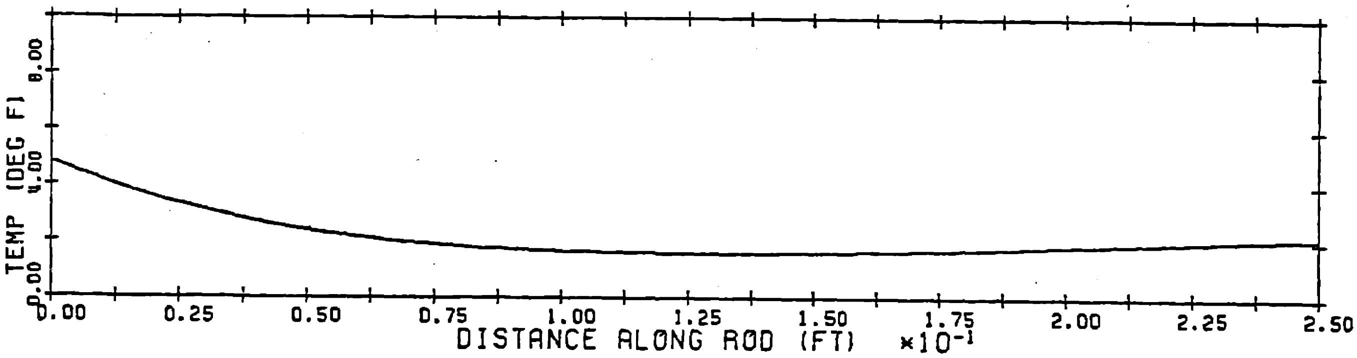
$$\begin{aligned} h &\simeq \frac{0.0263\, k_w R_e^{0.805}\, P_r^{0.31}}{D_0} \\ &= 1180 \text{ Btu/(sq ft hr F)} \quad . \end{aligned}$$

We will assume a process noise variance parameter $Q = 1\ F^2/\text{ft}$ and a measurement noise variance parameter $R = 1\ F^2/\text{ft}$. Table 3.6.1 lists the uncertainties associated with the boundary conditions for the three examples.

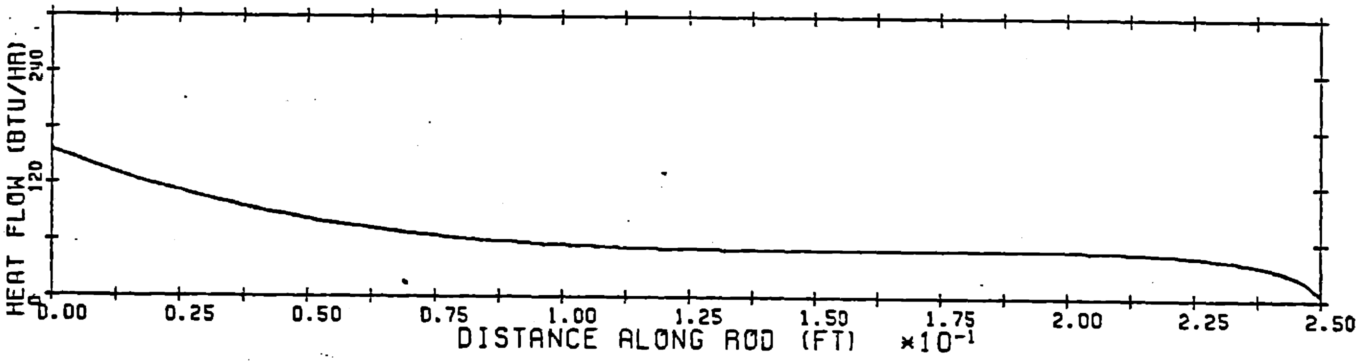
Example	$\sigma_t(0)$ (F)	$\sigma_t(L)$ (F)	$\sigma_q(L)$ (Btu/hr)	ρ
1. Thin rod	10.0	-	5.0	-
2. Pin-fin	10.0	10.0	-	0.0
3. Pin-fin	10.0	10.0	-	0.99

Table 3.6.1 Boundary Condition Standard Deviations

Plots of the results of the covariance analyses are presented in Figures 3.6.2, 3.6.3 and 3.6.4. Part a) of each figure shows the standard deviation

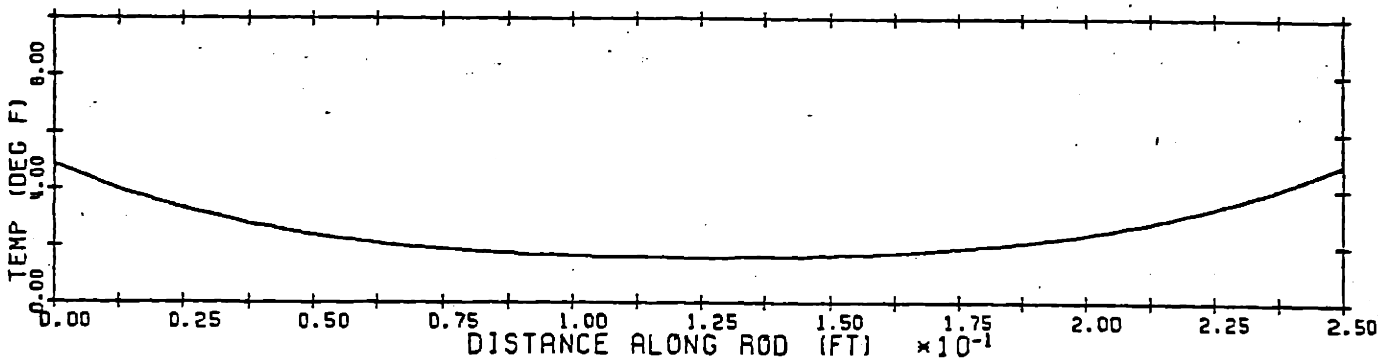


3.6.2(a)

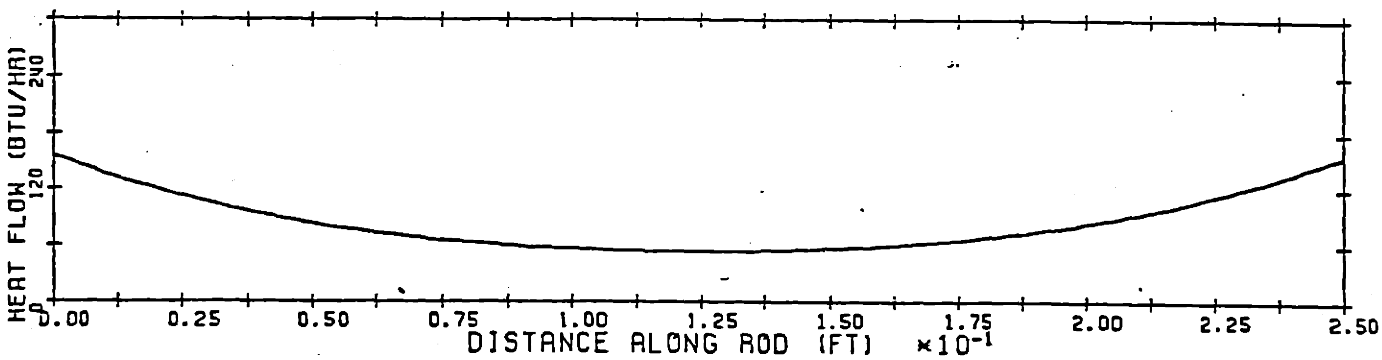


3.6.2(b)

Figure 3.6.2 Thin Rod Smoothing Error Standard Deviations: Example 1

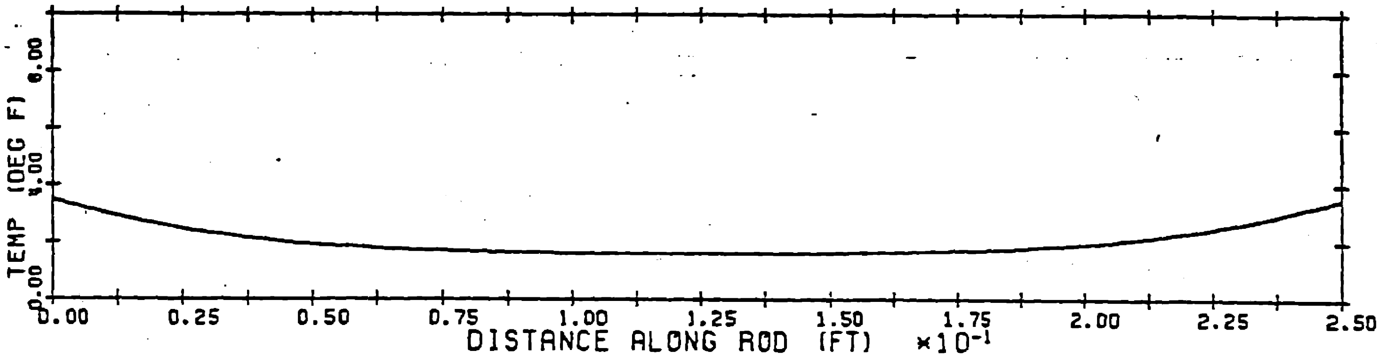


3.6.3(a)

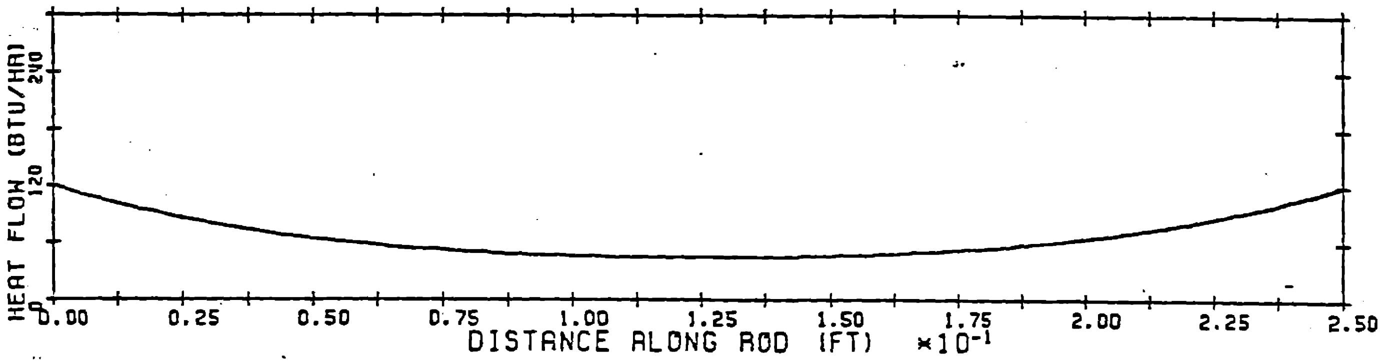


3.6.3(b)

Figure 3.6.3 Pin-Fin Smoothing Error Standard Deviations: Example 2, ρ=0



3.6.4(a)



3.6.4(b)

Figure 3.6.4 Pin-Fin Smoothing Error Standard Deviations: Example 3, ρ=0.99

in the smoothing error for temperature along the rod in degrees F. Part b) of each depicts the standard deviation of the heat flow in Btu/hr which has been calculated by scaling the uncertainty in $dt/d\ell$ as indicated in (3.6.5).

The results for the thin rod case in Figure 3.6.2 show that the heat flow uncertainty at the end of the rod, $\ell = 0.25$ ft, drops off to the boundary condition of 5 Btu/hr. In contrast, no such drop is seen for the pin-fin cases in Figures 3.6.3 and 3.6.4, for which the boundary condition is specified in terms of the temperature at both ends of the rod. Comparing between the pin-fin cases, we find that the highly correlated nonseparable case of example 3 has a larger reduction in uncertainty at the ends of the rod than does the separable case of example 2. In effect, the correlation allows the estimate at each end of the rod to utilize the information available at the opposite end. Comparing among all three examples, we find that the uncertainties at the midpoint of the rods, $\ell = 0.125$ ft, are about the same for all three cases. In fact, under the stabilizability and detectability conditions stated in Section 3.4, it can be shown for space-invariant cases and for very large smoothing intervals that the smoothing error covariance in the middle of the interval approaches

$$P_{ss} = \left[\Sigma^0_{f,ss} + \Sigma^0_{b,ss}\right]^{-1} ,$$

where ss denotes spatial steady-state values. Note that this expression for the steady-state error covariance is independent of both the structure and value of the smoother's boundary condition i.e. the steady-state covariance is the same for both causal and noncausal processes.

SECTION 3.7

CONCLUSIONS

An internal differential realization of the fixed-interval smoother for a 1-D, n^{th} order noncausal two-point boundary value stochastic process (TPBVP) has been obtained by applying the differential operator solution developed in Chapter 2. This representation for the TPBVP smoother has been shown to have the same $2n^{th}$ order Hamiltonian dynamics as the fixed-interval smoother for causal processes. The boundary condition for the TPBVP smoother, however, has been found to be more complex than that for the causal process smoother. By applying a time-varying diagonalizing transformation much like those employed by Kailath and Ljung [21] for causal processes, we have formulated a numerically stable n^{th} order two-filter implementation. The simplicity of this two-filter form is achieved by employing an information form for the diagonalizing transformation with carefully chosen boundary conditions for the differential equations governing its elements. The significant difference between our two-filter implementation and that for causal processes is that in the noncausal case the smoothed estimate at a given point in the interval is a noncausal function of each of the forward and backward processes (see (3.4.11) and (3.4.14)).

Our work in Chapter 2 has also provided a recipe for writing a differential realization for the smoothing error. Through an application of the same diagonalizing transformation, we have derived a two-filter representation for the smoothing error as well. From this representation, we have formulated an expression for the error covariance which is a function of the solutions of forward and backward Riccati equations (as in the causal process case) along with the solution of one additional matrix differential equation.

We have also discussed the application of the TPBVP smoother to a special class of noncausal processes which we refer to as separable, following the terminology introduced by Krener [23]. We have shown that separability can be interpreted in terms of the information contained in the two-point boundary condition v in (3.3.3b) and the boundary observation y_b in (3.3.2b). In particular, if the part of this information which pertains to the value of the process at the beginning of the smoothing interval, x(0), is uncorrelated with the information about the process value at the end of the interval, x(T), then the system is separable. The smoother for this class of systems is shown to be equivalent to a special form of a previously derived smoother for causal processes with "post-flight" measurements [27].

As discussed in Chapter 2, differential realizations for estimators of both discrete and continuous parameter multidimensional stochastic processes can be formulated as well by the method of complementary models. The problems associated with the implementation of those estimators are addressed in succeeding chapters of this thesis.

APPENDIX 3A

GENERALIZED BOUNDARY CONDITION

In this appendix we derive the Hilbert adjoint of a system whose boundary condition and boundary measurement are of the form:

$$v = Vx \tag{3.A.1a}$$

and

$$\phi_b = Wx \quad . \tag{3.A.1b}$$

From the spaces defined in Section 2.3.3 V, is the mapping $V: D(L) \to R(V)$ and W is the mapping $W: D(L) \to R(W)$. Previously we had considered boundary conditions and boundary observations which are linear functions of the boundary process x_b, i.e. $v = Vx_b$ and $\phi_b = Wx_b$, i.e. we had restricted the domains of W and V to H_b as defined in Section 2.3.3. Given a representation for the adjoint system, we formulate the smoother for 1-D processes with boundary condition of the form of (3.A.1a). Specifically that boundary condition is given by the integral:

$$v = \int_0^T V(s)x(s)ds \tag{3.A.1c}$$

In deriving the differential operator representation for the estimator in Chapter 2, we required an internal differential realization of the mapping we had denoted by H (see (2.5.3)) and its adjoint H^*. Recall that the complementary process Z was shown in Chapter 2 to be represented by the mapping

$$Z = [\,-I \;\vdots\; H^*]\Sigma_\zeta^{-1}\zeta \quad . \tag{3.A.2}$$

A differential realization for the estimator was obtained by inverting the system resulting from combining a differential realization for the complementary process Z with that for the observations. As discussed in Section 2.5.1 and as seen from (3.A.2), such a differential realization for the complementary process could be obtained directly from a differential realization for H^*. Here we construct a differential realization for that adjoint. We note that the mapping H is defined by (2.5.3) with the boundary condition and boundary output replaced by (3.A.1a) and (3.A.1b). Since we will rely on the developments in Section 2.5.1, a brief review of that section would be useful to the reader before proceeding here.

Our goal in this appendix is to present a detailed solution of the 1-D estimation problem introduced above and thereby establish a methodology which may be useful in solving generalizations of the smoothing problem for other classes of processes.

The starting point is (2.5.11) the inner-product identity through which the Hilbert adjoint map H^* with input $\{u_\lambda, v_\lambda\}$ and output $\{\psi_b, B^*\lambda\}$ is defined. Given the boundary condition (3.A.1a) and the output (observation) in (3.A.1b), the inner-product identity in (2.5.11) takes the form

$$\langle x, C^* u_\lambda \rangle + \langle x, W^* v_\lambda \rangle = \langle Lx, \lambda \rangle + \langle x, V^* \psi_b \rangle \tag{3.A.3a}$$

where $V^*: R(V) \to D(L)$ and $W^*: R(W) \to D(L)$. At this point, neither the dynamics of the internal process λ nor the relationship between the adjoint system boundary output ψ_b and the adjoint system inputs u_λ and v_λ has been specified. Their specification is the subject of what follows.

Substituting for the first term on the right hand side of (3.A.3a) from Green's identity (2.3.18), we get

$$\langle x, [C^* u_\lambda + W^* v_\lambda - L^\dagger \lambda - V^* \psi_b] \rangle = \langle x_b, E\lambda_b \rangle \quad . \tag{3.A.3b}$$

By choosing the dynamics of the internal process λ as

$$L^\dagger \lambda = C^* u_\lambda + W^* v_\lambda - V^* \psi_b \quad , \tag{3.A.4a}$$

the term in brackets on the left hand side of (3.A.3a) becomes zero. Given this choice, (3.A.3b) is satisfied (i.e. both sides are identically zero) for arbitrary x and x_b only if the adjoint boundary process λ_b in the inner product on the right hand side satisfies

$$E\lambda_b = 0 \quad . \tag{3.A.4b}$$

First note that the condition on λ_b in (3.A.4b) is in general an overspecification of the boundary condition for the dynamics in (3.A.4a). For example, in the 1-D case $L^\dagger$ is an n^{th} order differential operator and E is a full rank $2n\times 2n$ matrix so that (3.A.4b) provides twice as many constraints as are required for a properly posed boundary condition. Thus, to obtain a well-posed boundary value problem for λ, we will split (3.A.4b) into two conditions. The first, denoted as

$$E_1\lambda_b = 0 \quad , \tag{3.A.5a}$$

is chosen so that when it is combined with the dynamics (3.A.4a) the two comprise a well-posed boundary value problem. The second,

$$E_2\lambda_b = 0 \quad , \tag{3.A.5b}$$

is chosen so that E_1 and E_2 are linearly independent and should be viewed as an additional constraint on the solution of (3.A.4a) and (3.A.5a). Below we will use this additional constraint in determining the as yet unspecified relationship between the boundary output ψ_b and the inputs u_λ and v_λ.

Let the solution of (3.A.4a) with the boundary condition (3.A.5a) be denoted by

$$\lambda = \Gamma_{u_\lambda} u_\lambda + \Gamma_{v_\lambda} v_\lambda + \Gamma_{\psi_b}\psi_b \quad . \tag{3.A.6}$$

Define a boundary projection operator Δ_b by

$$\lambda_b = \Delta_b\lambda \quad . \tag{3.A.7}$$

Projecting (3.A.6) by (3.A.7) and applying the constraint in (3.A.5b) gives the following condition for ψ_b in terms of u_λ and v_λ:

$$E_2\Delta_b\Gamma_{\psi_b}\psi_b = -E_2\Delta_b\left[\Gamma_{u_\lambda} \vdots\ \Gamma_{v_\lambda}\right]\begin{bmatrix} u_\lambda \\ v_\lambda \end{bmatrix} \quad . \tag{3.A.8}$$

Here we assume that a left inverse for $E_2\Delta_b\Gamma_{\psi_b}$ exists (specifically, we will find that it exists for the 1-D case we consider later) and denote that left inverse by

$$\left(E_2\Delta_b\Gamma_{\psi_b}\right)^L E_2\Delta_b\Gamma_{\psi_b} = I \quad . \tag{3.A.9}$$

Finally, define the partitioned operator $[W_u \vdots W_v]$ as

$$[W_u \vdots W_v] = -\left(E_2\Delta_b\Gamma_{\psi_b}\right)^L E_2\Delta_b\left[\Gamma_{u_\lambda} \vdots \Gamma_{v_\lambda}\right] , \tag{3.A.10}$$

so that combining (3.A.8), (3.A.9) and (3.A.10) we have the relation we seek

$$\psi_b = [W_u \vdots W_v]\begin{bmatrix} u_\lambda \\ v_\lambda \end{bmatrix} . \tag{3.A.11}$$

This completes the specification of the internal differential realization of the Hilbert adjoint system:

Dynamics: $$L^\dagger\lambda = \left[C^* - V^*W_u\right]u_\lambda + \left[W^* - V^*W_v\right]v_\lambda \tag{3.A.12a}$$

Boundary Condition: $$E_1\lambda_b = 0 \tag{3.A.12b}$$

Outputs: $$\psi = B^*\lambda \tag{3.A.12c}$$

$$\psi_b = [W_u \vdots W_v]\begin{bmatrix} u_\lambda \\ v_\lambda \end{bmatrix} , \tag{3.A.12d}$$

or as an input-output map:

$$\begin{bmatrix} \psi \\ \psi_b \end{bmatrix} = H^*\begin{bmatrix} u_\lambda \\ v_\lambda \end{bmatrix} . \tag{3.A.12e}$$

Next we apply this form of the Hilbert adjoint system to define the smoother for a 1-D process with an integral form for its boundary condition.

Example:

Consider the $n\times 1$ vector 1-D stochastic process x satisfying

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{3.A.13a}$$

where u is an $m\times 1$ white noise process with covariance parameter $Q(t)$. The boundary condition for (3.A.13a) is an $n\times 1$ vector v with covariance matrix

Π_V and represents an integral of the process x over the interval [0,T]:

$$v = \int_0^T V(s)x(s)\, ds \quad . \tag{3.A.13b}$$

(Note that this is of the general form defined by (3.A.1))
In a derivation similar to that performed in obtaining a general solution for the two-point boundary condition, it can be shown that the general solution for (3.A.13a) and (3.A.13b) is given by

$$x(t) = \Phi(t,0)F^{-1}\{v - \xi^0(T)\} + x^0(t) \tag{3.A.13c}$$

where x^0 is the zero-initial-condition solution of (3.A.13a), F is the n×n matrix

$$F = \int_0^T V(s)\Phi(s,0)\, ds \tag{3.A.13d}$$

and ξ^0 is given by a running integral of x^0:

$$\xi^0(t) = \int_0^t V(s)x^0(s)\, ds \quad . \tag{3.A.13e}$$

As in the two-point boundary value problem, the invertibility of F is the well-posedness condition for this problem.

The smoothing problem we investigate here is to find the minimum variance estimate of x(t) as described above given the p×1 observation

$$y(t) = C(t)x(t) + r(t) \tag{3.A.14}$$

on the interval [0,T] where r is a white observation noise with covariance parameter R(t).

We begin by defining the complementary process by way of the Hilbert adjoint system as specified in (3.A.2). Invoking the duality between the input of H^* and the output of H, we note that since there is no boundary observation (no constant vector component of the output of H) there will be no constant vector component of the input to the adjoint system (i.e. no v_λ). Recalling that $(L^\dagger\lambda)(t) = -d\lambda(t)/dt - A'(t)\lambda(t)$ and substitituting into

(3.A.12), we have that the internal process of the adjoint system is governed by

$$\dot{\lambda}(t) = -A(t)\lambda(t) - C'(t)u_{\lambda}(t) + V'(t)\psi_b \tag{3.A.15a}$$

with output

$$\psi(t) = B'(t)\lambda(t) \tag{3.A.15b}$$

and where ψ_b is as yet undetermined (i.e. W_u has not yet been found).

To specify the functional form for ψ_b, we start by decomposing $E\lambda_b = 0$ into a boundary condition for (3.A.15a) and a constraint as discussed earlier. For the 1-D case, we have seen that E is a $2n \times 2n$ matrix and λ_b is a $2n \times 1$ vector given by

$$E = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix}$$

and

$$\lambda_b = \begin{bmatrix} \lambda(0) \\ \lambda(T) \end{bmatrix} \quad .$$

The condition that $E\lambda_b = 0$ is decomposed as follows:

<u>Boundary Condition:</u> $\lambda(0) = 0$ (3.A.16a)

and

<u>Constraint:</u> $\lambda(T) = 0$. (3.A.16b)

The solution of (3.A.13a) with boundary condition (3.A.16a) is

$$\lambda(t) = -\int_0^t \Phi_{\lambda}(t,s)C'(s)u_{\lambda}(s)ds + \int_0^t \Phi_{\lambda}(t,s)V'(s)ds \;\; \psi_b \quad . \tag{3.A.17a}$$

(Φ_{λ} is the transition matrix associated with -A'.)

Noting that

$$\Phi_{\lambda}(t,s) = \Phi'(s,t) \tag{3.A.17b}$$

we can rewrite (3.A.17a) as

$$\lambda(t) = -\int_0^t \Phi'(s,t)C'(s)u_\lambda(s)ds + \int_0^t \Phi'(s,t)V'(s)ds\ \psi_b \quad . \tag{3.A.17c}$$

To apply the constraint (3.A.16b), recall the definition of the matrix F in (3.A.13d), so that (3.A.17c) can be written at t = T as

$$\lambda(T) = -\int_0^T \Phi'(s,T)C'(s)u_\lambda(s)ds + \Phi'(0,T)F'\psi_b \quad . \tag{3.A.18}$$

The constraint in (3.A.16b) specifies that $\lambda(T)$ is zero. Thus, if we define W_u as the operator

$$W_u\alpha = F^{-1'}\Phi'(T,0)\int_0^T \Phi'(\tau,T)C'(\tau)\alpha(\tau)d\tau \tag{3.A.19}$$

then combining (3.A.16b) and (3.A.18) and inverting, we can solve for ψ_b as a function of the input $u\lambda$:

$$\psi_b = W_u u_\lambda \quad . \tag{3.A.20}$$

This completes the description of the internal differential realization for the Hilbert adjoint map H^*.

As stated earlier, the complementary process Z is defined in terms of H^* as

$$Z = [-I \ \vdots \ H^*]\Sigma_\zeta^{-1}\zeta \tag{3.A.21a}$$

where for the 1-D problem considered here the underlying process ζ is

$$\zeta = \begin{bmatrix} u \\ \hline v \\ \hline r \end{bmatrix} \tag{3.A.21b}$$

and the kernel of the correlation operator Σ_ζ is the covariance of ζ

$$\Sigma_\zeta(t,s) = \left[\begin{array}{c:c:c} Q(t)\delta(t-s) & 0 & 0 \\ 0 & \Pi_v & 0 \\ \hdashline 0 & 0 & R(t)\delta(t-s) \end{array}\right] \quad . \tag{3.A.21c}$$

As in Chapter 2, we denote the components of the complementary process as

$$Z = \begin{bmatrix} z \\ z_b \end{bmatrix} \quad . \tag{3.A.21d}$$

Given the definition (3.A.21) and the internal differential realization of the adjoint map H^* in (3.A.15) and (3.A.20), we can write the complementary process as the output of the following system:

$$\dot{\lambda} = -A'\lambda - C'R^{-1}r + V'W_u(R^{-1}r) \tag{3.A.22a}$$

$$\lambda(0) = 0 \tag{3.A.22b}$$

$$z = B'\lambda - Q^{-1}u \tag{3.A.22c}$$

$$z_b = W_u(R^{-1}r) - \Pi_v^{-1}v \quad . \tag{3.A.22d}$$

Augmenting (3.A.22) and (3.A.13) and inverting so that $\{y, z, z_b\}$ are inputs yields

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A & BQB' \\ C'R^{-1}C & -A' \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} + \begin{bmatrix} 0 \\ -C'R^{-1} \end{bmatrix} y + \begin{bmatrix} -BQ & 0 \\ 0 & V' \end{bmatrix} \begin{bmatrix} z \\ z_b + \Pi_v^{-1}\int_0^T V(s)x(s)ds \end{bmatrix} \tag{3.A.23a}$$

with boundary condition

$$-\Pi_v z_b = \int_0^T V(s)x(s)ds - \Pi_v F'^{-1}\Phi'(T,0)\int_0^T \Phi'(s,T)C'R^{-1}(y - Cx)ds \tag{3.A.23b}$$

and

$$0 = \lambda(0) \quad . \tag{3.A.23c}$$

The smoother is the solution of (3.A.23) conditioned on the observation y (i.e. with $z = 0$ and $z_b = 0$):

$$\begin{bmatrix} \dot{\hat{x}} \\ \dot{\hat{\lambda}} \end{bmatrix} = \begin{bmatrix} A & BQB' \\ C'R^{-1}C & -A' \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{\lambda} \end{bmatrix} + \begin{bmatrix} 0 \\ -C'R^{-1} \end{bmatrix} y + \begin{bmatrix} 0 \\ V'\Pi_v^{-1}\int_0^T V(s)\hat{x}(s)ds \end{bmatrix} \tag{3.A.24a}$$

with boundary condition

$$0 = \int_0^T V(s)\hat{x}(s)ds - \Pi_V F'^{-1}\Phi'(T,0)\int_0^T \Phi'(s,T)C'R^{-1}(y - C\hat{x})ds \qquad (3.A.24b)$$

and

$$0 = \hat{\lambda}(0) \quad . \qquad (A.24c)$$

The integro-differential equation and boundary condition in (3.A.24) can be transformed to an ordinary differential equation as follows. First define a constant process by:

$$\dot{\hat{v}}(t) = 0 \qquad (3.A.25a)$$

with boundary condition

$$\hat{v}(0) = \int_0^T V(s)\hat{x}(s)ds \qquad (\text{i.e., } \hat{v}(t) = \hat{v}(0) = \hat{v}\ ). \qquad (3.A.25b)$$

Given the initial condition (3.A.24c) and substituting from (3.A.25), we can write

$$\hat{\lambda}(t) = -\int_0^t \Phi'(s,t)C'R^{-1}\left[C\hat{x}(s) - y(s)\right]ds + \int_0^t \Phi'(s,t)V'\Pi_V^{-1}ds\ \hat{v} \quad . \qquad (3.A.26a)$$

Evaluating (3.A.26a) at t = T, premultiplying by $\Pi_V F'^{-1}\Phi'(T,0)$ and recalling that (see (3.A.18))

$$F' = \int_0^T \Phi'(s,0)V'(s)ds \quad , \qquad (3.A.26b)$$

it can be shown that (here we use the constraint (3.A.16b) that $\lambda(T) = 0$)

$$0 = -\Pi_V F'^{-1}\Phi'(T,0)\int_0^T \Phi'(s,T)C'R^{-1}\left[C\hat{x} - y\right]ds + \hat{v} \quad . \qquad (3.A.26c)$$

Substituting from (3.A.26c) into the boundary condition (3.A.24b) and augmenting the system in (3.A.24) with that in (3.A.25), gives the smoother as

$$\begin{bmatrix} \dot{\hat{x}} \\ \dot{\hat{\lambda}} \\ \dot{\hat{v}} \end{bmatrix} = \left[\begin{array}{c:c:c} A & BQB' & 0 \\ \hdashline C'R^{-1}C & -A' & V'\Pi_v^{-1} \\ \hdashline 0 & 0 & 0 \end{array}\right] \begin{bmatrix} \hat{x} \\ \hat{\lambda} \\ \hat{v} \end{bmatrix} + \begin{bmatrix} 0 \\ \hdashline -C'R^{-1} \\ \hdashline 0 \end{bmatrix} y \qquad (3.A.27a)$$

with boundary condition

$$\begin{bmatrix} 0 \\ \hdashline 0 \\ \hdashline 0 \end{bmatrix} = \left[\begin{array}{c:c} 0 & I \\ \hdashline I & 0 \\ \hdashline 0 & 0 \end{array}\right] \begin{bmatrix} \hat{\lambda}(0) \\ \hat{\lambda}(T) \end{bmatrix} + \begin{bmatrix} 0 \\ \hdashline 0 \\ \hdashline I \end{bmatrix} \hat{v} + \begin{bmatrix} 0 \\ \hdashline 0 \\ \hdashline -\int_0^T V(s)\hat{x}(s)ds \end{bmatrix} . \qquad (3.A.27b)$$

We remark that the boundary condition (3.A.27b) results from premultiplying the boundary condition obtained directly from the augmentation described above by the nonsingular matrix:

$$\left[\begin{array}{c:c:c} -\Pi_v F'^{-1}\Phi'(0,T) & 0 & 0 \\ \hdashline 0 & I & 0 \\ \hdashline 0 & 0 & I \end{array}\right] .$$

Since $\hat{v}$ is constant, a forward/backward two-filter implementation can be obtained by applying a diagonalizing transformation to the upper left $2n \times 2n$ of (3.A.27a).

It is likely that with further work this approach can be refined. In particular, a primary goal would be to reduce or eliminate the manipulations required to obtain the operator $[W_u \vdots W_v]$ whose action on u_λ and v_λ defines the adjoint system boundary output ψ_b. This simplification may be possible by a more judicious choice of decomposition of $E\lambda_b$ into a boundary condition and constraint. In our example, we have somewhat arbitrarily chosen the decomposition in (3.A.16).

APPENDIX 3B

THE CLASS OF HAMILTONIAN DIAGONALIZING TRANSFORMATIONS

To specify the entire class of transformations which lead to two-filter forms for the smoothing equations, we proceed as follows. Omitting specific reference to the independent variable t, the dynamics of the smoother for an $n\times 1$ vector process x on the interval [0,T], are given in the notation of Section 3.4.1 by

$$\begin{bmatrix} \dot{\hat{x}} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A & BQB' \\ C'R^{-1}C & -A' \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{\lambda} \end{bmatrix} + \begin{bmatrix} 0 \\ -C'R^{-1} \end{bmatrix} y \tag{3.B.1a}$$

or in abbreviated notation

$$\dot{X} = HX + Gy \quad . \tag{3.B.1b}$$

Although the principal subject of this appendix is the description of the complete class of diagonalizing transformations, we will also discuss the estimator boundary conditions in terms of those for the smoother for causal processes. We have chosen this case for notational convenience. For a causal process, the boundary condition for (3.B.1a) is written as

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} I & -\Pi_0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}(0) \\ \lambda(0) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \hat{x}(T) \\ \lambda(T) \end{bmatrix} \tag{3.B.1c}$$

where Π_0 is the covariance matrix of x(0). The objective is to find a system which is equivalent to (3.B.1) but which has been totally decoupled (i.e. both dynamics and boundary conditions) into two n^{th} order processes, one forward and one backward. Of course, total decoupling is only possible for separable cases such as causal processes (see Section 3.5.1). That is, we seek an equivalence transformation T such that the transformed process q:

$$q = \begin{bmatrix} q_f \\ q_b \end{bmatrix} = T \begin{bmatrix} \hat{x} \\ \hat{\lambda} \end{bmatrix} \tag{3.B.2a}$$

has dynamics of the form

$$\dot{q} = \begin{bmatrix} \dot{q}_f \\ \dot{q}_b \end{bmatrix} = \begin{bmatrix} A_f & 0 \\ 0 & A_b \end{bmatrix} \begin{bmatrix} q_f \\ q_b \end{bmatrix} + \begin{bmatrix} B_f \\ B_b \end{bmatrix} y \tag{3.B.2b}$$

and a boundary condition of the form

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} L_0^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} q_f(0) \\ q_b(0) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & M_T^{-1} \end{bmatrix} \begin{bmatrix} q_f(T) \\ q_b(T) \end{bmatrix} \quad . \tag{3.B.2c}$$

The reason for choosing the above notation for the boundary condition will become clear soon. The only real requirement is that the matrices denoted by L_0^{-1} and M_T^{-1} be nonsingular. This guarantees that the initial value for q_f and the final value for q_b are completely specified. It is straightforward to show that under (3.B.2a) the dynamics in (3.B.2b) are given by

$$\begin{bmatrix} A_f & 0 \\ 0 & A_b \end{bmatrix} = \dot{T}T^{-1} + THT^{-1} \quad . \tag{3.B.3a}$$

By equating the boundary conditions in (3.B.1c) and (3.B.2c), we obtain a two-point boundary condition for (3.B.3a):

$$\begin{bmatrix} L_0^{-1} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I & -\Pi_0 \\ 0 & 0 \end{bmatrix} T(0)^{-1} \tag{3.B.3b}$$

and

$$\begin{bmatrix} 0 & 0 \\ 0 & M_T^{-1} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} T(T)^{-1} \quad . \tag{3.B.3c}$$

With T partitioned as

$$T(t) = \begin{bmatrix} L_f(t) & M_f(t) \\ L_b(t) & M_b(t) \end{bmatrix} \tag{3.B.4a}$$

and rewriting (3.B.3a) as

$$\dot{T} = \begin{bmatrix} A_f & 0 \\ 0 & A_b \end{bmatrix} T - TH \quad , \tag{3.B.4b}$$

it can be shown that the elements of T are governed by two sets of coupled matrix differential equations:

$$\dot{L}_f = A_f L_f - L_f A - M_f C' R^{-1} C \qquad (3.B.5a)$$

$$\dot{M}_f = A_f M_f - L_f BQB' + M_f A' \qquad (3.B.5b)$$

and

$$\dot{L}_b = A_b L_b - L_b A - M_b C' R^{-1} C \qquad (3.B.6a)$$

$$\dot{M}_b = A_b M_b - L_b BQB' + M_b A' \quad . \qquad (3.B.6b)$$

The boundary conditions for the differential equations can be obtained from (3.B.3b) and (3.B.3c) as

$$L_f(0) = L_0 \quad , \text{ and } \quad M_f(0) = -L_0 \Pi_0 \qquad (3.B.7a,b)$$

$$L_b(T) = 0 \quad , \text{ and } \quad M_b(T) = M_T \quad . \qquad (3.B.8a,b)$$

Thus, (3.B.5), (3.B.6), (3.B.7) and (3.B.8) completely specify the class of transformations which lead to a two-filter implementation of the smoother. However, there are other considerations that we can state but for which we have no complete solution. The following properties are desirable to ensure numerical stability in the implementation of the two-filter solution:

1) Stable forward and backward dynamics for A_f and A_b respectively.
2) Stable coupled differential equations for computing the elements of T since they enter as gains on the observations:

$$\dot{q} = \begin{bmatrix} A_f & 0 \\ 0 & A_b \end{bmatrix} q + \begin{bmatrix} -M_f \\ -M_b \end{bmatrix} C' R^{-1} y \quad .$$

It would be nice to be able to specify the class of stable A_f and A_b for which the second property holds. Of course, one can investigate specific cases such as choosing $M_f = M_b$ = constant = Identity matrix. In this case, A_f and A_b can be determined directly from (3.B.5b) and (3.B.6b) and substituted into (3.B.5a) and (3.B.6a), yielding Riccati equations for L_f and L_b.

CHAPTER 4: **ALTERNATIVE APPROACHES TO THE TPBVP SMOOTHER**

SECTION 4.1
INTRODUCTION

A solution to the smoothing problem for an n^{th} order TPBVP was obtained in Chapter 3 by an application of the estimator developed from the method of complementary models. That solution was shown to be a $2n^{th}$ order boundary value process, and a two-filter implementation was formulated by diagonalizing the $2n^{th}$ order dynamics. In this chapter we show that constructing a Markov model for the TPBVP $x(t)$ by the methodology introduced by Castanon et al [37] provides an alternative means for formulating the smoothing equations. In particular, given a Markov model which we construct by this approach, filtering and smoothing can be accomplished by the classical Kalman filtering and associated smoothing algorithms. We will find that the Markov model obtained directly from this procedure has order 2n. A condition for model order reduction is established and related to Krener's separabiltiy condition discussed earlier in Chapter 3. In addition, we investigate two ways of incorporating the boundary measurement y_b into the Markov modelling framework. One is to include y_b as a post-flight measurement into a smoothed solution for $x(t)$. The other is to include it in a filtered solution by establishing a Markov model which incorporates y_b as a priori information.

As stated above, the two-filter implementation of the smoother developed in Chapter 3 has been derived by diagonalizing the $2n^{th}$ order smoother dynamics. In the second part of this chapter, we study this smoothing solution from a scattering viewpoint as has been done previously for the smoother for causal processes [38]. Our investigations in the scattering framework provide an alternative derivation of a two-filter implementation of the smoother. In addition, we construct a scattering diagram for the TPBVP smoother which, unlike that for the smoother for causal processes, is found to require feedback of information from each end of the interval [0,T]. It is anticipated that this diagram will provide a starting point for pictorial derivations of new results pertaining to the TPBVP smoother similar to those derived for the smoother for causal processes in [9].

SECTION 4.2

THE 1-D SMOOTHER VIA MARKOV MODELLING

4.2.1 Introduction

In Chapter 3 it was demonstrated that the n^{th} order model for the TPBVP x(t) is not a Markov model. In particular, it was shown that the correlation between the initial value of the process x(0) and any future value of the input white noise process u(t) is nonzero. Since most of the theory for estimation and control of stochastic processes has been developed for Markov models, the algorithms developed to implement this theory are not directly applicable to the TPBVP model described in Chapter 3. For this reason it is desirable to also have a Markov representation for this noncausal process. In particular, a Markov model would provide an alternative to the TPBVP smoothing procedure studied in Chapter 3.

In this section we define a class of linear Markov models in state space form and show that the TPBVP x(t) can be represented as a linear function of the internal state of a model in this class. The procedure we follow for constructing the Markov model is based on the methodology introduced by Castanon et al in [37] for causal processes with predictive information. Given the Markov model obtained from this constructive procedure, we demonstrate how in certain cases the order of the model can be reduced. In particular, the conditions for model order reduction are shown to be related to the concept of separability that was first introduced by Krener [23] and that was discussed earlier in Chapter 3 in reference to special two-filter forms of the TPBVP smoother solution.

In general, two-point measurements such as the boundary measurement y_b defined in (3.3.26) are not accomodated by Kalman filtering and associated smoothing algorithms. However, because the state vector for the Markov model that we construct for the TPBVP x(t) contains both the process itself x(t) and its initial condition x(0), we will find that the boundary measurement y_b

can be viewed as a post-flight measurement as considered in [27]. In this manner the information in y_b can be incorporated into the smoothed estimate of x(t) on the interval [0,T]. Alternatively, if we wish to include the information in this boundary measurement into a filtered estimate, this requires the construction of a different Markov model which incorporates y_b as a priori information. Again, employing the methodology developed in [37], we are also able to derive this second type of Markov model.

We begin this section by introducing some notation and mathematical preliminaries and by stating the form of the linear Markov model which will be used throughout the rest of the section.

4.2.2 Markov Models

The notation and terminology used in this subsection are taken from Wong [39]. Let the sigma field generated by a collection of random variables $\{s_i, i \in I\}$ be denoted by

$$\mathbf{S} = \sigma\{s_i, i \in I\} \quad . \tag{4.2.1}$$

A random variable s is said to be adapted to **S** if s is completely determined by (measurable with respect to) the collection of events contained in **S**, i.e. s is adapted to **S** if

$$E[s \mid S] = s \quad . \tag{4.2.2}$$

An increasing family of sigma fields $\{S_t\}$ is one for which information is cumulative, i.e. for $t \geq \tau \Rightarrow S_t \supseteq S_\tau$.

Definition Given an increasing family of sigma fields $\{S_t\}$, a stochastic process s(t) is an S_t-martingale if (i) s(t) is adapted to S_t, i.e. $s(t) = E[s(t) \mid S_t]$ and (ii) S_t contains no information about the forward increments[1] of s(t), i.e. $E[ds(t) \mid S_t] = 0$.

[1] Here all increments are defined by forward differences, i.e. $ds(t) = s(t+dt) - s(t)$, $dt > 0$.

Definition The model

$$dz(t) = F(t)z(t)dt + G(t)dw(t) \tag{4.2.3a}$$

$$x(t) = H(t)z(t) \tag{4.2.3b}$$

is a Markov model for x(t) with respect to an increasing family of sigma fields $\{F_t\}$ if

(i) z(t) is adapted to F_t

and

(ii) w(t) is an F_t-martingale.

Note: Since F_t is increasing and since z(t) is adapted to F_t, then x(t) is also adapted to F_t and

$$X_t \equiv \sigma\{x(\tau),\ 0 \leq \tau \leq t\} \subseteq F_t \quad . \tag{4.2.4}$$

Clearly, F_t can be no smaller than X_t. In the following, we will choose either $F_t = X_t$, or we will choose F_t as the union of X_t and the sigma field generated by the boundary measurement, $F_t = X_t \vee \sigma\{y_b\}$.

The dynamics and output equation in (4.2.3a) and (4.2.3b) provide a sample path description of x(t) for a given initial condition z(0). To complete the probabilistic description of x(t), we require the a priori distribution of z(0). We will delay the discussion of this prior distribution until after we have established the nature of the equations corresponding to (4.2.3a) and (4.2.3b) for the TPBVP. Note that the linearity in (4.2.3) represents a very specific from of Markov model. However, by construction, we will show that this structure is sufficient for a Markov representation of the TPBVP x(t).

4.2.3 A Markov Model for the TPBVP x(t) by Decomposition of dw(t)

In this section, x(t) represents the n×1 TPBVP which was introduced earlier in Chapter 3. In that chapter the dynamics of x(t) were formally expressed as a linear differential equation driven by white noise. Here we will write this process as the following stochastic differential equation driven by Wiener increments dw(t):

$$dx(t) = A(t)x(t)dt + B(t)dw(t) \tag{4.2.5a}$$

($E[dw(t)dw'(t)] = Q(t)dt$) with two-point boundary condition

$$v = V^0 x(0) + V^T x(T) \quad . \tag{4.2.5b}$$

The boundary value v with covariance matrix Π_v is assumed to be orthogonal to the increments $dw(t)$ for $t \in [0,T]$. Following the approach taken in Chapter 3, we will express the solution to (4.2.5a) and (4.2.5b) in terms of $x^0(t)$, the zero initial condition solution to (4.2.5a):

$$x^0(t) = \int_0^t \Phi(t,s)B(s)dw(s) \quad . \tag{4.2.6a}$$

Given (4.2.6a), the solution to (4.2.5a) and (4.2.5b) is

$$x(t) = \Phi(t,0)F^{-1}\left[v - V^T x^0(T)\right] + x^0(t) \tag{4.2.6b}$$

where F is the n×n matrix

$$F = V^0 + V^T \Phi(T,0) \quad . \tag{4.2.6c}$$

This form of the solution will be referred to often in our development of a Markov models for $x(t)$.

As a preview we will outline the general procedure to be followed in constructing a Markov model for the TPBVP $x(t)$. Given a sigma field F_t to which $x(t)$ is adapted, the Markov model with respect to F_t is constructed in the following steps:

(i) Adapt the Wiener increments $dw(t)$ to F_t by computing

$$d\hat{w}(t) = E[dw(t)|F_t] \quad . \tag{4.2.7}$$

(ii) Invoke the Doob-Meyer martingale decomposition theorem [40] to establish that

$$d\tilde{w}(t) = dw(t) - d\hat{w}(t) \tag{4.2.8a}$$

is an F_t-martingale increments process i.e.($\tilde{w}(t)$ is an F_t-martingale) whose quadratic variation is identical to that of the original Wiener increments $dw(t)$:

$$E[d\tilde{w}(t)d\tilde{w}'(t)] = E[dw(t)dw'(t)] = Q(t)dt \quad . \tag{4.2.8b}$$

(iii) Replace dw(t) in the differential equation (4.2.5a) by

$$dw(t) = d\tilde{w}(t) + d\hat{w}(t) \tag{4.2.9a}$$

to get the alternative representation

$$dx(t) = A(t)x(t)dt + B(t)d\hat{w}(t) + B(t)d\tilde{w}(t). \tag{4.2.9b}$$

(iv) Finally, we will find for the particular case considered here, i.e. where x(t) is the TPBVP and where we choose $F_t = X_t$, that the second term on the right hand side of (4.2.9b) can be expressed in the form

$$B(t)d\hat{w}(t) = \tilde{B}_x(t)x(t)dt + \tilde{B}(t)x(0)dt \quad . \tag{4.2.10}$$

If we define $z'(t) = [x'(t),x'(0)]$ and note that z(t) is adapted to $F_t = X_t$ and that $\tilde{w}(t)$ is an F_t-martingale, then the following is a Markov model for the TPBVP x(t)

$$dz(t) = \begin{bmatrix} \tilde{A}(t) & \tilde{B}(t) \\ 0 & 0 \end{bmatrix} z(t)dt + \begin{bmatrix} B(t) \\ 0 \end{bmatrix} d\tilde{w}(t) \tag{4.2.11a}$$

$$x(t) = [\ I \ \vdots \ 0 \ ]z(t) \tag{4.2.11b}$$

where

$$\tilde{A}(t) = A(t) + \tilde{B}_x(t) \quad . \tag{4.2.11c}$$

For the case when we choose to construct a Markov model with respect to the larger sigma field $F_t = X_t \vee \sigma\{y_b\}$, the expression in (4.2.10) will contain an additional term which is linear in y_b.

For the case $F_t = X_t$, the prior distribution for $z'(0) = [x'(0),x'(0)]$ is determined from the prior distributions for the boundary condition v and the Wiener process w(t) which define x(0) through (4.2.6b) evaluated at t = 0. Later when adding the boundary measurement to F_t, the prior distribution for x(0) will be determined in a Bayesian manner as suggested in [37]. In particular, in this second case the prior distribution for x(0) is obtained by updating the distribution for x(0) based

Recall that before any of the steps in the construction of the Markov model can be taken, the sigma field F_t must be identified. Our first choice will be the smallest possible sigma field, i.e. $F_t = X_t$. To determine the sigma field X_t, define the increasing sigma field generated by the Wiener process w(t) on the interval [0,t] as

$$W_t = \sigma\{w(\tau) \; ; \; \tau \in [0,t]\} \tag{4.2.12}$$

and define the sigma field generated by the value of x at t = 0 by

$$X_0 = \sigma\{x(0)\} \quad . \tag{4.2.13}$$

By reference to the initial value solution to (4.2.5a) given by the variation of constants formula ($x(t) = \Phi(t,0)x(0) + x^0(t)$), it is easy to see that x(t) is adapted to the sigma field generated by the combined events in W_t and X_0. The sigma field generated by this union of events is written as

$$X_t = W_t \vee X_0 \quad . \tag{4.2.14}$$

With F_t equal to X_t in (4.2.14), it can be shown that the model for x(t) given by the stochastic differential equation in (4.2.5a) with boundary condition (4.2.5b) is not Markov by showing that dw(t) is not an F_t-Martingale, i.e. by computing the expectation

$$E[dw(t) \,|\, F_t\,] \tag{4.2.15}$$

and showing that it is nonzero. An explicit expression for the nonzero value of this expectation will be computed later in the course of our construction of a Markov model for x(t) (see step (i) above). In particular, steps (i) and (ii) comprise the decomposition of dw(t) into an F_t-Martingale and a part which is predictable with respect to F_t. The decomposition is achieved through the computation in (4.2.7).

As discussed below, the mechanics of the decomposition of dw(t) are simplified if F_t is expressed in orthogonal components. In particular, by a Gram-Schmidt procedure we will orthogonalize X_0 with respect to W_t so that F_t can be written as the following direct sum

$$F_t = W_t \oplus \tilde{X}_{0,t} \tag{4.2.16}$$

where the random variables which generate $\tilde{X}_{0,t}$ are orthogonal to those which generate W_t (i.e. $\{w(\tau) \; ; \; \tau \in [0,t]\}$. The orthogonalization is performed by computing the conditional expectation of x(0) given W_t and then subtracting that computed value from x(0). This leaves a difference which is orthogonal to all elements of W_t. Let the part of x(0) which is predictable with respect to W_t be denoted by

$$\hat{x}(0;t) = E[x(0) \mid W_t] \quad . \tag{4.2.17a}$$

Evaluating the expression for x(t) in (4.2.6b) at t = 0 and recalling that the boundary value v is orthogonal to w(t), it is straightforward to show that

$$\hat{x}(0;t) = -F^{-1}V^T\Phi(T,t)x^0(t) \quad . \tag{4.2.17b}$$

The component of x(0) which is orthogonal to W_t is given by the "error":

$$\tilde{x}(0;t) = x(0) - \hat{x}(0;t) \quad . \tag{4.2.18a}$$

Substituting (4.2.17b) into (4.2.18a) gives

$$\tilde{x}(0;t) = x(0) + F^{-1}V^T\Phi(T,t)x^0(t) \quad . \tag{4.2.18b}$$

Thus,

$$\tilde{X}_{0,t} = \sigma\{\tilde{x}(0;t)\} \quad . \tag{4.2.19}$$

Given the orthogonality of the components of F_t in (4.2.16), the expectation in (4.2.7) can be written as the sum of two terms:

$$d\hat{w}(t) = E[dw(t) \mid F_t] = E[dw(t) \mid W_t] + E[dw(t) \mid \tilde{X}_{0,t}] \quad . \tag{4.2.20a}$$

Since Wiener increments are orthogonal, the first of these two terms is zero. Replacing the sigma field in the second term by the single random variable from which it is generated, (4.2.20a) simplifies to:

$$d\hat{w}(t) = E[dw(t) \mid \tilde{x}(0;t)] \quad . \tag{4.2.20b}$$

Thus the orthogonalization of the sigma fields has allowed us to express (4.2.7) as the expectation in (4.2.20b) which is conditioned on a single term.

The evaluation of this and other conditional expectations in this section are all made under the Gaussian assumption stated in Chapter 1. Under the Gaussian assumption, (4.2.20b) is given by

$$\hat{dw}(t) = E[dw(t)\tilde{x}'(0;t)]\left[E[\tilde{x}(0;t)\tilde{x}'(0;t)]\right]^{-1}\tilde{x}(0;t) \quad . \qquad (4.2.20c)$$

To simplify the notation, denote the expectations in (4.2.20c) by

$$K_w(4.t)dt \equiv E[dw(t)\tilde{x}'(0;t)]$$

$$= -Q(t)B'(t)\Phi'(T,t)V^{T'}F^{-1'}dt \qquad (4.2.21a)$$

and

$$\Sigma_0(t) \equiv E[\tilde{x}(0;t)\tilde{x}'(0;t)]$$

$$= F^{-1}\left[\Pi_V + V^T\left\{\Pi^0(T) - \Phi(T,t)\Pi^0(t)\Phi'(T,t)\right\}V^{T'}\right]F^{-1'} \qquad (4.2.21b)$$

where Π^0, the covariance of x^0, is governed by

$$\dot{\Pi}^0 = A\Pi^0 + \Pi^0 A' + BQB' \quad ; \; \Pi^0(0) = 0 \quad . \qquad (4.2.21c)$$

By noting that

$$I = F^{-1}V^0 + F^{-1}V^T\Phi(T,0) \quad , \qquad (4.2.22a)$$

we can premultiply x(0) in (4.2.18b) by this expression for the identity to get

$$\tilde{x}(0;t) = F^{-1}V^0x(0) + F^{-1}V^T\Phi(T,t)x(t) \quad . \qquad (4.2.22b)$$

Combining (4.2.20c), (4.2.21a), (4.2.21b) and (4.2.22a), the expectation in (4.2.20b) can be written as

$$\hat{dw}(t) = K_w(t)\Sigma_0^{-1}(t)F^{-1}\left[V^0x(0) + V^T\Phi(T,t)x(t)\right]dt \quad . \qquad (4.2.23)$$

This is the linear form described by (4.2.10) in step (iv) of the outline of the method for constructing our linear Markov model.

The Markov model is obtained by substituting the expression for the predictable part of the Wiener increments in (4.2.23) into (4.2.5b). If we define

$$\tilde{A}(t) = A(t) + B(t)K_w(t)\Sigma_0^{-1}(t)F^{-1}V^T\Phi(T,t) \quad , \tag{4.2.24a}$$

$$\tilde{B}(t) = B(t)K_w(t)\Sigma_0^{-1}(t)F^{-1}V^0 \tag{4.2.24b}$$

and

$$\xi = x(0) \quad , \tag{4.2.24c}$$

then the dynamics of the Markov model for (4.2.5a) and (4.2.5b) are given by

$$\begin{bmatrix} dx(t) \\ d\xi \end{bmatrix} = \begin{bmatrix} \tilde{A}(t) & \tilde{B}(t) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \xi \end{bmatrix} dt + \begin{bmatrix} B(t) \\ 0 \end{bmatrix} d\tilde{w}(t) \quad . \tag{4.2.25}$$

The initial condition for (4.2.25) has mean and covariance

$$E\begin{bmatrix} x(0) \\ \xi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad E\begin{bmatrix} x(0) \\ \xi \end{bmatrix}\left[x(0)' \vdots \xi'\right] = \begin{bmatrix} \Pi_0 & \Pi_0 \\ \Pi_0 & \Pi_0 \end{bmatrix} \tag{4.2.26a}$$

where Π_0 is the covarance of x(0). An expression for Π_0 is found from (4.2.6b) evaluated at t = 0 and the prior statistics of v and w(t) to be

$$\Pi_0 = F^{-1}\left[\Pi_v + V^T\Pi^0(T)V^{T'}\right]F^{-1'} \quad . \tag{4.2.26b}$$

4.2.4 <u>A Reduced Order Markov Model</u>

The order of the Markov model given by (4.2.25) is 2n, twice the dimension of the TPBVP x(t). Krener [23] has shown that under a "separable-stationary" condition there exists a Markov model of order n. Below, under general conditions, we derive a reduced order model whose order lies between n and 2n. Specifically, the Markov model order n_r can be reduced to n plus the rank of the matrix product $V^{T'}\Pi_v^{-1}V^0$. In the course of the derivation we generalize Krener's condition for separability and and show that is can be interpreted soley in terms of this product.

The basis for the model order reduction we study here is the recognition that if the coupling term $\tilde{B}(t)$ in (4.2.25) is not full rank, then we need not include the full n-vector ξ in the state of the Markov model. Denote the rank of this coupling term by n_C, and assume that we can write the $n \times n$ coupling matrix as the product of an $n \times n_C$ matrix $B_L(t)$ and a constant $n_C \times n$ matrix B_R as

$$\tilde{B}(t) = B_L(t)B_R \quad . \tag{4.2.27}$$

(The subscripts L and R signify left and right respectively.) In a moment we will investigate the mechanism for obtaining this type of decomposition. For now, we assume that one exists. If we define an $n_C \times 1$ random vector γ as

$$\gamma = B_R \xi \quad , \tag{4.2.28a}$$

then we can immediately write an $(n+n_C)$ order representation of the Markov model in (4.2.25) as

$$\begin{bmatrix} dx(t) \\ d\gamma \end{bmatrix} = \begin{bmatrix} \tilde{A}(t) & B_L(t) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \gamma \end{bmatrix} dt + \begin{bmatrix} B(t) \\ 0 \end{bmatrix} d\tilde{w}(t) \quad . \tag{4.2.28b}$$

To achieve the decomposition in (4.2.27), first recall the expression for the coupling matrix given earlier in (4.2.24b):

$$\tilde{B}(t) = B(t)K_w(t)\Sigma_0^{-1}(t)F^{-1}V^0 \quad . \tag{4.2.29a}$$

If we define

$$\Gamma(t) = \Pi^0(T) + \Phi(T,t)\Pi^0(t)\Phi'(T,t) \quad , \tag{4.2.29b}$$

then by substituting for each of the terms in (4.2.29a) it can be shown that

$$\tilde{B}(t) = -B(t)Q(t)B'(t)\Phi'(T,t)V^{T'}\left[\Pi_V + V^T\Gamma(t)V^{T'}\right]^{-1}V^0$$

$$= -B(t)Q(t)B'(t)\Phi'(T,t)\left[I + V^{T'}\Pi_V^{-1}V^T\Gamma(t)\right]^{-1}V^{T'}\Pi_V^{-1}V^0 \quad . \tag{4.2.29c}$$

Let θ_c denote the product

$$\theta_c = V^{T'} \Pi_v^{-1} V^0 \tag{4.2.30a}$$

and denote its rank by n_c

$$n_c = \text{rank}(\theta_c) \quad . \tag{4.2.30b}$$

Clearly, θ_c can be expressed as the product of full rank matrices θ_L and θ_R of dimension $n \times n_c$ and $n_c \times n$ respectively

$$\theta_c = \theta_L \theta_R \quad . \tag{4.2.30b}$$

This leads to the form of decompostion of $\tilde{B}(t)$ in (4.2.27) that is required for the reduced order model in (4.2.28). Specifically, this decomposition is given by:

$$B_L(t) = -B(t)Q(t)B'(t)\Phi'(T,t)\left[I + V^{T'} \Pi_v^{-1} V^T \Gamma(t)\right]^{-1} \theta_L \tag{4.2.31a}$$

and

$$B_R = \theta_R \quad . \tag{4.2.31b}$$

Substituting these expressions into (4.2.28a) and (4.2.28b) gives the reduced order ($n_r = n + n_c$) Markov model we seek.

Separability

Recall that a TPBVP is referred to as separable if it has an n^{th} order Markov model representation. We obtain such a model by the decomposition described above when $n_c = 0$ (or equivalently when $\theta_c = 0$). Thus, here we will say that a system is separable when θ_c is zero. Krener's condition for separability (in addition to stationarity) is that V^T, V^0 and Π_v are of the form (with identical partitioning for each matrix)

$$V^0 = \begin{bmatrix} V_1^0 & V_2^0 \\ 0 & 0 \end{bmatrix} , \qquad V^T = \begin{bmatrix} 0 & 0 \\ V_1^T & V_2^T \end{bmatrix}$$

and

$$\Pi_v = \begin{bmatrix} \Pi_v^0 & 0 \\ 0 & \Pi_v^T \end{bmatrix} \quad . \tag{4.2.32}$$

Thus, Krener's condition for separability is a special case of the condition $\theta_c = 0$ (with no stationarity restriction). In addition, note that this condition is the same condition we obtained in Chapter 3 for the case of no boundary measurement.

4.2.5 Markov Models and the Boundary Measurement y_b

Given the Markov model derived in the previous section, Kalman filtering and associated smoothing algorithms can be employed to optimally process observations of the form $y(t) = C(t)x(t) + r(t)$. In our statement of the smoothing problem for the noncausal TPBVP in Chapter 3 we also included an additional "boundary" observation of the form

$$y_b = W^0 x(0) + W^T x(T) + r_b \quad . \tag{4.2.33a}$$

Although, this type of two-point observation is not accomodatad within the Kalman filtering framework when dealing with a Markov model whose internal state contains x(t) alone, because the internal state of the Markov model that was developed earlier in this section contains both x(t) and x(0), we can include (4.2.33a) as a post-flight measurement. That is, if we recall that $\xi = x(0)$, we can simply rewrite (4.2.33a) as

$$y_b = [\, W^T \vdots\, W^0 \,] \begin{bmatrix} x(T) \\ \xi \end{bmatrix} + r_b \quad . \tag{4.2.33b}$$

Thus, y_b can be viewed as an observation of the state of the Markov model in (4.2.26) at the endpoint of the interval [0,T].

If we are to include the information in y_b in the estimate of x(t) through the post-flight measurement (4.2.33b), we find that the Markov model order reduction discussed in Section 4.2.4 must be reconsidered. Recall that in reducing the model order we replaced the $n\times 1$ vector state element ξ by the lower dimensional vector γ as defined in (4.2.28a). With this substitution, we will not in general be able to write y_b as a post-flight measurement in terms of the reduced order state. However, if we constrain the choice of the reduced order state element γ as discused below, then y_b can be written as a post-flight measurement. Specifically, the matrix B_R which defines the augmented state component γ in (4.3.28a) must be chosen in such a way that W^0 is a linear combination of its rows, i.e.,

$$W^0 x(0) = W^0 \xi = W^0_\gamma B_R \xi = W^0_\gamma \gamma \quad . \tag{4.2.34a}$$

This ensures that the boundary measurement y_b can be written as a linear combination of the reduced order state vector at t = T:

$$y_b = \left[W^T \ \vdots \ W^0_\gamma\right] \begin{bmatrix} x(T) \\ \gamma \end{bmatrix} + r_b \tag{4.2.34b}$$

where

$$W^0 = W^0_\gamma B_R \qquad (\text{i.e. } W^0_\gamma \gamma = W^0 \xi \ ) \quad . \tag{4.2.35}$$

Thus, for a reduced order model B_R must satisfy both (4.2.35) and (4.2.30b).

In the next subsection, we investigate an alternative approach for handling the boundary measurement whereby it is included as a priori information when forming the Markov model. This approach has been motivated by the work of Castanon et al in [37]. In particular, by including y_b as a priori information, we no longer need to consider it as a separate measurement.

4.2.6 A Markov Model w.r.t X_t Plus the Boundary Observation

There may be cases when the boundary observation is known as part of the a priori information. For instance, in [37] x(t) is a causal process modelling the trajectory of a vehicle, and the boundary observation represents some predictive information with respect to the vehicles destination at a known time T, i.e. $y_b = x(T) + r_b$. In order to include that information into a filtered estimate of the vehicle's trajectory, a Markov model was constructed to include this predictive information. In this section we will extend those results and consider the case where x(t) is a boundary value process and the predictive information y_b is a two-point boundary measurement as in (4.2.33a).

Here we construct a Markov model for the TPBVP x(t) with respect to the expanded sigma field containing both X_t and the sigma field generated by the boundary observation. Let

$$Y_b = \sigma\{y_b\} \quad . \tag{4.2.36}$$

The expanded sigma field of interest here will be denoted by

$$G_t = X_t \vee Y_b \tag{4.2.37a}$$

or substituting from (4.2.16)

$$G_t = \{\tilde{X}_{0,t} \vee W_t\} \vee Y_b \quad . \tag{4.2.37b}$$

Following the steps for the construction of a Markov model, we transform the TPBVP dynamics in (4.2.5a) to those of a Markov model by decomposing the Wiener increments process dw(t) into a G_t-predictable process and a G_t-martingale increments process. First, we continue the Gram-Schmidt procedure and orthogonalize Y_b in (4.2.37b) with respect to the two orthogonalized sigma fields W_t and $\tilde{X}_{0,t}$. As before, this is accomplished by removing from Y_b its $\{W_t \vee \tilde{X}_{0,t}\}$-predictable part. Specifically, that predictable part is given by

$$\hat{y}_b(t) = E\left[y_b \,\middle|\, \tilde{X}_{0,t} \vee W_t\right] \quad . \tag{4.2.38}$$

The difference

$$\tilde{y}_b(t) = y_b - \hat{y}_b(t) \tag{4.2.39a}$$

represents the information in y_b not contained in $\{\tilde{X}_{0,t} \vee W_t\}$ and generates a sigma field

$$\tilde{Y}_{b,t} = \sigma\{\tilde{y}_b(t)\} \tag{4.2.39b}$$

which is orthogonal to both $\tilde{X}_{0,t}$ and W_t. Thus, given $\tilde{y}_b(t)$ we can write G_t in orthogonal components as

$$G_t = \tilde{X}_{0,t} + W_t + \tilde{Y}_{b,t} \quad . \tag{4.2.40}$$

Before computing the decomposition of dw(t) with respect to G_t, we must derive an expression for $\tilde{y}_b(t)$. The first step is to compute $\hat{y}_b(t)$ in (2.38). Due to the orthogonality of the sigma fields in (4.2.38), we can write

$$\hat{y}_b(t) = E[y_b \mid \tilde{x}(0;t)] + E[y_b \mid W_t] \quad . \tag{4.2.41}$$

To make the computations in (4.2.41) more manageable, we will write y_b in orthogonal components as follows. Replacing x(0) and x(T) in (4.2.33a) by (4.2.6b) evaluated at t = 0 and t = T, it can be shown that

$$y_b = \left[F_b F^{-1} \; \vdots \; W^T - F_b F^{-1} V^T\right] \begin{bmatrix} v \\ - - - \\ x^0(T) \end{bmatrix} + r_b \tag{4.2.42a}$$

where

$$F_b = W^0 + W^T \Phi(T,0) \quad . \tag{4.2.42b}$$

With $\tilde{x}(0;t)$ given by (4.2.18b) and using the expression for y_b above, it can be shown after some manipulation that

$$\hat{y}_b(t) = \left[F_b \Sigma_0(t) - W^T \Gamma(t) V^{T'} F^{-1'}\right] \Sigma_0^{-1}(t) \tilde{x}(0;t)$$

$$+ \left[W^T - F_b F^{-1} V^T\right] \Phi(T,t) x^0(t) \tag{4.2.43}$$

where $\Sigma_0(t)$, the covariance of $\tilde{x}(0;t)$, is given in (4.2.21b) and $\Gamma(t)$ is defined in (4.2.29b).

The difference $\tilde{y}_b(t)$ is found by employing the expression for y_b in (4.2.42), the expression for $\hat{y}_b(t)$ in (4.2.43) and the expression for $\tilde{x}(0;t)$ in (4.2.18b) to give by (after considerable manipulation)

$$\tilde{y}_b(t) = W^T \left\{\Lambda(t) v + \left[I - \Lambda(t) V^T\right]\left(x^0(T) - \Phi(T,t) x^0(t)\right)\right\} + r_b \tag{4.2.44a}$$

where

$$\Lambda(t) = \Gamma(t) V^{T'} \left[\Pi_v + V^T \Gamma(t) V^{T'}\right]^{-1}$$

$$= \left[I + \Gamma(t) V^{T'} \Pi_v^{-1} V^T\right]^{-1} \Gamma(t) V^{T'} \Pi_v^{-1} \quad . \tag{4.2.44b}$$

If we note that $\Gamma(t)$ in (4.2.29b) is the variance of $x^0(T) - \Phi(T,t)x^0(t)$, then the variance of $\tilde{y}_b(t)$ can be written directly from (4.2.44a) as

$$\Sigma_b(t) = E[\tilde{y}_b(t)\tilde{y}_b'(t)] \tag{4.2.45}$$

$$= W^T\Lambda(t)\Pi_v\Lambda(t)W^{T'} + [I - \Lambda(t)V^T]\Gamma(t)[I - \Lambda(t)V^T]' + \Pi_b \quad .$$

Given the expressions for $\tilde{y}_b(t)$ and its covariance, we are ready to compute the G_t-predictable part of $dw(t)$. Using the superscript asterisk to differentiate this estimate from $\hat{dw}(t)$ in (4.2.7), the G_t-predictable part of dw(t) is

$$\hat{dw}^*(t) = E[dw(t) \mid \tilde{X}_{0,t}] + E[dw(t) \mid W_t] + E[dw(t) \mid \tilde{Y}_{b,t}]. \tag{4.2.46a}$$

Noting that the second term is zero and replacing the sigma fields in the remaining two nonzero terms by their associated random variables, (4.2.46a) becomes

$$\hat{dw}^*(t) = E[dw(t) \mid \tilde{x}(0;t)] + E[dw(t) \mid \tilde{y}_b(t)] \quad . \tag{4.2.46b}$$

By employing the expression for $\tilde{x}(0;t)$ in (4.2.22b), the first term in (4.2.46b) can be shown to be given by

$$E[dw(t) \mid \tilde{x}(0;t)] = -Q(t)B'(t)\Phi'(T,t)V^{T'}\left[\Pi_v + V^T\Gamma(t)V^{T'}\right]$$

$$\cdot\left\{V^0x(0) + V^T\Phi(T,t)x(t)\right\}dt \quad . \tag{4.2.47}$$

(This expression is, in fact, the same as (4.2.23) but written in slightly different notation.) The second term of (4.2.46b) requires a bit more effort. However, with a modicum of perseverance, it can be shown that

$$E[dw(t) \mid \tilde{y}_b(t)] = E[dw(t)\tilde{y}_b'(t)]\Sigma_b^{-1}(t)\tilde{y}_b(t) \tag{4.2.48}$$

$$= Q(t)B'(t)\Phi'(T,t)\left[I - \Lambda(t)V^T\right]'\Sigma_b^{-1}(t)$$

$$\cdot\left\{y_b - \left[W^0 - W^T\Lambda(t)V^0\right]x(0) - W^T\left[I - \Lambda(t)V^T\right]\Phi(T,t)x(t)\right\}dt$$

In order to express $B(t)d\tilde{w}^*(t)$ in more compact notation, define (see (4.2.10))

$$\tilde{B}^*(t) = -B(t)Q(t)B'(t)\Phi'(T,t)\{V^{T'}[\Pi_v + V^T\Gamma(t)V^{T'}]^{-1}V^0$$

$$+ [I - \Lambda(t)V^T]'\Sigma_b^{-1}(t)[W^0 - W^T\Lambda(t)V^0]\} \quad , \quad (4.2.49a)$$

$$\tilde{B}_x^*(t) = -B(t)Q(t)B'(t)\Phi'(T,t)\{V^{T'}[\Pi_v + V^T\Gamma(t)V^{T'}]^{-1}V^T$$

$$+ [I - \Lambda(t)V^T]'\Sigma_b^{-1}(t)W^T[I - \Lambda(t)V^T]\}\Phi(T,t) \quad (4.2.49b)$$

and

$$K_b^*(t) = B(t)Q(t)B'(t)\Phi'(T,t)[I - \Lambda(t)V^T]'\Sigma_b^{-1}(t) \quad (4.2.49c)$$

so that

$$B(t)d\hat{w}^*(t) = \{\tilde{B}_x^*(t)x(t) + \tilde{B}^*(t)x(0) + K_b^*(t)y_b\}dt \quad . \quad (4.2.49d)$$

Finally, substituting

$$dw(t) = d\tilde{w}^*(t) + d\hat{w}^*(t) \quad (4.2.50a)$$

into the dynamics of the TPBVP $x(t)$ in (4.2.4a) gives the dynamics for the Markov model of $x(t)$ with respect to the expanded sigma field G_t in (4.2.37) as

$$\begin{bmatrix} dx(t) \\ d\xi \end{bmatrix} = \begin{bmatrix} \tilde{A}^*(t) & \tilde{B}^*(t) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \xi \end{bmatrix} dt + \begin{bmatrix} K_b^*(t) \\ 0 \end{bmatrix} y_b dt + \begin{bmatrix} B(t) \\ 0 \end{bmatrix} d\tilde{w}^*(t) \quad (4.2.50b)$$

where

$$\tilde{A}^*(t) = A(t) + \tilde{B}_x^*(t) \quad (4.2.50c)$$

and

$$\xi = x(0) \quad . \quad (4.2.50d)$$

As discussed earlier, the prior distribution for the initial condition of the internal state of the Markov model in (4.2.50b), i.e. $[x'(0),x'(0)]'$, is the distribution of $x(0)$ conditioned on y_b. Under the Gaussian assumption, this distribution is completely defined by the conditional mean and conditional variance which are computed as follows. Let

$$\nu = \begin{bmatrix} v \\ \hline x^0(t) \end{bmatrix} \tag{4.2.51a}$$

where from the prior distributions for v and w(t)

$$E[\nu] = 0 \tag{4.2.51b}$$

and

$$\Pi_\nu \equiv E[\nu\nu']$$

$$= \begin{bmatrix} \Pi_v & 0 \\ 0 & \Pi^0(T) \end{bmatrix} \quad . \tag{4.2.51c}$$

If we define

$$H_0 \equiv F^{-1}[I \;\vdots\; -V^T] \quad , \tag{4.2.52}$$

then it can be shown from (4.2.8b) that

$$x(0) = H_0\nu \quad . \tag{4.2.53}$$

With

$$H_b \equiv [F_bF^{-1} \;\vdots\; W^T - F_bF^{-1}V^T] \quad , \tag{4.2.54}$$

it can be shown directly from (4.2.42a) that the boundary measurement can be expressed as a linear observation of ν:

$$y_b = H_b\nu + r_b \quad . \tag{4.2.55}$$

Since $x(0)$ is a linear function of ν, the conditional mean and variance of $x(0)$ given y_b can be computed from the conditional mean and variance of ν. The conditional mean of ν given y_b is

$$\hat{\nu} = K_\nu y_b \tag{4.2.56a}$$

where

$$K_\nu = \Pi_\nu H_b' \left[H_b \Pi_\nu H_b' + \Pi_b\right]^{-1} \quad . \tag{4.2.56b}$$

The conditional variance of ν is

$$\Pi_{\nu|y_b} = \left[\Pi_\nu^{-1} + H_b' \Pi_b^{-1} H_b\right]^{-1} \quad . \tag{4.2.57}$$

Therefore, the distribution of x(0) conditioned on y_b is

$$N\left(H_0 \hat{\nu} \; , \; H_0 \Pi_{\nu|y_b} H_0'\right) \quad . \tag{4.2.58}$$

This completes the description of the Markov model for the TPBVP x(t) with respect to the expanded sigma field G_t.

To obtain a reduced order model, we proceed as we had done earlier. That is, we seek a decomposition of the coupling term $\tilde{B}^*(t)$ in (4.2.49a). That is, we seek a decomposition of the form

$$\tilde{B}^*(t) = \tilde{B}_L^*(t) \theta_L \theta_R \tag{4.2.59a}$$

where

$$\theta_c = \theta_L \theta_R \quad . \tag{4.2.59b}$$

Unfortunately, due to the complexity of the terms in (4.2.49a), we have been unable to achieve a factorization of this form.

In summary, the Markov model developed in this subsection makes it possible to include the information in the boundary measurement y_b in a forward filtered estimate of the TPBVP x(t). This has been accomplished by augmenting the sigma field with respect to which the Markov model is derived to include the sigma field generated by y_b. In contrast, the Markov model which was developed in Section 4.2.3 required that y_b be included as a post-flight measurement so that the information in this measurement is incorporated in the smoothed estimate of x(t).

SECTION 4.3

A SCATTERING APPROACH

4.3.1 Introduction

Scattering theory refers to the study of the propagation (reflection and transmission) of waves through a medium. Ljung et al. [38] found that the linear smoother for causal stochastic processes has a natural interpretation in the framework of linear scattering theory. This discovery has inspired a number of studies [9,33,41] which have produced both new results and insights into old results related to the smoother for causal systems. As explained in [9], many of these results are obtained through simple pictorial derivations using the basic superposition property of linear systems.

After discussing the basics of linear scattering theory in Section 4.3.2, we show in Section 4.3.3 that any two-point boundary value system of the type discussed in Chapter 3 can be viewed in the scattering framework. In general, these scattering pictures will contain feedback paths. Since both the causal system smoother and the noncausal system smoother can be represented as two-point boundary value systems (cf. Chapter 3), both have scattering interpretations. However, due to the simple structure of the boundary conditions for the causal system smoother, feedback is not required, as evidenced in the previously mentioned studies. By studying the properties of the TPBVP smoother in the scattering framework, we find that the two-filter type of implementation developed by Hamiltonian diagonalization in Chapter 3, is a natural form of solution obtained from the scattering representation. In addition, by formulating a scattering picture for the TPBVP smoother, we provide a starting point for pictorial derivations of new results of the type preformed in [9] for the smoother for causal processes.

4.3.2 Preliminaries for Linear Scattering

The form of 1-D linear scattering picture that we will use is described as follows. Let r(t) be a rightward propagating wave and $\ell(t)$ be a leftward propagating wave which interact within a scattering medium as depicted in the block diagram in Figure 3.3.1. The input/output relation for this scattering interaction will be denoted by

$$\begin{bmatrix} r(\tau) \\ \ell(t) \end{bmatrix} = S(\tau,t) \begin{bmatrix} r(t) \\ \ell(\tau) \end{bmatrix} + \begin{bmatrix} q^+(\tau,t) \\ q^-(\tau,t) \end{bmatrix} \qquad (4.3.1)$$

where q^+ and q^- are internal sources and S is the scattering matrix:

$$S(\tau,t) = \begin{bmatrix} A(\tau,t) & B(\tau,t) \\ C(\tau,t) & D(\tau,t) \end{bmatrix} \qquad (4.3.2)$$

with A and D representing transmission operators, and B and C representing reflection operators. Later in Section 3.3.3 we will show that Figure 3.3.1 must be augmented to include feedback when illustrating the scattering picture for a TPBVP.

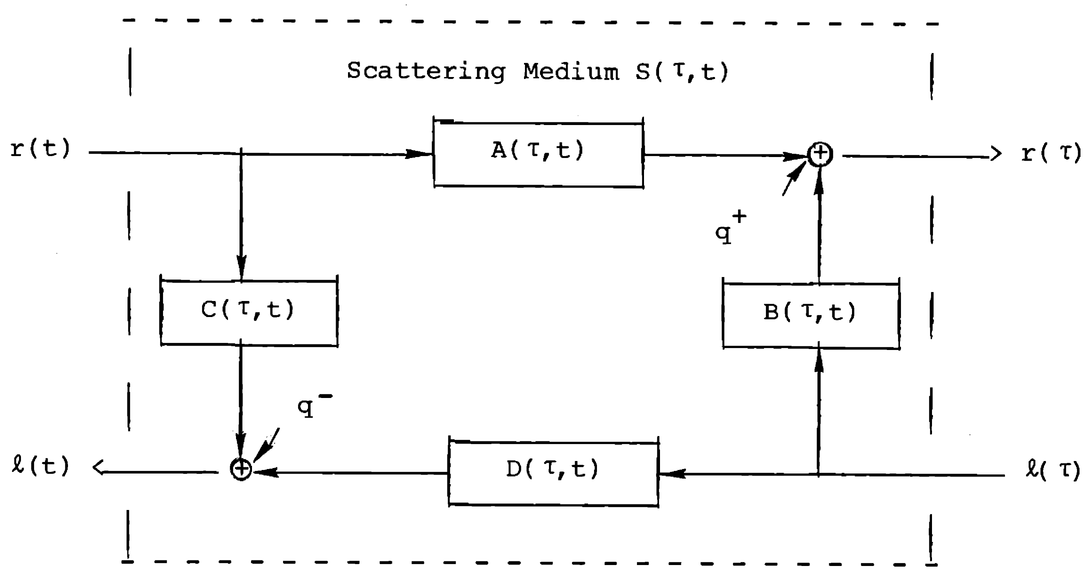


Figure 4.3.1 Linear Scattering Picture

The medium in Figure 4.3.1 represents only a single layer of a composite scattering medium. To compute the effect of scattering through two cascaded layers as depicted in 3.3.2, we introduce the star product [42] and the dot sum [41]. Let S_1 and S_2 and q_1 and q_2 be the scattering matrices and internal sources, respectively, of layers 1 and 2. The combined scattering of these two layers will be represented by

$$\begin{bmatrix} r^+ \\ \ell^- \end{bmatrix} = S_{12} \begin{bmatrix} r^- \\ \ell^+ \end{bmatrix} + q_{12} , \qquad (4.3.3)$$

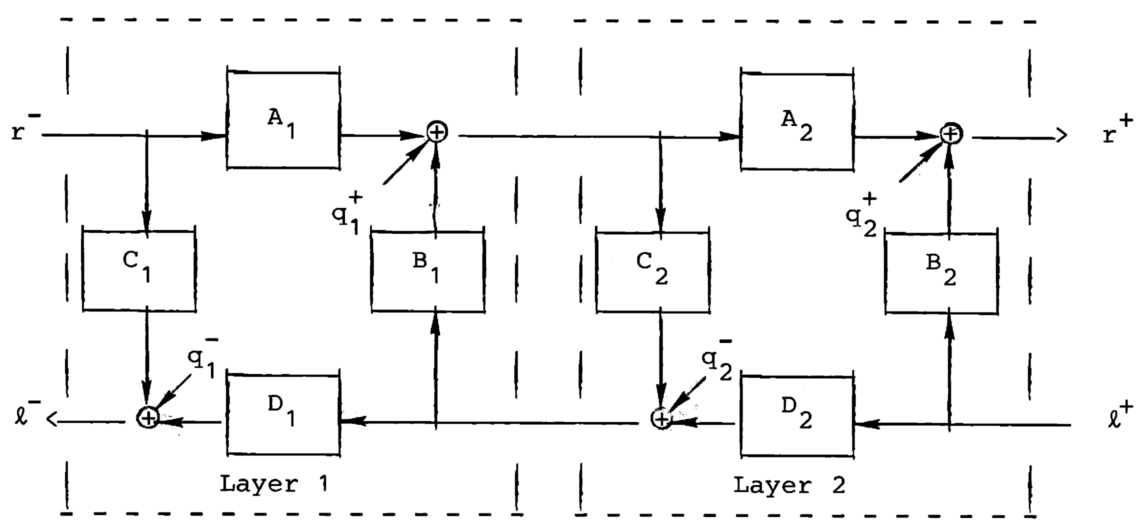


Figure 4.3.2. Cascade of Two Layers

where the combined scattering matrix S_{12} is computed as the star product of S_1 and S_2:

$$S_{12} = S_2 * S_1 \qquad (4.3.4)$$

$$= \left[\begin{array}{c:c} A_2(I - B_1C_2)^{-1}A_1 & B_2 + A_2B_1(I - C_2B_1)^{-1}D_2 \\ \hdashline C_1 + D_1C_2(I - B_1C_2)^{-1}A_1 & D_1(I - C_2B_1)^{-1}D_2 \end{array}\right]$$

and the combined internal source vector q_{12} is given by the assembly sum of q_1 and q_2 which we denote by

$$q_{12} = q_2 \circ q_1 \tag{4.3.5a}$$

and which can be written as the following linear combination of q_1 and q_2

$$q_{12} = \left[\begin{array}{c:c} A_2(I - B_1C_2)^{-1} & 0 \\ \hdashline -D_1(I - C_2B_1)^{-1}C_2 & I \end{array}\right] q_1 + \left[\begin{array}{c:c} I & A_2(I - B_1C_2)^{-1}B_1 \\ \hdashline 0 & D_1(I - C_2B_1)^{-1} \end{array}\right] q_2 \ . \tag{4.3.5b}$$

4.3.3 Scattering for the TPBVP

Here we present a general discussion of scattering for a two-point boundary value process. The results we obtain are applicable to all linear two-point boundary value problems. Later we will apply these results to the special case of the smoother derived in Chapter 3. In this section a two-point boundary value process will be represented by an $n_{x\lambda}\times 1$ vector partitioned into $n_x\times 1$ and $n_\lambda\times 1$ vectors $x(t)$ and $\lambda(t)$ as

$$\begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix}$$

$(n_{x\lambda} = n_x + n_\lambda)$. The TPBVP is denoted in this way rather than simply by $x(t)$ because later the partitions will be identified with those of the TPBVP smoother (see (3.4.2)). The linear differential equation defining the dynamics of the TPBVP will be written in partitioned form as

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A_x & A_{x\lambda} \\ A_{\lambda x} & A_\lambda \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} + \begin{bmatrix} B_x \\ B_\lambda \end{bmatrix} u \tag{4.3.6a}$$

with two-point boundary condition denoted by

$$v_{x\lambda} = [V_x^0 \ \vdots \ V_\lambda^0] \begin{bmatrix} x(0) \\ \lambda(0) \end{bmatrix} + [V_x^T \ \vdots \ V_\lambda^T] \begin{bmatrix} x(T) \\ \lambda(T) \end{bmatrix} . \tag{4.3.6b}$$

Here we assume that the boundary condition meets the well-posedness conditions for the existence of a unique solution $[x(t), \lambda(t)]$. In the following two subsections our discussion of the scattering interpretation of this process is separated into two parts: one part for the dynamics in (4.3.6a) and the other for the boundary condition in (4.3.6b).

The Dynamic Layers

Following the approach taken in [9], we show how the dynamics in (4.3.6a) can be manipulated to derive corresponding differential equations for the scattering matrices and internal sources depicted in the scattering diagram of Figure 4.3.3. Temporarily, it will be assumed that the inputs $x(0)$ and $\lambda(T)$ in Figure 4.3.3 are known. Later we will describe how these inputs are determined by considering the boundary value v.

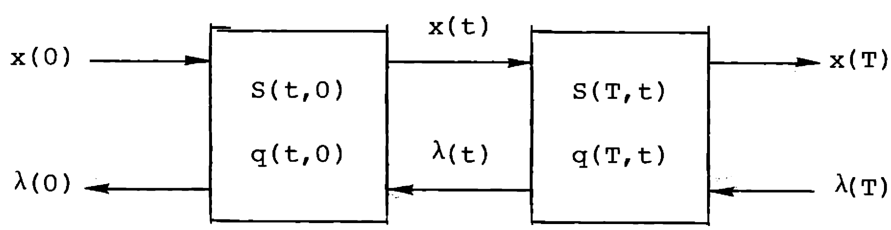


Figure 4.3.3. TPBVP Scattering (Dynamic Layers)

To obtain initial-value differential equations for $S(t,0)$ and $q(t,0)$, first consider the following discrete-step approximation for (4.3.6a)

$$\underbrace{\begin{bmatrix} x(t+\Delta) \\ \lambda(t) \end{bmatrix}}_{} = \underbrace{\begin{bmatrix} I+A_x\Delta & A_{\lambda x}\Delta \\ -A_{\lambda x}\Delta & I-A_\lambda\Delta \end{bmatrix}}_{S(t+\Delta,t)} \begin{bmatrix} x(t) \\ \lambda(t+\Delta) \end{bmatrix} + \underbrace{\begin{bmatrix} B_x \\ -B_\lambda \end{bmatrix}}_{q(t+\Delta,t)} u(t)\Delta + O(\Delta^2) \tag{4.3.7}$$

If we define the partitions of S as

$$S(t,\tau) = \begin{bmatrix} S_x(t,\tau) & S_{x\lambda}(t,\tau) \\ S_{\lambda x}(t,\tau) & S_\lambda(t,\tau) \end{bmatrix} \qquad (4.3.8)$$

and note that

$$S(t+\Delta,\tau) = S(t+\Delta,t) * S(t,\tau) \quad , \qquad (4.3.9)$$

then differential equations for the elements of the partitions of $S(t,\tau)$ in (4.3.8) can be found from the definition of the derivative:

$$\frac{d}{dt}\left[S(t,\tau)\right] = \lim_{\Delta \to 0} \frac{S(t+\Delta,\tau) - S(t,\tau)}{\Delta} \quad . \qquad (4.3.10)$$

Carrying out the star product operation in (4.3.9), substituting the result into (4.3.10) and taking limits yields the coupled matrix differential equations for the partitions of $S(t,\tau)$ [43](also see Appendix 4B for the details of this kind of derivation):

$$\dot{S}_x(t,\tau) = [A_x(t) - S_{x\lambda}(t,\tau)A_{\lambda x}(t)]\, S_x(t,\tau) \qquad ; \quad S_x(\tau,\tau) = I,$$

$$\dot{S}_{x\lambda}(t,\tau) = A_x(t)S_{x\lambda}(t,\tau) - S_{x\lambda}(t,\tau)A_\lambda(t) + A_{x\lambda}(t)$$
$$- S_{x\lambda}(t,\tau)A_{\lambda x}(t)S_{x\lambda}(t,\tau) \qquad ; \quad S_{x\lambda}(\tau,\tau) = 0,$$

$$\dot{S}_{\lambda x}(t,\tau) = -S_\lambda(t,\tau)A_{\lambda x}(t)S_x(t,\tau) \qquad ; \quad S_{\lambda x}(\tau,\tau) = 0$$

and

$$\dot{S}_\lambda(t,\tau) = -S_\lambda(t,\tau)\,[A_\lambda(t) + A_{\lambda x}(t)S_{x\lambda}(t,\tau)] \qquad ; \quad S_\lambda(\tau,\tau) = I \quad .$$

(4.3.11a,b,c,d)

Differential equations for the sources are found in a similar fashion. Compute $q(t+\Delta,\tau)$ via the assembly sum

$$q(t+\Delta,\tau) = q(t+\Delta,t) \circ q(t,\tau) \quad . \qquad (4.3.12)$$

Then, taking limits (see (4.3.10)) gives the following vector differential equations for the internal sources

$$\dot{q}^{+}(t,\tau) = \left[A_x(t) - S_{x\lambda}(t,\tau)A_{\lambda x}(t)\right]q^{+}(t,\tau) + \left[B_x(t) - S_{x\lambda}(t,\tau)B_{\lambda}(t)\right]u(t)$$

$$; \quad q^{+}(\tau,\tau) = 0 \qquad (4.3.13a)$$

and

$$\dot{q}^{-}(t,\tau) = -S_{\lambda}(t,\tau)\left[A_{\lambda x}(t)q^{+}(t,\tau) + B_{\lambda}(t)u(t)\right]$$

$$; \quad q^{-}(\tau,\tau) = 0 \quad . \qquad (4.3.13b)$$

Values for $S(t,0)$ and $q(t,0)$ in Figure 4.3.3 are found by solving (4.3.11) and (4.3.13), with the initial conditions specified at $\tau = 0$. Derivations similar to those used to formulate (4.3.11) and (4.3.13) can be used to obtain final-value differential equations for $S(\tau,t)$ and $q(\tau,t)$ in the second layer of Figure 4.3.3. These derivations are carried out in Appendix 4A.

Before moving to the discussion of the boundary layers, expressions for $x(t)$ and $\lambda(t)$ are derived from the scattering picture in Figure 4.3.3. That is, we solve for $x(t)$ and $\lambda(t)$ as a function of the inputs $x(0)$ and $\lambda(T)$ and the sources $q(t,0)$ and $q(T,t)$. First, consider the scattering equations implied by the left and right media:

$$\begin{bmatrix} x(t) \\ \lambda(0) \end{bmatrix} = S(t,0) \begin{bmatrix} x(0) \\ \lambda(t) \end{bmatrix} + q(t,0) \qquad (4.3.14a)$$

$$\begin{bmatrix} x(T) \\ \lambda(t) \end{bmatrix} = S(T,t) \begin{bmatrix} x(t) \\ \lambda(T) \end{bmatrix} + q(T,t) \quad . \qquad (4.3.14b)$$

These two equations can be combined into a higher dimensional equation which we will represent by

$$\begin{bmatrix} x(t) \\ \lambda(0) \\ x(T) \\ \lambda(t) \end{bmatrix} = \Psi \left[\Psi_i \begin{vmatrix} x(0) \\ \lambda(T) \end{vmatrix} + \Psi_q \begin{bmatrix} q^+(t,0) \\ q^-(t,0) \\ q^+(T,t) \\ q^-(T,t) \end{bmatrix} \right] \quad . \qquad (4.3.14c)$$

If we define

$$\Gamma_x(t) = I - S_{x\lambda}(t,0) S_{\lambda x}(T,t) \qquad (4.3.14d)$$

and

$$\Gamma_\lambda(t) = I - S_{\lambda x}(T,t) S_{x\lambda}(t,0) \quad , \qquad (4.3.14e)$$

then it can be shown that the coefficient matrices in (4.3.14c) are

$$\Psi = \left[\begin{array}{c:c:c:c} \Gamma_x(t)^{-1} & 0 & 0 & 0 \\ \hdashline S_\lambda(t,0) S_{\lambda x}(T,t) \Gamma_x(t)^{-1} & I & 0 & 0 \\ \hdashline 0 & 0 & I & S_x(T,t) S_{x\lambda}(t,0) \Gamma_\lambda(t)^{-1} \\ \hdashline 0 & 0 & 0 & \Gamma_\lambda(t)^{-1} \end{array}\right]$$

(4.3.14f)

$$\Psi_i = \left[\begin{array}{c:c} S_x(t,0) & S_{x\lambda}(t,0) S_\lambda(T,t) \\ \hdashline S_{\lambda x}(t,0) & S_\lambda(t,0) S_\lambda(T,t) \\ \hdashline S_x(T,t) S_x(t,0) & S_{x\lambda}(T,t) \\ \hdashline S_{\lambda x}(T,t) S_x(t,0) & S_\lambda(T,t) \end{array}\right] \qquad (4.3.14g)$$

and

$$\Psi_q = \begin{bmatrix} I & 0 & 0 & S_{x\lambda}(t,0) \\ 0 & I & 0 & S_{\lambda}(t,0) \\ S_x(T,t) & 0 & I & 0 \\ S_{\lambda x}(T,t) & 0 & 0 & I \end{bmatrix} . \qquad (4.3.14h)$$

Substituting these into (4.3.14c) gives the following expressions for $x(t)$ and $\lambda(t)$:

$$x(t) = \Gamma_x(t)^{-1}\{S_x(t,0)x(0) + q^+(t,0) + S_{x\lambda}(t,0)[S_\lambda(T,t)\lambda(T) + q^-(T,t)]\}$$

$$\lambda(t) = \Gamma_\lambda(t)^{-1}\{S_\lambda(T,t)\lambda(T) + q^-(T,t) + S_{\lambda x}(T,t)[S_x(t,0)x(0) + q^+(t,0)]\}$$

(4.3.15a,b)

Later when we identify the elements of the scattering matrix and the internal sources with variables associated with the smoother, it will become clear that equation (4.3.15a) for $x(t)$ corresponds to a form of two-filter smoother solution. Thus the two-filter form of the solution is a natural one associated with the scattering point of view.

The existence of the solution (4.3.14c) hinges on the invertibility of $\Gamma_x(t)$ and $\Gamma_\lambda(t)$ in (4.3.14d,c). In the smoother case considered in Section 4.3.4 , existence is guaranteed by uniform complete controllability and observability. That is, the Riccati equation solutions $S_{x\lambda}(t,0)$ and $S_{x\lambda}(T,t)$ will be symmetric and strictly negative definite and strictly positive definite, respectively, under the controllability and observability assumptions.

The Boundary Layers

Up to this point we have ignored the contribution of the boundary condition in (4.3.6b). Actually, close inspection of the solution for $x(t)$ and $\lambda(t)$ in (4.3.15) reveals that we have assumed knowledge of both $x(0)$ and $\lambda(T)$, e.g. a special case of the two-point boundary condition. In general, these two values will not be known explicitly, and, as indicated below, we must use the boundary condition to solve for them.

In solving for $[x(0), \lambda(T)]$ two cases will be considered. We start by first rewriting the boundary condition (4.3.6b) as

$$[\, V_x^0 \; \vdots \; V_\lambda^T \,] \begin{bmatrix} x(0) \\ \lambda(T) \end{bmatrix} = v_{x\lambda} - [\, V_x^T \; \vdots \; V_\lambda^0 \,] \begin{bmatrix} x(T) \\ \lambda(0) \end{bmatrix} \qquad (4.3.16)$$

Case 1

If $[V_x^0 \; \vdots \; V_\lambda^T]$ is invertible, then (4.3.16) can be solved directly for $[x(0), \lambda(T)]$ as

$$\begin{bmatrix} x(0) \\ \lambda(T) \end{bmatrix} = \begin{bmatrix} M_x & M_{x\lambda} \\ M_{\lambda x} & M_\lambda \end{bmatrix} \begin{bmatrix} x(T) \\ \lambda(0) \end{bmatrix} + \begin{bmatrix} M_{xv} \\ M_{\lambda v} \end{bmatrix} v_{x\lambda} \qquad (4.3.17a)$$

where

$$\begin{bmatrix} M_x & M_{x\lambda} \\ M_{\lambda x} & M_\lambda \end{bmatrix} \equiv -[V_x^0 \; \vdots \; V_\lambda^T]^{-1} [V_x^T \vdots \; V_\lambda^0] \quad \text{and} \quad \begin{bmatrix} M_{xv} \\ M_{\lambda v} \end{bmatrix} \equiv [V_x^0 \vdots \; V_\lambda^T]^{-1} . \qquad (4.3.17b)$$

This allows us to complete Figure 4.3.3 with left and right boundary layers and feedback of $x(T)$ and $\lambda(0)$ as depicted in Figure 4.3.4. Alternatively, these two boundary layers could be combined into a single left or right boundary layer directly from (4.3.17a) which is already in the scattering form of (4.3.1).

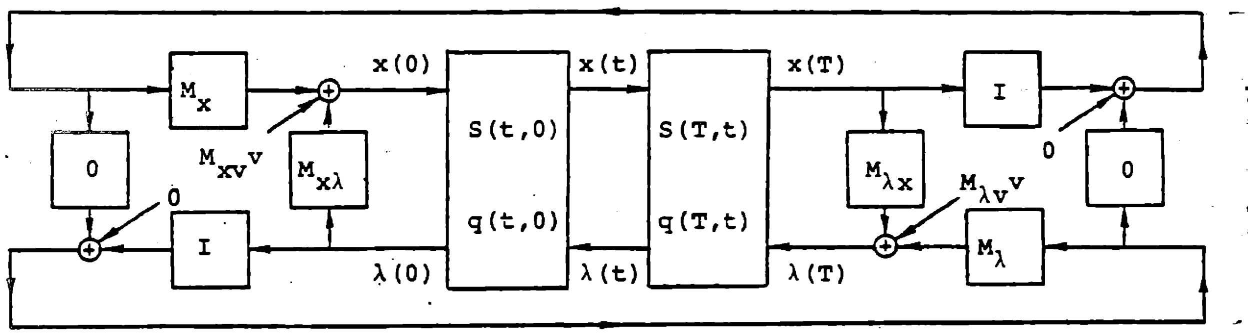


Figure 4.3.4. TPBVP Scattering Picture for $[V_x^0 \vdots V_\lambda^T]$ Invertible

Case 2

When $[V_x^0 \vdots V_\lambda^T]$ is not invertible, there are two approaches to solving for $[x(0), \lambda(T)]$. The first approach discussed is the most direct. Although, it does not readily lend itself to a scattering diagram, it does lead to a form of the forward/backward two-filter solution for the TPBVP as developed in Chapter 3. The second approach, although indirect, allows us to complete the scattering picture with feedback loops as we had done in Figure 4.3.4 for Case 1. Indeed, the second approach is actually a method of transforming a Case 2 problem into a Case 1 problem.

First write $[x(T), \lambda(0)]$ as the output of the scattering equation for the entire interval [0, T].

$$\begin{bmatrix} x(T) \\ \lambda(0) \end{bmatrix} = \underbrace{\begin{bmatrix} S_x(T,0) & S_{x\lambda}(T,0) \\ S_{\lambda x}(T,0) & S_\lambda(T,0) \end{bmatrix}}_{S(T,0)} \begin{bmatrix} x(0) \\ \lambda(T) \end{bmatrix} + \underbrace{\begin{bmatrix} q^+(T,0) \\ q^-(T,0) \end{bmatrix}}_{q(T,0)} . \quad (4.3.18)$$

Using (4.3.18) to replace $[x(T), \lambda(0)]$, (4.3.16) can be written as

$$\left\{ [V_x^0 \,\vdots\, V_\lambda^T] + [V_x^T \,\vdots\, V_\lambda^0] S(T,0) \right\} \begin{bmatrix} x(0) \\ \lambda(T) \end{bmatrix} = v_{x\lambda} - [V_x^T \,\vdots\, V_\lambda^0] q(T,0). \tag{4.3.19}$$

The matrix in brackets on the left hand side of (4.3.19) will be invertible if the boundary value problem (4.3.6) is well-posed. That is, this matrix is the scattering equivalent of the matrix F in (3.2.36). Premultiplying (4.3.19) by the inverse of this matrix inverse yields a solution for $[x(0), \lambda(T)]$ as a linear function of the internal sources and the boundary value $v_{x\lambda}$. Denote the inverse by

$$F_s^{-1} \equiv \left\{ [V_x^0 \,\vdots\, V_\lambda^T] + [V_x^T \,\vdots\, V_\lambda^0] S(T,0) \right\}^{-1} \tag{4.3.20a}$$

and define

$$\begin{bmatrix} F_{xv} \\ F_{\lambda v} \end{bmatrix} \equiv F_s^{-1} \quad \text{and} \quad \begin{bmatrix} F_{xq}^+ & F_{xq}^- \\ F_{\lambda q}^+ & F_{\lambda q}^- \end{bmatrix} \equiv -F_s^{-1} [V_x^T \,\vdots\, V_\lambda^0] \,. \tag{4.3.20b}$$

Inverting (4.3.19) gives:

$$\begin{bmatrix} x(0) \\ \lambda(T) \end{bmatrix} = \begin{bmatrix} F_{xv} \\ F_{\lambda v} \end{bmatrix} v_{x\lambda} + \begin{bmatrix} F_{xq}^+ & F_{xq}^- \\ F_{\lambda q}^+ & F_{\lambda q}^- \end{bmatrix} \begin{bmatrix} q^+(T,0) \\ q^-(T,0) \end{bmatrix} \,. \tag{4.3.20c}$$

Substituting from (4.3.20c) into (4.3.15a) we obtain the following expression for $x(t)$

$$\begin{aligned} x(t) = \Gamma_x^{-1}(t) \Big\{ & [S_x(t,0)F_{xv} + S_{x\lambda}(t,0)S_\lambda(T,t)F_{\lambda v}] v_{x\lambda} \\ & + [S_x(t,0)F_{xq}^+ + S_{x\lambda}(t,0)S_\lambda(T,t)F_{\lambda q}^+] q^+(T,0) \\ & + [S_x(t,0)F_{xq}^- + S_{x\lambda}(t,0)S_\lambda(T,t)F_{\lambda q}^-] q^-(T,0) \\ & + S_{x\lambda}(t,0) q^-(T,t) \\ & + q^+(t,0) \Big\} \end{aligned} \tag{4.3.21}$$

If we solve for $q^{+}(t,0)$ in the forward direction from the initial value problem in (4.3.13a) with $q^{+}(T,0)$ its value at the endpoint T, and if we solve for $q^{-}(T,t)$ as the backward solution to (4.A.11b) in Appendix 4B with $q^{-}(T,0)$ its value at the initial point 0, then (4.3.21) represents a forward/backward two-filter solution for the TPBVP. Unfortunately, this solution is not posed directly in terms of the outputs $[x(T), \lambda(0)]$ of the dynamic scattering layer (depicted in Figure 4.3.3); and thus, this form of the solution provides no straightforward way to augment Figure 4.3.3 as we had done for Case 1 (Figure 4.3.4).

Although it does not directly lead to a scattering picture, the form of the solution in (4.3.21) has a number of notable properties. First, it provides the mechanism for a forward/backward implementation of the general solution for a TPBVP without having to explicitly determine a decoupling transformation. Of course, solving the Riccati equations for $S_{\lambda x}$ and $S_{x\lambda}$ represents the identical computational problem to that of solving the Riccati equations for the elements of the decoupling transformation (see (3.4.6)). Second, the solution in (4.3.21) makes it clear that the inputs u(t) can be processed independently of any knowledge of the boundary conditions. Specifically, the solution of the differential equations for the forward and backward processes $q^{+}(t,0)$ and $q^{-}(T,t)$ makes no use of the boundary value v or the form of the boundary condition.

An alternative solution from which we can construct a scattering picture is derived indirectly using the approach taken for Case 1. Recall that the solution in Case 1 required that $[V_x^0 \;\vdots\; V_\lambda^T]$ be invertible. In general, well-posedness only guarantees that the partitioned matrix:

$$[V^0 \;\vdots\; V^T] = [V_x^0 \;\vdots\; V_\lambda^0 \;|\; V_x^T \;\vdots\; V_\lambda^T] \tag{4.3.22}$$

is full rank n so that the ranks of V^0 and V^T, n_0 and n_T, satisfy

$$n_0 + n_T \geq n_{x\lambda} . \tag{4.3.23}$$

Until now, no assumption has been made concerning the dimensions of the partitioning $[x(t), \lambda(t)]$ of the TPBVP in (4.3.6). Indeed, any vector TPBVP could be arbitrarily partitioned to obtain the form (4.3.6). We will show that an appropriate choice of partitioning of the TPBVP will guarantee that $[x(t), \lambda(t)]$ can be transformed to an equivalent process insuch a way that the invertibility condition is met for the boundary condition for that transformed process. First note that we can always partition the $n_{x\lambda}\times 1$ boundary value process so that the dimensions of x and λ are less than the ranks of V^0 and V^T

$$n_x = \dim(x) \leq n_0 \quad , \tag{4.3.24a}$$

$$n_\lambda = \dim(\lambda) \leq n_T \tag{4.3.24b}$$

and so that the overall dimension of the TPBVP is the sum of the two

$$n_{x\lambda} = \dim(x) + \dim(\lambda) \quad . \tag{4.3.24c}$$

Next consider a time-varying equivalence transformation

$$\begin{bmatrix} \tilde{x}(t) \\ \tilde{\lambda}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} T_x(t) & T_{x\lambda}(t) \\ T_{\lambda x}(t) & T_\lambda(t) \end{bmatrix}}_{T_b(t)} \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} \tag{4.3.25a}$$

with the inverse denoted by

$$T_b(t) = T_b^{-1}(t) \quad . \tag{4.3.25b}$$

Let

$$\tilde{V}^0 \equiv [V_x^0 \; \vdots \; V_\lambda^0] T_b(0) \equiv [\tilde{V}_x^0 \; \vdots \; \tilde{V}_x^0] \tag{4.3.26a}$$

and

$$\tilde{V}^T \equiv [V_x^T \; \vdots \; V_\lambda^T] T_b(T) \equiv [\tilde{V}_x^T \; \vdots \; \tilde{V}_\lambda^T] \quad . \tag{4.3.26b}$$

Then under T_b, the boundary condition is transformed to

$$v_{x\lambda} = \tilde{V}^0 \begin{bmatrix} \tilde{x}(0) \\ \tilde{\lambda}(0) \end{bmatrix} + \tilde{V}^T \begin{bmatrix} \tilde{x}(T) \\ \tilde{\lambda}(T) \end{bmatrix} \quad . \tag{4.3.27a}$$

The key is to choose T_b at t = 0 and t = T to shift the linearly independent columns of V^0 and V^T (see (4.3.22)) so that

$$[\ \tilde{V}_x^0 \ \vdots\ \tilde{V}_\lambda^T\ ]$$

is invertible. Thus, the problem can be transformed to Case 1. In particular, if we reorganize terms in (4.3.27a) to get an equation analogous to (4.3.16), i.e.

$$[\tilde{V}_x^0 \ \vdots\ \tilde{V}_\lambda^T]\begin{bmatrix}\tilde{x}(0)\\ \tilde{\lambda}(T)\end{bmatrix} = v_{x\lambda} - [\tilde{V}_x^T \ \vdots\ \tilde{V}_\lambda^0]\begin{bmatrix}\tilde{x}(T)\\ \tilde{x}(0)\end{bmatrix}, \quad (4.3.27b)$$

then the invertibility of the transformed equation allows us to write (4.3.27b) as

$$\begin{bmatrix}\tilde{x}(0)\\ \tilde{\lambda}(T)\end{bmatrix} = \begin{bmatrix}\tilde{M}_x & \tilde{M}_{x\lambda}\\ \tilde{M}_{\lambda x} & \tilde{M}_\lambda\end{bmatrix}\begin{bmatrix}\tilde{x}(T)\\ \tilde{\lambda}(0)\end{bmatrix} + \begin{bmatrix}\tilde{M}_{xv}\\ \tilde{M}_{\lambda v}\end{bmatrix} v_{x\lambda} \quad . \quad (4.3.27c)$$

Note that an application of the equivalence transformation in (4.3.25a) for all times t will require a transformation of the dynamics as well. However, we now show that it is not actually necessary to transform the dynamics.

We start by developing a scattering representation for the transformation in (4.3.25a). Consider (4.3.25a) at t=0 and its inverse which we will write in partitioned form as

$$\begin{bmatrix}x(0)\\ \lambda(0)\end{bmatrix} = \begin{bmatrix}T_x & \vdots & T_{x\lambda}\\ - - & \vdots & - -\\ T_{\lambda x} & \vdots & T_\lambda\end{bmatrix}\begin{bmatrix}\tilde{x}(0)\\ - -\\ \tilde{\lambda}(0)\end{bmatrix} \quad . \quad (4.3.28)$$

Selecting the first row of (4.3.28) and the second row of (4.3.25a) gives

$$x(0) = T_x(0)\tilde{x}(0) + T_{x\lambda}(0)\tilde{\lambda}(0) \quad (4.3.29a)$$

$$\tilde{\lambda}(0) = T_{\lambda x}(0)x(0) + T_\lambda(0)\lambda(0) \quad (4.3.29b)$$

which can be represented by the two scattering layers in Figure 4.3.5.

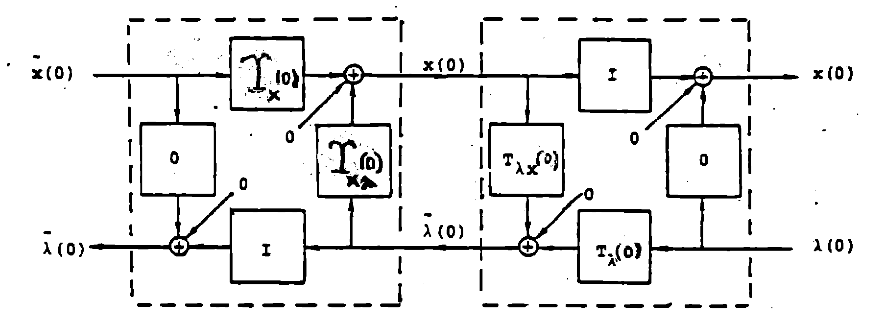


Figure 4.3.5 $\Sigma_b(0)$, The Scattering Representation of $T_b(0)$.

This has a simple interpretation as the transformation of a transmission operator to a scattering operator. That is, if the transformation (4.3.25a) is viewed as a transmission operation [9], then Figure 4.3.5 is the scattering representation of its transmission matrix. A similar picture can be developed for this transformation at t = T, $T_b(T)$. As a shorthand, these scattering representations will be denoted by

$$\begin{bmatrix} x(0) \\ \tilde{\lambda}(0) \end{bmatrix} = \Sigma_b(0) \begin{bmatrix} \tilde{x}(0) \\ \lambda(0) \end{bmatrix} \qquad (4.3.30a)$$

and

$$\begin{bmatrix} \tilde{x}(T) \\ \lambda(T) \end{bmatrix} = \Sigma_b(T) \begin{bmatrix} x(T) \\ \tilde{\lambda}(T) \end{bmatrix} \quad . \qquad (4.3.30b)$$

Employing (4.3.30a) and (4.3.30b) leads to the two equivalent scattering pictures for the transformed dynamics as shown in Figure 4.3.6. Combining the dynamic layers from Figure 4.3.6 and the feedback representation of the effect of the transformed boundary condition from Figure 4.3.4, we get the scattering picture for the transformed process $(\tilde{x},\tilde{\lambda})$ in Figure 4.3.7. Of course, $\Sigma_b(0)$ and $\Sigma_b(T)$ could be combined with the other boundary layers via the star product to simplify Figure 4.3.7.

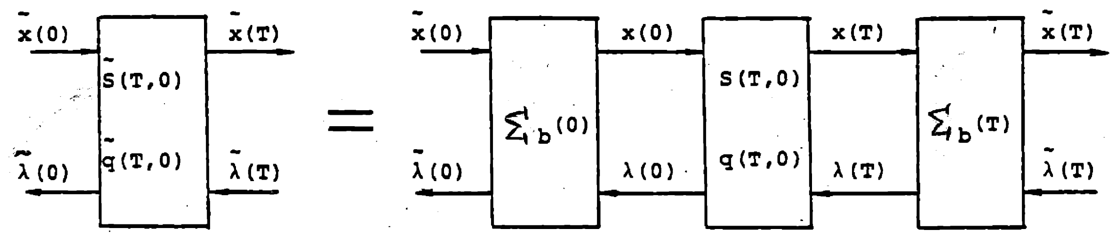


Figure 4.3.6. Equivalent Pictures for $[\tilde{S}(T,0), \tilde{q}(T,0)]$.

We have gone to the trouble of developing a scattering representation for a TPBVP so that in the next section we can view the special case of the noncausal smoother in this framework. In particular, we have developed a picture (Figure 4.3.7) which maintains the dynamics of the original, untransformed process so that the dynamics and boundary conditions for the TPBVP smoother can be analyzed separately. Having a complete scattering picture for the TPBVP smoother provides a starting point for pictorial derivations of decentralized processing, map combining and map updating as performed for the smoother for causal processes in [9].

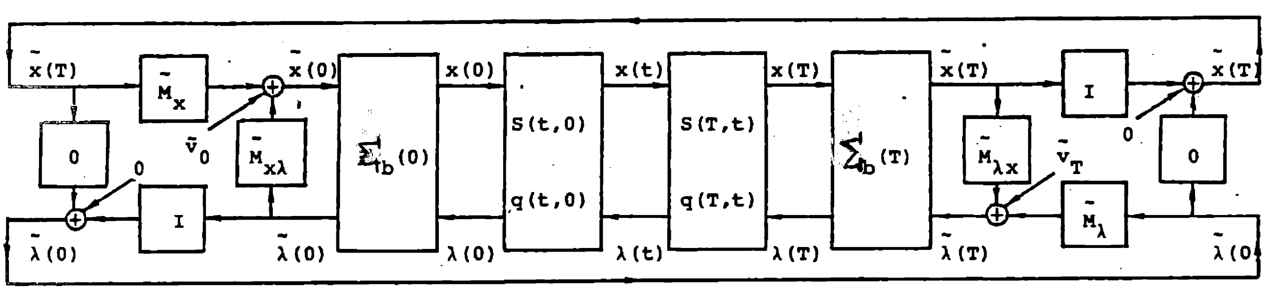


Figure 4.3.7. Scattering for the general TPBVP: $\begin{bmatrix} \tilde{v}_0 \\ \tilde{v}_T \end{bmatrix} = \begin{bmatrix} \tilde{M}_{xv} \\ \tilde{M}_{\lambda v} \end{bmatrix} v_{x\lambda}$

4.3.4 Scattering Picture for the TPBVP Smoother

Here we specialize the scattering results obtained in the previous subsections for a TPBVP written in the general partitioned form in (4.3.6) to the special case of the smoother derived in Chapter 3. In the same way that the discussions of the dynamical contribution and the boundary value contribution for the scattering representation of a general TPBVP were separated, we will separate the discussions of the Hamiltonian dynamics and the boundary conditions for the smoother. Since the Hamiltonian dynamics are identical for both the noncausal and causal system smoothers, we can simply review some previously published results pertaining to the dynamics which were derived in the context of the causal system smoother (see e.g. [9, 38 and 41]). The discussion of the boundary layers includes both the boundary contribution to the two-filter solution in (4.3.21) and a transformation of the type given in (4.3.25) which leads to a scattering diagram for the smoother in the form of Figure 4.3.7.

Dynamic Contribution

In Chapter 3 the smoother dynamics were shown to satisfy

$$\begin{bmatrix} \dot{\hat{x}} \\ \dot{\hat{\lambda}} \end{bmatrix} = \begin{bmatrix} A & BQB' \\ C'R^{-1}C & -A' \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{\lambda} \end{bmatrix} + \begin{bmatrix} 0 \\ -C'R^{-1} \end{bmatrix} y \quad . \qquad (4.3.31a)$$

The partitions in (4.3.31a) are associated with those in (4.3.6) as

$$A_x = A, \quad A_\lambda = -A', \quad A_{x\lambda} = BQB' \quad \text{and} \quad A_{\lambda x} = C'R^{-1}C \quad . \qquad (4.3.31b)$$

Denote the elements of the scattering matrix in (4.3.8) whose differential equations are given in (4.3.11) by

$$\Phi_f^0(t,0) = S_x(t,0) \qquad (4.3.32a)$$

$$P_f^0(t) = S_{x\lambda}(t,0) \qquad (4.3.32b)$$

$$W_f^0(t) = -S_{\lambda x}(t,\tau) \qquad (4.3.32c)$$

and

$$\psi_f^0(t,\tau) = S_\lambda(t,\tau) = \Phi_f^{0\prime}(t,\tau) \quad . \qquad (4.3.23d)$$

Substituting this notation into the differential equations in (4.3.11) gives

$$\dot{P}_f^0(t) = P_f^0 A' + AP_f^0 + BQB' - P_f^0 C'R^{-1}CP_f^0 \quad ; \quad P_f^0(0) = 0 \qquad (4.3.33a)$$

$$\dot{\Phi}_f^0(t,0) = (A - K_f^0 C)\Phi_f^0 \quad ; \quad \Phi_f^0(0,0) = I \qquad (4.3.33b)$$

and

$$\dot{W}_f^0(t) = \Phi_f^{0\prime} CR^{-1} C\Phi_f^0 \quad ; \quad W_f^0(0) = 0 \qquad (4.3.33c)$$

where

$$K_f^0(t) = P_f^0 C'R^{-1} \quad . \qquad (4.3.34)$$

The solutions to (4.3.33) correspond to the covariance matrix P_f^0 , the transition matrix Φ_f^0 and the observability matrix W_f^0 for the Kalman filter whose initial condition is known to be exactly zero at $t = 0$. The internal sources also have an interpretation in this setting.

Substituting from (4.3.31), (4.3.32) and (4.3.33) into (4.3.13a) and (4.3.13b) gives differential equations for the internal sources:

$$\dot{q}_f^+(t) = Aq_f^+ + K_f^0(y - Cq_f^+) \quad ; \quad q_f^+(0) = 0 \qquad (4.3.35a)$$

$$\dot{q}_f^-(t) = -\Phi_f^{0\prime} C'R^{-1}(y - Cq_f^+) \quad ; \quad q_f^-(0) = 0 \qquad (4.3.35b)$$

where

$$q_f^+(t) \equiv q^+(t,0) \quad \text{and} \quad q_f^-(t) \equiv q^-(t,0) \quad . \qquad (4.3.35c)$$

Thus $q_f^+(t)$ is the estimate for the Kalman filter with zero initial condition, and $q_f^-(t)$ is a running integral of its innovations process. We emphasize again that no matter what the boundary conditions are for the smoother dynamics (4.3.31), the forward scattering dynamics can be represented by this zero-initial-condition Kalman filter. The reverse scattering dynamics have a similar interpretation in terms of a backward Kalman filter in information form [41] as follows. Denote two of the elements of the scattering matrix $S(T,t)$ by

$$\theta_b^0(t) = -S_{\lambda x}(T,t) \qquad (4.3.36a)$$

$$\Phi_b^0(T,t) = S_{\lambda}(T,t) \qquad (4.3.36b)$$

and the lower partition of the internal source $q(T,t)$ by

$$q^-(t) \equiv q^-(T,t) \quad . \qquad (4.3.36c)$$

Substituting this notation into the final-value differential equations derived in Appendix 4A results in

$$-\dot{\theta}_b^0 = A'\theta_b^0 + \theta_b^0 A + C'R^{-1}C - \theta_b^0 BQB'\theta_b^0 \quad ; \quad \theta_b^0(T) = 0 \qquad (4.3.37a)$$

$$\dot{\Phi}_b^0 = -[A' - \theta_b^0 BQB']\Phi_b^0 \quad ; \quad \Phi_b^0(T,T) = I \qquad (4.3.37b)$$

and

$$\dot{q}_b^- = -[A' - \theta_b^0 BQB']q_b^- - C'R^{-1}y \quad ; \quad q_b^-(T) = 0 \quad . \qquad (4.3.37c)$$

Reviewing the two-filter formula in (4.3.21), we find that the solutions to these differential equations and those in (4.3.33a), (4.3.33b) and (4.3.35a) represent all of the dynamical components of the two-filter solution. Comparing these dynamical parts to the two-filter solution in Chapter 3 obtained by diagonalization of the Hamiltonian, it can be shown that the solution developed here corresponds to the case when the the diagonalizing transformation is chosen to be

$$T(t) = \begin{bmatrix} I & -P_f^0(t) \\ \theta_b^0(t) & I \end{bmatrix} , \tag{4.3.38}$$

which is similar to the diagonalizing transformation suggested in the proposal for this thesis [2]. In the next section, we use the smoother's boundary condition to complete the specification of the two-filter solution and also to construct a scattering picture for the smoother.

<u>The Boundary Value Contribution</u>

The boundary condition for the TPBVP smoother is given in (3.4.8a). Associating that boundary condition with the notation in (4.3.6b), we have

$$v_{x\lambda} = W' \Pi_b^{-1} y_b \tag{4.3.39a}$$

$$V_x^0 = \begin{bmatrix} \theta_f(0) \\ \hline \theta_c \end{bmatrix} , \qquad V_\lambda^0 = \begin{bmatrix} -I \\ \hline 0 \end{bmatrix} \tag{4.3.39b}$$

and

$$V_x^T = \begin{bmatrix} \theta_c' \\ \hline \theta_b(T) \end{bmatrix} , \qquad V_\lambda^T = \begin{bmatrix} 0 \\ \hline I \end{bmatrix} \tag{4.3.39c}$$

where, employing the notation introduced in Chapter 3,

$$\theta_f(0) = V^{0'} \Pi_v^{-1} V^0 + W^{0'} \Pi_b^{-1} W^0 \tag{4.3.40a}$$

$$\theta_b(T) = V^{T'} \Pi_v^{-1} V^T + W^{T'} \Pi_b^{-1} W^T \tag{4.3.40c}$$

and

$$\theta = V^{T'} \Pi_v^{-1} V^0 + W^{T'} \Pi_b^{-1} W^0 \quad . \tag{4.3.40c}$$

Recall that the form of the two-filter solution in (4.3.21) is dependent on the inverse of the matrix F_s in (4.3.20a):

$$F_s = \{[V_x^0 \vdots V_\lambda^T] + [V_x^T \vdots V_\lambda^0]S(T,0)\} \quad . \tag{4.3.41}$$

Employing the notation from (4.3.26) and (4.3.41), S(T,0) for the case of the smoother can be written as

$$S(T,0) = \left[\begin{array}{c:c} \Phi_f^0(T,0) & P_f^0(T) \\ \hdashline -\theta_b^0(0) & \Phi_b^0(T,0) \end{array}\right] \quad . \tag{4.3.42}$$

Substituting (4.3.42) and (4.3.39) into (4.3.39) results in the following expression for F_s

$$F_s = \left[\begin{array}{c:c} \theta_f(0) + \theta_c'\Phi_f^0(T,0) + \theta_b^0(0) & \theta_c'P_f^0(T) - \Phi_b^0(T,0) \\ \hdashline \theta_c + \theta_b(T)\Phi_f^0(T,0) & I + \theta_b(T)P_f^0(T) \end{array}\right] . \tag{4.3.43a}$$

Using the notation from (4.3.20b), define

$$\begin{bmatrix} F_{xv} \\ F_{\lambda v} \end{bmatrix} = F_s^{-1} \quad \text{and} \quad \begin{bmatrix} F_{xq}^+ & F_{xq}^- \\ F_{\lambda q}^+ & F_{\lambda q}^- \end{bmatrix} = -F_s^{-1} \begin{bmatrix} \theta_c' & -I \\ \theta_b(T) & 0 \end{bmatrix} . \tag{4.3.43b}$$

Combining (4.3.35), (4.3.37) and the expression for Γ_x in (4.3.14d), the two-filter solution in (4.3.21) for the special case of the TPBVP smoother is

$$\hat{x}(t) = \left[I + P_f^0(t)\theta_b^0(t)\right]^{-1}\Big\{ \left[\Phi_f^0(t,0)F_{xv} + P_f^0(t)\Phi_b^0(T,t)F_{\lambda v}\right]W'\Pi_b^{-1}y_b$$

$$+ \left[\Phi_f^0(t,0)F_{xq}^+ + P_f^0(t)\Phi_b^0(T,t)F_{\lambda q}^+\right]q_f^+(T)$$

$$+ \left[\Phi_f^0(t,0)F_{xq}^- + P_f^0(t)\Phi_b^0(T,t)F_{\lambda q}^-\right]q_b^-(0)$$

$$+ P_f^0(t)q_b^-(t)$$

$$+ q_f^+(t) \quad . \tag{4.3.44}$$

Thus, by working entirely within the scattering framework, we have been able to derive a two-filter implementation of the TPBVP smoother. Note that this form of the solution as represented by (4.3.44) shows that a change in the value of the covariance of either the boundary value v, Π_v, or the covariance of the error in the boundary observation r_b, Π_b, is easily incorporated into the solution simply by a change in the value of the matrix F_s in (4.3.42). This is analogous to the methods for incorporating a change in the value of the covariance of the initial condition for causal process smoothers derived in [1] and [38]. Again, the solution in (4.3.44) corresponds to the two-filter solution developed in Chapter 3 for the special case that the diagonalizing transformation is chosen to be T(t) in (4.3.38).

As discussed earlier, although it has been derived from a scattering perspective, the two-filter solution in (4.3.44) does not lend itself to a simple pictorial description via a scattering diagram. Recall that in order to construct a scattering picture for a TPBVP (in this case the TPBVP smoother) it is required that the partitioned matrix (see (4.3.16))

$$V_B = \left[V_x^0 \;\vdots\; V_\lambda^T\right] \tag{4.3.45}$$

(or a transformed version as in (4.3.26)) be invertible. Substituting from (4.3.39), the untransformed matrix V_B for the smoother is

$$V_B = \begin{bmatrix} \theta_f(0) & 0 \\ \theta_c & I \end{bmatrix} \tag{4.3.46}$$

which is not in general invertible since $\theta_f(0)$ is not necessarily full rank. Thus, the smoother falls into Case 2 as defined earlier in this section, and we must find a transformation which transforms V_B as described in (4.3.26) so that the transformed version is invertible. To obtain a transformed version which is invertible, consider the transformations

$$T_b(0) = \begin{bmatrix} \theta_f(0) & -I \\ \theta_b(0) & I \end{bmatrix} \tag{4.3.47a}$$

$$T_b(T) = I \tag{4.3.47b}$$

where it happens that $T_b(0)$ is the value of the Hamiltonian diagonalizing transformation in (3.4.4a) at t = 0. Note that these transformations have not been chosen by any constructive procedure. They are merely one of many possible choices which result in a transformed version of V_B which is invertible as described below.

Applying these transformations as indicated in (4.3.26a) and (4.3.26b) with the inverse of $T_b(0)$ in (4.3.47a) given by

$$T_b^{-1}(0) = \begin{bmatrix} I & I \\ -\theta_b(0) & \theta_f(0) \end{bmatrix} \begin{bmatrix} P_s(0) & 0 \\ 0 & P_s(0) \end{bmatrix} , \tag{4.3.48}$$

we find that the transformed version of (4.3.46) is

$$\tilde{V}_B = [V_x^0 T_b^{-1}(0) \;\vdots\; V_\lambda^T]$$

$$= [\tilde{V}_x^0 \;\vdots\; \tilde{V}_\lambda^T]$$

$$= \begin{bmatrix} I & 0 \\ \theta_c P_s(0) & I \end{bmatrix} \tag{4.3.49a}$$

which is invertible with inverse

$$\tilde{V}_B^{-1} = \begin{bmatrix} I & 0 \\ -\theta_c P_s(0) & I \end{bmatrix} . \qquad (4.3.49b)$$

Substituting into (4.3.27b) for the special case of the smoother and with the transformations given by (4.3.47a) and 4.3.47b), the matrices in (4.3.27c) can be shown to be

$$\begin{bmatrix} \tilde{M}_{xv} \\ \tilde{M}_{\lambda v} \end{bmatrix} = \begin{bmatrix} I & 0 \\ -\theta_c P_s(0) & I \end{bmatrix} \qquad (4.3.50a)$$

and

$$\begin{bmatrix} \tilde{M}_{x} & \tilde{M}_{x\lambda} \\ \tilde{M}_{\lambda x} & \tilde{M}_{\lambda} \end{bmatrix} = \left[\begin{array}{c:c} -\theta'_c & 0 \\ \hdashline \theta_c P_s(0)\theta'_c - \theta_b(T) & -\theta_c P_s(0) \end{array}\right] . \qquad (4.3.50b)$$

Finally, substituting the partitions of (4.3.50a) and (4.3.50b) into the scattering picture in (4.3.7) gives a scattering picture for the TPBVP smoother.

APPENDIX 4A

DIFFERENTIAL EQUATIONS FOR ELEMENTS OF THE SCATTERING MATRIX AND INPUT SOURCES

In this appendix we derive final value differential equations which can be solved backwards to obtain elements of the scattering matrix $S(T,t)$ and input source $q(T,t)$. Following the same procedures taken here, one can derive the initial value differential equations for $S(t,0)$ and $q(t,0)$ in (4.3.11) and (4.3.13). The notation used in this appendix has been defined previously in Section 4.3.3.

The differential equation for $S(T,t)$ will be obtained from the limit (overdot implies differentiation with respect to t)

$$\dot{S}(\tau,t) = \lim_{\Delta \to 0} \frac{S(\tau,t) - S(\tau,t - \Delta)}{\Delta} \tag{4.A.1}$$

We start by formulating an expression for $S(\tau,t-\Delta)$ as follows. Consider the scattering picture in Figure 4.A.1. The cascade of layers depicted there is

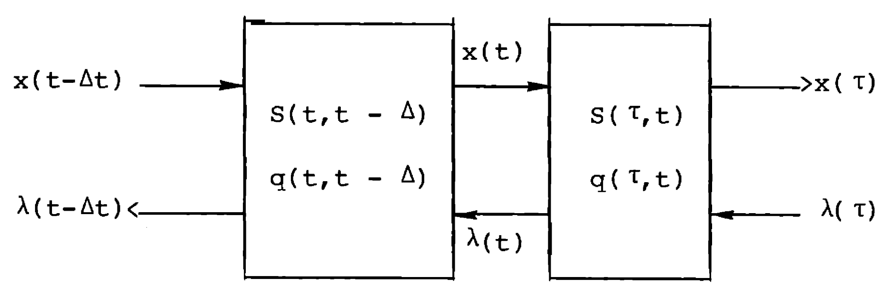


Figure 4.A.1 Scattering Picture for $S(\tau,t-\Delta)$

is computed by the star product operation as

$$S(\tau,t - \Delta) = S(\tau,t) * S(t,t - \Delta) \tag{4.A.2a}$$

with source given by the assembly sum

$$q(\tau,t - \Delta) = q(\tau,t) \circ q(t,t - \Delta) \quad . \tag{4.A.2b}$$

To obtain an expression for the infinitesimal layer $S(t,t-\Delta)$ consider the approximations of the the derivatives in (4.3.6a):

$$x(t) = x(t-\Delta) + \Delta\left[A_x(t)x(t) + A_{x\lambda}(t)\lambda(t) + B_x(t)u(t)\right] + O(\Delta^2) \qquad (4.A.3a)$$

and

$$\lambda(t-\Delta) = \lambda(t) - \Delta\left[A_{\lambda x}(t)x(t) + A_\lambda(t)\lambda(t) + B_\lambda(t)u(t)\right] + O(\Delta^2)\ . \qquad (4.A.3b)$$

If we replace $x(t)$ in the right hand side of each of these expressions by

$$x(t) = \Delta\dot{x}(t) + x(t-\Delta) + O(\Delta^2)\ , \qquad (4.A.3c)$$

then to order Δ we can write

$$\begin{bmatrix} x(t) \\ \lambda(t-\Delta) \end{bmatrix} = \begin{bmatrix} I + A_x\Delta & A_{x\lambda}\Delta \\ -A_{\lambda x}\Delta & I - A_\lambda\Delta \end{bmatrix} \begin{bmatrix} x(t-\Delta) \\ \lambda(t) \end{bmatrix} + \begin{bmatrix} B_x \\ -B_\lambda \end{bmatrix} u(t)\Delta + O(\Delta^2) \qquad (4.A.4a)$$

which is the scattering representation we seek:

$$\begin{bmatrix} x(t) \\ \lambda(t-\Delta) \end{bmatrix} = S(t,t-\Delta) \begin{bmatrix} x(t-\Delta) \\ \lambda(t) \end{bmatrix} + q(t,t-\Delta) + O(\Delta^2)\ . \qquad (4.A.4b)$$

If we denote the partitions of $S(\tau,t)$ by

$$S(\tau,t) = \begin{bmatrix} S_x & S_{x\lambda} \\ S_{\lambda x} & S_\lambda \end{bmatrix}\ , \qquad (4.A.5a)$$

then with the expression for $S(t,t-\Delta)$ in (4.A.4a) the star product in (4.A.2a) can be shown to be given by

$$S(\tau,t-\Delta) = \qquad (4.A.5b)$$

$$\begin{bmatrix} S_x\left[I - A_{x\lambda}S_{\lambda x}\Delta\right]^{-1}(I + A_x\Delta) & S_{\lambda x} + S_x A_{x\lambda}\left[I - S_{\lambda x}A_{x\lambda}\Delta\right]^{-1}S_\lambda\Delta \\ -A_{\lambda x}\Delta + (I - A_\lambda\Delta)S_{\lambda x}\left[I - A_{x\lambda}S_{\lambda x}\Delta\right]^{-1}(I + A_x\Delta) & (I - A_\lambda\Delta)\left[I - S_{\lambda x}A_{x\lambda}\Delta\right]^{-1}S_\lambda \end{bmatrix}$$

Substituting (4.A.5a) and (4.A.5b) into (4.A.1) and employing the approximation

$$[I - M\Delta]^{-1} \simeq I + M\Delta \tag{4.A.6}$$

for each of the inverses in (4.A.5a), it can be shown that the elements of $S(\tau,t)$ are governed by

$$\dot{S}_x(\tau,t) = -S_x(\tau,t)\left[A_x(t) + A_{x\lambda}(t)S_{\lambda x}(\tau,t)\right] \qquad ;S_x(\tau,\tau) = I \tag{4.A.7a}$$

$$\dot{S}_{x\lambda}(\tau,t) = -S_x(\tau,t)A_{x\lambda}(t)S_\lambda(\tau,t) \qquad ;S_{x\lambda}(\tau,\tau) = 0 \tag{4.A.7b}$$

$$\dot{S}_{\lambda x}(\tau,t) = A_\lambda(t)S_{\lambda x}(\tau,t) - S_{\lambda x}(\tau,t)A_x(t) - S_{\lambda x}(\tau,t)A_{x\lambda}(t)S_{\lambda x}(\tau,t) + A_{\lambda x}(t) \qquad ;S_{\lambda x}(\tau,\tau) = 0 \tag{4.A.7c}$$

and

$$\dot{S}_\lambda(\tau,t) = \left[A_\lambda(t) - S_{\lambda x}(\tau,t)A_{x\lambda}(t)\right] \qquad ;S_\lambda(\tau,\tau) = I\ . \tag{4.A.7d}$$

Similarly, denoting the partitions of $q(\tau,t)$ by

$$q(\tau,t) = \begin{bmatrix} q^+(\tau,t) \\ q^-(\tau,t) \end{bmatrix} \tag{4.A.8}$$

and computing the assembly sum in (4.A.2b) with $q(t,t-\Delta)$ as in (4.A.4a), it can be shown that

$$q(\tau,t-\Delta) \simeq \begin{bmatrix} [I - A_{x\lambda}S_{\lambda x}\Delta]^{-1}(I + A_x\Delta) & \vdots & 0 \\ - - - - - - - - - - - & \vdots & - - \\ -(I - A_\lambda\Delta)[I - S_{\lambda x}A_{x\lambda}\Delta]^{-1}S_{\lambda x} & \vdots & I \end{bmatrix} \begin{bmatrix} B_x \\ - - \\ -B_\lambda \end{bmatrix} u(t)\Delta$$

$$+ \begin{bmatrix} I & \vdots & [I - A_{x\lambda}S_{\lambda x}\Delta]^{-1}(I + A_x\Delta)A_{x\lambda}\Delta \\ - - & \vdots & - - - - - - - - - - - \\ 0 & \vdots & (I - A_\lambda\Delta)[I - S_{\lambda x}A_{x\lambda}\Delta]^{-1} \end{bmatrix} q(\tau,t)\ . \tag{4.A.9}$$

Differential equations for the partitions of the source are found by taking the limit

$$\dot{q}(\tau,t) = \lim_{\Delta \to 0} \frac{q(\tau,t) - q(\tau,t - \Delta)}{\Delta} \quad , \tag{4.A.10}$$

yielding

$$\dot{q}^{+}(\tau,t) = -A_{x\lambda}(t)q^{-}(\tau,t) - B_{x}(t)u(t) \qquad ;q^{+}(\tau,\tau) = 0 \tag{4.A.11a}$$

$$\dot{q}^{-}(\tau,t) = \left[A_{\lambda}(t) - S_{\lambda x}(\tau,t)A_{x\lambda}(t)\right]q^{-}(\tau,t) + \left[S_{\lambda x}(\tau,t)B_{x}(t) + B_{\lambda}(t)\right]u(t)$$

$$;q^{-}(\tau,t) = 0. \tag{4.A.11b}$$

For the case of a TPBVP defined on the interval [0,T], these differential equations could be initialized at $\tau = T$ and solved backwards to $t = 0$. This would provide the variables required in the two-filter form of the solution in (4.3.21).

CHAPTER 5: DISCRETE PARAMETER BOUNDARY VALUE STOCHASTIC PROCESSES

SECTION 5.1
INTRODUCTION

The class of discrete parameter linear two-point boundary value processes that we study in this chapter was previously introduced as an example in Chapter 2 to illustrate linear estimation by the method of complementary models. In that earlier chapter the dynamics, boundary condition and measurements were defined, and the difference equation and boundary condition for the smoother were established. In this chapter we review the discrete TPBVP model description, study the properties of its solution, formulate recursions for computing its covariance matrix, and develop a two-filter implementation for its smoother. For ease of reference, some of the material previously introduced in Chapter 2 to describe the 1-D discrete TPBVP and its smoother is repeated in this chapter.

The material in this chapter for the discrete TPBVP parallels that of Chapter 3 for the continuous parameter TPBVP. We begin by presenting the dynamics and boundary condition for the n^{th} order discrete stochastic TPBVP and then derive a forward/backward form of the general solution. Employing this form of the general solution, we develop matrix difference equations which are used to compute the process covariance. We saw earlier in Chapter 2 that the Green's identity is a critical element in the specification of the differential realization of our estimator. In this chapter we derive the Green's identity for this model which we had stated and used in Chapter 2 in defining the smoother.

Given Green's identity, we showed in Chapter 2 that the smoother for an n^{th} order descrete TPBVP is a $2n^{th}$ order boundary value problem. In the continuous parameter case we found in Chapter 3 that a numerically stable implementation of the smoother could be obtained by diagonalizing the smoother dynamics into an n^{th} order forward process and an n^{th} order backward process. In this chapter we investigate this type of dynamical decoupling for the discrete process smoother from two points of view: one is based on the so-called descriptor representation [35] for the smoother dynamics and the other has its foundations in the scattering framework introduced in Chapter 4.

We will find that the derivation of a diagonal or decoupled representation for this discrete process smoother is somewhat more complex than the corresponding derivation for the continuous case presented in Appendix 3B. The two cases differ in that the smoother dynamics for the continuous case are in the standard state-space form, and consequently, the entire class of equivalent dynamical representations (including the decoupled form) can be obtained through an equivalence transformation of the underlying process. On the other hand, the dynamics for the discrete process smoother are not in state-space form but are in descriptor form as mentioned above. As we will see, the specification of the complete class of equivalent dynamical representations for the descriptor form requires an equivalence transformation of the underlying process along with one other transformation which is applied directly to the difference equation. Nevertheless, we will find that an equivalence transformation similar in form to the one used to obtain a diagonal representation for the continuous case is applicable in the dynamical decoupling for the discrete case as well.

SECTION 5.2

1-D DISCRETE LINEAR STOCHASTIC TPBVP

5.2.1 The Model

In this section we repeat the description of the model for the 1-D discrete linear stochastic TPBVP which was introduced earlier in Chapter 2 as an example. Let u_k be an $m\times 1$ white sequence on [0,K-1] with covariance matrix Q_k. Let A_k and B_k be sequences of $n\times n$ and $n\times m$ matrices respectively on [0,K-1]. Let V be an $n\times 2n$ matrix written in $n\times n$ partitions as $[V^0 : V^K]$, and let v be an $n\times 1$ random vector uncorrelated with u and with covariance matrix Π_v. Then the discrete TPBVP satisfies the difference equation:

$$x_{k+1} = A_k x_k + B_k u_k \tag{5.2.1a}$$

with two-point boundary condition

$$v = V^0 x_0 + V^K x_K \quad . \tag{5.2.1b}$$

The differential realization of the smoother developed in Chapter 2 requires an operator representation of the process dynamics and boundary condition. Here we repeat the operator description given earlier in Chapter 2. Let the set of points between 0 and K-1 be represented by $\Omega_1 = [0,K-1]$, and let the the boundary of this region be defined as $\partial\Omega_1 = \{0,K\}$. With D representing the unit delay, the dynamics of the $n\times 1$ vector process are given by the first order difference operator:

$$L: l_2^n(\Omega_1 \cup \partial\Omega_1) \to l_2^n(\Omega_1)$$

defined notationally as

$$L = (D^{-1}I - A)$$

and operationally as

$$(Lx)_k = x_{k+1} - A_k x_k \quad . \tag{5.2.2a}$$

Viewing the matrix V as the operator

$$V: R^{2n} \to R^{n} \quad , \tag{5.2.2b}$$

the dynamics and boundary condition in (5.2.1a) and (5.2.1b) can be expressed as

$$Lx = Bu \tag{5.2.3a}$$

$$Vx_b = v \tag{5.2.3b}$$

where

$$x_b = \begin{bmatrix} x_0 \\ x_K \end{bmatrix} \quad . \tag{5.2.3c}$$

5.2.2 The General Solution

The general solution for (5.2.1a) with boundary condition (5.2.1b) is derived in much the same manner as the general solution for the continuous parameter TPBVP in Chapter 3. That is, we start by defining x^0 as the solution to (5.2.1a) with a known zero initial condition:

$$x_k^0 = \sum_{j=0}^{K-1} \Phi(k,j+1)B_j u_j \tag{5.2.5}$$

where Φ is the transition matrix which can be computed by the recursion

$$\Phi(k+1,j) = A_k \Phi(k,j) \ ; \qquad \Phi(j,j) = I \quad . \tag{5.2.6}$$

Next, note that the solution for x given the initial value x_0 can be written in terms of x^0 in (5.2.5) as

$$x_k = \Phi(k,0)x_0 + x_k^0 \quad . \tag{5.2.7}$$

If we substitute for x_K from (5.2.7) evaluated at k = K into the boundary condition in (5.2.1b), then an expression for x_0 can be found in terms of the boundary value v and x_k^0 at the endpoint k = K as

$$x_0 = F^{-1}\left[v - V^K x_K^0\right] \tag{5.2.8a}$$

where the $n \times n$ matrix F is given by[1]

$$F = V^0 + V^K \Phi(K,0) \quad . \tag{5.2.8b}$$

[1] As in the continuous case, invertibility of the matrix F is the well-posedness condition for the boundary value problem in (2.1a) and (2.1b).

Combining (5.2.7) and (5.2.8a) gives an expression for the general solution as

$$x_k = \Phi(k,0)F^{-1}\left[v - V^K x_K^0\right] + x_k^0 \quad . \tag{5.2.9}$$

Just as in the continuous case, there will be systems for which the dynamics in (5.2.1a) are not stable, and consequently, the implementation of the general solution in (5.2.9) will be susceptible to numerical error buildup for large intervals [0,K]. These numerical errors can be avoided if we can find an equivalence transformation of the type

$$\begin{bmatrix} x_{f,k} \\ x_{b,k} \end{bmatrix} \equiv T_k x_k \tag{5.2.10a}$$

which decouples the dynamics in (5.2.1a) as

$$x_{f,k+1} = A_{f,k} x_{f,k} + B_{f,k} u_k \tag{5.2.10b}$$

and

$$x_{b,k} = A_{b,k} x_{b,k+1} + B_{b,k} u_k \tag{5.2.10c}$$

in such a way that A_f is forward stable and A_b is backward stable. The construction of such a decoupling transformation for the smoother dynamics is the subject of Sections 5.3.2 and 5.3.3 of this chapter. Under a transformation as in (5.2.10a), the boundary condition in (5.2.4b) takes the form

$$v = \left[V_f^0 \vdots\ V_b^0\right] \begin{bmatrix} x_{f,0} \\ x_{b,0} \end{bmatrix} + \left[V_f^K \vdots\ V_b^K\right] \begin{bmatrix} x_{f,K} \\ x_{b,K} \end{bmatrix} \tag{5.2.10d}$$

where

$$\left[V_f^0 \vdots\ V_b^0\right] \equiv V^0 T_0^{-1} \qquad \text{and} \qquad \left[V_f^K \vdots\ V_b^K\right] \equiv V^K T_K^{-1} \quad . \tag{5.2.10e}$$

Employing this forward/backward representation of the TPBVP, we can derive an alternative to the general solution in (5.2.9). In particular, define $x_{f,k}^0$ as the solution to (5.2.10b) with zero intitial condition and $x_{b,k}^0$ as the solution to (5.2.10c) with a zero final condition. With Φ_f the transition matrix associated with A_f and with Φ_b the transition matrix associated with A_b, define

$$\Phi_{fb}(k) = \begin{bmatrix} \Phi_f(k,0) & \vdots & 0 \\ - - - & \vdots & - - - \\ 0 & \vdots & \Phi_b(k,K) \end{bmatrix} \tag{5.2.11a}$$

and

$$F_{fb} = \left[V_f^0 + V_f^K \Phi_f(K,0) \ \vdots \ V_b^0 \Phi_b(0,K) + V_b^K\right] \quad . \tag{5.2.11b}$$

Then, in a derivation similar to that used to obtain (5.2.9), it can be shown that the solution to (5.2.10) is given by

$$\begin{bmatrix} x_{f,k} \\ x_{b,k} \end{bmatrix} = \Phi_{fb}(k) F_{fb}^{-1} \left\{ v - V_f^K x_{f,K}^0 - V_b^0 x_{b,0}^0 \right\} + \begin{bmatrix} x_{f,k}^0 \\ x_{b,k}^0 \end{bmatrix} \quad . \tag{5.2.11c}$$

Applying the inverse of the transformation in (5.2.10a), the original process x_k is recoverd by

$$x_k = T_k^{-1} \begin{bmatrix} x_{f,k} \\ x_{b,k} \end{bmatrix} \quad . \tag{5.2.11d}$$

In this way, we have have constructed a stable, forward/backward two-filter recursive implementation of the general solution for the model in (5.2.1a) and (5.2.1b).

The Green's Function Form

Traditionally, solutions for two-point boundary value problems are given in the Green's function form. The Green's function form of the general solution for our discrete TPBVP can be found from (5.2.9) by combining the summation expressions for x_k^0 and x_K^0. In particular, with the discrete Green's function given by

$$G(k,j) = \begin{cases} \left[I - \Phi(k,0)F^{-1}V^K\Phi(K,k)\right]\Phi(k,j+1)B_j \ ; & 0 \leq j \leq k-1 \\ -\Phi(k,0)F^{-1}V^K\Phi(k,j+1)B_j \ ; & k \leq j \leq K-1 \end{cases} \quad , \tag{5.2.12a}$$

the general solution for (5.2.4a) and (5.2.4b) can be shown to be

$$x_k = \Phi(k,0)F^{-1}v + \sum_{j=0}^{K-1} G(k,j)u_j \quad . \tag{5.2.12b}$$

If the A_k are invertible at each point in [0,K-1], then the discrete Green's function G in (5.2.12a) can be put into a form which parallels that of the continuous case as given in [17]. Specifically, under this invertibility condition the term in brackets in (5.2.12a) can be written as

$$\begin{aligned}\left[I - \Phi(k,0)F^{-1}V^K\Phi(K,k)\right] &= \Phi(k,0)F^{-1}\left[F\Phi^{-1}(k,0) - V^K\Phi(K,k)\right] \\ &= \Phi(k,0)F^{-1}V^0\Phi^{-1}(k,0) \quad .\end{aligned} \tag{5.2.13a}$$

Substituting this expression back into (5.2.12a), the top term becomes

$$G(k,j) = \Phi(k,0)F^{-1}V^0\Phi(0,j+1)B_j \quad ; \quad 0 \leq j \leq k-1 \quad . \tag{5.2.13b}$$

This term plus the term for $k \leq j \leq K-1$ in (5.2.12a) gives a discrete Green's function which is in a form similar to that of the Green's function for the continuous case as given in [17].

5.2.3 Process Variance

It will be shown later in Section 3 of this chapter that the smoothing error dynamics can be transformed to the forward/backward form of equation (5.2.10). In this subsection we present equations which allow us to compute the covariance matrix for processes represented in that dynamically decoupled form. Our starting point is the expression for the general solution in (5.2.11c). Recall that the boundary value v is orthogonal to u_k throughout [0,K-1]. Thus, v is also orthogonal to each of the x_f^0 and x_b^0 terms in that expression for the general solution. With this in mind, it is straightforward to see that the covariance of x_k can be written as a linear combination of Π_v, the covariance of v, and the following three covariances:

(1) $$P_f^0(n,k) \equiv E\left[x_{f,n}^0 x_{f,k}^{0\,\prime}\right] \quad , \tag{5.2.14a}$$

(2) $$P_b^0(n,k) \equiv E\left[x_{b,n}^0 x_{b,k}^{0\,\prime}\right] \tag{5.2.14b}$$

and

(3) $$P^0_{fb}(n,k) \equiv E\left[x^0_{f,n} x^{0\,\prime}_{b,k}\right] \quad . \qquad (5.2.14c)$$

Difference equations are established for each of these three covariances in Appendix 6A for a general multi-point boundary condition (of which this two-point condition is a special case). These difference equations are derived by substituting the summation representations for x^0_f and x^0_b (e.g. see (5.2.5)) into each of the expectations in (5.2.14). From that appendix, we have that:

(1) $$P^0_f(n,k) = \Phi_f(n,k)P^0_f(k,k) \qquad (5.2.15a)$$

where

$$P^0_f(k+1,k+1) = \Phi_f(k+1,k)P^0_f(k,k)\Phi'_f(k+1,k) + B_{f,k}Q_k B'_{f,k} \;;\; P^0_f(0,0) = 0 \qquad (5.2.15b)$$

(2) $$P^0_b(n,k) = P^0_b(k,k)\Phi'_b(k,n) \qquad (5.2.16a)$$

where

$$P^0_b(k-1,k-1) = \Phi_b(k-1,k)P^0_b(k,k)\Phi'_b(k-1,k) + B_{b,k}Q_k B'_{b,k} \;;\; P^0_b(K,K) = 0 \qquad (5.2.16b)$$

and (3) for $n > k$

$$P^0_{fb}(n,k) = \Pi^0_{fb,n}\Phi'_b(k,n) - \Phi_f(n,k)\Pi^0_{fb,k} \qquad (5.2.17a)$$

where

$$\Pi^0_{fb,k+1} = \Phi_f(k+1,k)\Pi^0_{fb,k}\Phi'_b(k+1,k) + B_{f,k}Q_k B'_{b,k} \;;\; \Pi^0_{fb,0} = 0 \quad . \qquad (5.2.17b)$$

For $n \leq k$,

$$P^0_{fb}(n,k) = 0 \quad . \qquad (5.2.17c)$$

Given these three covariances, the covariance of x_k:

$$P_k = E[x_k x_k']$$

$$= T_k^{-1}\left\{E\begin{bmatrix} x_{f,k} \\ x_{b,k} \end{bmatrix}\left[x_{f,k}' : x_{b,k}'\right]\right\}T_k^{-1'} \tag{5.2.18a}$$

can be written as

$$P_k = T_k^{-1}\left\{\Phi_{fb}(k)F_{fb}^{-1}\Pi_v F_{fb}^{-1}\Phi_{fb}'(k) + \Xi(k) + \Xi(k)' + \Theta(k) + \begin{bmatrix} P_f^0(k,k) & \vdots & 0 \\ \hline 0 & \vdots & P_b^0(k,k) \end{bmatrix}\right\}T_k^{-1'} \tag{5.2.18b}$$

where

$$\Xi(k) = -\Phi_{fb}(k)F_{fb}^{-1}\left\{V_f^K\left[P_f^0(K,k) + P_{fb}^0(K,k)\right] + V_b^0\left[P_{fb}^{0'}(k,0) + P_b^0(0,k)\right]\right\} \tag{5.2.18c}$$

and

$$\Theta(k) = \Phi_{fb}(k)F_{fb}^{-1}\left[V_f^K : V_b^0\right]\begin{bmatrix} P_f^0(K,K) & \vdots & P_{fb}^0(K,0) \\ \hline P_{fb}^{0'}(K,0) & \vdots & P_b^0(0,0) \end{bmatrix}\begin{bmatrix} V_f^{K'} \\ \hline V_b^{0'} \end{bmatrix} \quad . \tag{5.2.18d}$$

Thus, the covariance of the process x_k can be computed for any point k given the solution of the three matrix difference equations (5.2.15), (5.2.16) and (5.2.17).

5.2.4 Green's Identity

When the 1-D discrete TPBVP was introduced in Chapter 2, it was stated that the Green's identity for discrete processes could be obtained from summation by parts of the inner product[2]

$$\langle \gamma, Lx \rangle_{l_2^n[0,K-1]} = \sum_{k=0}^{K-1} \gamma_k'(Lx)_k$$

$$= \sum_{k=0}^{K-1} \gamma_k'(x_{k+1} - A_k x_k) \quad . \tag{5.2.19}$$

[2] The reason for using γ instead of λ, as we have used in earlier discussions of Green's identity, will be made clear shortly.

Summation by parts can be interpreted as the counterpart of integration by parts as follows:

Integration by Parts	Summation by Parts
$\int_0^T u\,dv = uv\Big\vert_0^T - \int_0^T v\,du$	$\sum_{k=0}^{K-1} u_{k+1}(v_{k+1} - v_k) = (u_K v_K - u_0 v_0) - \sum_{k=0}^{K-1} v_k(u_{k+1} - u_k)$ (5.2.2)

The term on the right hand side of the identity for summation by parts in (5.2.20) has been obtained simply by shifting the index of summation on the left hand side and adding $(u_K v_K - u_0 v_0)$ to account for the shift. To put (5.2.19) into the form of (5.2.20), we perform the same type of index shifting to write

$$\sum_{k=0}^{K-1} (x_{k+1} - A_k x_k)' \gamma_k = \sum_{k=0}^{K-1} x_k'(\gamma_{k-1} - A_k \gamma_k) - x_0'\gamma_{-1} + x_K'\gamma_{K-1} \quad . \qquad (5.2.21)$$

Defining

$$L^\dagger = DI - A' \,, \qquad (5.2.22a)$$

$$x_b = \begin{bmatrix} x_0 \\ x_K \end{bmatrix}, \quad \gamma_b = \begin{bmatrix} \gamma_{-1} \\ \gamma_{K-1} \end{bmatrix} \quad \text{and} \quad E = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} \quad . \qquad (5.2.22c,d,e)$$

Green's identity can be written directly form (5.2.21) as

$$\langle Lx, \gamma \rangle_{l_2^n[0,K-1]} = \langle L^\dagger \gamma, x \rangle_{l_2^n[0,K-1]} + \langle x_b, E\gamma_b \rangle_{R^{2n}} \qquad (5.2.23)$$

We have used γ here in deriving Green's identity rather than λ because, as we will see later (and as we saw earlier in Chapter 2), the smoother can be expressed in a simpler form if written in terms of a shifted version of γ, which we denote by λ:

$$\lambda_{k+1} \equiv \gamma_k \quad (\text{i.e. } \lambda = D\gamma) \quad . \qquad (5.2.24a)$$

In terms of the shifted process λ, γ_b is given by

$$\lambda_b \equiv \gamma_b = \begin{bmatrix} \lambda_0 \\ \lambda_K \end{bmatrix} \quad . \tag{5.2.24b}$$

Given Green's identity as expressed in (5.2.22a) through (5.2.23), in the next section we establish an internal difference realization of the smoother for the 1-D discrete TPBVP.

SECTION 5.3

THE SMOOTHER AND TWO-FILTER IMPLEMENATION

5.3.1 The Smoother and Smoothing Error

The observations for the discrete 1-D fixed-interval smoothing problem have been described earlier when we introduced the 1-D discrete problem as an example in Chapter 2. For convenience, we repeat that description here. Let C_k be a $p\times n$ matrix on $[0,K-1]$, and let W be a full rank $q\times 2n$ matrix with $q\leq n$, with the rows of W linearly independent of those of V in (2.3) and with $q\times n$ partitions:

$$W = [\, W^0 \; \vdots \; W^K \,] \quad . \tag{5.3.1}$$

Let r_k be a $p\times 1$ white noise process over $[0,K-1]$ whose covariance matrix R_k is nonsingular on $[0,K-1]$. Let r_b be a $q\times 1$ random vector with nonsingular covariance matrix Π_b. In addition, u, v, r and r_b are assumed to be mutually orthogonal. The observations are defined by a process y_k

$$y_k = C_k x_k + r_k \qquad \text{on } [0,K-1] \tag{5.3.2a}$$

and a $q\times 1$ boundary observation

$$y_b = Wx_b + r_b \quad . \tag{5.3.2b}$$

The minimum variance estimator of x given the observations y and y_b can be written by substituting the notation defined in this chapter into the operator form for the estimator in (2.5.25). In this case, the adjoints of B, C, W and V are all simply given by matrix transpositions. The formal adjoint difference operator $L^\dagger$, the matrix E, and x_b and γ_b (temporarily γ will be used in place of λ) have all been determined in the derivation of Green's identity and are given in (5.2.22a) through (5.2.22d). The resulting smoother dynamics are given by

$$\begin{bmatrix} \hat{x}_{k+1} \\ \hat{\gamma}_{k-1} \end{bmatrix} = \begin{bmatrix} A_k & \vdots & B_k Q_k B_k' \\ - - - & \vdots & - - - \\ -C_k' R_k^{-1} C_k & \vdots & A_k' \end{bmatrix} \begin{bmatrix} \hat{x}_k \\ \hat{\gamma}_k \end{bmatrix} + \begin{bmatrix} 0 \\ - - \\ C_k' R_k^{-1} \end{bmatrix} y_k \tag{5.3.3a}$$

with boundary condition

$$W'\Pi_b^{-1}y_b = [W'\Pi_b^{-1}W + V'\Pi_v^{-1}V]\begin{bmatrix}\hat{x}_0 \\ \hat{x}_K\end{bmatrix} + E\begin{bmatrix}\hat{\gamma}_{-1} \\ \hat{\gamma}_{K-1}\end{bmatrix} \tag{5.3.3b}$$

As mentioned earlier, it will be convenient to write the smoother in terms of the shifted process λ defined in (5.2.24). In this way the apparent four-point boundary condition in (5.3.3b) becomes a two-point boundary condition. Furthermore, when we specialize to the case of causal processes as in the example in Section 2.6.2, the smoother takes the traditional form of the discrete fixed-interval smoother (see e.g. [33]). Thus, in terms of λ and λ_b (5.3.3a) and (5.3.3b) become

$$\begin{bmatrix}\hat{x}_{k+1} \\ \hat{\lambda}_k\end{bmatrix} = \left[\begin{array}{c:c} A_k & B_kQ_kB_k' \\ \hdashline -C_k'R_k^{-1}C_k & A_k' \end{array}\right]\begin{bmatrix}\hat{x}_k \\ \hat{\lambda}_{k+1}\end{bmatrix} + \begin{bmatrix}0 \\ -\,- \\ C_k'R_k^{-1}\end{bmatrix}y_k \tag{5.3.4a}$$

and

$$\begin{bmatrix}W^{0'} \\ W^{K'}\end{bmatrix}\Pi_b^{-1}y_b = \left[\begin{array}{c:c} V^{0'}\Pi_v^{-1}V^0 + W^{0'}\Pi_b^{-1}W^0 & -I \\ \hdashline V^{K'}\Pi_v^{-1}V^0 + W^{K'}\Pi_b^{-1}W^0 & 0 \end{array}\right]\begin{bmatrix}\hat{x}_0 \\ \hat{\lambda}_0\end{bmatrix} + \left[\begin{array}{c:c} V^{0'}\Pi_v^{-1}V^K + W^{0'}\Pi_b^{-1}W^K & 0 \\ \hdashline V^{K'}\Pi_v^{-1}V^K + W^{K'}\Pi_b^{-1}W^K & I \end{array}\right]\begin{bmatrix}\hat{x}_K \\ \hat{\lambda}_K\end{bmatrix} \tag{5.3.4b}$$

In the next two subsections we investigate methods for decoupling (5.3.4a) into a forward/backward two-filter form similar to that obtained in Chapter 3 for the continuous case.

The smoothing error dynamics and boundary condition are defined by the operator equations (2.5.36) and (2.5.33). When written in terms of the shifted variable λ, these equations become

$$\begin{bmatrix}\tilde{x}_{k+1} \\ -\tilde{\lambda}_k\end{bmatrix} = \left[\begin{array}{c:c} A_k & B_kQ_kB_k' \\ \hdashline -C_k'R_k^{-1}C_k & A_k' \end{array}\right]\begin{bmatrix}\tilde{x}_k \\ -\tilde{\lambda}_{k+1}\end{bmatrix} + \begin{bmatrix}B_ku_k \\ -\,-\,- \\ C_k'R_k^{-1}r_k\end{bmatrix} \tag{5.3.5a}$$

with boundary condition

$$v_e = \left[\begin{array}{c:c} V^{0'}\Pi_v^{-1}V^0 + W^{0'}\Pi_b^{-1}W^0 & -I \\ \hdashline V^{K'}\Pi_v^{-1}V^0 + W^{K'}\Pi_b^{-1}W^0 & 0 \end{array}\right]\begin{bmatrix} \tilde{x}_0 \\ -\tilde{\lambda}_0 \end{bmatrix} + \left[\begin{array}{c:c} V^{0'}\Pi_v^{-1}V^K + W^{0'}\Pi_b^{-1}W^K & 0 \\ \hdashline V^{K'}\Pi_v^{-1}V^K + W^{K'}\Pi_b^{-1}W^K & I \end{array}\right]\begin{bmatrix} \tilde{x}_K \\ -\tilde{\lambda}_K \end{bmatrix} \quad (5.3.5b)$$

where

$$v_e = \left[\begin{array}{c:c} V^{0'} & -W^{0'} \\ \hdashline V^{K'} & -W^{K'} \end{array}\right]\left[\begin{array}{c:c} \Pi_v^{-1} & 0 \\ \hdashline 0 & \Pi_b^{-1} \end{array}\right]\begin{bmatrix} v \\ r_b \end{bmatrix} \quad (5.3.5c)$$

The same techniques that we develop for decoupling the estimator dynamics in (5.3.4a) can also be applied to decouple the error dynamics in (5.3.5a). Given this decoupled form, the error covariance can be computed via the matrix recursions formulated in the previous subsection.

5.3.2 Decoupling via The Descriptor Representation

Descriptor Form Representations

The first of the two approaches which we investigate for decoupling the smoother dynamics is based on the smoother dynamics written in the so-called descriptor [35] or generalized state-space [44] form. In this subsection we introduce this dynamical representation and discuss the class of equivalent representations for dynamics which are written in this form. Eventually, we will show that one such equivalent representation of the smoother dynamics is the decoupled form we seek.

Consider an $n \times 1$ vector process x_k whose dynamics are governed by [1]

$$H_{1,k}x_{k+1} = H_{2,k}x_k + G_k u_k \quad (5.3.6)$$

where u_k is an $m \times 1$ input, G_k is an $n \times m$ matrix and $H_{1,k}$ and $H_{2,k}$ are $n \times n$ matrices neither of which is necessarily invertible. Therefore, (5.3.6) cannot in general be reduced to the standard state-space form by premultiplying by the inverse of either H_1 or H_2.

---

[1] In this subsection x and u will be used to represent a generic internal state process and input process, respectively.

Next we consider the class of dynamical representations which are equivalent to (5.3.6). The form of equivalent representation we present below can be inferred from Luenberger's study of descriptor forms in [35]. In particular, consider the following equivalence transformation defining the transformed process q:

$$q_k = T_k x_k \tag{5.3.7a}$$

where T_k is invertible for all k of interest. Under this transformation, the dynamics in (5.3.6) become

$$H_{1,k} T_{k+1}^{-1} q_{k+1} = H_{2,k} T_k^{-1} q_k + G_k u_k \quad . \tag{5.3.7b}$$

However, unlike the case for standard state-space representations, this transformation does not lead to the most general equivalent representation for (5.3.6). To see this, consider premultiplying (5.3.7b) by an $n \times n$ matrix F_k which is invertible for all k. This results in an equivalent descriptor form representation of the system in (5.3.6) of the following general form:

$$\tilde{H}_{1,k} q_{k+1} = \tilde{H}_{2,k} q_k + \tilde{G}_k u_k \tag{5.3.8a}$$

where

$$\tilde{H}_{1,k} = F_k H_{1,k} T_{k+1}^{-1} \quad , \tag{5.3.8b}$$

$$\tilde{H}_{2,k} = F_k H_{2,k} T_k^{-1} \tag{5.3.8c}$$

and

$$\tilde{G}_k = F_k G_k \quad . \tag{5.3.8d}$$

In the next section we determine the matrix sequences F_k and T_k which transform the TPBVP smoother dynamics into the desired decoupled form.

The Smoother in Descriptor Form

The smoother dynamics can be written in descriptor form by a simple rearrangement of the terms in (5.3.4a):

$$\begin{bmatrix} I & \vdots & -B_k Q_k B_k' \\ - & - & - \\ 0 & \vdots & A_k' \end{bmatrix} \begin{bmatrix} \hat{x}_{k+1} \\ \hat{\lambda}_{k+1} \end{bmatrix} = \begin{bmatrix} A_k & \vdots & 0 \\ - & - & - \\ C_k' R_k^{-1} C_k & \vdots & I \end{bmatrix} \begin{bmatrix} \hat{x}_k \\ \hat{\lambda}_k \end{bmatrix} + \begin{bmatrix} 0 \\ - - - \\ -C_k' R_k^{-1} \end{bmatrix} y_k \tag{5.3.9a}$$

or identifying this representation with the notation of the previous section:

$$H_{1,k} \begin{bmatrix} \hat{x}_{k+1} \\ \hat{\lambda}_{k+1} \end{bmatrix} = H_{2,k} \begin{bmatrix} \hat{x}_k \\ \hat{\lambda}_k \end{bmatrix} + G_k y_k \quad . \tag{5.3.9}$$

In order to decouple (5.3.9a) into forward and backward recursions, we must find sequences F_k and T_k which transform (5.3.9a) into the form

$$\begin{bmatrix} I & 0 \\ 0 & A_{b,k} \end{bmatrix} \begin{bmatrix} q_{f,k+1} \\ q_{b,k+1} \end{bmatrix} = \begin{bmatrix} A_{f,k} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} q_{f,k} \\ q_{b,k} \end{bmatrix} + \begin{bmatrix} B_{f,k} \\ B_{b,k} \end{bmatrix} y_k \tag{5.3.10a}$$

so that q_f satisfies a forward recursion

$$q_{f,k+1} = A_{f,k} q_{f,k} + B_{f,k} y_k \tag{5.3.10b}$$

and q_b satisfies a backward recursion

$$q_{b,k} = A_{b,k} q_{b,k+1} - B_{b,k} y_k \quad . \tag{5.3.10c}$$

More specifically, we seek F_k and T_k where

$$\begin{bmatrix} q_{f,k} \\ q_{b,k} \end{bmatrix} = T_k \begin{bmatrix} \hat{x}_k \\ \hat{\lambda}_k \end{bmatrix} \tag{5.3.11a}$$

$$\begin{bmatrix} B_{f,k} \\ B_{b,k} \end{bmatrix} = F_k \begin{bmatrix} 0 \\ -C_k R_k^{-1} \end{bmatrix} \tag{5.3.11b}$$

and such that the dynamics are decoupled:

$$F_k \left[\begin{array}{c:c} I & -B_kQ_kB_k' \\ \hdashline 0 & -A_k' \end{array}\right] T_{k+1}^{-1} = \begin{bmatrix} I & 0 \\ 0 & A_{b,k} \end{bmatrix} \tag{5.3.11c}$$

$$F_k \left[\begin{array}{c:c} A_k & 0 \\ \hdashline C_k'R_k^{-1}C_k & I \end{array}\right] T_k^{-1} = \begin{bmatrix} A_{f,k} & 0 \\ 0 & I \end{bmatrix} \quad . \tag{5.3.11d}$$

The relations which define F_k and T_k are derived from the decoupling constraints in (5.3.11c) and (5.3.11d) as follows. Denote the partitions of F and T by

$$F_k = \begin{bmatrix} F_{11}^k & F_{12}^k \\ F_{21}^k & F_{22}^k \end{bmatrix} \quad \text{and} \quad T_k = \begin{bmatrix} T_{11}^k & T_{12}^k \\ T_{21}^k & T_{22}^k \end{bmatrix} \quad . \tag{5.3.12}$$

Postmultiplying (5.3.11c) by T_{k+1} and (5.3.11d) by T_k and then working through the algebra, it can be shown that the partitions of F_k, T_k and T_{k+1} satisfy the following eight relations:

$$T_{12}^{k+1} = -T_{11}^{k+1} B_kQ_kB_k' + A_{f,k}T_{12}^kA_k' \tag{5.3.13a}$$

$$A_{b,k}T_{22}^{k+1} = -A_{b,k}T_{21}^{k+1} B_kQ_kB_k' + T_{22}^kA_k' \tag{5.3.13b}$$

$$T_{21}^k = A_{b,k}T_{21}^{k+1} A_k + T_{22}^kC_k'R_k^{-1}C_k \tag{5.3.13c}$$

$$A_{f,k}T_{11}^k = T_{11}^{k+1} A_k + A_{f,k}T_{12}^kC_k'R_k^{-1}C_k \tag{5.3.13d}$$

$$F_{11}^k = T_{11}^{k+1} \tag{5.3.13e}$$

$$F_{12}^k = A_{f,k}T_{12}^k \tag{5.3.13f}$$

$$F_{21}^k = A_{b,k}T_{21}^{k+1} \tag{5.3.13g}$$

and

$$F_{22}^{k} = T_{22}^{k} \quad . \qquad (5.3.13h)$$

Of course, these equations define the entire class of decoupling transformations, and are functions of some desired forward and backward dynamics A_f and A_b. As we had done in the continuous case in Chapter 3, we can find stable A_f and A_b by recalling a known stable forward/backward two-filter form of the smoother for the causal case. In particular, if we choose two of the partitions of T to be constant:

$$T_{12}^{k} = T_{12}^{k+1} = -I \qquad (5.3.14a)$$

$$T_{22}^{k} = T_{22}^{k+1} = I \qquad (5.3.14b)$$

and denote the varying partitions by

$$\theta_{f,k} \equiv T_{11}^{k} \qquad (5.3.14c)$$

and

$$\theta_{b,k} \equiv T_{21}^{k} \quad , \qquad (5.3.14d)$$

then it can be shown by substituting these values into the eight equations in (5.3.13) that

$$\theta_{f,k+1} = \left[A_k(\theta_{f,k} + C_k'R_k^{-1}C_k)^{-1}A_k' + B_kQ_kB_k'\right]^{-1} \qquad (5.3.15a)$$

$$A_{f,k} = \theta_{f,k+1}A_k(\theta_{f,k} + C_k'R_k^{-1}C_k)^{-1} \qquad (5.3.15b)$$

$$\theta_{b,k} = A_k'(I + \theta_{b,k+1}B_kQ_kB_k')^{-1}\theta_{b,k+1}A_k + C_k'R_k^{-1}C_k \qquad (5.3.16a)$$

$$A_{b,k} = A_k'(I + \theta_{b,k+1}B_kQ_kB_k')^{-1} \qquad (5.3.16b)$$

$$F_{11}^{k} = \theta_{f,k+1} \tag{5.3.17a}$$

$$F_{12}^{k} = -A_{f,k} \tag{5.3.17b}$$

$$F_{21}^{k} = A_{b,k}\theta_{b,k+1} \tag{5.3.17c}$$

and

$$F_{22}^{k} = I \quad . \tag{5.3.17d}$$

Substituting for F_k from (5.3.17) into (5.3.11b), we have for this special case that

$$\begin{bmatrix} B_{f,k} \\ -- \\ B_{b,k} \end{bmatrix} = \begin{bmatrix} \theta_{f,k+1}A_k(\theta_{f,k} + C_k'R_k^{-1}C_k)^{-1}C_k'R_k^{-1} \\ ------------ \\ -C_k'R_k^{-1} \end{bmatrix} \quad . \tag{5.3.18}$$

In summary, with T_k given by (5.3.14) and F_k as in (5.3.17), the transformed smoother process is given by

$$\begin{bmatrix} q_{f,k} \\ q_{b,k} \end{bmatrix} = \begin{bmatrix} \theta_{f,k} & -I \\ \theta_{b,k} & I \end{bmatrix} \begin{bmatrix} \hat{x}_k \\ \hat{\lambda}_k \end{bmatrix} \tag{5.3.19a}$$

with dynamics

$$q_{f,k+1} = A_{f,k}q_{f,k} + B_{f,k}y_k \tag{5.3.19b}$$

$$q_{b,k} = A_{b,k}q_{b,k+1} - B_{b,k}y_k \tag{5.3.19c}$$

where A_f, A_b, θ_f, θ_b, B_f and B_b are defined in (5.3.15), (5.3.16) and (5.3.18).

The boundary conditions for the forward and backward processes are obtained by applying the transformation (5.3.19a) to the smoother boundary condition (5.3.4b). If we rewrite (5.3.4b) exactly as we had done for the boundary condition in the continuous case (see (3.4.8a)), i.e.

$$W'\Pi_b^{-1}y_b = V_{x\lambda}^{0}\begin{bmatrix} \hat{x}_0 \\ \hat{\lambda}_0 \end{bmatrix} + V_{x\lambda}^{K}\begin{bmatrix} \hat{x}_K \\ \hat{\lambda}_K \end{bmatrix} \quad . \tag{5.3.20}$$

Then with

$$V_q^0 = V_{x\lambda}^0 T_0^{-1} \tag{5.3.21a}$$

and

$$V_q^K = V_{x\lambda}^K T_K^{-1} \quad , \tag{5.3.21b}$$

the coupled boundary condition for q_f and q_b is

$$W' \Pi_b^{-1} y_b = V_q^0 \begin{bmatrix} q_{f,0} \\ q_{b,0} \end{bmatrix} + V_q^K \begin{bmatrix} q_{f,K} \\ q_{b,K} \end{bmatrix} \quad . \tag{5.3.21c}$$

Again, following the development of the boundary condition for the continuous case, we choose the boundary conditions for the recursions for θ_f and θ_b in (5.3.15a) and (5.3.16a) as

$$\theta_{f,0} = V^{0'} \Pi_v^{-1} V^0 + W^{0'} \Pi_b^{-1} W^0 \tag{5.3.22a}$$

$$\theta_{b,K} = V^{K'} \Pi_v^{-1} V^K + W^{K'} \Pi_b^{-1} W^K \quad . \tag{5.3.22b}$$

Also define

$$\theta_c = V^{K'} \Pi_v^{-1} V^0 + W^{K'} \Pi_b^{-1} W^0 \tag{5.3.22c}$$

and

$$P_{s,k} = \left(\theta_{f,k} + \theta_{b,k}\right)^{-1} \quad . \tag{5.3.22d}$$

Then the coefficient matrices in (5.3.21a) and (5.3.21b) can be written as

$$V_q^0 = [\ V_f^0 : V_b^0\ ] \tag{5.3.23a}$$

where it can be shown that

$$V_f^0 = \begin{bmatrix} I \\ \hline \theta_c P_{s,0} \end{bmatrix} \quad , \qquad V_b^0 = \begin{bmatrix} 0 \\ \hline \theta_c P_{s,0} \end{bmatrix} \tag{5.3.23b}$$

and

$$V_q^K = [\ V_f^K : V_b^K\ ] \tag{5.3.24a}$$

where one can show that

$$V_f^K = \begin{bmatrix} \underline{\theta' P_c S,K} \\ 0 \end{bmatrix} , \qquad V_b^K = \begin{bmatrix} \underline{\theta' P_c S,K} \\ I \end{bmatrix} . \tag{5.3.24b}$$

In summary, the discrete smoother is implemented in a two-filter form by solving for the dynamically decoupled forward and backward processes q_f and q_b as described by the general solution in (5.2.11c). Note that the boundary condition for q_f and q_b required by that solution is given by (5.3.21c). In order to compute that general solution and in order to invert the transformation in (5.3.19a) to obtain the smoothed estimate from the computed values for q_f and q_b, we must also solve the Riccati equations in (5.3.15a) and (5.3.16a) for θ_f and θ_b and compute the transition matrices for A_f and A_b in (5.3.15b) and (5.3.16b). Each of these latter computations can be performed off-line.

The Smoothing Error

The smoothing error dynamics in (5.3.5a) can be put into a descriptor form which is similar to that for the smoother dynamics (5.3.9a). With the error dynamics in descriptor form, the same transformations (T_k and F_k) described in (5.3.14) through (5.3.18) which were used to decouple the smoother dynamics (see (5.3.19) can also be applied to decouple the error dynamics. Of course, these same transformations would also be applied to the smoothing error boundary condition in (5.3.5b). Given this decoupled form for the error dynamics, the smoothing error covariance can be computed via the matrix difference equations in (5.2.14) through (5.2.18). In particular, with θ_f and θ_b satisfying the dynamics in (5.3.15a) and (5.3.16a) and boundary conditions in (5.3.22a) and (5.3.22b), define the transformed error process by

$$\begin{bmatrix} e_{f,k} \\ e_{b,k} \end{bmatrix} = \begin{bmatrix} \theta_{f,k} & -I \\ \theta_{b,k} & I \end{bmatrix} \begin{bmatrix} \tilde{x}_k \\ -\tilde{\lambda}_k \end{bmatrix} . \tag{5.3.25}$$

The dynamics of e_f and e_b are decoupled in precisely the same way as the transformed smoother in (5.3.19b) and (5.3.19c). Performing the algebra, it is readily shown that transformed error processes e_f and e_b satisfy

$$e_{f,k+1} = A_{f,k}e_{f,k} + \theta_{f,k+1}B_k u_k + A_k' C_k' R_k^{-1} r_k \tag{5.3.26a}$$

and

$$e_{b,k} = A_{b,k}e_{b,k+1} + A_{b,k}\theta_{b,k+1}B_k u_k - C_k' R_k^{-1} r_k \quad . \tag{5.3.26b}$$

The two-point boundary condition in (5.2.29) is transformed to

$$v_e = [V_f^0 \vdots V_b^0]\begin{bmatrix} e_{f,0} \\ e_{b,0} \end{bmatrix} + [V_f^K \vdots V_b^K]\begin{bmatrix} e_{f,K} \\ e_{b,K} \end{bmatrix} \tag{5.3.27}$$

where the coefficient matrices are the same as those found in the smoother boundary condition in (5.3.23) and (5.3.24).

Thus, by working with the descriptor form representation for the smoother and smoothing error equations we have been able to parallel all of the decoupling operations that were carried out for the continuous case in Chapter 3. Although decoupling the discrete smoother dynamics in descriptor form requires not only an equivalence transformation (which was sufficient for decoupling in the continuous case) but also the transformation F_k in (5.3.17) which operates directly on the dynamics as described by (5.3.8b) and (5.3.8c), the resulting two-filter smoother solution and filter error covariance computations are nearly identical in form to their continuous counterparts. In the next section we consider an alternative representation for the smoother, namely the scattering form, and show that the decoupling can also be achieved from that point of view.

5.3.3 Decoupling via Scattering

Both the smoother and the smoothing error dynamics as written in (5.3.4a) and (5.3.5a) are in the scattering form described by (4.3.1). In the case of the smoother (smoothing error) the forward[2] moving process is $\hat{x}$ ($\tilde{x}$) and the backward moving process is $\hat{\lambda}$ ($-\tilde{\lambda}$). Thus, an alternative to investigating the dynamical decoupling in the descriptor form as we have

[2] In keeping with the forward/backward terminology, we will use forward and backward instead of leftward and rightward.

done in the previous section is to investigate the decoupling within the scattering framework. To this end, we begin this section by discussing equivalence transformations and show that they can be represented as identity layers in the scattering framework. Given this representation, we illustrate how these transformations can be employed to achieve the desired dynamical decoupling of the forward and backward processes. We note that in contrast to our investigation of the scattering representation for the continuous TPBVP smoother, we do not seek to derive a scattering picture for the discrete process smoother. Rather, our intent here is to study equivalence transformations from a scattering point of view. Nevertheless, one could follow the same developements as those used to obtain a scattering diagram in the continuous case in Section 4.3 to formulate a scattering diagram for the discrete smoother as well.

Equivalence Transformations as Scattering Layers

Consider a process with dynamics written in the scattering form as

$$\begin{bmatrix} x_{k+1} \\ \lambda_k \end{bmatrix} = H(k) \begin{bmatrix} x_k \\ \lambda_{k+1} \end{bmatrix} + \rho(k) \qquad (5.3.28)$$

where $H(k)$ is the scattering matrix and $\rho(k)$ is the input source. As discussed previously in Chapter 4, the scattering picture for (5.3.28) is depicted as in Figure 5.3.1.

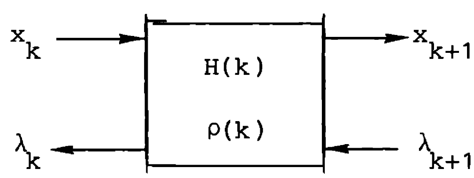


Figure 5.3.1. Scattering Picture for (5.3.28)

Next, consider an equivalence transformation of the form

$$\begin{bmatrix} q_{1,k} \\ q_{2,k} \end{bmatrix} = T_k \begin{bmatrix} x_k \\ \lambda_k \end{bmatrix} \qquad (5.3.29)$$

(i.e. T_k is invertible for all k of interest). A scattering representation for the transformation in (5.3.29) is developed as follows. Since the input to (5.3.29) is $\{x_k, \lambda_k\}$ and the output is $\{q_{1,k}, q_{2,k}\}$, the transformation must be represented as a cascade of scattering layers as pictured in Figure 5.3.2. The issue now is to express S(k) in Figure 5.3.2 in terms of T(k). As a first

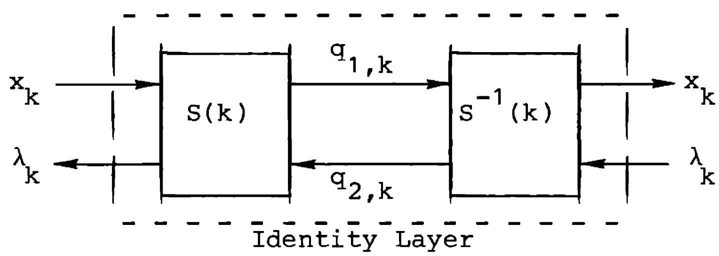


Figure 5.3.2. Scattering Picture for the Equivalence Transformation (5.3.29)

step, note that the scattering matrix $S^{-1}(k)$ in the figure is simply the matrix inverse of S(k), i.e.

$$\begin{bmatrix} q_{1,k} \\ \lambda_k \end{bmatrix} = S(k) \begin{bmatrix} x_k \\ q_{2,k} \end{bmatrix} \tag{5.3.30a}$$

and inverting

$$\begin{bmatrix} x_k \\ q_{2,k} \end{bmatrix} = S^{-1}(k) \begin{bmatrix} q_{1,k} \\ \lambda_k \end{bmatrix} \quad . \tag{5.3.30b}$$

In each of (5.3.30a) and (5.3.30b) the sources are zero. Thus the cascaded action of S(k) and $S^{-1}(k)$ is an identity layer as indicated in the figure. If we denote partitions of T_k by

$$T_k = \begin{bmatrix} T_{11}^k & T_{12}^k \\ T_{21}^k & T_{22}^k \end{bmatrix} \tag{5.3.31a}$$

and following the notation introduced in Chapter 4 denote the partitions of S(k) by

$$S(k) = \begin{bmatrix} S_A(k) & S_B(k) \\ S_C(k) & S_D(k) \end{bmatrix} \quad , \tag{5.3.31b}$$

then from (5.3.29), (5.3.30a) and (5.3.30b), it can be shown after some algebra (also see Redheffer [43]) that T, S and S^{-1} are related by

$$T_k = \left[\begin{array}{c:c} S_A(k) - S_B(k)S_D^{-1}(k)S_C(k) & S_B(k)S_D^{-1}(k) \\ \hdashline -S_D^{-1}(k)S_C(k) & S_D^{-1}(k) \end{array}\right] , \qquad (5.3.32a)$$

$$S(k) = \left[\begin{array}{c:c} T_{11}^k - T_{12}^k {T_{22}^k}^{-1} T_{21}^k & T_{12}^k {T_{22}^k}^{-1} \\ \hdashline -{T_{22}^k}^{-1} T_{21}^k & {T_{22}^k}^{-1} \end{array}\right] \qquad (5.3.32b)$$

and

$$S^{-1}(k) = \left[\begin{array}{c:c} {T_{11}^k}^{-1} & -{T_{11}^k}^{-1} T_{12}^k \\ \hdashline T_{21}^k {T_{11}^k}^{-1} & -\left(T_{21}^k {T_{11}^k}^{-1} T_{12}^k + T_{22}^k\right) \end{array}\right] . \qquad (5.3.32c)$$

To determine the effect of an equivalence transformation on the scattering dynamics, consider the cascade of layers depicted in Figure 5.3.3a. By inserting an identity layer of the type depicted in Figure 5.3.2 between each of the layers in Figure 5.3.3a, we get the "equivalent" layers shown in Figure 5.3.3b. Using the star product notation for combining layers which was previously

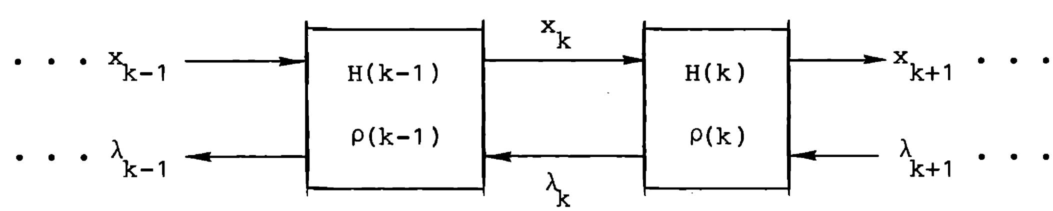


Figure 5.3.3a. Untransformed Cascade of H(k) and H(k-1)

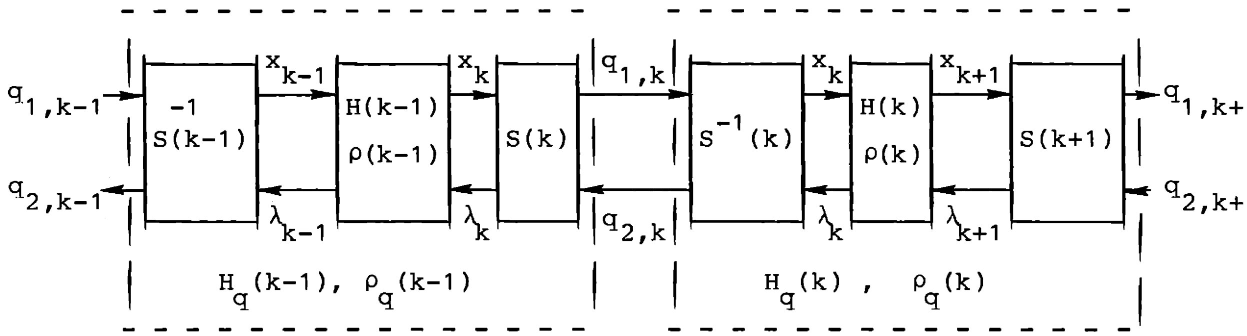


Figure 5.3.3b. Equivalent Layers $H_q(k)$ and $H_q(k-1)$

introduced in Chapter 4, the equivalent layer $H_q(k)$ can be expressed as either

$$H_q(k) = \left[S(k+1) * H(k)\right] * S^{-1}(k) \tag{5.3.33a}$$

or

$$H_q(k) = S(k+1) * \left[H(k) * S^{-1}(k)\right] \quad . \tag{5.3.33b}$$

Of course, both (5.3.33a) and (5.3.33b) are equally valid representations for $H_q(k)$. However, performing the star product operation in different sequences leads to vastly different expressions for $H_q(k)$ in terms of the partitions of $S(k)$, $S^{-1}(k)$ and $H(k)$. Later in our investigation of decoupling of the smoother dynamics we will find both forms useful. Similarly, the source $\rho_q(k)$ can be expressed by the assembly sum (see (4.3.5)) of the zero sources in the transformation layers $S(k)$ and $S^{-1}(k)$ and the source $\rho(k)$:

$$\rho_q(k) = \left[\emptyset\left(S(k+1)\right) \circ \rho(k)\right] \circ \emptyset\left(S^{-1}(k)\right) \tag{5.3.34a}$$

or

$$\rho_q(k) = \emptyset\left(S(k+1)\right) \circ \left[\rho(k) \circ \emptyset\left(S^{-1}(k)\right)\right] \quad . \tag{5.3.34b}$$

Here the zero sources (denoted by slashed zeros) are represented with explicit arguments $S(k+1)$ and $S^{-1}(k)$ in order to avoid confusion when performing the assembly sum operations.

Diagonalization of H(k)

In this section we determine the form of the transformation layers S(k) which decouple the smoother dynamics into forward and backward processes. If we define

$$H(k) = \left[\begin{array}{c:c} A_k & B_k Q_k B'_k \\ \hdashline -C'_k R_k^{-1} C_k & A'_k \end{array}\right] \tag{5.3.35a}$$

and

$$\rho(k) = \left[\begin{array}{c} 0 \\ \hdashline C'_k R_k^{-1} \end{array}\right] y_k \quad , \tag{5.3.35b}$$

then the smoother dynamics in (5.3.4a) are written in scattering form as

$$\begin{bmatrix} \hat{x}_{k+1} \\ \hat{\lambda}_k \end{bmatrix} = H(k) \begin{bmatrix} \hat{x}_k \\ \hat{\lambda}_{k+1} \end{bmatrix} + \rho(k) \quad . \tag{5.3.36}$$

These dynamics can be diagonlized by an application of an equivalence transformation of the type discussed in the previous subsection as follows. Let S(k) be a transformation layer which defines a new process via

$$\begin{bmatrix} q_{f,k} \\ \hat{\lambda}_k \end{bmatrix} = S(k) \begin{bmatrix} \hat{x}_k \\ q_{b,k} \end{bmatrix} \quad . \tag{5.3.37}$$

From (5.3.33a) (or alternatively from (5.3.33b)) the scattering matrix for this new process is given by

$$H_q(k) = \big(S(k+1) * H(k) \big) * S^{-1}(k) \quad . \tag{5.3.38a}$$

The two processes q_f and q_b will be decoupled forward and backward processes repectively if we can find a sequence of S(k) such that H_q is of the form

$$H_q(k) = \begin{bmatrix} H_f(k) & 0 \\ 0 & H_b(k) \end{bmatrix} \quad . \tag{5.3.38b}$$

We can construct such a sequence as follows. Consider S(k) partitioned as in (5.3.31b):

$$S(k) = \begin{bmatrix} S_A(k) & S_B(k) \\ S_C(k) & S_D(k) \end{bmatrix} \quad . \qquad (5.3.39a)$$

If we assume that $S_A(k)$ is invertible, then the inverse of S(k) can be written in partitioned form as

$$S^{-1}(k) = \left[\begin{array}{c:c} S_A^{-1}(k)\left(I + S_B(k)\Lambda(k)S_C(k)S_A^{-1}(k)\right) & S_A^{-1}(k)S_B(k)\Lambda(k) \\ \hdashline -\Lambda(k)S_C(k)S_A^{-1}(k) & \Lambda(k) \end{array}\right] \qquad (5.3.39b)$$

where

$$\Lambda(k) = \left(S_D(k) - S_C(k)S_A^{-1}(k)S_B(k) \right)^{-1} \quad . \qquad (5.3.39c)$$

The relations which define the dynamics of the sequence S(k) are found by carrying out the indicated star product operations in (5.3.38a) and setting each of the resulting partitions of H_q equal to the partitions of the desired diagonal form in (5.3.38b). Unfortunately, due to the nature of the expressions resulting from the star product operations, this results in four unwieldy nonlinear algebraic equations. For instance, equating the lower left partition to zero gives the relation

$$0 = \Lambda(k)S_C(k)S_A^{-1}(k) + \Lambda(k)\Psi\left(I + S_A^{-1}(k)S_B(k)\Psi\right)^{-1}S_A^{-1}(k)\left(I + S_B(k)\Lambda(k)S_C(k)S_A^{-1}(k)\right) \qquad (5.3.40a)$$

where

$$\Psi = \left[-C'_k R_k^{-1} C_k + A_k' S_C(k+1)\left(I - B_k Q_k B_k' S_C(k+1)\right)^{-1} A_k\right] \quad . \qquad (5.3.40b)$$

Efforts to simplify the expression in (5.3.40) along with the three other relations obtained from the remaining three partitions of H_q have met with little success. However, if we investigate a special case as we had done in the continuous case and again in the case of the descriptor form in Section

5.3.2, then progress can be made. Specifically, we employ the same expression for T_k used in Section 5.3.2, e.g.

$$T_k = \begin{bmatrix} \theta_{f,k} & -I \\ \theta_{b,k} & I \end{bmatrix} . \qquad (5.3.41a)$$

In this case, the relations between the transformation matrix T_k and the transformation layer S(k) and its inverse $S^{-1}(k)$ in (5.3.32a) and (5.3.32b) respectively give expressions for S(k) and its inverse as

$$S(k) = \left[\begin{array}{c:c} \theta_{f,k} + \theta_{b,k} & -I \\ \hdashline -\theta_{b,k} & I \end{array}\right] \qquad (5.3.41b)$$

and

$$S^{-1}(k) = \left[\begin{array}{c:c} I & I \\ \hdashline \theta_{b,k} & \theta_{f,k} + \theta_{b,k} \end{array}\right] \begin{bmatrix} \theta_{f,k}^{-1} & 0 \\ 0 & \theta_{f,k}^{-1} \end{bmatrix} . \qquad (5.3.41c)$$

Recursions for θ_f and θ_b are found by employing the expressions in (5.3.41a) and (5.3.41b) and (1) computing the upper two partitions of H_q by the star product sequence in (5.3.33b) and (2) computing the lower two partitions by (5.3.33a). Following (1) and (2), it is straightforward to show that these recursions are identical to those found earlier when working with the descriptor form in (5.3.15a) and (5.3.16a). In addition, as one would expect, H_f and H_b take the same form as A_f and A_b in (5.3.15b) and (5.3.16b).

An expression for the source ρ_q is computed by a similar procedure. That is, compute the upper partition of the source by (5.3.34b) and the lower partition by (5.3.34a). In this way, one obtains expressions for the partitions of the source which are identical to $B_{f,k}y_k$ and $-B_{b,k}y_k$ with B_f and B_b given by the partitions of (5.3.18).

SECTION 5-4

CONCLUSIONS

As is true of smoothers for causal processes, we have seen that the smoother for the discrete noncausal TPBVP is in many ways similar to the smoother for the continuous case. However, decoupling of the smoother dynamics to obtain a two-filter implementation for the discrete case has been shown to be quite a different problem than its continuous counterpart. The reason for this difference is that the smoother dynamics in the discrete case can be written in either descriptor form or scattering form but not the standard state-space form. We have shown that the standard methods for transforming state-space dynamics by means of equivalence transformations are not sufficient for attaining the most general form of equivalent descriptor form dynamics. However, by investigating equivalent systems in the descriptor and scattering forms, we have been able to determine decoupling transformations.

In studying equivalent systems in the descriptor and scattering forms, we found that working with the descriptor form led to much more manageable algebra than working in the scattering framework. This is primarily due to the complexity of the star product operation used in combining scattering layers. In particular, working with the descriptor form, we have been able to completely specify difference equations defining the entire class of transformations for decoupling the smoother dynamics. On the other hand, we found when considering these transformations as scattering layers, that the resulting nonlinear algebraic relations were unmanageable. Nevertheless, in each of the two cases when we reverted to a special structure for the transformation (as we had also done in the continuous case), we were able to completely specify difference equations defining the decoupling transformation sequence.

CHAPTER 6: 2-D DISCRETE PARAMETER BOUNDARY VALUE PROCESSES

SECTION 6.1
INTRODUCTION

The nearest neighbor model (NNM) for discrete 2-D random processes describes a class of first order linear 2-D difference equations and is discussed in detail in Section 6.2. This model has proved useful in modeling a variety of 2-D stochastic processes. For example, Jain and Angel [46], found it applicable for modelling 2-D images whose brightness levels have a nonseparable covariance. More generally, by considering vector processes, it will be shown in Section 6.2 that virtually all linear 2-D difference equations, including finite difference approximations of hyperbolic, parabolic and elliptic linear partial differential equations, can be put into the NNM form. Although the dynamics represented by the NNM are inherently noncausal, we will see that this model can be used to model "space-time" 2-D discrete dynamics which are causal in one index and noncausal in the other.

In Section 6.2 we present various forms of the general solution for the NNM. Each of these forms of the general solution suggests a different method for computing the solution of the NNM. The discussion begins with what we will refer to as the matrix inversion method. Although this form of the solution does not lead to the most efficient implementation, it is conceptually the simplest and as such provides a straightforward way in which to state a well-posedness condition for any given boundary condition for the NNM. In establishing this well-posedness condition, we will find that the Dirichlet condition (knowledge of the process on the boundary of the region of interest) plays a fundamental role which is analagous to the role played by the initial value for 1-D processes in determining the well-posedness condition for a two-point boundary condition (see Sections 3.2 and 5.2). In particular, recall that in the 1-D case the well-posedness condition stated that knowledge of both the specified two-point boundary condition and the input over the interval of interest should uniquely determine the initial value of the process. In this chapter, we consider boundary conditions that are not necessarily of the Dirichlet type (e.g. they may be initial values or values of the first spatial differences along parts of the boundary). What we will find, however, is that well-posedness for such a boundary condition for the NNM implies that given the boundary condition and the value of the input over the region of interest, one can compute the Dirichlet condition.

Having employed the matrix inversion method to establish a well-posedness condition, we study other methods which lead to computationally more efficient solutions. Each of these solutions is based on solving the 2-D model in a 1-D fashion. In particular, under fairly general conditions, the 2-D discrete NNM dynamics can be written as a 1-D recursion of large vector dimension. Given the dynamics in this 1-D form, we then must transform the boundary condition for the 2-D NNM into a compatible 1-D description. It is shown that a 2-D boundary condition for the NNM becomes a multi-point boundary condition for the 1-D process, i.e. it is a condition on the process at many points within its interval of definition. A stable forward/backward implementation of this 1-D boundary value problem is discussed. Under slightly more restrictive conditions we find that the 1-D process can be decoupled into a system of low (vector) dimensional processes by an FFT-based transformation. This decoupling is achieved by an extension of a technique that was introduced by Hockney for discrete elliptic 2-D processes [47] and later was applied by Angel and Jain to a 2-D estimation problem [46].

Applying the estimator solution developed in Chapter 2, we obtain the optimal linear smoother for processes described by NNMs. In particular, we will show that the smoother for an $n \times 1$ vector process obeying a NNM (i.e. an nth order discrete process) takes the form of a $2n$th order NNM. We also discuss the conditions for which the smoother dynamics can be transformed to 1-D recursive form. In Section 6.3 we present two examples of the discrete 2-D smoother. The first is an example of a process whose dynamics are given by the finite difference approximation of Poisson's equation with a Dirichlet boundary condition. We show how the discrete 2-D dynamics of the smoother for this process can be, as discussed above, put into 1-D recursive form and subsequently decoupled into a system of lower dimensional 1-D problems. The second process that we study is governed by 2-D dynamics that are causal in one index and noncausal in the other. After writing these dynamics in the nearest neighbor model form, we immediately obtain the smoothing equations in NNM form. For this second example, we will find that no 1-D recursive form can be written for the 2-D NNM smoother dynamics because certain invertibility conditions are not met. However, with some manipulation, we are able to rewrite the 2-D NNM smoothing dynamics as a high (vector) dimension 1-D scattering form. It is shown that this 1-D scattering form is amenable to the same FFT-based decoupling transformation as applied to the 1-D recursive form of the first example. An application of this transformation results in a system of low order 1-D scattering dynamics of the type studied in Chapter 5. By applying the diagonalizing methods developed in Chapter 5, these 1-D scattering dynamics can be transformed to decoupled stable forward and backward recursive forms. Thus, although we are not able to transform the original NNM smoother dynamics for this mixed causal/noncausal process to a 1-D recursive form, we are able to achieve a 1-D scattering form which we can ultimately manipulate into a low order stable forward/backward recursive form.

SECTION 6.2

THE NEAREST NEIGHBOR MODEL

6.2.1 The Nearest Neighbor Model

Let u_{ij} be an m×1 vector 2-D white noise process with nonsingular covariance

$$E\{u_{ij}u'_{kl}\} = Q\,\delta_{ik}\,\delta_{jl} \quad . \tag{6.2.1a}$$

A discrete n×1 vector 2-D stochastic process, x, will be said to satisfy a nearest neighbor model if its dynamics are of the form[1]

$$x_{ij} = A_1 x_{i-1,j} + A_2 x_{i,j-1} + A_3 x_{i+1,j} + A_4 x_{i,j+1} + B\,u_{ij} \quad ; \tag{6.2.1b}$$

$$(i,j) \in S \equiv [1,I-1]\times[1,J-1] \tag{6.2.1c}$$

where I and J are integers which define the extent of the lattice S. Figure 6.2.1 depicts the depedence of x_{ij} on its neighbors as described in (6.2.1b)

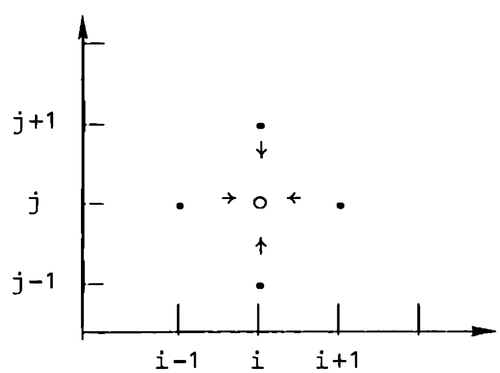


Figure 6.2.1 Nearest Neighbor Dependence

In discussing the various types of boundary conditions we will consider for (6.2.1b), it will be useful to define the boundary region ∂S as the four edges of the lattice S as shown in Figure 6.2.2. In particular, the set of

[1] Here we assume that the A_k are constant. Allowing variations in i and j of these coefficient matrices would require more than the already overwhelming notation used in this chapter.

points comprising ∂S is given by the union of the points in these four edges:

$$\partial S = \{(i,j) : i = 0,\ j = 1,2,\dots,J-1\} \cup \{(i,j) : i = I,\ j = 1,2,\dots,J-1\}$$
$$\cup \{(i,j) : j = 0,\ i = 1,2,\dots,I-1\} \cup \{(i,j) : j = J,\ i = 1,2,\dots,I-1\} \qquad (6.2.2)$$

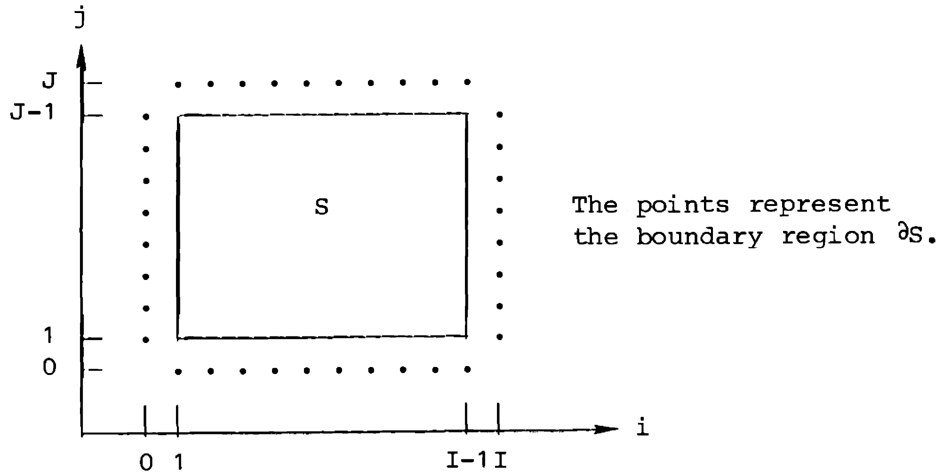


Figure 6.2.2 The Boundary Region ∂S

We will denote the values of the process x on ∂S by

$$x_L = \begin{bmatrix} x_{0,1} \\ x_{0,2} \\ \vdots \\ x_{0,J-1} \end{bmatrix}, \quad x_R = \begin{bmatrix} x_{I,1} \\ x_{I,2} \\ \vdots \\ x_{I,J-1} \end{bmatrix}, \quad x_B = \begin{bmatrix} x_{1,0} \\ x_{2,0} \\ \vdots \\ x_{I-1,0} \end{bmatrix}, \quad x_T = \begin{bmatrix} x_{1,J} \\ x_{2,J} \\ \vdots \\ x_{I-1,J} \end{bmatrix} . \qquad (6.2.3)$$

(L - left, R - right, B - bottom, T - top)

Before considering other forms of boundary condition for the NNM, it will be useful to discuss the Dirichlet condition for this model. As we will see later in this section in discussing the general solution for this model, the Dirichlet condition plays a key role in establishing a well-posedness condition for other forms of boundary condition. The classical definition of the Dirichlet condition for (6.2.1b) is given as the values of the process on the boundary region ∂S as defined in (6.2.3). However, as shown in Appendix 6A, if the rank n_k of any of the four $n \times n$ coefficient matrices A_k in (6.2.1b) is less than n, then specifying the entirety of a corresponding one of the boundary vectors in (6.2.3) leads to an unnecessary overspecification of the boundary conditions. In this case this boundary vector can be replaced by a reduced-order boundary vector which is sufficient to specify the boundary condtions for (6.2.1b). These "replacement" processes are shown in Appendix 6A to be linear combinations of the corresponding components of x on ∂S in (6.2.3) and are listed below along with their dimensions. Note that the dimensions are expressed in terms of the ranks of the A_k:

$$x_L \leftrightarrow d_L \qquad n_1(J-1)\times 1 \tag{6.2.4a}$$

$$x_B \leftrightarrow d_B \qquad n_2(I-1)\times 1 \tag{6.2.4b}$$

$$x_R \leftrightarrow d_R \qquad n_3(J-1)\times 1 \tag{6.2.4c}$$

and

$$x_T \leftrightarrow d_T \qquad n_4(I-1)\times 1 \quad . \tag{6.2.4d}$$

Note that when any of the n_k is equal to n (i.e. A_k is nonsingular), then the replacement process and the corresponding process in (6.2.3) which it replaces are identical.

The general form of boundary condition for the NNM that we will consider here is expressed as a linear combination of the process x on S and the replacement processes as follows

$$v = V_x x + V_d d \tag{6.2.5a}$$

where

$$d = \begin{bmatrix} d_L \\ d_R \\ d_B \\ d_T \end{bmatrix} \tag{6.2.5b}$$

and where we have represented the value of the process on S simply as x. Note that if $V_x = 0$ and $V_d = I$, the boundary condition in (6.2.5a) is precisely the Dirichlet condition. We will consider the more general case in this chapter and, in particular, the well-posedeness condition for the general boundary condition (6.2.5a) is discussed in the next subsection.

In the following example we illustrate how a non-NNM difference equation can be written in NNM form. In this particular example, the boundary conditions are taken to be Dirichlet. After illustrating the versatility of the NNM model in some additional examples, we investigate several methods for obtaining the general solution for processes governed by this model given a boundary condition of the general form (6.2.5a).

Example: (NNM Modeling of a 2-D Difference Equation not in the NNM form)
Consider the noncausal dynamics for a scalar process q_{ij} satisfying

$$q_{ij} = q_{i+1,j} + q_{i-1,j} + q_{i,j+1} + q_{i,j-1} + q_{i+1,j-1} + q_{i,j-2} + \varepsilon_{ij} \quad . \tag{6.2.6}$$

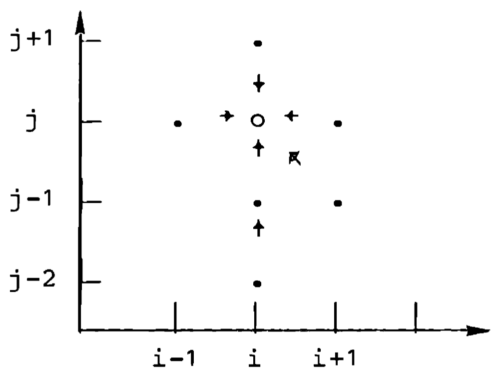


Figure 6.2.3 Dependence Relation for q_{ij} in (6.2.6)

Figure 6.2.3 clearly illustrates that (6.2.6), as written, does not have the precise nearest neighbor dependence depicted in Figure 6.2.1. However, if we consider a 2×1 vector process x_{ij} defined by

$$x_{ij} = \begin{bmatrix} q_{ij} \\ q_{i,j-1} \end{bmatrix} \tag{6.2.7a}$$

then its dynamics can be expressed in nearest neighbor form as

$$x_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x_{i-1,j} + \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} x_{i,j-1} + \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} x_{i+1,j} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x_{i,j+1} + \begin{bmatrix} 0 \\ \varepsilon_{ij} \end{bmatrix} . \tag{6.2.7b}$$

In this case the ranks of the coefficient matrices are $n_1 = n_3 = n_4 = 1$ and $n_2 = 2$. We can see that the value of x along the entire boundary ∂S would not be required to specify x on S by considering the trivial case where I = J = 2, so that S is simply the point (1,1) and ∂S consists of $\{(0,1), (2,1), (1,0), (1,2)\}$. In order to specify $x_{1,1}$, we find from (6.2.7b) that we need to know the input $\varepsilon_{1,1}$, and the following values corresponding to the Dirichlet condition in (6.2.5b):

$$d_L = [1 : 0]x_{0,1} \tag{6.2.8a}$$

$$d_R = [1 : 1]x_{2,1} \tag{6.2.8b}$$

$$d_B = x_{1,0} \tag{6.2.8c}$$

and

$$d_T = [1 : 0]x_{1,2} \quad . \tag{6.2.8d}$$

It is easy to see that the values in (6.2.8) are sufficient to specify $x_{1,1}$ if we rewrite (6.2.7b) for (i,j) = (1,1) as

$$x_{1,1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} d_L + \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} d_B + \begin{bmatrix} 1 \\ 0 \end{bmatrix} d_R + \begin{bmatrix} 1 \\ 0 \end{bmatrix} d_T + \begin{bmatrix} 0 \\ \varepsilon_{1,1} \end{bmatrix} \quad . \tag{6.2.9}$$

Note that if there were a higher order input to (6.2.6) (for example, $\varepsilon_{ij} + \varepsilon_{i,j-1}$ instead of ε_{ij} alone), then a NNM could be achieved by additionally augmenting x_{ij} in (6.2.7a) to include ε_{ij}. ∎

This example illustrates two important points. The first is that the Dirichlet boundary condition can be specified by a process of dimension lower than that of the process x on ∂S as defined in (6.2.3). The second is that from this simple example one can infer a method for transforming a general discrete noncausal dynamical representation into a vector nearest neighbor form. Indeed, this method of transforming higher order 2-D processes to a first order vector NNM process is analogous to the transformation of 1-D discrete ARMA models to first order vector state-space models [48]. The following examples show how standard finite difference approximations of second order partial differential equations can be put into a NNM form.

Examples(NNM Forms for Finite Difference Approximations of PDEs)

For each of the examples the continuous independent variables are denoted by t and s and these correspond to discretized variables i and j, respectively. The finite difference approximations are all based on a grid spacing which is equal in each variable: $h = \Delta t = \Delta s$.

1)Wave Equation

PDE: $$\frac{\partial^2 x}{\partial t^2} - c^2 \frac{\partial^2 x}{\partial s^2} + \alpha x = u$$

The finite difference approximation is:

$$x_{i+1,j} = c^2 x_{i,j-1} - c^2 x_{i,j+1} - h^2 \alpha_1 x_{ij} - x_{i-1,j} + h^2 u_{ij}$$

where

$$\alpha_1 = \alpha + \frac{2}{h^2}(1 - c^2) \quad .$$

The NNM is obtained by dividing by α_1 and rearranging terms:

$$x_{ij} = \frac{1}{\alpha_1 h^2} \{-x_{i-1,j} + c^2 x_{i,j-1} - x_{i+1,j} + c^2 x_{i,j+1}\} + \frac{1}{\alpha_1} u_{ij} \quad .$$

2)Poisson's Equation

PDE: $\nabla^2 x = u$

The finite difference approximation is in the form of a NNM:

$$x_{ij} = (1/4)[x_{i-1,j} + x_{i,j-1} + x_{i+1,j} + x_{i,j+1}] - (h^2/4)u_{ij}$$

3)Biharmonic Equation

PDE: $\nabla^4 x = u$

Finite difference approximation: A direct finite difference approximation of the fourth order biharmonic equation yields a 13 term difference equation. However, by introducing an auxillary variable z defined by

$\nabla^2 x = z$, we can express $\nabla^4 x = u$ as $\nabla^2 z = u$. This leads to a finite difference approximation given by the coupled difference equations:

$$x_{ij} = (1/4)\left[x_{i-1,j} + x_{i,j-1} + x_{i+1,j} + x_{i,j+1}\right] - (h^2/4)z_{ij}$$

$$z_{ij} = (1/4)\left[z_{i-1,j} + z_{i,j-1} + z_{i+1,j} + z_{i,j+1}\right] - (h^2/4)u_{ij}$$

Combining x and z into the vector process

$$X_{ij} = \begin{bmatrix} x_{ij} \\ z_{ij} \end{bmatrix} \quad ,$$

it can be shown that X has the following dynamics

$$X_{ij} = \begin{bmatrix} 1 & \frac{h^2}{4} \\ 0 & 1 \end{bmatrix} \left\{ \frac{1}{4}\left(X_{i-1,j} + X_{i,j-1} + X_{i+1,j} + X_{i,j+1}\right) + \begin{bmatrix} 0 \\ -\frac{h^2}{4} \end{bmatrix} u_{ij} \right\} \quad .$$

∎

The finite difference approximation of the wave equation in example 1 is clearly causal in the index i. So although it can be put into the NNM form, one must be careful to properly specify its boundary conditions. In this case one would expect that an initial vale would be specified for the process and its derivative, and this boundary condition will itself have to be discretized [14] and then expressed in the form (6.2.5a).

6.2.2 The General Solution for the NNM

As in our previous studies of 1-D boundary value problems, we investigate various methods of writing the general solution for the 2-D discrete processes studied here in an attempt to find stable and efficient methods for implementing the solution. Similar to the 1-D cases, our interest in efficient methods for implementing the solution of the NNM is based on the fact that, as we will see, the dynamics of the smoother for processes defined by the NNM are also in the form of this model. We remark that the methods for solving the NNM derived in this section should be of interest to a wider audience than simply those interested in the implementation of the solution to our smoothing problem. In particular, these methods will be useful in implementing numerical solutions of finite difference approximations of PDEs.

Three methods for obtaining the solution to (6.2.1b) will be discussed. The first method, which we refer to as the matrix inversion method [49], is conceptually the simplest of the three. However, we will see that for a large lattice S, a direct matrix inversion becomes numerically impractical. Consequently, this method is rarely used in practice, and here it serves mainly to introduce some vector notation that we use in later discussions and also to provide a means for establishing well-posedness conditions for boundary conditions other than the Dirichlet condition.

The second method falls into the class of what are called marching methods [49, 50]. We will see that a sufficient condition for writing 1-D marching method representations for the dynamics of vector 2-D processes described by the NNM is the requirement that at least one of the A_k in (6.2.1b) be invertible. Basically, these 1-D methods represent a transformation of the noncausal 2-D dynamics into causal 1-D dynamics of higher order that can be solved recursively. In the past, a major criticism of marching methods has been that they are numerically unstable [49] and, therefore, that they also are not useful for solving the NNM on large lattices. However, we develop here a variation of the classical marching methods which avoids these numerical problems. This is accomplished by introducing 1-D recursions in two directions (forward and backward), each stable in its particular direction.

The third method for obtaining a general solution for the NNM is a special case of the aforementioned forward/backward marching methods. In particular, the method is modified to realize further computational efficiencies. The basis for this modification is an application of the fast Fourier transform (FFT) which transforms the dynamics of the 1-D process associated with the marching method into a system of low order decoupled 1-D processes. This decoupling transformation was first suggested by Hockney [47] for improving the classical marching method solution for Poisson's equation, and later by Jain and Angel [48] in solving a 2-D estimation problem. Here we extend its application to vector processes satsifying a NNM. This extension requires, in addition to the invertibility mentioned above, a symmetry property. For example, if A_3 were invertible, then the symmetry constraint would require that A_2 and A_4 be identical. This type of symmetry is not uncommon in NNM descriptions of physical systems. For instance, the NNMs for each of the three PDEs in the examples above possess this property.

The Matrix Inversion Method and Well-Posedness

The entire process x over $S = [1, I-1] \times [1, J-1]$ can be written as one large vector by first forming the $n(J-1)\times 1$ vector x_i and the $m(J-1)\times 1$ vector u_i:

$$x_i = \begin{bmatrix} x_{i,1} \\ x_{i,2} \\ \vdots \\ x_{i,J-1} \end{bmatrix}, \qquad u_i = \begin{bmatrix} u_{i,1} \\ u_{i,2} \\ \vdots \\ u_{i,J-1} \end{bmatrix} \tag{6.2.10a}$$

and then stacking these into the $n(J-1)(I-1)\times 1$ vector x and the $m(J-1)(I-1)\times 1$ vector u:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{i-1} \end{bmatrix}, \qquad u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{i-1} \end{bmatrix} \quad . \tag{6.2.10b}$$

The linear relationship between the vector x, the input u and boundary processes d_L, d_R, d_B and d_T is developed as follows. Let a_0 be the $n(J-1)\times n(J-1)$ matrix in $n\times n$ partitions:

$$a_0 = \begin{bmatrix} 0 & A_4 & 0 & \cdots & & 0 \\ A_2 & 0 & A_4 & & 0 & \vdots \\ 0 & A_2 & 0 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ \vdots & 0 & & \ddots & 0 & A_4 \\ 0 & \cdots & & 0 & A_2 & 0 \end{bmatrix} \tag{6.2.11}$$

or in Kronecker product notation [51]

$$a_0 = (Z \otimes A_4) + (Z' \otimes A_2) \tag{6.2.11b}$$

where Z is the $(J-1)\times(J-1)$ matrix

$$Z = \begin{bmatrix} 0 & 1 & 0 & \cdots & & & 0 \\ 0 & 0 & 1 & 0 & \cdots & & 0 \\ \cdot & & \cdot & \cdot & & 0 & \cdot \\ \cdot & & & \cdot & \cdot & & \cdot \\ \cdot & & & & \cdot & \cdot & \cdot \\ \cdot & & 0 & & & \cdot\ 1 & 0 \\ \cdot & & & & & 0 & 1 \\ 0 & \cdots & & & & & 0 \end{bmatrix} \qquad (6.2.12)$$

With $I_{(J-1)}$ denoting the $(J-1)\times(J-1)$ identity matrix, define the $n(J-1)\times n(J-1)$ matrices a_1 and a_3 as:

$$a_k = \mathrm{Diag}(A_k) = I_{(J-1)} \otimes A_k \quad , k = 1,3 \quad ; \qquad (6.2.13a)$$

the $n(I-1)\times n(I-1)$ matrices a_2 and a_4 as:

$$a_k = \mathrm{Diag}(A_k) = I_{(I-1)} \otimes A_k \quad , k = 2,4 \quad ; \qquad (6.2.13b)$$

and the $n(J-1)\times m(J-1)$ matrix b as

$$b = \mathrm{Diag}(B) = I_{(J-1)} \otimes B \quad . \qquad (6.2.14)$$

Also define H_L and H_R as the $n(J-1)(I-1)\times n(J-1)$ matrices

$$H_L = \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{and} \quad H_R = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I \end{bmatrix} \quad . \qquad (6.2.15)$$

Next define the $n(I-1)\times n(I-1)$ block-circulant permutation matrix R in $n\times n$ blocks as:

$$R = \begin{bmatrix} 0 & I & 0 & \cdots & & & 0 \\ 0 & 0 & I & 0 & \cdots & & 0 \\ \cdot & & \cdot & \cdot & & 0 & \cdot \\ \cdot & & & \cdot & \cdot & & \cdot \\ \cdot & & & & \cdot & \cdot & \cdot \\ \cdot & & 0 & & & \cdot\ I & 0 \\ \cdot & & & & & 0 & I \\ I & \cdots & & & & & 0 \end{bmatrix} \quad . \qquad (6.2.16)$$

Note that postmultiplying a matrix by R causes the columns of that matrix to be circularly shifted or rotated to the right. Define I_B and I_T as the $n(I-1)\times n(I-1)$ matrices

$$I_B = \begin{bmatrix} I & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & 0 & \\ 0 & & & \end{bmatrix} \quad \text{and} \quad I_T = \begin{bmatrix} 0 & & & \\ \vdots & & 0 & \\ 0 & & & \\ I & 0 & \cdots & 0 \end{bmatrix} \quad . \tag{6.2.17}$$

Combining these two matrices with R we define two $n(J-1)(I-1)\times n(I-1)$ matrices

$$H_B = \begin{bmatrix} I_B \\ I_B R \\ I_B R^2 \\ \vdots \\ I_B R^{I-2} \end{bmatrix} \quad \text{and} \quad H_T = \begin{bmatrix} I_T \\ I_T R \\ I_T R^2 \\ \vdots \\ I_T R^{I-2} \end{bmatrix} \quad . \tag{6.2.18}$$

Given these definitions we can write the relationship between the process x on S and x on the boundary ∂S as defined in (6.2.3) and the input u:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{I-1} \end{bmatrix} = \begin{bmatrix} a_0 & a_3 & & & 0 \\ a_1 & a_0 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & a_3 \\ 0 & & & a_1 & a_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{I-1} \end{bmatrix} + (I \otimes b) \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{I-1} \end{bmatrix}$$

$$+ \left[H_L \,\vdots\, H_R \,\vdots\, H_B \,\vdots\, H_T \right] \begin{bmatrix} a_1 x_L \\ a_3 x_R \\ a_2 x_B \\ a_4 x_T \end{bmatrix} \quad . \tag{6.2.19}$$

Employing the definition of x and u in (6.2.10b) and the relations between x_L, x_R, x_B, x_T and d_L, d_R, d_B, d_T in (6.A.9) in Appendix 6A, this relation can be written in more compact notation as

$$x = A_x x + B_x u + Hd \tag{6.2.20a}$$

where

$$H = \left[H_L f_1 \,\vdots\, H_R f_3 \,\vdots\, H_B f_2 \,\vdots\, H_T f_4 \right] \tag{6.2.20b}$$

and d is as in (6.2.5b). In Kronecker product notation, A_x and B_x can be expressed as:

$$A_x = (I \otimes a_0) + (Z \otimes a_3) + (Z' \otimes a_1) \qquad (6.2.20c)$$

and

$$B_x = (I \otimes b) \quad . \qquad (6.2.20d)$$

Note that we have not yet specified the boundary condition, so that d in (6.2.20a) should not be confused as being a given boundary condition. Rather, the representation (6.2.20a) and the specified boundary conditions (6.2.5a) must be solved simultaneously in order to express x in terms of the input and the boundary value v.

Before pursuing the general solution of (6.2.20a) for arbitrary boundary conditions, it will be useful to first consider the special case in which the boundary condition is the Dirichlet condition, i.e.

$$v = d \quad . \qquad (6.2.21)$$

Given the Dirichlet condition, we can solve for x from (6.2.20a) as:

$$x = (I - A_x)^{-1}\left[B_x u + Hd\right] \quad . \qquad (6.2.22)$$

As we will see below, given this solution we are in a position to study more general boundary conditions of the form (6.2.5a).

Using this same approach, we establish the well-posedness condition for the NNM given boundary conditions of the form (6.2.5a). In particular, recall that in deriving a general solution for a 1-D two-point boundary value problem that we first used the variation of constants formula to write an expression for the process in terms of the inputs and the unknown initial value of the process. Next this expression for the process was substituted into the two-point boundary condition to solve for the unknown initial value in terms of the inputs and the boundary value. Finally, we again used the variation of constants formula but with this new expression determined for the initial condition, yielding the general solution. In our derivations below, the Dirichlet condition d plays a similar role to that of the initial value in the

1-D derivations. Specifically, from (6.2.22), we can write an expression for the process x given the input u and the unknown value of d as

$$x = (I - A_x)^{-1}[B_x u + Hd] \quad . \tag{6.2.23a}$$

To determine the value of d in terms of the known values of the input u and the boundary value v, substitute the expression for x in (6.2.23a) into the boundary condition (6.2.5a):

$$v = V_x(I - A_x)^{-1}B_x u + [V_x(I - A_x)^{-1}H + V_d]d \quad . \tag{6.2.23b}$$

Solving for d, we have

$$d = F^{-1}\{v - V_x(I-A_x)^{-1}B_x u\} \tag{6.2.23c}$$

where

$$F = V_x(I - A_x)^{-1}H + V_d \quad . \tag{6.2.23d}$$

Therefore, the well-posedness condition is the invertibility of F. Finally, the general solution for (6.2.20a) with boundary condition (6.2.5a) is obtained as a linear combination of the boundary value v and the unput u by substituting the expression for d in (6.2.23c) into (6.2.23a):

$$x = (I - A_x)^{-1}\{HF^{-1}v + [I - HF^{-1}V_x(I - A_x)^{-1}]B_x u\} \quad . \tag{6.2.24}$$

Although this method of solving for x and determining well-posedness is straightforward, its actual implementation may be difficult. In particular, note that $(I - A_x)$ is an $n(I-1)(J-1) \times n(I-1)(J-1)$ matrix so that applying numerical matrix inversion algorithms to obtain its inverse when I and J are large can be impractical even when exploiting its highly structured form. Indeed, research on the problem of explicitly inverting this matrix is ongoing (see [49] for a survey). Of course, in any solution of the NNM (such as those that follow) this inversion is performed implicitly.

The 1-D Marching Method

Marching methods were originally established when it was first recognized that the 2-D discrete approximation of Poisson's equation for scalar processes could be transformed to a vector 1-D process (for a survey of early work in this area see [52]). This kind of transformation has been applied by Jain and Angel [46] in the derivation and implementation of an estimator for a particular scalar 2-D discrete process. In this section we show that extending this idea to writing 1-D models for the more general class of 2-D processes obeying the vector NNM is straightforward. We will see that the only requirement for this extension is that one of the A_k is invertible. Assuming that this requirement is met, we can ,without loss of generality, choose A_3 as the invertible one. First we write a 1-D dynamical representation for the 2-D NNM dynamics, and then show how the general form of the NNM boundary condition (6.2.5a) is transformed to a boundary condition for the 1-D representation of the process.

Given that A_3 is invertible, define

$$\tilde{A}_3 = A_3^{-1} \quad , \tag{6.2.25a}$$

$$\tilde{A}_i = -A_3^{-1}A_i \qquad ; \; i = 1,2,4 \tag{6.2.25b}$$

and

$$\tilde{B} = -A_3^{-1}B \quad . \tag{6.2.25c}$$

Block-diagonal matrices with the matrices in (6.2.25) as their diagonal elements can be written using the Kronecker product as (see (6.2.13) and (6.2.14))

$$\tilde{a}_3 = (I \otimes \tilde{A}_3) \tag{6.2.26a}$$

$$\tilde{a}_k = (I \otimes \tilde{A}_k)$$

$$= -\tilde{a}_3 a_k \qquad ; \; k = 1,2,4 \tag{6.2.26b}$$

and

$$\tilde{b} = (I \otimes \tilde{B}) = -\tilde{a}_3 b \quad . \tag{6.2.26c}$$

Given these definitions, the NNM dynamics in (6.2.1b) can be written as (we have simply premultiplied (6.2.1b) by the inverse of A_3):

$$x_{i+1,j} = \tilde{A}_3 x_{ij} + \tilde{A}_2 x_{i,j-1} + \tilde{A}_1 x_{i-1,j} + \tilde{A}_4 x_{i,j+1} + \tilde{B}\, u_{ij} \quad . \tag{6.2.27}$$

In the same manner we can premultiply the rows of the matrix representation of the NNM in (6.2.19) by the inverse of a_3, to obtain a recursion for x_i as

$$\begin{aligned} x_{i+1} = {} & \left[\, \tilde{a}_3 + (Z \otimes \tilde{A}_4) + (Z' \otimes \tilde{A}_2) \right] x_i - \tilde{a}_3 a_1 x_{i-1} - \tilde{a}_3 b u_i \\ & - \tilde{a}_3 \left[I_B R^{i-1} f_2 d_B + I_T R^{i-1} f_4 d_T \right] \quad ; \; i = 1,2,\ldots I-1 \quad . \end{aligned} \tag{6.2.28a}$$

with the following constraints (also see (6.A.9) of Appendix 6A)

(1) $x_I = x_R = d_R$ (the second equality holding since A_3 is nonsingular) (6.2.28b)

and

(2) $a_1 x_0 = a_1 x_L = f_1 d_L$. (6.2.28c)

Here I_T and I_B are defined in (6.2.17), Z in (6.2.12) and R in (6.2.16). These constraints are added to make (6.2.28a) evaluated for i = 1 and I-1 compatible with the matrix equation (6.2.19). Therefore, under the single condition that one of the A_k be invertible (here we assume it is A_3), we can transform the 2-D NNM dynamics to a 1-D model.

The boundary condition for the 1-D dynamics (6.2.28) is obtained from the general form of the NNM boundary condition (6.2.22) as follows. First recall (6.2.5a):

$$v = V_x x + V_d d \quad .$$

Define partitions of V_x and V_d with dimensions compatible with the partitions of x and d in (6.2.10b) and (6.2.5b), respectively as

$$V_x = \left[V_x^1 \;\vdots\; V_x^2 \;\vdots\; \cdot \;\; \cdot \;\; \cdot \;\vdots\; V_x^{I-1} \right] \tag{6.2.29a}$$

and

$$V_d = \left[V_L \vdots V_R \vdots V_B \vdots V_T\right] \qquad (6.2.29b)$$

so that the NNM boundary condition can be expressed as

$$v = \sum_{i=1}^{I-1} V_x^i x_i + V_L d_L + V_R d_R + V_B d_B + V_T d_T \quad . \qquad (6.2.30)$$

Using the constraints on the dynamics in (6.2.28b) and (6.2.28c) we can rewrite (6.2.30) as follows. First, from (6.2.28b) we can make the substitution

$$V_R d_R = V_x^0 x_I \qquad (6.2.31a)$$

where

$$V_x^0 = V_R \quad . \qquad (6.2.31b)$$

To replace $V_L d_L$ in (6.2.30), we require an expression for d_L in terms of x_0. In particular, the defintion of d_L given in (6.A.4a) is compatible with the constraint (6.2.28c) (see (6.A.9)):

$$d_L = \left[I_{J-1} \otimes [I_{n_1} \vdots 0]\right](I \otimes \phi_1)x_0 \qquad (6.2.32a)$$

where we have used $x_0 = x_L$. Thus, the term $V_L d_L$ can be written as

$$V_L d_L = V_x^0 x_0 \qquad (6.2.32b)$$

where

$$V_x^0 = V_L\left[I_{J-1} \otimes [I_{n_1} \vdots 0]\right](I \otimes \phi_1) \quad . \qquad (6.2.32c)$$

Substituting from (6.2.31b) and (6.2.32c) into the expression for the boundary condition in (6.2.30), we have (note that the summation now ranges over [0,I])

$$v = \sum_{i=0}^{I} V_x^i x_i + V_{TB} d_{TB} \qquad (6.2.33a)$$

where

$$d_{TB} = \begin{bmatrix} d_B \\ d_T \end{bmatrix} \qquad \text{and} \qquad V_{TB} = [V_B \; \vdots \; V_T] \quad . \tag{6.2.33b}$$

Our final objective in this subsection is to write the dynamics in (6.2.28) as a first order 1-D process. Let χ_i be defined as

$$\chi_i = \begin{bmatrix} x_{i+1} \\ x_i \end{bmatrix} \quad . \tag{6.2.34}$$

Then the dynamics for χ_i can be written as

$$\chi_i = A\chi_{i-1} + G_u u + G_{TBi} d_{TB} \tag{6.2.35a}$$

where

$$A = \left[\begin{array}{c:c} \tilde{a}_3 + (Z \otimes \tilde{A}_4) + (Z' \otimes \tilde{A}_2) & \tilde{a}_1 \\ \hdashline I & 0 \end{array} \right] , \tag{6.2.35b}$$

$$G_u = \begin{bmatrix} \tilde{b} \\ 0 \end{bmatrix} , \tag{6.2.35c}$$

and

$$G_{TBi} = \left[\begin{array}{c:c} -\tilde{a}_3 I_B R^{i-1} f_2 & -\tilde{a}_3 I_T R^{i-1} f_4 \\ \hdashline 0 & 0 \end{array} \right] \quad . \tag{6.2.35d}$$

The boundary condition (6.2.33) is written in terms of the process χ_i as

$$v = \sum_{i=0}^{I} [0 \; \vdots \; V_x^i \,]\chi_i + V_{TB} d_{TB} \quad . \tag{6.2.36}$$

We will refer to this as a multi-point boundary condition in contrast to the two-point boundary condition for the 1-D discrete process in Chapter 5.

As an example, consider the special case for which all of the A_k are nonsingular and v is the Dirichlet condition in (6.2.5b). Since all A_k are nonsingular, $d_R = x_R = x_I$, $d_L = x_L = x_0$, $d_B = x_B$ and $d_T = x_T$, and d_{TB} in (6.2.33b) becomes:

$$d_{TB} = \begin{bmatrix} x_T \\ x_B \end{bmatrix} \quad .$$

By defining

$$V_x^0 = \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad V_x^I = \begin{bmatrix} 0 \\ I \\ 0 \\ 0 \end{bmatrix} \quad V_{TB} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ I & 0 \\ 0 & I \end{bmatrix} \quad \text{and} \quad V_x^i = 0 \ , \ i = 1,2..I-1 \ ,$$

the Dirichlet boundary condition can be written in the form (6.2.36) as required for the marching method.

Stable Two-Filter (Forward/Backward) Solution

Recall that in formulating the general solution for the 1-D discrete two-point boundary value problem in Chapter 5, we diagonalized the dynamics into a forward stable part and a backward stable part. In this way we were able to develop a forward/backward two-filter implementation of the solution, each filter stable in its own direction. The differences between that 1-D discrete TPBVP and the 1-D discrete process χ_i whose dynamics are given in (6.2.35a) and boundary condition in (6.2.36) are (1) the presence of d_{TB} in both the dynamics and boundary condition and (2) the multi-point nature of the boundary condition. Common to both is that, in general, the poles of the dynamics will lie both inside and outside of the unit circle. Thus, some modes will be stable for increasing i (forward stable) and some stable for decreasing i (backwards stable). To separate these modes, consider the class of similarity transformations which decouple the dynamics in (6.2.35a) into the form:

$$TAT^{-1} = \left[\begin{array}{c:c} A_f & 0 \\ \hdashline 0 & A_b \end{array}\right] \tag{6.2.37}$$

where A_b contains only backward stable modes and A_f contains all other modes, i.e. forward stable, marginally stable and zero modes. With partitions compatitble with those in (6.2.37), define

$$\begin{bmatrix} x_{f,i} \\ x_{b,i} \end{bmatrix} = T\chi_i \ , \qquad \begin{bmatrix} B_f \\ B_b \end{bmatrix} = TG_u \quad \text{and} \quad \begin{bmatrix} M_{fi} \\ M_{bi} \end{bmatrix} = TG_{TBi} \ . \tag{6.2.38}$$

so that the following system is equivalent to (6.2.35a) and (6.2.36) but with stable decoupled dynamics:

$$x_{f,i} = A_f x_{f,i-1} + B_f u_i + M_{fi} d_{TB} \tag{6.2.39a}$$

and

$$x_{b,i-1} = A_b^{-1} x_{b,i} - A_b^{-1} B_b u_i - A_b^{-1} M_{bi} d_{Tb} \tag{6.2.39b}$$

and coupled boundary conditions:

$$v = \sum_{i=0}^{I} [V_f^i \vdots V_b^i] \begin{bmatrix} x_{f,i} \\ x_{b,i} \end{bmatrix} + V_{TB} d_{TB} \tag{6.2.39c}$$

where

$$[V_f^i \vdots V_b^i] = [0 \vdots V_x^i] T^{-1} \ . \tag{6.2.39d}$$

The general solution for (6.2.39) is derived in Appendix 6B in the same way that we derived the forward/backward solution for the two-point boundary value problem in Chapter 5. That is, for the case of $d_{TB} = 0$ we define x_f^0 as the solution to (6.2.39a) with a zero initial condition:

$$x_{f,0}^0 = 0 \tag{6.2.40a}$$

and x_b^0 as the solution to (6.2.39b) with zero final condition

$$x_{b,I}^0 = 0. \tag{6.2.40b}$$

The general solution is given by (6.B.11) in Appendix 6B as a linear combination of these processes and the boundary value v:

$$\begin{bmatrix} x_{f,n} \\ x_{b,n} \\ d_{TB} \end{bmatrix} = \Phi(n) F_{fb}^{-1} \{ v - \sum_{i=0}^{I} [V_f^i \vdots V_b^i] \begin{bmatrix} x_{f,i}^0 \\ x_{b,i}^0 \end{bmatrix} \} + \begin{bmatrix} x_{f,n}^0 \\ x_{b,n}^0 \\ 0 \end{bmatrix} \ . \tag{6.2.41}$$

where $\Phi(n)$ is defined in (6.B.10) and the invertibility of F_{fb} (in (6.B.8)) is the well-posedness condition. Later we will see that being able to write the 2-D NNM dynamics in a 1-D marching method form is only a sufficient condition for the existence of a forward/backward 1-D representation. In particular, in an example in Section 6.3, it is shown that a 1-D forward/backward form can also be obtained from a 1-D scattering representation of the NNM dynamics.

The two major computational problems associated with this form of the general solution are 1) the inversion of the $n_f \times n_f$ matrix F_{fb} $\left(n_f = (n_2+n_4)(I-1) + 2n(J-1) \right)$ and 2) the determination of the diagonalizing transformation T. However, in comparison to the inversion of $(I - A_x)$ (whose size is of the order of the product of I and J as opposed to their sum) in (6.2.20d) these computations are minor for large I and J . As a final remark, we note that in addition to deriving the general solution for x_f and x_b in Appendix 6B, we also develop equations for computing the process variance when the input and boundary conditions are random processes.

A More Efficient Marching Method

Although the stable marching method developed in the previous section offers a substantial savings in computation over the matrix inversion method, the determination of the diagonalizing transformation T (as well as the computation of the transition matrices and recursions for the $2n(J-1)\times 1$ processes x_f and x_b) can still be a considerable computational task when I and J are large. In this subsection it is shown for a large class of NNMs that the computational burden can be further reduced. The reduction is based on an extension of the work of Jain and Angel in [46] where they considered a scalar process satisfying a discretized version of Poisson's equation.

Consider the class of NNMs which have both the invertibility property (6.2.25a), as required for the implementation of any marching method, and a symmetry property as described below. If the index of the invertible matrix A_m is odd (even), then the even (odd) indexed A_k must be identical. For example, if A_3 is to be inverted to achieve the marching method form, then the symmetry property is satisfied if

$$A_2 = A_4 \quad . \tag{6.2.41}$$

Given (6.2.41) and defining the symmetric tridiagonal matrix P as the sum of Z in (6.2.12) and its transpose:

$$P = Z' + Z \quad , \tag{6.2.42}$$

the dynamics in (6.2.28a) can be written as

$$\chi_i = \left[\begin{array}{c|c} \tilde{a}_3 + (P \otimes \tilde{A}_2) & \tilde{a}_1 \\ \hline I & 0 \end{array}\right] \chi_{i-1} + \left[G_u \vdots G_{TBi}\right]\left[\begin{array}{c} u_i \\ \hline d_{TB} \end{array}\right] \quad . \tag{6.2.43}$$

It can be shown that the the upper left partition of the dynamics matrix has a special tridiagonal block-Toeplitz form that we will take advantage of shortly. In particular, its upper and lower diagonal blocks are identical. The action of this block-Toeplitz matrix can be viewed as that of convolving the upper partition of χ_{i-1} with a finite impulse response (FIR) filter [62]. It is, in part, this property that allows us to decouple the dynamics in (6.2.43) by the FFT-based transformation as discussed below.

The dynamics in (6.2.43) can be decoupled into J-1 subsystems by transforming the process in χ_i with the transformation matrix defined as follows. Let D be the (J-1)×(J-1) matrix with elements

$$D_{ij} = \sqrt{(2/J)}\ \sin(ij\pi/J) \quad . \tag{6.2.44}$$

The matrix D has two important properties [46] of which we will make use:

1) D is symmetric and orthonormal, i.e.

$$DD' = D'D = DD = I \quad . \tag{6.2.45a}$$

2) The matrix P in (6.2.42) is diagonalized by D:

$$\Lambda = DPD' = \mathrm{Diag}\{\lambda_j\}$$
$$\lambda_j = 2\cos(j\pi/J) \quad ; \quad j = 1,2, \ldots .\ J-1 \quad . \tag{6.2.45b}$$

Employing D, define a new n(J-1)×1 process q_i via the equivalence transformation:

$$q_i = (D \otimes I)x_i \qquad \text{or} \qquad \begin{bmatrix} q_{i+1} \\ q_i \end{bmatrix} = \left[\begin{array}{c|c} D\otimes I & \\ \hline 0 & D\otimes I \end{array}\right] \chi_i \quad . \tag{6.2.46a}$$

and denote the elements of q_i by the $n\times1$ partitions q_{ij}:

$$q_i = \begin{bmatrix} q_{i,1} \\ q_{i,2} \\ \vdots \\ q_{i,J-1} \end{bmatrix} \quad . \tag{6.2.46b}$$

Below it is shown that the dynamics of each of the q_{ij} are decoupled via the transformation in (6.2.46a). First we state the following basic identities for Kronecker products [51]:

$$(A \otimes B)^{-1} = (A^{-1} \otimes B^{-1}) \qquad \{A,B \text{ invertible}\} \quad , \tag{6.2.47a}$$

$$(A \otimes B)' = (A' \otimes B') \tag{6.2.47b}$$

and

$$(A \otimes B)(C \otimes D) = (AC \otimes BD) \quad . \tag{6.2.47c}$$

A trivial application of these identities allows us to establish two properties of $(D \otimes I)$, the transformation in (6.2.46a):

<u>Property 1</u> For any $n\times n$ matrix M, $(I \times M)$ is invariant under the following transformation

$$\begin{aligned}(D \otimes I)(I \otimes M)(D \otimes I)^{-1} &= (D \otimes M)(D \otimes I) \\ &= (I \otimes M) \quad .\end{aligned} \tag{6.2.48}$$

<u>Property 2</u> For any $n\times n$ matrix M, $(P \otimes M)$ is block-diagonalized by $(D \otimes I)$:

$$\begin{aligned}(D \otimes I)(P \otimes M)(D \otimes I)^{-1} &= (DP \otimes M)(D \otimes I) \\ &= (\Lambda \otimes M) \\ &= \begin{bmatrix} \lambda_1 M & & & 0 \\ & \lambda_2 M & & \\ & & \ddots & \\ 0 & & & \lambda_{J-1} M \end{bmatrix} \quad .\end{aligned} \tag{6.2.49}$$

The dynamics of the transformed process q_i in (6.2.46a) are found in the usual way by applying the transformation to the dynamics in (6.2.43):

$$\begin{bmatrix} q_{i+1} \\ q_i \end{bmatrix} = \begin{bmatrix} D \otimes I & 0 \\ 0 & D \otimes I \end{bmatrix} \begin{bmatrix} (I \otimes \tilde{A}_3) + (P \otimes \tilde{A}_2) & I \otimes \tilde{A}_1 \\ I & 0 \end{bmatrix} \begin{bmatrix} D \otimes I & 0 \\ 0 & D \otimes I \end{bmatrix}^{-1} \begin{bmatrix} q_i \\ q_{i-1} \end{bmatrix}$$

$$+ \begin{bmatrix} D \otimes I & 0 \\ 0 & D \otimes I \end{bmatrix} \left[G_u : G_{TBi}\right] \begin{bmatrix} u_i \\ \bar{d}_{TB} \end{bmatrix} \quad . \tag{6.2.50a}$$

If we recall the definitions of G_u and G_{TBi} in (6.2.35c) and (6.2.35d) and let

$$U_{q_i} = \tilde{b}\, u_i - \tilde{a}_3\left(I_B R^{i-1} f_2 d_B + I_T R^{i-1} f_4 d_T\right) \quad , \tag{6.2.50b}$$

then the input term in (6.2.50a) can be written as

$$\begin{bmatrix} D \otimes I \\ 0 \end{bmatrix} U_{q_i} \quad . \tag{6.2.50c}$$

Noting that $(D \times I)$ is its own inverse and invoking the two properties established above, (6.2.50a) can be expressed as

$$\begin{bmatrix} q_{i+1} \\ q_i \end{bmatrix} = \begin{bmatrix} (I \otimes \tilde{A}_3) + (\Lambda \otimes \tilde{A}_2) & (I \otimes \tilde{A}_1) \\ I & 0 \end{bmatrix} \begin{bmatrix} q_i \\ q_{i-1} \end{bmatrix} + \begin{bmatrix} D \otimes I \\ 0 \end{bmatrix} U_{q_i} \tag{6.2.51a}$$

If we partition the product $(D \otimes I)U_{q_i}$ into $n \times 1$ vectors $U_{i,j}$ as

$$\begin{bmatrix} U_{i,1} \\ U_{i,2} \\ \vdots \\ U_{i,J-1} \end{bmatrix} \equiv (D \otimes I) U_{q_i} \quad , \tag{6.2.51b}$$

and note that each of the partitions in the dynamics matrix in (6.2.51a) is block-diagonal, we can write the dynamics of q_i as J-1 sets of decoupled equations representing the dynamics of its elements $q_{i,j}$ (see (6.2.46b))

$$\begin{bmatrix} q_{i+1,j} \\ q_{i,j} \end{bmatrix} = \left[\begin{array}{c:c} \tilde{A}_3 + \lambda_j \tilde{A}_2 & \tilde{A}_1 \\ \hdashline I & 0 \end{array}\right] \begin{bmatrix} q_{i,j} \\ q_{i-1,j} \end{bmatrix} + \begin{bmatrix} U_{i,j} \\ 0 \end{bmatrix} \quad ; \; j = 1,2,\ldots J-1 \quad .(6.2.52)$$

The boundary conditions for the q_{ij} in (6.2.52) are obtained by an application of the transformation in (6.2.46a) to the boundary condition in (6.2.36).

In this decoupled representation, the task of transforming the dynamics into stable forward/backward diagonal forms is considerably simplified. Specifically, computational efficiencies are realized in two ways. First, the decomposition into forward and backward processes is reduced from a 2n(J-1)-dimensional problem for the entire vector χ_i to the J-1 decoupled 2n-dimensional problems for the q_{ij}. The details of splitting each of the q_{ij} into a forward stable component and a backward stable component with appropriately transformed boundary conditions are discussed in Appendix 6C. The second is that the decoupling transformation matrix $(D \times I)$ in (6.2.46a) is the same for all processes in NNM form with the stated symmetry condition. Furthermore, the matrix D is related to the matrix representation of the FFT [53], and consequently, many of the transformations required in decoupling χ_i into the q_{ij} can be executed efficiently via the FFT. For details of the FFT implementation also see Appendix 6C. In the next section we demonstrate by way of two simple examples how the NNM dynamics of 2-D smoothers can be simplified via this decoupling transformation.

SECTION 6.3

THE NNM SMOOTHER

In this section we present the dynamics and boundary conditions for the smoother for processes governed by an NNM and show that the smoother dynamics themselves can be written in the NNM form. Next, conditions are established for writing the 2-D smoother dynamics in a 1-D marching method form, and the corresponding 1-D multi-point representation of the smoother boundary condition is presented. The section ends with the application of the smoothing equations to two processes governed by an NNM.

6.3.1 The Smoother

Here we consider the smoother for a 2-D discrete process governed by the NNM in (6.2.1b) with boundary condition which we discuss shortly. The observations are given by the p×1 vector process y:

$$y_{ij} = Cx_{ij} + r_{ij} \qquad ; \ (i,j) \in S \tag{6.3.1a}$$

$$E\{r_{ij}r'_{k\ell}\} = R\delta_{ik}\delta_{j\ell} \quad . \tag{6.3.1b}$$

In an obvious operator notation, the process dynamics in (6.2.1b) and the observations in (6.3.1) can be expressed as

$$Lx = Bu \tag{6.3.2a}$$

and

$$y = Cx + r \quad . \tag{6.3.2b}$$

The operator representation for the estimator dynamics has been shown in (2.5.25a) to be

$$\begin{bmatrix} L & \vdots & -BQB^* \\ - & - & - \\ C^*R^{-1}C & \vdots & L^\dagger \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{\lambda} \end{bmatrix} = \begin{bmatrix} 0 \\ - \\ C^*R^{-1} \end{bmatrix} y \quad . \tag{6.3.3}$$

The formal adjoint difference operator $L^\dagger$ appearing in the smoother dynamics (6.3.3) is shown in Appendix 6D to take the form

$$(L^\dagger\lambda)_{ij} = \lambda_{ij} - A_3'\lambda_{i-1,j} - A_4'\lambda_{i,j-1} - A_1'\lambda_{i+1,j} - A_2'\lambda_{i,j+1} \quad . \tag{6.3.4}$$

The smoother dynamics in (6.3.3) can be put into the NNM form as follows. Define

$$\Delta_{x\lambda} = \begin{bmatrix} I & -BQB' \\ C'R^{-1}C & I \end{bmatrix}, \qquad B_{x\lambda} = \begin{bmatrix} 0 \\ C'R^{-1} \end{bmatrix}, \tag{6.3.5a}$$

and

$$A_{x\lambda_1} = \begin{bmatrix} A_1 & 0 \\ 0 & A_3' \end{bmatrix}, \quad A_{x\lambda_2} = \begin{bmatrix} A_2 & 0 \\ 0 & A_4' \end{bmatrix}, \quad A_{x\lambda_3} = \begin{bmatrix} A_3 & 0 \\ 0 & A_1' \end{bmatrix}, \quad A_{x\lambda_4} = \begin{bmatrix} A_4 & 0 \\ 0 & A_2' \end{bmatrix} \tag{6.3.5b}$$

Using these definitions we can rewrite (6.3.3) as

$$\Delta_{x\lambda}\begin{bmatrix} \hat{x}_{ij} \\ \hat{\lambda}_{ij} \end{bmatrix} = A_{x\lambda_1}\begin{bmatrix} \hat{x}_{i-1,j} \\ \hat{\lambda}_{i-1,j} \end{bmatrix} + A_{x\lambda_2}\begin{bmatrix} \hat{x}_{i,j-1} \\ \hat{\lambda}_{i,j-1} \end{bmatrix} + A_{x\lambda_3}\begin{bmatrix} \hat{x}_{i+1,j} \\ \hat{\lambda}_{i+1,j} \end{bmatrix} + A_{x\lambda_4}\begin{bmatrix} \hat{x}_{i,j+1} \\ \hat{\lambda}_{i,j+1} \end{bmatrix} + B_{x\lambda}y_{ij}. \tag{6.3.5b}$$

While this is not in the NNM form, it is in the descritpor equivalent and can be put in NNM form by premultiplying by the inverse of $\Delta_{x\lambda}$. Using the matrix inversion lemma, it can be seen that this inverse exists with:

$$\Delta_{x\lambda}^{-1} = \begin{bmatrix} I & -BQB' \\ C'R^{-1}C & I \end{bmatrix}^{-1} = \begin{bmatrix} I & BQB' \\ -C'R^{-1}C & I \end{bmatrix}\begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix} \tag{6.3.6}$$

where

$$\Delta_1 = \left(I + BQB'C'R^{-1}C\right)^{-1} \quad \text{and} \quad \Delta_2 = \left(I + C'R^{-1}CBQB'\right)^{-1} . \tag{6.3.7}$$

Thus, the NNM form for the smoother dynamics is

$$\begin{bmatrix} \hat{x}_{ij} \\ \hat{\lambda}_{ij} \end{bmatrix} = \Delta_{x\lambda}^{-1}\left\{ A_{x\lambda_1}\begin{bmatrix} \hat{x}_{i-1.j} \\ \hat{\lambda}_{i-1,j} \end{bmatrix} + A_{x\lambda_2}\begin{bmatrix} \hat{x}_{i,j-1} \\ \hat{\lambda}_{i,j-1} \end{bmatrix} + A_{x\lambda_3}\begin{bmatrix} \hat{x}_{i+1,j} \\ \hat{\lambda}_{i+1,j} \end{bmatrix} + A_{x\lambda_4}\begin{bmatrix} \hat{x}_{i,j+1} \\ \hat{\lambda}_{i,j+1} \end{bmatrix} + B_{x\lambda}y_{ij}\right\} \tag{6.3.8}$$

Now we consider the boundary conditions for the process to be estimated and the resulting boundary conditions for the NNM form of the estimator. The boundary condition for the process to be estimated is assumed to be in the form prescribed in Chapter 2, i.e. $v = Vx_b$, where x_b is determined from Green's identity. In what follows we present the expression for x_b which is determined in Appendix 6D and show that $v = Vx_b$ represents a restricted class of the general form of NNM boundary condition (6.2.5a), $v = V_x x + V_d d$. In addition to x_b we also require an expression for λ_b (which is also determined from Green's identity) in order to write the smoother boundary condition as specified in (2.5.25b) of Chapter 2. An expression for λ_b is also given below after a short discussion of the Dirichlet condition for the NNM smoother dynamics (6.3.8).

As discussed in Appendix 6A, for each process satisfying a NNM there exists a minimim dimension replacement for the process on the boundary ∂S. This replacement process plays the role of the Dirichlet condition and has been shown in Section 6.2 to be useful in writing 1-D representations for the 2-D NNM dynamics (see (6.2.28)). In Appendix 6D it is shown that the replacement process for $\{x, \lambda\}$ in (6.3.8) is given by the replacement processs d in (6.2.5b) for x and the process (see (6.D.18))

$$\xi = \begin{bmatrix} \xi_L \\ \xi_R \\ \xi_B \\ \xi_T \end{bmatrix} \tag{6.3.9}$$

which replaces λ on ∂S. Indeed, the elements of ξ are shown in (6.D.18) to be linear combinations of λ on ∂S. As we will see below, these replacement processes are also used in the definitions of x_b and λ_b which have been determined from Green's identity in Appendix 6D.

In order to show that the boundary condition for the process to be estimated, $v = Vx_b$, can be written in the form of (6.2.5a) and in order to write the boundary condition for the NNM smoother, we need the following definitions from (6.D.19a), (6.D.19b) and (6.D.21) of Appendix 6D:

$$D_L = \begin{bmatrix} d_L \\ x_{L+1} \end{bmatrix} \quad D_R = \begin{bmatrix} x_{R-1} \\ d_R \end{bmatrix} \quad D_B = \begin{bmatrix} d_B \\ x_{B+1} \end{bmatrix} \quad D_T = \begin{bmatrix} x_{T-1} \\ d_T \end{bmatrix} \tag{6.3.10a}$$

$$\Xi_L = \begin{bmatrix} \xi_L \\ \lambda_{L+1} \end{bmatrix} \quad \Xi_R = \begin{bmatrix} \lambda_{R-1} \\ \xi_R \end{bmatrix} \quad \Xi_B = \begin{bmatrix} \xi_B \\ \lambda_{B+1} \end{bmatrix} \quad \Xi_T = \begin{bmatrix} \lambda_{T-1} \\ \xi_T \end{bmatrix} \tag{6.3.10b}$$

and from the derivation of Green's identity in that appendix

$$x_b = \begin{bmatrix} D_L \\ D_R \\ D_B \\ D_T \end{bmatrix} \quad \text{and} \quad \lambda_b = \begin{bmatrix} \Xi_L \\ \Xi_R \\ \Xi_B \\ \Xi_T \end{bmatrix} \quad . \tag{6.3.10c}$$

As indicated in (6.D.13), the processes with subscripts L+1, R-1, B+1 and T-1 represent the values of x and λ on the four edges (left, right, bottom and top) contained within the lattice S (not ∂S). Thus, they are elements of x and λ on S. As discussed previously, the elements of d and ξ represent the replacement processes for x and λ respectively along the boundary ∂S. Given these definitions, it is clear that the boundary condition $v = Vx_b$ can be expressed as:

$$\begin{aligned} v &= Vx_b \\ &= V_x \begin{bmatrix} x_{L+1} \\ x_{R-1} \\ x_{B+1} \\ x_{T-1} \end{bmatrix} + V_d \begin{bmatrix} d_L \\ d_R \\ d_B \\ d_T \end{bmatrix} \end{aligned} \tag{6.3.11}$$

which is in the form of (6.2.5a). Note, however, that this form of boundary condition required for the process to be estimated is restricted to a linear combination of the process on ∂S and the process on the four interior edges of S. This can be contrasted with the more general form in (6.2.5a) which allowed the boundary condition to be specified in terms of x on ∂S and x throughout S. An extension to include the most general boundary condition (6.2.5a) should be considered in the future. This would be similar to the extension of our 1-D smoother results to include integral boundary conditions (see Appendix 3A).

Given that the boundary condition for x is defined as $v = Vx_b$ with $E[vv'] = \Pi_v$, the smoother boundary condition is written directly from (2.5.25b) as

$$0 = \left[V'\Pi_v^{-1}V \vdots E\right] \begin{bmatrix} \hat{x}_b \\ \hat{\lambda}_b \end{bmatrix} \tag{6.3.12}$$

where x_b and λ_b are defined in (6.3.10c) and E is the matrix in the boundary term of Green's Identity which is given explicitly in (6.D.25) of Appendix 6D. Next we investigate 1-D marching method representations of the 2-D discrete smoother.

1-D Marching Method Dynamics

It will be possible to transform the smoother dynamics from the NNM form (6.3.8) to the marching method form if the invertibility condition stated in Section 6.2.2 is met. This will occur when, for instance, A_1 and A_3 are both invertible, i.e. $A_{x\lambda_3}$ and $A_{x\lambda_1}$ are invertible (see (6.3.5a)). Assuming the invertibility of these two matrices the smoother dynamics can be written as

$$\begin{bmatrix} \hat{x}_{i+1,j} \\ \hat{\lambda}_{i+1,j} \end{bmatrix} = \tilde{A}_{x\lambda_3} \begin{bmatrix} \hat{x}_{i,j} \\ \hat{\lambda}_{i,j} \end{bmatrix} + \tilde{A}_{x\lambda_2} \begin{bmatrix} \hat{x}_{i,j-1} \\ \hat{\lambda}_{i,j-1} \end{bmatrix} + \tilde{A}_{x\lambda_1} \begin{bmatrix} \hat{x}_{i-1,j} \\ \hat{\lambda}_{i-1,j} \end{bmatrix} + \tilde{A}_{x\lambda_4} \begin{bmatrix} \hat{x}_{i,j+1} \\ \hat{\lambda}_{i,j+1} \end{bmatrix} + \tilde{B}_{x\lambda} y_{ij} \tag{6.3.13a}$$

where

$$\tilde{A}_{x\lambda_3} = A^{-1}_{x\lambda_3} \Delta_{x\lambda}, \quad \tilde{A}_{x\lambda_k} = -A^{-1}_{x\lambda_3} A_{x\lambda_k} \; ; \; k = 1,2,4 \quad \text{and} \quad \tilde{B}_{x\lambda} = -A^{-1}_{x\lambda_3} B_{x\lambda} \tag{6.3.13b}$$

Following the derivation of the 1-D model in (6.2.35a), it can be shown that the 1-D dynamics corresponding to (6.3.13a) can be expressed as

$$\begin{bmatrix} \hat{x}_{i+1} \\ \hat{\lambda}_{i+1} \\ \hline \hat{x}_{i} \\ \hat{\lambda}_{i} \end{bmatrix} = \left[\begin{array}{c|c} (I \otimes \tilde{A}_{x\lambda_3}) + (Z \otimes \tilde{A}_{x\lambda_4}) + (Z' \otimes \tilde{A}_{x\lambda_2}) & (I \otimes \tilde{A}_{x\lambda_1}) \\ \hline I & 0 \end{array} \right] \begin{bmatrix} \hat{x}_{i} \\ \hat{\lambda}_{i} \\ \hline \hat{x}_{i-1} \\ \hat{\lambda}_{i-1} \end{bmatrix}$$

$$+ \; G_{x\lambda_{TBi}} \begin{bmatrix} \hat{d}_T \\ \hat{\xi}_T \\ \hline \hat{d}_B \\ \hat{\xi}_B \end{bmatrix} + (I \otimes \tilde{B}_{x\lambda}) y_i \quad . \tag{6.3.13c}$$

If, in addition, A_2 and A_4 were equal, then $A_{x\lambda_2}$ and $A_{x\lambda_4}$ would be equal and the more efficient FFT-based decoupled marching method could be applied to decouple thse dynamics into lower dimensional subsystems.

Boundary Condition for the Marching Method Representation

The purpose of this subsection is to express the boundary condition for the NNM smoother in a form compatible with the 1-D marching method dynamics described in the previous subsection. From (6.3.12) we have

$$0 = V'\Pi_v^{-1} V\hat{x}_b + E\hat{\lambda}_b \quad . \tag{6.3.14}$$

As discussed in Section 6.2.2, when considering the marching method form, the boundary condition in (6.3.14) should be transformed to a multi-point form which we will express as

$$0 = \sum_{i=0}^{I} V^i_{x\lambda} \begin{bmatrix} \hat{x} \\ \hat{\lambda} \end{bmatrix}_i + V_{x\lambda_{TB}} \begin{bmatrix} \hat{d} \\ \hat{\xi} \end{bmatrix}_{TB} \tag{6.3.15a}$$

where

$$\begin{bmatrix} \hat{x} \\ \hat{\lambda} \end{bmatrix}_i \equiv \begin{bmatrix} \hat{x}_{i,1} \\ \hat{\lambda}_{i,1} \\ \hline \hat{x}_{i,2} \\ \hat{\lambda}_{i,2} \\ \hline \vdots \\ \hline \hat{x}_{i,J-1} \\ \hat{\lambda}_{i,J-1} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \hat{x} \\ \hat{\lambda} \end{bmatrix}_{TB} \equiv \begin{bmatrix} \hat{d}_T \\ \hat{\xi}_T \\ \hline \hat{d}_B \\ \hat{\xi}_B \end{bmatrix} \quad . \tag{6.3.15b}$$

The formulation of explicit expressions for $V^i_{x\lambda}$ and $V_{x\lambda_{TB}}$ is conceptually straightforward but requires a great deal of additional notation. Consequently, we have relegated the details to Appendix E. Combining this description for the boundary condition with the 1-D dynamics in (6.3.13c), gives a complete 1-D specification of the 2-D smoother as a multi-point 1-D discrete boundary value problem for which a forward/backward two-filter implementation is derived in Appendix 6B.

6.3.2 The Smoothing Error

The operator representation for the smoothing error is rewritten here from (2.5.36):

$$\begin{bmatrix} L & \vdots -BQB^* \\ - - - & \vdots - - \\ C^*R^{-1}C & \vdots L^\dagger \end{bmatrix} \begin{bmatrix} \tilde{x} \\ -\hat{\lambda} \end{bmatrix} = \begin{bmatrix} B & \vdots & 0 \\ - & \vdots & - - \\ 0 & \vdots & C^*R^{-1} \end{bmatrix} \begin{bmatrix} u \\ r \end{bmatrix} \qquad (6.3.16a)$$

with boundary condition from (2.5.33)

$$[V^*\Pi_v^{-1}v] = [V^*\Pi_v^{-1}V \vdots E] \begin{bmatrix} \tilde{x}_b \\ -\hat{\lambda}_b \end{bmatrix} \quad . \qquad (6.3.16b)$$

The 2-D difference equation which corresponds to the operator expression (6.3.16a) for the smoothing error dynamics is identical to that in (6.3.8) except for the input term:

$$\begin{bmatrix} \tilde{x}_{ij} \\ -\tilde{\lambda}_{ij} \end{bmatrix} = \Delta_{x\lambda}^{-1}\{ A_{x\lambda_1} \begin{bmatrix} \tilde{x}_{i-1,j} \\ -\tilde{\lambda}_{i-1,j} \end{bmatrix} + A_{x\lambda_2} \begin{bmatrix} \tilde{x}_{i,j-1} \\ -\tilde{\lambda}_{i,j-1} \end{bmatrix} + A_{x\lambda_3} \begin{bmatrix} \tilde{x}_{i+1,j} \\ -\tilde{\lambda}_{i+1,j} \end{bmatrix} + A_{x\lambda_4} \begin{bmatrix} \tilde{x}_{i,j+1} \\ -\tilde{\lambda}_{i,j+1} \end{bmatrix}$$

$$+ \begin{bmatrix} B & \vdots & 0 \\ - - & \vdots & - - - \\ 0 & \vdots & C'R^{-1} \end{bmatrix} \begin{bmatrix} u_{i,j} \\ r_{i,j} \end{bmatrix} \} \quad . \qquad (6.3.17)$$

Given the invertibility of both of either A_1 and A_3 or A_2 and A_4, one can readily obtain the 1-D marching method representation of the error dynamics such as those for the smoother in (6.3.13c).

Because of the similarity of the error boundary condition (6.3.16b) and the smoother boundary condition (6.3.16), we can write the 1-D form for (6.3.16b) directly from (6.3.15) as

$$V'\Pi_v^{-1}v = \sum_{i=0}^{I} V_{x\lambda}^i \begin{bmatrix} \tilde{x} \\ -\tilde{\lambda} \end{bmatrix}_i + V_{x\lambda_{TB}} \begin{bmatrix} \tilde{d} \\ -\tilde{\xi} \end{bmatrix}_{TB} \quad . \qquad (6.3.18)$$

Given the 1-D representation of the smoothing error in (6.3.17) and (6.3.18), one can compute the error variance using the matrix difference equations developed in Appendix 6B for computing the covariance of processes with such a 1-D represenation.

6.3.3 2-D Discrete Smoothing Examples

In this subsection we apply the results developed earlier in this chapter to formulate equations for implementing the smoother for two examples. In particular, we will concentrate on the development of 1-D dynamical representations of the smoothers. Once the transformations leading to these representations have been established, the same transformations must also be applied to rewrite the smoother boundary conditions in a 1-D multi-point form. As seen in Appendices 6C and 6E, formulating explicit expressions for these boundary conditions can be quite tedious and we will not display them explicitly here. The first example is a discrete 2-D process whose dynamics are given by the finite difference approximation of Poisson's equation (see Section 6.2.1). It is shown that the estimator for this problem can be written in the NNM form with both the invertibility and symmetry conditions satisfied. Therefore, the FFT decoupling can be applied to a 1-D marching method representation of the smoother dynamics. The other example is a 2-D discrete process whose dynamics are causal in one index and noncausal in the other. This process can be interpreted as obeying the finite difference approximation of the 1-D heat equation. We will find that because the invertibility condition is not satisfied for the NNM representation of the smoother dynamics. In this case, no 1-D recursive marching method representation can be obtained. However, we will see that if we first manipulate the smoother dynamics into a new form. In particular, rather than a 1-D recursive marching method form, the smoother dynamics are put into a 1-D scattering form (see Chapter 5) of high (vector) dimension. It is shown that the structure of this scattering form is such that we can apply the FFT-based decoupling transformation to obtain a decoupled system of 1-D dynamics each in scattering form. Then, each of these low order scattering form models can be split into stable forward and backward models by the same method used to diagonalize the scattering form dynamics of the discrete 1-D smoother in Chapter 5.

Example 1: Discrete Poisson's Equation

The dynamics of the process to be estimated are given by

$$x_{ij} = \frac{1}{4}\left[x_{i-1,j} + x_{i,j-1} + x_{i+1,j} + x_{i,j+1}\right] + u_{ij} \tag{6.3.19}$$

where u is a unit variance white noise (Q = I). The boundary condition is the Dirichlet form described in (6.3.10b). The observations are simply the process itself plus an additive noise r of unit variance:

$$y_{ij} = x_{ij} + r_{ij} \quad ; \text{ on } S_{IJ} \quad . \tag{6.3.20}$$

Therefore, for this problem each of the A_k, B, C, Q and R are scalars, and in particular,

$$A_1 = A_2 = A_3 = A_4 = 1/4 \tag{6.3.21a}$$

and

$$B = C = Q = R = 1 \quad . \tag{6.3.21b}$$

From (6.3.8), the NNM form of the estimator dynamics are defined by way of:

$$\Delta_{x\lambda} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \qquad \Delta_{x\lambda}^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}, \quad B_{x\lambda} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{6.3.22a}$$

and

$$A_{x\lambda_k} = \begin{bmatrix} 1/4 & 0 \\ 0 & 1/4 \end{bmatrix} \quad ; \quad k = 1,2,3,4 \quad . \tag{6.3.22b}$$

Since each of the $A_{x\lambda}$ is invertible, we can invert any one of them to obtain smoother dynamics which are causal in one index, leading to a marching method form. As we had done earlier, we will invert $A_{x\lambda_3}$. This choice results in (see (6.3.13b))

$$\tilde{A}_{x\lambda_3} = A_{x\lambda_3}^{-1}\Delta_{x\lambda} = \begin{bmatrix} 4 & -4 \\ 4 & 4 \end{bmatrix} \tag{6.3.23a}$$

$$\tilde{A}_{x\lambda_k} = -A_{x\lambda_3}^{-1}A_{x\lambda_k} = -I \quad ; \; k = 1,2,4 \tag{6.3.23b}$$

and

$$\tilde{B}_{x\lambda} = -A_{x\lambda_3}^{-1} B_{x\lambda} = \begin{bmatrix} 0 \\ -4 \end{bmatrix} \quad . \tag{6.3.23c}$$

These values are substituted into (6.3.13c) to obtain the 1-D marching method dynamics.

Since the symmetry condition is also met for these dynamics, i.e.

$$\tilde{A}_{x\lambda_2} = \tilde{A}_{x\lambda_4} \ , \tag{6.3.24}$$

we can apply the FFT-based decoupling transformation ($D \times I$) to obtain dynamically decoupled lower-dimensional systems. First define the transformed process:

$$\begin{bmatrix} q_{i+1} \\ q_i \end{bmatrix} = \left[\begin{array}{c:c} (D \otimes I) & 0 \\ \hdashline 0 & (D \otimes I) \end{array}\right] \begin{bmatrix} \hat{x}_{i+1} \\ \hat{\lambda}_{i+1} \\ \hat{x}_i \\ \hat{\lambda}_i \end{bmatrix} \tag{6.3.25}$$

and matrices

$$M_{x\lambda_4} = \begin{bmatrix} 0 & & & & \\ \vdots & & 0 & & \\ 0 & & & & \\ -I & 0 & \cdot & \cdot & 0 \end{bmatrix} \quad \text{and} \quad M_{x\lambda_2} = \begin{bmatrix} -I & 0 & \cdot & \cdot & 0 \\ 0 & & & & \\ \vdots & & 0 & & \\ 0 & & & & \end{bmatrix} , \tag{6.3.26}$$

then the dynamics of the decoupled systems are given by (see (6.2.52))

$$\begin{bmatrix} q_{i+1,j} \\ \hdashline q_{i,j} \end{bmatrix} = \left[\begin{array}{c:c:c} 4 - 2\cos(j\pi/J) & -4 & \\ \hdashline 4 & 4 - 2\cos(j\pi/J) & -I \\ \hdashline & I & 0 \end{array}\right] \begin{bmatrix} q_{i,j} \\ \hdashline q_{i-1,j} \end{bmatrix} + \begin{bmatrix} U_{x\lambda_{i,j}} \\ \hdashline 0 \end{bmatrix}$$

$$; \quad j = 1,2,\ldots J-1 \tag{6.3.27a}$$

where the input is given by the transformed process (see (6.2.50b))

$$\begin{bmatrix} U_{x\lambda_{i,1}} \\ U_{x\lambda_{i,2}} \\ \vdots \\ U_{x\lambda_{i,J-1}} \end{bmatrix} = U_{x\lambda_i} = (D \otimes I)\left\{ (I \otimes \tilde{B}_{x\lambda})y_i + M_{x\lambda_4} R^{i-1} \begin{bmatrix} \hat{x}_T \\ \hat{\lambda}_T \end{bmatrix} + M_{x\lambda_2} R^{i-1} \begin{bmatrix} \hat{x}_B \\ \hat{\lambda}_B \end{bmatrix} \right\}. \tag{6.3.27b}$$

Equivalent forward/backward stable representations can be found for each value of j by first finding the similarity transformation which diagonalizes the dynamics matrix in (6.3.27a) for each value of j. The process obtained by applying that transformation for each value of j to $[q'_{i+1,j}, q'_{i,j}]'$ is then comprised of four dynamically decoupled scalar processes. These four processes would then be split into forward stable and backward stable groups, yielding a forward/backward representation of the type in (6.2.39a) and (6.2.39b).

The multi-point boundary condition for the 1-D form of the smoother dynamics defined by (6.3.23) is formulated by following the developements in Appendix 6E. The boundary condition for the system of FFT-decoupled processes in (6.3.27a) and the subsequent forward/backward representation are obtained from this 1-D multi-point boundary condition by the procedure discussed in Section 2 of Appendix 6C. Finally, given that the boundary condition and dynamics are in the 1-D forward/backward form, the estimates can be computed by implementing the general solution derived in Appendix 6B and applying the inverse of each of the above mentioned transformations.

Example 2: A Mixed Causal/Noncausal Process

Consider a scalar 2-D discrete process x governed by the 2-D difference equation

$$x_{i+1,j} = ax_{i,j} + b\left[x_{i,j-1} + x_{i,j+1}\right] + u_{i,j} \qquad (i,j) \in S. \tag{6.3.28a}$$

These dynamics are causal in the index i and noncausal in j. The input process u is a discrete 2-D white process with support S and variance Q. The boundary condition for this process will be given by

$$v = \begin{bmatrix} x_1 \\ x_T \\ x_B \end{bmatrix} \quad . \tag{6.3.28b}$$

Here x_1 is the vector representation (6.2.10b) for the process x at i=1 and x_T and x_B are the values of the process at the top and bottom boundaries of S as defined in (6.2.3).

To see why we have included x_1 and not x_0 in the boundary condition (6.3.28b), consider the dynamics in (6.3.28a) for i=1, the smallest value of i in S. To compute x_{i+1} for i=1, we require knowledge of x_1 and not x_0. Finally, we note that since $x_1 = x_{L-1} \in x_b$ (see (6.3.10)), this boundary condition is of the form $v = Vx_b$ which is required for an application of our estimator.

A physical interpretation of the dynamics and boundary condition described above is that of a finite difference approximation of the 1-D heat equation for a homogeneous rod:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial \ell^2} + f \quad . \tag{6.3.29}$$

Upper case T represents temperature; lower case t, time; ℓ a spatial variable and f a forcing function. This forcing function could represent spatial variations of the thermal properties of the rod from nominal and temporal variations in external temperature.

The observations of the process x are given by

$$y_{ij} = Cx_{ij} + r_{ij} \qquad \text{on } S \tag{6.3.30}$$

where r is a white observation noise with covariance R. The estimator for x given y on S could be formulated by deriving Green's identity for the dynamics in (6.3.28a) (which is not in the NNM form) and applying the operator solution developed in Chapter 2. Alternatively, we could rewrite the dynamics in NNM form and apply the machinery developed in this chapter. We will pursue this second approach here.

We will consider two forms of nearest neighbor model for (6.3.28a). The first, the simplest of the two, requires that the coefficient a in (6.3.28a) be nonzero. The other is valid for the more general case of arbitrary values for **a**. Since the second model is more widely applicable, we will merely present the first model and will develop the smoother for the second. The first model is obtained by dividing by a to get

$$x_{ij} = -a^{-1}bx_{i,j-1} + a^{-1}x_{i+1,j} - a^{-1}bx_{i+1,j} - a^{-1}u_{ij} \quad . \tag{6.3.31a}$$

In this case

$$A_1 = 0, \quad A_2 = A_4 = -a^{-1}b, \quad A_3 = -a^{-1} \quad \text{and} \quad B = -a^{-1} \quad . \tag{6.3.31b}$$

The second model requires an increase in dimension (see Section 6.2.1). Specifically, define the 2×1 process X_{ij} as[1]

$$X_{ij} = \begin{bmatrix} x_{i+1,j} \\ x_{i,j} \end{bmatrix} \quad . \tag{6.3.32a}$$

It is straightforward to show that the following define a NNM model for X

$$A_1 = \begin{bmatrix} a & 0 \\ 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix},$$

and

$$B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad . \tag{6.3.32b}$$

In terms of the new process X, the observation is

$$y_{ij} = [0 \; : \; C]X_{ij} + r_{ij} \quad . \tag{6.3.32c}$$

If we denote the 2×1 adjoint process by Λ_{ij}, then it can be shown from (6.3.5) that the smoother dynamics for the model in (6.3.32) are given by

$$\left[\begin{array}{cc:cc} 1 & 0 & -Q & 0 \\ 0 & 1 & 0 & 0 \\ \hdashline 0 & 0 & 1 & 0 \\ 0 & C^2R^{-1} & 0 & 1 \end{array}\right] \begin{bmatrix} \hat{X}_{ij} \\ \hat{\Lambda}_{ij} \end{bmatrix} = \left[\begin{array}{cc:cc} a & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} \hat{X}_{i-1,j} \\ \hat{\Lambda}_{i-1,j} \end{bmatrix} + \left[\begin{array}{cc:cc} 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 \end{array}\right] \begin{bmatrix} \hat{X}_{i,j-1} \\ \hat{\Lambda}_{i,j-1} \end{bmatrix}$$
$$+ \left[\begin{array}{cc:cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & a & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} \hat{X}_{i+1,j} \\ \hat{\Lambda}_{i+1,j} \end{bmatrix} + \left[\begin{array}{cc:cc} 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 \end{array}\right] \begin{bmatrix} \hat{X}_{i,j+1} \\ \hat{\Lambda}_{i,j+1} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ CR^{-1} \end{bmatrix} y_{ij} \tag{6.3.33}$$

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[1] The use of X and Λ in this example should not be confused with their use elsewhere in the thesis.

The first thing that we notice about these dynamics is that none of the coefficient matrices that we have previously referred to as the $A_{x\lambda}$ is invertible. Thus, no 1-D dynamics in the recursive form of a marching method can be obtained. However, as we demonstrate below, the 2-D dynamics can be manipulated into a 1-D scattering form which can be decoupled into a system of lower-dimensional processes which are also in 1-D scattering form. First denote the partitions of Λ by

$$\hat{\Lambda}_{ij} = \begin{bmatrix} \hat{\lambda}_{ij} \\ \hat{\gamma}_{ij} \end{bmatrix} \tag{6.3.34}$$

and note that the first, third and fourth rows of (6.3.33) can be written as (with some shifting of indices)

<u>first:</u>
$$\hat{x}_{i+1,j} = Q\hat{\lambda}_{ij} + a\hat{x}_{ij} + b\hat{x}_{i,j-1} + b\hat{x}_{i,j+1} \tag{6.3.35a}$$

<u>third:</u>
$$\hat{\gamma}_{ij} = \hat{\lambda}_{i-1,j} - a\hat{\lambda}_{ij} \tag{6.3.35b}$$

and

<u>fourth:</u>
$$\hat{\gamma}_{ij} = -C^2R^{-1}\hat{x}_{ij} + b\hat{\lambda}_{i,j-1} + b\hat{\lambda}_{i,j+1} + CR^{-1}y_{ij} \quad . \tag{6.3.35c}$$

Substituting the expression for the third row into the fourth to eliminate γ, we have

$$\hat{\lambda}_{i-1,j} = a\hat{\lambda}_{ij} + b\hat{\lambda}_{i,j-1} + b\hat{\lambda}_{i,j+1} - C^2R^{-1}\hat{x}_{ij} + CR^{-1}y_{ij} \quad . \tag{6.3.36}$$

Combining (6.3.36) with (6.3.35a) gives the smoother dynamics in a form which is reminiscent of the scattering form of the 1-D discrete smoother studied in Chapter 5:

$$\begin{bmatrix} \hat{x}_{i+1,j} \\ \hat{\lambda}_{i-1,j} \end{bmatrix} = \begin{bmatrix} a & Q \\ -C^2R^{-1} & a \end{bmatrix} \begin{bmatrix} \hat{x}_{ij} \\ \hat{\lambda}_{ij} \end{bmatrix} + \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} \hat{x}_{i,j-1} \\ \hat{\lambda}_{i,j-1} \end{bmatrix} + \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} \hat{x}_{i,j+1} \\ \hat{\lambda}_{i,j+1} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 \\ CR^{-1} \end{bmatrix} y_{ij} \quad . \tag{6.3.37}$$

In fact, the dynamics in (6.3.37) do result in a 1-D scattering form when written in terms of the stacked vector processes x_i and λ_i (except for a shift in the index of λ):

$$\begin{bmatrix} \hat{x}_{i+1} \\ \hat{\lambda}_{i-1} \end{bmatrix} = \begin{bmatrix} \text{TRIDIAG}\{b,a,b\} & \text{DIAG}\{Q\} \\ \text{DIAG}\{-C^2R^{-1}\} & \text{TRIDIAG}\{b,a,b\} \end{bmatrix} \begin{bmatrix} \hat{x}_i \\ \hat{\lambda}_i \end{bmatrix} + MR^{i-1} \begin{bmatrix} \hat{x} \\ \hat{\lambda} \end{bmatrix}_{TB} + \begin{bmatrix} 0 \\ \text{DIAG}\{CR^{-1}\} \end{bmatrix} y_i$$

where, as usual, M is a matrix which accounts for the contribution of the process on the boundary of S, TRIDIAG$\{b,a,b\}$ is the tridiagonal matrix with a on the diagonal and b on the upper and lower off-diagonals. We remark that this form for the dynamics could have been obtained more directly from the first form of the nearest neighbor model (6.3.31a). However, as we have seen, when using that model an intermediate step requires the inversion of the parameter a which may be zero for some models.

Finally, we show that the FFT-based decoupling transformation can be applied to these dynamics as well. In particular, define

$$q_{1,i} = (D \otimes I)\hat{x}_i \tag{6.3.38a}$$

$$q_{2,i} = (D \otimes I)\hat{\lambda}_{i-1} \tag{6.3.38b}$$

and

$$U_i = \begin{bmatrix} (D \otimes I) & 0 \\ 0 & (D \otimes I) \end{bmatrix} MR^{i-1} \begin{bmatrix} \hat{x} \\ \hat{\lambda} \end{bmatrix}_{TB} + \begin{bmatrix} 0 \\ (D \otimes I)\text{DIAG}\{CR^{-1}\} \end{bmatrix} y_i \tag{6.3.38c}$$

where the index in (6.3.38b) has been shifted so that the dynamic model below is in a form identical to the 1-D scattering dynamics studied in Chapter 5. Recalling that $(D \times I)$ diagonalizes a symmetric tridiagonal matrix (see (6.2.43)), we can write the transformed smoother dynamics as

$$\begin{bmatrix} q_{1,i+1} \\ q_{2,i} \end{bmatrix} = \begin{bmatrix} \text{DIAG}\{a + b\lambda_j\} & \text{DIAG}\{Q\} \\ \text{DIAG}\{-C^2R^{-1}\} & \text{DIAG}\{a + b\lambda_j\} \end{bmatrix} \begin{bmatrix} q_{1,i} \\ q_{2,i+1} \end{bmatrix} + U_i \tag{6.3.38d}$$

where the λ_j are the eigenvalues given in (6.2.45d).

Since each of the partitions of the dynamics matrix in (6.3.38d) is diagonal, we have J-1 decoupled processes with the following scattering form dynamics:

$$\begin{bmatrix} q_{1_{i+1,j}} \\ q_{2_{i,j}} \end{bmatrix} = \left[\begin{array}{c:c} a + b\rho_j & Q \\ \hdashline -C^2R^{-1} & a + b\rho_j \end{array}\right] \begin{bmatrix} q_{1_{i,j}} \\ q_{2_{i+1,j}} \end{bmatrix} + U_{i,j} \; ; \; j = 1,2 \ldots J-1. \quad (6.3.39)$$

In Chapter 5 we derived a method for transforming dynamics in this scattering form into stable forward and backward components. The final step is to rewrite the estimator boundary conditions given by (6.3.24) as boundary conditions for the transformed forward/backward processes. The same methodology discussed in Appendices 6C and 6E can be followed in rewriting the boundary condition. The estimates are obtained by solving for the forward/backward processes as described in Appendix 6B and then successively transforming back to the low-dimensional scattering variables, to the high-dimensional scattering variables (via the FFT) and finally to the original processes $\hat{x}$ and $\hat{\lambda}$.

APPENDIX 6A

THE DIRICHLET CONDITION FOR SINGULAR A_k

In this appendix we show that if any of the $n \times n$ coefficient matrices A_k in the 2-D dynamics (6.2.1b) is singular, then the minimum dimension of the boundary condition required for a well-posed problem is less than that of the classical Dirichlet boundary condition defined by knowledge of the processes x_L, x_R, x_B and x_T in (6.2.3). That is, we will show that it is not necessary to know the entire process x_{ij} on the lattice boundary ∂S. Below we determine the minimum dimension boundary value and redefine the Dirichlet condition in terms of it.

Denote the ranks of each of the A_k as

$$n_k = \text{rank}(A_k) \quad ; \; k = 1,2,3,4 \quad . \tag{6.A.1}$$

First consider the case for which A_1 is singular, i.e. $n_1 < n$. From (6.2.1b), the values of the process for $i = 1$, $j = 1,2,\ldots J-1$ are given by

$$x_{1,j} = A_1 x_{0,j} + A_2 x_{1,j-1} + A_3 x_{2,j} + A_4 x_{1,j+1} + B\, u_{1,j} \quad . \tag{6.A.2a}$$

With A_1 singular, there exists an invertible $n \times n$ matrix, call it ϕ_1 whose inverse compresses A_1 into a full rank $n \times n_1$ matrix F_1 as

$$[F_1 : 0] = A_1 \phi_1^{-1} \quad . \tag{6.A.2b}$$

(Of course, ϕ_1 and the corresponding F_1 are not unique.) Employing ϕ_1, $x_{0,j}$ can be transformed for $j = 1,2,\ldots J-1$ (i.e., along the left edge of ∂S as depicted in Figure 6.2.2) as

$$\tilde{d}_{0,j} = \phi_1 x_{0,j} \tag{6.A.2c}$$

and (6.A.2a) can be written in terms of this transformed process as

$$\begin{aligned} x_{1,j} &= A_1 \phi_1^{-1} \tilde{d}_{0,j} + A_2 x_{1,j-1} + A_3 x_{2,j} + A_4 x_{1,j+1} + B\, u_{1,j} \\ &= [F_1 : 0] \tilde{d}_{0,j} + A_2 x_{1,j-1} + A_3 x_{2,j} + A_4 x_{1,j+1} + B\, u_{1,j} \end{aligned} \tag{6.A.2d}$$

If we define $d_{0,j}$ as the first n_1 components of $\tilde{d}_{0,j}$ and substitute from (6.A.26), then (6.A.2d) can be expressed as

$$x_{1,j} = F_1 d_{0,j} + A_2 x_{1,j-1} + A_3 x_{2,j} + A_4 x_{1,j+1} + B\, u_{1,j} \quad . \tag{6.A.3}$$

Recalling the definition of x_L in (6.2.2) and employing the Kronecker matrix product [51], it can be shown that $d_{0,j}$ for $j = 1,2,..J-1$ can be expressed as a linear combination of x on the left edge of ∂S:

$$d_L = \left[I_{J-1} \otimes [I_{n_1} \vdots\ 0]\right] (I \otimes \phi_1) x_L \tag{6.A.4a}$$

where, for instance, I_{J-1} is the $J-1 \times J-1$ identity and d_L is given by the $(J-1)n_1 \times 1$ vector with $n_1 \times 1$ partitions

$$d_L = \begin{bmatrix} d_{0,1} \\ d_{0,2} \\ \vdots \\ d_{0,J-1} \end{bmatrix} \quad . \tag{6.A.4b}$$

This clearly demonstrates that we only need to specify the $(J-1)n_1$ values of d_L as the portion of the Dirichlet condition for the left edge of ∂S. A similar argument holds for the remaining three edges of ∂S so that in the same way that we defined d_L we can also define

$$d_B = \left[I_{I-1} \otimes [I_{n_2} \vdots\ 0]\right] (I \otimes \phi_2) x_B \tag{6.A.5a}$$

$$d_R = \left[I_{J-1} \otimes [I_{n_3} \vdots\ 0]\right] (I \otimes \phi_3) x_R \tag{6.A.5b}$$

and

$$d_T = \left[I_{I-1} \otimes [I_{n_4} \vdots\ 0]\right] (I \otimes \phi_4) x_T \quad . \tag{6.A.5c}$$

where the ϕ_k are chosen as in (6.A.2b), i.e.

$$\left[F_k \vdots\ 0\right] = A_k \phi_k^{-1} \quad . \tag{6.A.5d}$$

Thus, to completely specify the $n \times 1$ process x on the lattice S, we can redefine the Dirichlet boundary condition as

$$v_d = \begin{bmatrix} d_L \\ d_R \\ d_B \\ d_T \end{bmatrix} \tag{6.A.6}$$

and the total dimension of the boundary value v is

$$n_v = \dim(v) = (J - 1)(n_1 + n_3) + (I - 1)(n_2 + n_4) \quad . \tag{6.A.7}$$

Note that in the case where one of the A_k is full rank n, we will choose ϕ_k to be the identity so that $F_k = A_k$ and the corresponding edge process d_* is is precisely equal to its counterpart in (6.2.2). Also when $A_k = 0$, then $F_k = 0$, and the corresponding d_* does not appear in the boundary condition (6.A.6).

Throughout Chapter 6 we make use of some relations which follow directly from the developments above. First define

$$a_k = (I \otimes A_k) \tag{6.A.8a}$$

and

$$f_k = (I \otimes F_k) \qquad ; k = 1,2,3,4 \quad . \tag{6.A.8b}$$

By substituting, for instance, for d_L from (6.A.4a) and using the relation between A_1, ϕ_1 and F_1 in (6.A.2b), it can be shown that x_L and d_L are related by:

$$a_1 x_L = f_1 d_L \quad . \tag{6.A.9a}$$

A similar argument establishes:

$$a_2 x_B = f_2 d_B \quad , \tag{6.A.9b}$$

$$a_3 x_R = f_3 d_R \tag{6.A.9c}$$

and

$$a_4 x_T = f_4 d_T \quad . \tag{6.A.9d}$$

APPENDIX 6B

THE GENERAL SOLUTION OF THE FORWARD/BACKWARD 1-D MULTI-POINT BOUNDARY VALUE REPRESENTATION OF THE NNM

6.B.1 The General Solution

Consider a process governed by the NNM dynamics in (6.2.1b) which have been transformed into the forward/backward decoupled 1-D dynamics in (6.2.39a) and (6.2.39b). For convenience we rewrite those equations here:

$$x_{f,i} = A_f x_{f,i-1} + B_f u_i + M_{fi} d_{TB} \tag{6.B.1a}$$

and

$$x_{b,i-1} = A_b^{-1} x_{b,i} - A_b^{-1} B_b u_i - A_b^{-1} M_{bi} d_{TB} \quad . \tag{6.B.1b}$$

Recall that the process x_f is forward stable (stable for increasing i), that the process x_b is backward stable (stable for decreasing i) and that the original process x is related to x_f and x_b by a known dynamical decoupling transformation T (see (6.2.37) and (6.2.38):

$$T\begin{bmatrix} x_{i+1} \\ x_i \end{bmatrix} = \begin{bmatrix} x_{f,i} \\ x_{b,i} \end{bmatrix} \tag{6.B.2}$$

where x_i is the representation of x given in (6.2.10a). In this section of the appendix, we derive the general solution for the process with dynamics given by (6.B.1) and with a multi-point boundary condition of the form (see (6.2.39c)):

$$v = \sum_{i=0}^{I} [\, V_f^i \; \vdots \; V_b^i] \begin{bmatrix} x_{f,i} \\ x_{b,i} \end{bmatrix} + V_{TB} d_{TB} \quad . \tag{6.B.3}$$

To derive the general solution for (6.B.1) given the boundary condition (6.B.3), let Φ_f and Φ_b be the transition matrices for A_f and A_b^{-1} respectively and consider the processes

$$x_{f,n}^0 = \sum_{i=0}^{n} \Phi_f(n,i) B_f u_i \tag{6.B.4a}$$

and

$$x^0_{b,n} = \sum_{i=I}^{n} \Phi_b(n,i)A_b^{-1}B_b u_i \quad . \tag{6.B.4b}$$

Specifically, these two processes are the solutions of (6.B.1a) with a zero initial value and (6.B.1b) with a zero final value, respectively and with $d_{TB} = 0$. Next, define the following matrices which account for the contribution of a nonzero d_{TB}:

$$G_{f,n} = \sum_{i=0}^{n} \Phi_f(n,i)M_{fi} \tag{6.B.5a}$$

and

$$G_{b,i} = \sum_{i=I}^{n} \Phi_b(n,i)A_b^{-1}M_{bi} \quad . \tag{6.B.5b}$$

If d_{TB}, $x_{f,0}$ and $x_{b,I}$ were known, we could write the solutions of (6.B.1a) and (6.B.1b) as

$$x_{f,n} = \Phi_f(n,0)x_{f,0} + x^0_{f,n} + G_{f,n}d_{TB} \tag{6.B.6a}$$

and

$$x_{b,n} = \Phi_b(n,I)x_{b,I} + x^0_{b,n} + G_{b,n}d_{TB} \quad . \tag{6.B.6b}$$

However, these values will not be known unless we have a very special boundary condition (6.B.3)). To determine them, substitute (6.B.6a) and (6.B.6b) into the boundary condition (6.B.3) to get

$$v = \sum_{i=0}^{I} \left[V_f^i\Phi_f(i,0) \vdots V_b^i\Phi_b(i,I)\right] \begin{bmatrix} x_{f,0} \\ x_{b,I} \end{bmatrix} + \sum_{i=0}^{I} \left[V_f^i x^0_{f,i} + V_b^i x^0_{b,i}\right]$$

$$+ \left(V_{TB} + \sum_{i=0}^{I} \left[V_f^i G_{f,i} + V_b^i G_{b,i}\right]\right) d_{TB} \quad . \tag{6.B.7}$$

Define the matrix F_{fb} as

$$F_{fb} = \left(\sum_{i=0}^{I} \left[V_f^i\Phi_f(i,0) \vdots V_b^i\Phi_b(i,I)\right] \;\vdots\; V_{TB} + \sum_{i=0}^{I} \left[V_f^i \vdots V_b^i\right] \begin{vmatrix} G_{f,i} \\ G_{b,i} \end{vmatrix} \right) \quad . \tag{6.B.8}$$

When the problem is well-posed, F_{fb} will be invertible and we can solve for the initial and final conditions $x_{f,0}$ and $x_{b,I}$ and the top and bottom edge term d_{TB} as

$$\begin{bmatrix} x_{f,0} \\ x_{b,I} \\ d_{TB} \end{bmatrix} = F_{fb}^{-1}\left\{ v - \sum_{i=0}^{I} \left[V_f^i \vdots V_b^i \right] \begin{bmatrix} 0 \\ x_{f,i} \\ 0 \\ x_{b,i} \end{bmatrix} \right\} \quad . \tag{6.B.9}$$

Combining (6.B.9) and (6.B.6) and defining

$$\Phi(n) = \begin{bmatrix} \Phi_f(n,0) & \vdots & 0 & \vdots & G_{f,n} \\ \hline 0 & \vdots & \Phi_b(n,I) & \vdots & G_{b,n} \\ \hline 0 & \vdots & 0 & \vdots & I \end{bmatrix} \quad , \tag{6.B.10}$$

we can write the general solution as

$$\begin{bmatrix} x_{f,n} \\ x_{b,n} \\ d_{TB} \end{bmatrix} = \Phi(n) F_{fb}^{-1}\left\{ v - \sum_{i=0}^{I} \left[V_f^i \vdots V_b^i \right] \begin{bmatrix} 0 \\ x_{f,i} \\ 0 \\ x_{b,i} \end{bmatrix} \right\} + \begin{bmatrix} x_{f,n}^0 \\ x_{b,n}^0 \\ 0 \end{bmatrix} \quad . \tag{6.B.11}$$

The original process x is recovered from the computed values for x_f and x_b by inverting the transformation in (6.B.2).

6.B.2 Process Variance

In this section we formulate expressions for computing the variance of the process for which we derived a general solution in the previous section. It is assumed that the input process is a 2-D discrete white noise:

$$E\{ u_{ij} u_{k\ell}' \} = Q_{ij} \delta_{ik} \delta_{j\ell} \quad . \tag{6.B.12a}$$

In the stacked vector representation of u given in (A.2a), we will write:

$$E\{ u_i u_k' \} = Q_i \delta_{ik} \tag{6.B.12b}$$

where

$$Q_i = \begin{bmatrix} Q_{i,1} & & & 0 \\ & Q_{i,2} & & \\ & & \ddots & \\ 0 & & & Q_{i,J-1} \end{bmatrix} \quad . \tag{6.B.12c}$$

The boundary vector v is assumed orthogonal to the input u with a given covariance

$$Evv' = \Pi_v \quad . \tag{6.B.13}$$

Given this orthogonality assumption, one can see from (6.B.11) that the variance of $\{x_{f,n}, x_{b,n}, d_{TB}\}$ can be formulated in terms of Π_v, the covariance of v, and the following three covariances (see (6.B.19)):

(1) $$P_f^0(n,k) \equiv E\{x_{f,n}^0 \; x_{f,k}^{0'}\} \quad , \tag{6.B.14a}$$

(2) $$P_b^0(n,k) \equiv E\{x_{b,n}^0 \; x_{b,k}^{0'}\} \tag{6.B.14b}$$

and

(3) $$P_{fb}^0(n,k) \equiv E\{x_{f,n}^0 \; x_{b,k}^{0'}\} \quad . \tag{6.B.14c}$$

Using the expressions for x_f^0 and x_b^0 in (6.B.4a) and (6.B.4b), difference equations are specified for each of these three covariances as follows. In each case it is assumed that n > k.

(1):

$$P_f^0(n,k) = \Phi_f(n,k)\Big\{ \sum_{i=0}^{k} \Phi_f(k,i)B_f Q_i B_f' \Phi_f'(k,i) \Big\}$$

$$= \Phi_f(n,k)\Pi_{f,k}^0 \tag{6.B.15a}$$

Where

$$\Pi_{f,k+1}^0 = \Phi_f(k+1,k)\Pi_{f,k}^0 \Phi_f'(k+1,k) + B_f Q_k B_f' \quad ; \; \Pi_{f,0}^0 = 0 \tag{6.B.15b}$$

$$= P_f^0(k,k)$$

For the case when n < k, an expression for P_f^0 can be obtained directly from the relationship

$$P_f^0(k,n) = P_f^{0'}(n,k) \quad . \tag{6.B.15c}$$

(2):

$$P_b^0(n,k) = \left\{ \sum_{i=I}^{n} \Phi_b(n,i) B_b Q_i B_b' \Phi_b'(n,i) \right\} \Phi_b'(k,n)$$

$$= \Pi_{b,n}^0 \Phi_b'(k,n) \quad . \qquad (6.B.16a)$$

Where

$$\Pi_{b,k-1}^0 = \Phi_b(k-1,k) \Pi_{b,k}^0 \Phi_b'(k-1,k) + B_b Q_k B_b' \quad ; \Pi_{b,I}^0 = 0 \qquad (6.B.16b)$$

$$= P_b^0(k-1,k-1) \quad .$$

Again, when $n < k$, we can use the relationship

$$P_b^0(k,n) = P_b^{0'}(n,k) \quad . \qquad (6.B.16c)$$

(3):

$$P_{fb}^0(n,k) = \left\{ \sum_{i=k}^{n} \Phi_f(n,i) B_f Q_i B_b' \Phi_b'(n,i) \right\} \Phi_b'(k,n) \qquad (6.B.17a)$$

If we define

$$\Pi_{fb,k+1}^0 = \Phi_f(k+1,k) \Pi_{fb,k}^0 \Phi_b'(k+1,k) + B_f Q_k B_b' \quad ; \Pi_{fb,0}^0 = 0 \qquad (6.B.17b)$$

and note that

$$\Pi_{fb,k}^0 = \sum_{i=0}^{k} \Phi_f(k,i) B_f Q_i B_b' \Phi_b'(k,i) \quad , \qquad (6.B.17c)$$

then by direct substitution it can be shown that

$$P_{fb}^0(n,k) = \Pi_{fb,n}^0 \Phi_b'(k,n) - \Phi_f(n,k) \Pi_{fb,k}^0 \qquad (6.B.18c)$$

Also, it is straightforward to show that

$$P_{bf}^0(n,k) = E\{ x_b^0(k) x_f^{0'}(n) \} = P_{fb}^{0'}(k,n) \qquad (6.B.18d)$$

and that when $n < k$

$$P^0_{fb}(n,k) = 0 \quad . \tag{6.B.18e}$$

Given these covariances we can write an expression for the variance of the process whose general solution is given by (6.B.11). First define

$$\Xi(n) = -\Phi(n)F^{-1}_{fb}\Big\{ \sum_{i=0}^{I} \left[v^i_f \vdots v^i_b \right] \left[\begin{array}{cc:c} P^0_f(i,n) & P^0_{fb}(i,n) & 0 \\ \hdashline P^{0'}_{fb}(n,i) & P^0_b(i,n) & 0 \end{array}\right] \Big\} \tag{6.B.19a}$$

and

$$\Psi(n) = \Phi(n)F^{-1}_{fb}\Big\{ \sum_{i=0}^{I} \sum_{j=0}^{I} \left[v^i_f \vdots v^i_b \right] \left[\begin{array}{c:c} P^0_f(i,j) & P^0_{fb}(i,j) \\ \hdashline P^{0'}_{fb}(j,i) & P^0_b(i,j) \end{array}\right] \left[\begin{array}{c} v^{i'}_f \\ \hdashline v^{i'}_b \end{array}\right] \Big\} F^{-1'}_{fb}\Phi(n)' \quad . \tag{6.B.19b}$$

then the covariance

$$P(n) = E\Big\{ \begin{bmatrix} x_{f,n} \\ x_{b,n} \\ d_{TB} \end{bmatrix} \left[x'_{f,n} : x'_{b,n} : d'_{TB} \right] \Big\}$$

can be expressed as

$$P(n) = \Phi(n)F^{-1}_{fb}\Pi_v F^{-1'}_{fb}\Phi'(n) + \Xi(n) + \Xi'(n) + \Psi(n) + \left[\begin{array}{c:c:c} P^0_f(n,n) & 0 & 0 \\ \hdashline 0 & P^0_b(n,n) & 0 \\ \hdashline 0 & 0 & 0 \end{array}\right] \quad . \tag{6.B.19c}$$

With minor modifications, both these results and those in the first section of this appendix, can be rewritten for processes satisfying either forward only or backward only dynamics.

APPENDIX 6C

IMPLEMENTATION OF THE DECOUPLED MARCHING METHOD DYNAMICS

6.C.1 Implementation via the FFT

Following the development in Jain and Jain [53], we show how the vector matrix product

$$\alpha = D\beta \tag{6.C.1}$$

where α and β are $(J-1)\times 1$ vectors, can be implemented by the FFT. Then, by employing an identity for Kronecker products, we extend the results of Jain and Jain and show that the transformation $(D \otimes I)$ can also be implemented via the FFT.

From the definition of D in (6.2.44), the elements of the product $D\beta$ in (6.C.1) can be written as

$$\alpha_k = \sqrt{(2/J)} \sum_{m=1}^{J-1} \beta_m \sin(mk\pi/J) \quad . \tag{6.C.2}$$

If we let

$$M = 2J \tag{6.C.3}$$

and define an $M\times 1$ vector with elements

$$\tilde{\beta}_m = \begin{cases} \beta_m & , \quad 1 \le m \le J-1 \\ 0 & , \quad \text{otherwise ,i.e. when } m=0 \text{ or } m \ge J \end{cases} \quad , \tag{6.C.4}$$

then by direct substitution it can be shown that

$$\alpha_k = 2 \, \mathrm{Im}\left\{ (1/M)^{1/2} \sum_{m=0}^{M-1} \tilde{\beta}_m \exp\left[(-1)^{1/2} \, 2\pi km/M\right] \right\} \tag{6.C.5a}$$

which can be expressed in terms of the kth element of the discrete Fourier transform (DFT) of $\tilde{\beta}$ as

$$\alpha_k = 2 \, \mathrm{Im}\left[\mathrm{DFT}\{\tilde{\beta}\}\right]_k \quad ; \quad 1 \le k \le J-1 \quad . \tag{6.C.5b}$$

Thus, the vector α in (6.C.1) can be computed by padding β with J+1 zeros and setting elements of the vector α equal to twice the imaginary part of the first J-1 elements of the DFT of this padded representation of β.

Now we consider the product of the $n(J-1)\times n(J-1)$ transformation $(D \times I)$ and an $n(J-1)$ vector g written in $n\times 1$ partitions as

$$g = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_{J-1} \end{bmatrix} \quad . \tag{6.C.6a}$$

The vector g can also be expressed as the lexicographic ordering of the array

$$G = [g_1 : g_2 : \ldots : g_{J-1}] \quad . \tag{6.C.6b}$$

We will write the relation between the vector g and the array G via the <u>Stacking operator</u> S( ) [54] (that is, g is formed by stacking the columns of G):

$$g = S(G) \quad . \tag{6.C.6c}$$

Using this notation, we can write the product

$$f = (D \otimes I)g \tag{6.C.7a}$$

as

$$f = (D \otimes I)\ S(G) \quad . \tag{6.C.7b}$$

By employing the following identity for Kronecker products [54]

$$S(ABC) = (C' \otimes A)\ S(B) \quad , \tag{6.C.8}$$

we can express f in (6.C.7b) as

$$\begin{aligned} f &= S(IGD') \\ &= S(GD') = S((DG')') \quad . \end{aligned} \tag{6.C.9}$$

Thus elements of f can be computed from the DFT of the rows of G appropriately padded with zeros (i.e. computing the columns of DG' via (6.C.5b)).

6.C.2 Boundary Conditions and Summary

The decoupled dynamics resulting from the FFT-based transformation of the 1-D marching dynamics are summarized. In addition, we present the result of applying the decoupling transformation to the boundary conditions. Then we further diagonalize each of the J-1 systems into stable forward and backward component processes. Our objective is to write the 1-D dynamics and boundary condition in the form of the forward/backward dynamics and boundary condition in (6.B.1) and (6.B.3) in Appendix 6B. Then the general solution formulated there can be applied to solve the decoupled system. We begin with a restatement of the basic NNM dynamics of the $n\times 1$ process x_{ij}:

$$x_{ij} = A_1 x_{i-1,j} + A_2 x_{i,j-1} + A_3 x_{i+1,j} + A_4 x_{i,j+1} + Bu_{ij} \qquad (6.C.10)$$

where here we assume that A_3 is invertible and that A_2 and A_4 are identical. The general form of the boundary condition for (6.C.10) can been written in terms of x_i the stacked vector representation of x_{ij} and is of the multi-point form of (6.B.3) (see (6.2.33a)):

$$v = \sum_{i=0}^{I} V^i x_i + V_{TB} \begin{bmatrix} d_B \\ d_T \end{bmatrix} \quad . \qquad (6.C.11)$$

The first of the two transformations performed on the 1-D marching method dynamics in (6.2.43) is the FFT-based decoupling transformation in (6.2.46a):

$$q_i = (D \otimes I) x_i \quad . \qquad (6.C.12)$$

Recall that q_i is the stacked representation of a 2-D process q_{ij} and it has been shown in (6.2.52) that the dynamics of this process are given by

$$\begin{bmatrix} q_{i+1,j} \\ q_i \end{bmatrix} = A_{qj} \begin{bmatrix} q_{i,j} \\ q_{i-1,j} \end{bmatrix} + \begin{bmatrix} U_{i,j} \\ 0 \end{bmatrix} \quad ; \; j = 1,2,\ldots J-1 \qquad (6.C.13a)$$

where

$$A_{qj} = \left[\begin{array}{c:c} \tilde{A}_3 + \lambda_j \tilde{A}_2 & \tilde{A}_1 \\ \hline I & 0 \end{array}\right] \qquad (6.C.13b)$$

and $U_{i,j}$ are the elements of the vector U_i in (6.2.51b):

$$U_i \equiv \begin{bmatrix} U_{i,1} \\ U_{i,2} \\ \vdots \\ U_{i,J-1} \end{bmatrix} = (D \otimes I)(I \otimes \tilde{B})u_i + M_{TBi}d_{TB} \quad . \qquad (6.C.13c)$$

where

$$M_{TBi} = -(D \otimes I)\tilde{a}_3\left(I_B R^{i-1} f_2 d_B + I_T R^{i-1} f_4 d_T\right) \quad . \qquad (6.C.23d)$$

Here I_B, I_T, and R are defined in (6.2.16) and (6.2.17), and f_2 and f_4 are defined in (6.A.8). The computation of the first term on the right hand side of (6.C.13c) is preformed via the FFT as follows. Define

$$w_i = \begin{bmatrix} w_{i,1} \\ w_{i,2} \\ \vdots \\ w_{i,J-1} \end{bmatrix} \equiv (D \otimes I) \begin{bmatrix} \tilde{B}\ u_{i,1} \\ \tilde{B}\ u_{i,2} \\ \vdots \\ \tilde{B}\ u_{i,J-1} \end{bmatrix} \qquad (6.C.14a)$$

Then U_i can be written as

$$U_i = w_i + M_{TBi}d_{TB} \qquad (6.C.14b)$$

and w_i can be computed via the FFT as described in the previous section. Given the sparseness of I_B, I_T and R^{i-1}, it is likely that the most efficient way to compute M_{TBi} in (6.C.23d) will make use of this sparseness rather than the FFT.

Under the transformation in (6.C.12), the boundary condition in (6.C.11) becomes

$$v = \sum_{i=0}^{I} V_q^i q_i + V_{TB}d_{TB} \qquad (6.C.15a)$$

where

$$V_q^i = V^i (D \otimes I)$$

$$= \left[(D \otimes I)\, V^{i'} \right]' \quad . \tag{6.C.15b}$$

The second form of (6.C.15b) is intended to illustrate how the V^i can be computed via the FFT as discussed in the first section of this appendix (see (6.C.9)).

Thus far we have invoked the invertibility and symmetry assumptions and the transformation in (6.C.12) to rewrite the original 2-D NNM dynamics of the $n \times 1$ process x_{ij} on S as J-1 decoupled forward 1-D problems (6.C.13a) with coupled boundary conditions (6.C.15a). Next we consider a second transformation which transforms each of the J-1 dynamical representations in (6.C.13a) into stable forward and backward dynamics. Thus, we seek a family of transformations $\{T_j\}$ which transform the A_{qj} in (6.C.13b) into the diagonal form

$$\begin{bmatrix} A_{f,j} & \vdots & 0 \\ \hdashline 0 & \vdots & A_{b,j} \end{bmatrix} = T_j A_{qj} T_j^{-1} \qquad ; \; j = 1, J-1 \tag{6.C.16a}$$

so that each of the A_f is forward stable and each of the A_b is backward stable. Since each of the A_{qj} is $2n \times 2n$, these transformations can easily be found by numerical eigen-decompositions of the A_{qj} (That is, we assume that n is small, e.g. <<100). The dimensions of the partitions in (6.C.16a) are given by

$$A_{f,j} - n_{f,j} \times n_{f,j} \quad , \quad A_{b,j} - n_{b,j} \times n_{b,j} \quad \text{and} \quad n_{f,j} + n_{b,j} = 2n \quad . \tag{6.C.16b}$$

Note that for some j, either $n_{f,j}$ or $n_{b,j}$ may be zero so that for these subsystems the dynamics will be entirely forward or entirely backward. The new processes resulting from each of these transformations will be denoted by

$$\begin{bmatrix} q_{f_{i,j}} \\ q_{b_{i,j}} \end{bmatrix} = T_j \begin{bmatrix} q_{i,j} \\ q_{i-1,j} \end{bmatrix} \quad . \tag{6.C.16c}$$

Their dynamics are decoupled and are written as

$$q_{f_{i+1,j}} = A_{f,j} q_{f_{i,j}} + u_{f_{i,j}} \tag{6.C.17a}$$

and

$$q_{b_{i,j}} = A_{b,j}^{-1} q_{b_{i+1,j}} + u_{b_{i,j}} \qquad ; \; j = 1, \; J-1 \quad . \tag{6.C.17b}$$

Expressions for $u_{f_{i,j}}$ and $u_{b_{i,j}}$ can be written in terms of the $U_{i,j}$ which drive the dynamics of $q_{i,j}$ in (6.C.13a) as follows. Denote the partitions of the T_j as

$$T_j = \begin{bmatrix} T_{f1,j} & \vdots & T_{f2,j} \\ - - - & : & - - - \\ T_{b1,j} & \vdots & T_{b2,j} \end{bmatrix} , \tag{6.C.18a}$$

then with $U_{i,j}$ given by (6.C.18c) u_f and u_b can be written as

$$u_{f_{i,j}} = T_{f1,j} U_{i,j} \tag{6.C.18b}$$

and

$$u_{b_{i,j}} = -A_{b,j}^{-1} T_{b1,j} U_{i,j} \quad . \tag{6.C.18c}$$

Finally, the dynamics can be written in the form of the process whose general solution is presented in Appendix 6B as follows. Denote the stacked vector representations for $q_{f_{i,j}}$ and $q_{b_{i,j}}$ as

$$q_{f,i} \equiv \begin{bmatrix} q_{f_{i,1}} \\ q_{f_{i,2}} \\ \vdots \\ q_{f_{i,J-1}} \end{bmatrix} \quad \text{and} \quad q_{b,i} \equiv \begin{bmatrix} q_{b_{i,1}} \\ q_{b_{i,2}} \\ \vdots \\ q_{b_{i,J-1}} \end{bmatrix} \quad . \tag{6.C.19a}$$

Then the dynamics in (6.C.17a) and (6.C.17b) can be written as

$$q_{f,i+1} = \text{DIAG}\{A_{f,j}\} q_{f,i} + \text{DIAG}\{T_{1f,j}\} U_i$$

$$= \text{DIAG}\{A_{f,j}\} q_{f,i} + \text{DIAG}\{T_{1f,j}\}[\, w_i + M_{TBi} d_{TB}] \tag{6.C.19b}$$

and

$$q_{b,i} = \mathrm{DIAG}\{A_{b,j}^{-1}\} q_{b,i+1} + \mathrm{DIAG}\{-A_{b,j}^{-1} T_{b1,j}\} U_i$$

$$= \mathrm{DIAG}\{A_{b,j}^{-1}\} q_{b,i+1} + \mathrm{DIAG}\{-A_{b,j}^{-1} T_{b1,j}\} [\, w_i + M_{TBi} d_{TB}] \qquad (6.C.19c)$$

where U_i has been replaced by the expression in (6.C.14b) and $\mathrm{DIAG}\{\cdot\}$ represents a diagonal matrix with diagonal elements given by its argument $\{\cdot\}$. The dynamics in (6.C.19b) and (6.C.19c) are in the form of (6.B.1a) and (6.B.1b). Next we must rewrite the boundary condition (6.C.15a) in terms of the transformed processes q_f and q_b.

First rewrite that boundary condition as

$$v = \sum_{i=0}^{I} \sum_{j=1}^{J-1} [\, V_q^{i,j} \vdots 0] \begin{bmatrix} q_{i,j} \\ q_{i-1,j} \end{bmatrix} + V_{TB} d_{TB} \qquad (6.C.20a)$$

where the $V_q^{i,j}$ are the partitions of V_q^i in (6.C.15b):

$$V_q^i = [\, V_q^{1,i} \vdots V_q^{i,2} \vdots \ldots \vdots V_q^{i,J-1}] \quad . \qquad (6.C.20b)$$

By substituting from the forward/backward decoupling transformation in (6.C.16c), (6.C.20a) becomes

$$v = \sum_{i=0}^{I} \sum_{j=1}^{J-1} [\, V_q^{i,j} \vdots 0] T_j^{-1} \begin{bmatrix} q_{f_{i,j}} \\ q_{b_{i,j}} \end{bmatrix} + V_{TB} d_{TB} \quad , \qquad (6.C.21a)$$

or defining

$$[\, V_f^{i,j} \vdots V_b^{i,j}] \equiv [\, V_q^{i,j} \vdots 0] T_j^{-1} \qquad (6.C.21b)$$

the boundary condition can be written as

$$v = \sum_{i=0}^{I} \sum_{j=1}^{J-1} [\, V_f^{i,j} \vdots V_b^{i,j}] \begin{bmatrix} q_{f_{i,j}} \\ q_{b_{i,j}} \end{bmatrix} + V_{TB} d_{TB} \quad . \qquad (6.C.21c)$$

Finally if we define

$$V_f^i \equiv [\, V_f^{i,1} \vdots V_f^{i,2} \vdots \ldots \vdots V_f^{i,J-1}] \qquad (6.C.22a)$$

and

$$V_b^i \equiv [\, V_b^{i,1} \; \vdots \; V_b^{i,2} \; \vdots \; . \; . \; . \; \vdots \; V_b^{i,J-1} \,] \qquad , \tag{6.C.22b}$$

then we get the form of (B.3), the boundary condition in Appendix B,

$$v = \sum_{i=0}^{I} [\, V_f^i \; \vdots \; V_b^i \,] \begin{bmatrix} q_{f,i} \\ q_{b,i} \end{bmatrix} + V_{TB} d_{TB} \tag{6.C.22c}$$

where q_f and q_b are defined in (6.C.19a). Thus, with the dynamics given by (6.C.19b) and (6.C.19c) and the boundary condition given by (6.C.22c), the problem is in the form of the multi-point boundary value problem whose general solution is derived in Appendix 6B. To recover the original process x from the computed values of q_f and q_b, one need only implement the inverse of the transformations in (6.C.16c) and (6.C.12) where the latter can be performed by the FFT as described in the previous section of this appendix.

APPENDIX 6D

GREEN'S IDENTITY FOR THE NNM

6.D.1 The Adjoint Difference Operator

The relationship between the formal difference operator representing the dynamics of the NNM in (6.2.1b) and its formal adjoint $L^\dagger$ is defined by way of the Green's identity. A similar approach to that taken in Chapter 5 for the 1-D discrete case yields an expression for the adjoint difference operator for the discrete 2-D case. First define the 2-D shift (delay) operators D_1 and D_2 as

$$(D_1 x)_{ij} = x_{i-1,j} \quad , \qquad (D_1^{-1} x)_{ij} = x_{i+1,j} \tag{6.D.1a}$$

and

$$(D_2 x)_{ij} = x_{i,j-1} \quad , \qquad (D_2^{-1} x)_{ij} = x_{i,j+1} \quad . \tag{6.D.1b}$$

The NNM dynamics in (6.2.1b) are described by the following first order difference operator (first order implies that the total shift in any term, that is the shift in i plus the shift in j, is one)

$$L = \left[I - A_1 D_1 - A_2 D_2 - A_3 D_1^{-1} - A_4 D_2^{-1} \right] \quad ,$$

$$L: \; l_2^n(S \cup \partial S) \rightarrow l_2^n(S) \quad . \tag{6.D.2}$$

Thus, in operator notation, the NNM dynamics in (6.2.1b) are

$$Lx = Bu \tag{6.D.3}$$

The Green's identity is a relation between L and its formal adjoint $L^\dagger$ of the form

$$\langle Lx, \lambda \rangle_{l_2^n(S)} = \langle L^\dagger \lambda , x \rangle_{l_2^n(S)} + \text{boundary term} \quad . \tag{6.D.4}$$

The formal adjoint $L^\dagger$ is a formal difference operator of the same order as L (first order) and is determined by manipulating the summations implied by the inner product $\langle Lx,\lambda\rangle$ as follows. For economy of notation, let

$$\sum_i = \sum_{i=1}^{I-1} \quad , \qquad \sum_j = \sum_{j=1}^{J-1} \tag{6.D.5a,b}$$

and

$$\sum_{ij}\sum = \sum_{i=1}^{I-1}\sum_{j=1}^{J-1} \quad . \tag{6.D.5c}$$

Given this notation, the inner product on the left hand side of Green's identity in (6.D.4) can be expressed as

$$\langle Lx,\lambda\rangle_{l_2^n(S_{IJ})} = \sum_{ij}\sum \lambda'_{ij}\left[x_{ij} - A_1x_{i-1,j} - A_2x_{i,j-1} - A_3x_{i+1,j} - A_4x_{i,j+1}\right] \tag{6.D.6}$$

It will be convenient to define each term in the summand in (6.D.6) by a separate variable:

$$S_1 = -\sum_{ij}\sum x'_{i-1,j}A'_1\lambda_{ij} \quad , \qquad S_2 = -\sum_{ij}\sum x'_{i,j-1}A'_2\lambda_{ij} \quad , \tag{6.D.7a,b}$$

$$S_3 = -\sum_{ij}\sum x'_{i+1,j}A'_3\lambda_{ij} \quad , \qquad S_4 = -\sum_{ij}\sum x'_{i,j+1}A'_4\lambda_{ij} \tag{6.D.7c,d}$$

and

$$S_0 = -\sum_{ij}\sum x'_{i,j}\lambda_{ij} \quad , \tag{6.D.7e}$$

so that the inner product in (6.D.6) is the sum of these five terms. Writing Green's identity in (6.D.4) as

$$\sum_{ij}\sum (Lx)'_{ij}\lambda_{ij} = \sum_{ij}\sum (L^\dagger\lambda)'_{ij}x_{ij} + \text{b.t.}$$

it can be seen that to find an expression for $L^\dagger$, we must make a change of indices in S_1 through S_4 so that x_{ij} appears in each (as on the right hand side of Green's identity) rather than λ_{ij} (as on the left hand side).

First define

$$\text{b.t.}_1 = -\sum_j \left[x'_{0,j}A'_1\lambda_{1,j} - x'_{I-1,j}A'_1\lambda_{I,j}\right] \tag{6.D.8a}$$

$$b.t._2 = -\sum_i \left[x'_{i,0}A'_2\lambda_{i,1} - x'_{i,J-1}A'_2\lambda_{i,J}\right] \qquad (6.D.8b)$$

$$b.t._3 = -\sum_j \left[x'_{I,j}A'_3\lambda_{I-1,j} - x'_{1,j}A'_3\lambda_{0,j}\right] \qquad (6.D.8c)$$

$$b.t._4 = -\sum_i \left[x'_{i,J}A'_4\lambda_{i,J-1} - x'_{i,1}A'_4\lambda_{i,0}\right] \qquad (6.D.8d)$$

Then, when the indices are shifted as suggested above, S_1 through S_4 become

$$S_1 = \langle x, -A_1'D_1^{-1}\lambda\rangle + b.t._1 \quad , \qquad (6.D.9a)$$

$$S_2 = \langle x, -A_2'D_2^{-1}\lambda\rangle + b.t._2 \quad , \qquad (6.D.9b)$$

$$S_3 = \langle x, -A_3'D_1\lambda\rangle + b.t._3 \qquad (6.D.9c)$$

and

$$S_4 = \langle x, -A_4'D_2\lambda\rangle + b.t._4 \quad . \qquad (6.D.9d)$$

With the formal adjoint given by ($L^\dagger$ has the same range and domain as L in (6.D.21))

$$L^\dagger = I - A_1'D_1^{-1} - A_2'D_2^{-1} - A_3'D_1 - A_4'D_2 \quad , \qquad (6.D.10a)$$

or

$$(L^\dagger x)_{ij} = x_{ij} - A_1'x_{i+1,j} - A_2'x_{i,j+1} - A_3'x_{i-1,j} - A_4'x_{i,j-1} \qquad (6.D.10b)$$

it is straightforward to show that the desired form for Green's identity

,i.e. (6.D.4), is achieved:

$$\langle Lx,\lambda\rangle = \sum_{k=0}^{4} S_k$$

$$= \langle x,L^{\dagger}\lambda\rangle + \sum_{k=1}^{4} b.t._k \quad . \tag{6.D.11}$$

Note that L is formally self-adjoint (i.e. $L = L^{\dagger}$) when

$$A_3' = A_1 \quad \text{and} \quad A_4' = A_2 \quad . \tag{6.D.12}$$

6.D.2 The Boundary Term

To express the sum of the boundary terms in (6.D.11) in more compact notation, let

$$X_L = \begin{bmatrix} x_{0,1} \\ x_{0,2} \\ \vdots \\ x_{0,J-1} \\ x_{1,1} \\ x_{1,2} \\ \vdots \\ x_{1,J-1} \end{bmatrix}, \quad X_R = \begin{bmatrix} x_{I-1,1} \\ x_{I-1,2} \\ \vdots \\ x_{I-1,J-1} \\ x_{I,1} \\ x_{I,2} \\ \vdots \\ x_{I,J-1} \end{bmatrix}, \quad X_B = \begin{bmatrix} x_{1,0} \\ x_{2,0} \\ \vdots \\ x_{I-1,0} \\ x_{1,1} \\ x_{2,1} \\ \vdots \\ x_{I-1,1} \end{bmatrix}, \quad X_T = \begin{bmatrix} x_{1,J-1} \\ x_{2,J-1} \\ \vdots \\ x_{I-1,J-1} \\ x_{1,J} \\ x_{2,J} \\ \vdots \\ x_{I-1,J} \end{bmatrix} \tag{6.D.13a}$$

$$\uparrow \; 2n(J-1)\times 1 \qquad \uparrow \; 2n(J-1)\times 1 \qquad \uparrow \; 2n(I-1)\times 1 \qquad \uparrow \; 2n(I-1)\times 1$$

with similar definitions for Λ_L, Λ_R, Λ_B and Λ_T given in terms of the λ_{ij}. It will be convenient to denote the partitions of the vectors in (6.D.13a) (and their Λ_L, Λ_R, etc. counterparts) by

$$X_L = \begin{bmatrix} x_L \\ -- \\ x_{L+1} \end{bmatrix}, \quad X_R = \begin{bmatrix} x_{R-1} \\ -- \\ x_R \end{bmatrix}, \quad X_B = \begin{bmatrix} x_B \\ -- \\ x_{B+1} \end{bmatrix}, \quad X_T = \begin{bmatrix} x_{T-1} \\ -- \\ x_T \end{bmatrix}, \tag{6.D.13b}$$

where x_L, x_R, x_B and x_T have been defined in (6.2.3). If we also define a_{13} and a_{24} as the following $2n(I-1)\times 2n(I-1)$ and $2n(J-1)\times 2n(J-1)$

matrices respectively

$$a_{13} = \left[\begin{array}{c:c} 0 & I\otimes A_3 \\ \hdashline -I\otimes A_1 & 0 \end{array}\right] \quad \text{and} \quad a_{24} = \left[\begin{array}{c:c} 0 & I\otimes A_4 \\ \hdashline -I\otimes A_2 & 0 \end{array}\right] , \qquad (6.D.14)$$

then the sum of boundary terms in (6.D.11) can be expressed as the sum of four inner products:

$$\sum_{k=1}^{4} b.t._k = \langle a'_{13}X_L,\Lambda_L\rangle_{\mathbf{R}^{2n(I-1)}} - \langle a_{13}X_R,\Lambda_R\rangle_{\mathbf{R}^{2n(I-1)}}$$

$$+ \langle a'_{24}X_B,\Lambda_B\rangle_{\mathbf{R}^{2n(J-1)}} - \langle a_{24}X_T,\Lambda_T\rangle_{\mathbf{R}^{2n(J-1)}} \quad . \qquad (6.D.15)$$

By defining lower dimensional boundary processes equivalent to:

$$\lambda_L = \begin{bmatrix} \lambda_{0,1} \\ \lambda_{0,2} \\ \vdots \\ \lambda_{0,J-1} \end{bmatrix}, \quad \lambda_R = \begin{bmatrix} \lambda_{I,1} \\ \lambda_{I,2} \\ \vdots \\ \lambda_{I,J-1} \end{bmatrix}, \quad \lambda_B = \begin{bmatrix} \lambda_{1,0} \\ \lambda_{2,0} \\ \vdots \\ \lambda_{I-1,0} \end{bmatrix}, \quad \lambda_T = \begin{bmatrix} \lambda_{1,J} \\ \lambda_{2,J} \\ \vdots \\ \lambda_{I-1,J} \end{bmatrix} \qquad (6.D.16)$$

similar to d_L, d_R etc. in Appendix 6A for x_L, x_R etc., we show below that the sum of boundary terms in (6.D.15) can be written in terms of these lower dimensional processes. Choose Γ_k so that A_k' is compressed as

$$A_k' G_k^{-1} = [g_k \vdots\ 0] \qquad (g_k - n\times n_k) \quad . \qquad (6.D.17)$$

Similar to d_L, d_R, d_B and d_T in Appendix 6A define

$$\xi_L = \left[I_{J-1} \otimes [I_{n_3} \vdots\ 0]\right](I\otimes \Gamma_3)\lambda_L \qquad (6.D.18a)$$

$$\xi_R = \left[I_{J-1} \otimes [I_{n_1} \vdots\ 0]\right](I\otimes \Gamma_1)\lambda_R \qquad (6.D.18b)$$

$$\xi_B = \left[I_{I-1} \otimes [I_{n_4} \vdots 0] \right] (I \otimes \Gamma_4) \lambda_B \tag{6.D.18c}$$

and

$$\xi_T = \left[I_{I-1} \otimes [I_{n_2} \vdots 0] \right] (I \otimes \Gamma_2) \lambda_T \quad . \tag{6.D.18d}$$

Next we show how the boundary term in Green's identity can be written in terms of these quantities. First redefine the variables in (6.D.13), X_L etc., and their counterparts Λ_L etc. by the equivalent (in the sense of boundary information as discuseed in Appendix 6A) expressions:

$$D_L = \begin{bmatrix} d_L \\ x_{L+1} \end{bmatrix} \qquad D_R = \begin{bmatrix} x_{R-1} \\ d_R \end{bmatrix} \qquad D_B = \begin{bmatrix} d_B \\ x_{B+1} \end{bmatrix} \qquad D_T = \begin{bmatrix} x_{T-1} \\ d_T \end{bmatrix} \tag{6.D.19a}$$

and

$$\Xi_L = \begin{bmatrix} \xi_L \\ \lambda_{L+1} \end{bmatrix} \qquad \Xi_R = \begin{bmatrix} \lambda_{R-1} \\ \xi_R \end{bmatrix} \qquad \Xi_B = \begin{bmatrix} \xi_B \\ \lambda_{B+1} \end{bmatrix} \qquad \Xi_T = \begin{bmatrix} \lambda_{T-1} \\ \xi_T \end{bmatrix} \quad . \tag{6.D.19b}$$

Recalling the defintion of the F_k in (6.A.5d) and substituting from (6.D.18) and (6.D.19), it can be shown that the sum of boundary terms in (6.D.8) can be expressed as

$$\sum_{i=1}^{4} b.t._i = \left\langle D_L, \begin{bmatrix} 0 & \vdots & -I \times F_1' \\ \hline I \times G_3 & \vdots & 0 \end{bmatrix} \Xi_L \right\rangle + \left\langle D_R, \begin{bmatrix} 0 & \vdots & I \times G_1 \\ \hline -I \times F_3' & \vdots & 0 \end{bmatrix} \Xi_R \right\rangle$$

$$+ \left\langle D_B, \begin{bmatrix} 0 & \vdots & -I \times F_2' \\ \hline I \times G_4 & \vdots & 0 \end{bmatrix} \Xi_B \right\rangle + \left\langle D_T, \begin{bmatrix} 0 & \vdots & I \times G_2 \\ \hline -I \times F_4' & \vdots & 0 \end{bmatrix} \Xi_T \right\rangle . \tag{6.D.20}$$

With λ_b and x_b defined in terms of the quantities in (6.D.19a) and (6.D.19b) as

$$x_b = \begin{bmatrix} D_L \\ D_R \\ D_B \\ D_T \end{bmatrix} \qquad \text{and} \qquad \lambda_b = \begin{bmatrix} \Xi_L \\ \Xi_R \\ \Xi_B \\ \Xi_T \end{bmatrix} \tag{6.D.21}$$

and with E the block-diagonal matrix:

$$E = \begin{bmatrix} e_{13} & 0 & 0 & 0 \\ 0 & e_{31} & 0 & 0 \\ 0 & 0 & e_{24} & 0 \\ 0 & 0 & 0 & e_{42} \end{bmatrix} \qquad (6.D.21a)$$

where (see (6.D.28))

$$e_{13} = \left[\begin{array}{c:c} 0 & -I \otimes F_1' \\ \hdashline I \otimes G_3 & 0 \end{array}\right], \qquad e_{31} = \left[\begin{array}{c:c} 0 & I \otimes G_1 \\ \hdashline -I \otimes F_3' & 0 \end{array}\right],$$

and

$$e_{24} = \left[\begin{array}{c:c} 0 & -I \otimes F_2' \\ \hdashline I \otimes G_4 & 0 \end{array}\right] \quad \text{and} \quad e_{42} = \left[\begin{array}{c:c} 0 & I \otimes G_2 \\ \hdashline I \otimes F_4' & 0 \end{array}\right] \qquad (6.D.21b)$$

the sum of boundary terms can be written as

$$\sum_{i=1}^{4} b.t._i = \langle x_b, E\lambda_b \rangle \quad . \qquad (6.D.22)$$

The product $E\lambda_b$ in terms of these lower dimensional vectors is

$$E\lambda_b = E_{L_1}\xi_L + E_{L_2}\lambda_{L+1} + E_{R_1}\lambda_{R-1} + E_{R_2}\xi_R + E_{B_1}\xi_B + E_{B_2}\lambda_{B+1} + E_{T_1}\lambda_{T-1} + E_{T_2}\xi_T \qquad (6.D.23)$$

where

$$E_{L_1} = \begin{bmatrix} 0 \\ I \otimes g_3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad E_{L_2} = \begin{bmatrix} -I \otimes f_1' \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad E_{R_1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -I \otimes f_3' \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad E_{R_2} = \begin{bmatrix} 0 \\ 0 \\ I \otimes g_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$E_{B_1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ I \otimes g_4 \\ 0 \\ 0 \end{bmatrix} \quad E_{B_2} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -I \otimes f'_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad E_{T_1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -I \otimes f'_4 \end{bmatrix} \quad E_{T_2} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ I \otimes g_2 \\ 0 \end{bmatrix} \tag{6.D.24}$$

These expressions will be useful in Appendix 6E where the boundary condition for the 2-D discrete estimator expressed in 1-D marching method form is written as a multi-point 1-D boundary condition.

6.D.3 The Replacement Processes for the NNM Smoother

Following the developments in Appendix 6A, we show that the reduced-order Dirichlet condition (the boundary replacement processes) for the smoother dynamics in NNM form is, in fact, comprised of d, the Dirichlet condition for the process to be estimated, and ξ which has been defined in (6.D.18) of the previous subsection. To show this, first note that the matrices ϕ_k and Γ_k in (6.A.5d) and (6.D.23a) have been chosen so that A_k and its transpose are compressed into full rank matrices:

$$[F_k \vdots 0] = A_k \phi_k^{-1} \tag{6.A.25a}$$

and

$$[G_k \vdots 0] = A'_k \Gamma_k^{-1} \quad . \tag{6.A.25b}$$

As shown below, these same matrices can be employed to compress the four $2n\times2n$ coefficient matrices $A_{x\lambda_k}$ of the smoother dynamics given in (6.3.5b). As in Appendix 6A, we will use k = 1 for an example. The exposition extends to all values of k. Multiplying $A_{x\lambda_1}$ on the right by the following block-diagonal matrix gives the compressed form:

$$A_{x\lambda_1} \begin{bmatrix} \phi_1^{-1} & 0 \\ 0 & \Gamma_3^{-1} \end{bmatrix} = \left[\begin{array}{c:c} [F_1 \vdots 0] & 0 \\ \hdashline 0 & [G_3 \vdots 0] \end{array}\right] \quad . \tag{6.A.26a}$$

Employing the definition of d_L in (6.A.4a) and ξ_L in (6.D.18a), it can be shown that

$$(I \otimes A_{x\lambda_1}) \begin{bmatrix} \hat{x}_L \\ \hat{\lambda}_L \end{bmatrix} = \left[\begin{array}{c:c} (I \otimes F_1) & 0 \\ \hdashline 0 & (I \otimes G_3) \end{array}\right] \begin{bmatrix} \hat{d}_L \\ \hat{\xi}_L \end{bmatrix} \tag{6.A.26b}$$

Thus, the information in $\hat{x}$ and $\hat{\lambda}$ on the left edge of ∂S is contained in $\hat{d}_L$ and $\hat{\xi}_L$. Following the arguments in the first part of this appendix, this establishes $\{\hat{d}_L, \hat{\xi}_L\}$ as an equivalent boundary process for $\{\hat{x}_L, \hat{\lambda}_L\}$, and we will correspondingly define $F_{x\lambda_1}$ as

$$F_{x\lambda_1} = \Delta_{x\lambda}^{-1} \begin{bmatrix} F_1 & 0 \\ 0 & G_3 \end{bmatrix} \tag{6.A.27a}$$

with the block-diagonal form used in defining 1-D models (see (6.2.28)):

$$f_{x\lambda_1} = (I \otimes F_{x\lambda_1}) \quad . \tag{6.A.27b}$$

Similar definitions for other values of k are formulated by considering $\{\hat{x}_R, \hat{\lambda}_R\}$, $\{\hat{x}_B, \hat{\lambda}_B\}$ and $\{\hat{x}_T, \hat{\lambda}_T\}$ and the corresponding definitions of $\{\hat{d}_R, \hat{\xi}_R\}$, $\{\hat{d}_B, \hat{\xi}_B\}$ and $\{\hat{d}_T, \hat{\xi}_T\}$. These are:

$$F_{x\lambda_2} = \Delta_{x\lambda}^{-1} \begin{bmatrix} F_2 & 0 \\ 0 & G_4 \end{bmatrix} \tag{6.D.28a}$$

$$F_{x\lambda_3} = \Delta_{x\lambda}^{-1} \begin{bmatrix} F_3 & 0 \\ 0 & G_1 \end{bmatrix} \tag{6.D.28b}$$

$$F_{x\lambda_4} = \Delta_{x\lambda}^{-1} \begin{bmatrix} F_4 & 0 \\ 0 & G_2 \end{bmatrix} \tag{6.D.28c}$$

and

$$f_{x\lambda_k} = (I \otimes F_{x\lambda_k}) \quad , k = 2,3,4 \quad . \tag{6.D.28d}$$

APPENDIX 6E

MULTI-POINT 1-D BOUNDARY CONDITION FOR THE NNM SMOOTHER

In this appendix we transform the boundary condition for the 2-D smoother as given in (6.3.14) to a 1-D multi-point form which is compatible with the 1-D marching method represenation of the smoother dynamics. In particular that form is

$$0 = \sum_{i=0}^{I} V^{i}_{x\lambda} \begin{bmatrix} \hat{x} \\ \hat{\lambda} \end{bmatrix}_i + V_{x\lambda_{TB}} \begin{bmatrix} \hat{d} \\ \hat{\xi} \end{bmatrix}_{TB} \tag{6.E.1}$$

where

$$\begin{bmatrix} \hat{x} \\ \hat{\lambda} \end{bmatrix}_i \equiv \begin{bmatrix} \hat{x}_{i,1} \\ \hat{\lambda}_{i,1} \\ \hline \hat{x}_{i,2} \\ \hat{\lambda}_{i,2} \\ \hline \vdots \\ \hline \hat{x}_{i,J-1} \\ \hat{\lambda}_{i,J-1} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \hat{d} \\ \hat{\xi} \end{bmatrix}_{TB} \equiv \begin{bmatrix} \hat{d}_T \\ \hat{\xi}_T \\ \hline \hat{d}_B \\ \hat{\xi}_B \end{bmatrix} . \tag{6.E.2}$$

Note that the first process in (6.E.2) can be written in terms of the processes $\hat{x}_i$ and $\hat{\lambda}_i$ (see (6.2.10b)) if we define the following $2n(J-1)\times 2n(J-1)$ matrix in $n\times n$ partitions

$$S_{x\lambda} \equiv \left[\begin{array}{cccccccc:ccccccc}
I & 0 & 0 & 0 & \cdots & & & 0 & 0 & 0 & 0 & 0 & \cdots & & 0 \\
0 & 0 & 0 & 0 & \cdots & & & 0 & I & 0 & 0 & 0 & \cdots & & 0 \\
0 & I & 0 & 0 & \cdots & & & 0 & 0 & 0 & 0 & 0 & \cdots & & 0 \\
0 & 0 & 0 & 0 & \cdots & & & 0 & 0 & I & 0 & 0 & \cdots & & 0 \\
0 & 0 & I & 0 & \cdots & & & 0 & 0 & 0 & 0 & 0 & \cdots & & 0 \\
0 & 0 & 0 & 0 & & & & 0 & 0 & 0 & I & 0 & \cdots & & \cdot \\
\cdot & & & I & & & & \cdot & 0 & & & \cdot & & & \cdot \\
\cdot & & & & \cdot & 0 & & \cdot & \cdot & & & & \cdot & 0 & \cdot \\
\cdot & & & & & \cdot & & \cdot & \cdot & & 0 & & \cdot & & \cdot \\
\cdot & & 0 & & & & \cdot & \cdot & \cdot & & & & & \cdot & \cdot \\
& & & & & & & \cdot & \cdot & & & & & & \cdot \\
0 & \cdots & & & & 0 & I & 0 & 0 & \cdots & & & & 0 & 0 \\
0 & \cdots & & & & 0 & 0 & 0 & 0 & \cdots & & & & 0 & I
\end{array}\right] \tag{6.E.3a}$$

so that

$$\begin{bmatrix} \hat{x} \\ \hat{\lambda} \end{bmatrix}_i = S_{x\lambda} \begin{bmatrix} \hat{x}_i \\ \hat{\lambda}_i \end{bmatrix} \quad . \qquad (6.E.3b)$$

To transform the smoother boundary condition in (6.3.14) to the form of (6.E.1), start by writing its first term as

$$V\hat{x}_b = \left[V_L \; \vdots V_R \; \vdots V_B \; \vdots V_T \right] \begin{bmatrix} \hat{D}_L \\ \hat{D}_R \\ \hat{D}_B \\ \hat{D}_T \end{bmatrix} \qquad (6.E.4)$$

Substituting expressions for $\hat{D}_L$, $\hat{D}_R$, $\hat{D}_B$ and $\hat{D}_T$ from (6.D.21) into (6.E.4) gives

$$V\hat{x}_b = V_L \begin{bmatrix} \hat{d}_L \\ \hat{x}_{L+1} \end{bmatrix} + V_R \begin{bmatrix} \hat{x}_{R-1} \\ \hat{d}_R \end{bmatrix} + V_B \begin{bmatrix} \hat{d}_B \\ \hat{x}_{B+1} \end{bmatrix} + V_T \begin{bmatrix} \hat{x}_{T-1} \\ \hat{d}_T \end{bmatrix} \quad . \qquad (6.E.5)$$

Next note that the vectors in (6.E.5) can be expressed in terms of the process $\hat{x}_i$ (see (6.D.13)) as

$$\hat{d}_L = \Psi_0 \hat{x}_0, \quad \hat{x}_{L+1} = \hat{x}_1, \quad \hat{x}_{R-1} = \hat{x}_{I-1}, \quad \hat{d}_R = \Psi_I \hat{x}_I \qquad (6.E.6a)$$

where from (6.A.4a) and (6.A.5b)

$$\Psi_0 = \left[I_{J-1} \otimes [I_{n_1} \vdots \; 0] \right] (I \otimes \phi_1) \qquad (6.E.6b)$$

and

$$\Psi_I = \left[I_{J-1} \otimes [I_{n_3} \vdots \; 0] \right] (I \otimes \phi_3) \qquad (6.E.6c)$$

and as

$$\hat{x}_{B+1} = \sum_{i=1}^{I-1} S_B^i \hat{x}_i \quad \text{and} \quad \hat{x}_{T-1} = \sum_{i=1}^{I-1} S_T^i \hat{x}_i \tag{6.E.6d}$$

where

$$S_B^i \equiv R^{1-i}\begin{bmatrix} I & 0 & 0 & \cdot & \cdot & 0 \\ 0 & & & & & \\ \vdots & & & 0 & & \\ 0 & & & & & \end{bmatrix} \qquad S_T^i \equiv R^{1-i}\begin{bmatrix} 0 & \cdot & \cdot & \cdot & 0 & 0 & I \\ & & & & & & 0 \\ & & 0 & & & & \vdots \\ & & & & & & 0 \end{bmatrix} . \tag{6.E.6e}$$

Denote the partitions of the matrices in (6.E.5) as

$$V_L = [V_{L_1} \vdots V_{L_2}], \quad V_R = [V_{R_1} \vdots V_{R_2}], \quad V_B = [V_{B_1} \vdots V_{B_2}] \text{ and } V_T = [V_{T_1} \vdots V_{T_2}] \tag{6.E.7}$$

where the partitions are compatible with the partitions of the vectors in (6.E.5). Combining the above, we can rewrite (6.E.5) as

$$V\hat{x}_b = V_{L_1}\Psi_0 x_0 + V_{L_2}\hat{x}_1 + V_{R_1}\hat{x}_{I-1} + V_{R_2}\Psi_I x_I + V_{B_1}\hat{d}_B + V_{T_2}\hat{d}_T$$

$$+ V_{B_2}\sum_{i=1}^{I-1} S_B^i \hat{x}_i + V_{T_1}\sum_{i=1}^{I-1} S_T^i \hat{x}_i \tag{6.E.8a}$$

or in more compact notation

$$V\hat{x}_b = \sum_{i=0}^{I} \tilde{V}_x^i \hat{x}_i + [V_{T_2} \vdots V_{B_1}]\begin{bmatrix} \hat{d}_T \\ \hat{d}_B \end{bmatrix} \tag{6.E.8b}$$

where

$$\tilde{V}_x^0 = V_{L_1}\Psi_0 \tag{6.E.9a}$$

$$\tilde{V}_x^1 = V_{L_2} + V_{B_2} S_B^1 + V_{T_1} S_T^1 \tag{6.E.9b}$$

$$\tilde{V}_x^i = V_{B_2} S_B^i + V_{T_1} S_T^i \qquad ; \ i = 2, 3, \ldots I-2 \tag{6.E.9c}$$

$$\tilde{V}_x^{I-1} = V_{R_1} + V_{B_2} S_B^{I-1} + V_{T_1} S_T^{I-1} \tag{6.E.9d}$$

and

$$\tilde{V}_x^{I} = V_{R_2} \Psi_I \ . \tag{6.E.9e}$$

Finally, if we define

$$V_x^i \equiv V' \Pi_v^{-1} \tilde{V}_x^i \qquad ; \quad i = 0, 1, \ldots I \tag{6.E.10a}$$

then (6.E.8b) can be expressed as

$$V' \Pi_v^{-1} V \hat{x}_b = \sum_{i=0}^{I} V_x^i \hat{x}_i + V' \Pi_v^{-1} \left[V_{T_2} \vdots V_{B_1} \right] \begin{bmatrix} \hat{d}_T \\ \hat{d}_B \end{bmatrix} \tag{6.E.10b}$$

which is the desired form for the first term of (6.3.14).

Part of the work of writing the second term of (6.3.14), $E\hat{\lambda}_b$, in the form of (6.E.1) is done in Appendix 6D. In particular, from (6.D.23) and (6.D.24), we have

$$E\hat{\lambda}_b = E_{L_1} \hat{\xi}_L + E_{L_2} \hat{\lambda}_{L+1} + E_{R_1} \hat{\lambda}_{R-1} + E_{R_2} \hat{\xi}_R + E_{B_2} \hat{\lambda}_{B+1} + E_{B_1} \hat{\xi}_B$$

$$+ E_{T_1} \hat{\lambda}_{T-1} + E_{T_2} \hat{\xi}_T \ . \tag{6.E.11}$$

Noting the similarities between the definitions of d_L, d_R, etc. and ξ_L, ξ_R, etc. (see (6.A.5) and (6.D.18)), we can immediately write

$$E\hat{\lambda}_b = \sum_{i=0}^{I} V_\lambda^i \hat{\lambda}_i + \left[E_{T_2} \vdots E_{B_1} \right] \begin{bmatrix} \hat{\xi}_T \\ \hat{\xi}_B \end{bmatrix} \tag{6.E.12}$$

where

$$V_\lambda^0 = E_{L_1} \left[I_{J-1} \otimes [I_{n_3} \vdots \ 0] \right] (I \otimes \Gamma_3) \tag{6.E.13a}$$

$$V_\lambda^1 = E_{L_2} + E_{B_2} S_B^1 + E_{T_1} S_T^1 \tag{6.E.13b}$$

$$V_\lambda^i = E_{B_2} S_B^i + E_{T_1} S_T^i \qquad ; \; i = 2, 3, \ldots I-2 \tag{6.E.13c}$$

$$V_\lambda^{I-1} = E_{R_1} + E_{B_2} S_B^{I-1} + E_{T_1} S_T^{I-1} \tag{6.E.13d}$$

and

$$V_\lambda^I = E_{R_2} \left[I_{J-1} \otimes [I_{n_1} \vdots \; 0] \right] (I \otimes \Gamma_1) \quad . \tag{6.E.13e}$$

Finally, we can combine the expressions in (6.E.10b) and (6.E.12) and the relationships in (6.E.2) and (6.E.3b) to write the entire smoother boundary condition (6.3.14) in the desired multi-point form:

$$0 = \sum_{i=0}^{I} V_{x\lambda}^i \begin{bmatrix} \hat{x} \\ \hat{\lambda} \end{bmatrix}_i + V_{x\lambda_{TB}} \begin{bmatrix} \hat{d} \\ \hat{\xi} \end{bmatrix}_{TB} \tag{6.E.14a}$$

where

$$V_{x\lambda}^i = [\, V_x^i \;\vdots\; V_\lambda^i \,] S_{x\lambda} \tag{6.E.14b}$$

and

$$V_{x\lambda_{TB}} = [\, V'\Pi_v^{-1} V_{T_2} \;\vdots\; E_{T_2} \;\vdots\; V'\Pi_v^{-1} V_{B_1} \;\vdots\; E_{B_1} \,] \quad . \tag{6.E.14c}$$

CHAPTER 7: OPERATOR DIAGONALIZATION OF THE ESTIMATOR DYNAMICS

SECTION 7.1
INTRODUCTION

Having derived an internal differential realization of the estimator for boundary value stochastic processes in Chapter 2, we have concentrated our efforts in Chapters 3, 5 and 6 on transforming that operator solution into representations which lend themselves to efficient and numerically stable implementations. In particular, for the 1-D continuous and 1-D discrete problems, we have sought diagonalized dynamical representations whose diagonal components were each stable with respect to some particular direction of recursion. These diagonalized representations were obtained by writing the estimator dynamics in either differential or difference form and investigating equivalent dynamical representations in those forms. In each case the diagonalized representation was shown to be one member of the class of equivalent representations. In this chapter, we unify the approach to diagonalization of the estimator dynamics in such a way that all cases are described within a single operator framework.

In Section 7.2 we describe some equivalent dynamical representations for the differential operator description of the estimator dynamics. The diagonal form we seek, if it exists, is in the class of equivalent differential operator representations, and we present the conditions which define the class of transformation operators which lead to such forms. As an alternative to diagonalization, we outline a method for triangularizing the dynamics which leads to a representation of the estimator which is similar to smoothers obtained for 1-D and 2-D causal processes by the innovations approach [55] and [6]. We show that this approach has some advantages over diagonalization. In particular, in contrast to diagonalization, no operator inversions are required in computing the estimate by the triangularization approach we suggest.

In Section 7.3, we turn to the continuous and discrete 1-D cases, for which we have already solved the diagonalization problem, and reconsider those problems within the operator framework. We are able to solve the operator diagonalization for these cases by restricting the class of transformation operators to those with a specific structure that we have chosen through hindsight of our previous work with the 1-D problems. With the insight gained from the 1-D solutions, we move on to estimators for processes governed by second order (2-D) partial differential operators in Section 7.4. Assuming the same structure for the transformation operator as employed in the 1-D case, we are able to solve the diagonalization problem for a distributed parameter process goverened by a parabolic partial differential equation. Given the solution for this specific process, we extend our investigation to include the general class of processes governed by second order partial differential equations. Although we lay the groundwork for solving the diagonalization problem for the general 2-D continuous case, some questions remain unanswered in this extended investigation so that it should be viewed in part as a suggestion for further research. Of particular interest is the result that, subject to the existence of the solution to Riccati operator equations [56], there exists a two-filter type implementation of the estimator solution in the 2-D case which is similar in many respects to two-filter solutions for the 1-D case as derived in Chapter 3. We also investigate the triangularization solution for the 2-D continuous case which, as discussed above, has certain advantages when compared to diagonalization.

Finally, in the last part of Section 7.4 we discuss the diagonalization problem for the estimator of a quadrant-causal 2-D discrete process governed by a model introduced by Roesser [57]. We investigate the existence of a particularly simple form of two-filter smoother (each quadrant-causal with different directions of causality, e.g. a northeast-southwest pair) for processes of this type. Employing the methodology for deriving diagonal forms developed in this chapter, we prove that no two-filter solution of this type exists. In closing we propose an approach to developing a four-filter implementation of the estimator for these processes by triangularization of the estimator dynamics.

SECTION 7.2

EQUIVALENT DIFFERENTIAL OPERATOR REPRESENTATIONS

Within the class of equivalent differential operator representations for the estimator dynamics is, if it exists, the stable decoupled form we seek. In this section we describe the class of equivalent representations for the differential operator form of the estimator derived in Chapter 2. The operator transformations which lead to these equivalent representations are quite similar to the matrix transformations which defined the class of equivalent 1-D discrete descriptor form representations in Chapter 2.

7.2.1 Equivalent Representations

The differential operator form of the estimator dynamics developed via the method of complementary models in Chapter 2 is given by (see (2.5.25a))

$$\begin{bmatrix} L & -BQB^* \\ C^*R^{-1}C & L^\dagger \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{\lambda} \end{bmatrix} = \begin{bmatrix} 0 \\ C^*R^{-1} \end{bmatrix} y \quad . \tag{7.2.1}$$

Our goal is, if possible, to diagonalize these dynamics into two decoupled systems, each of which is stable. The stability property is desirable for purposes of numerical implementation of the estimator. The operator diagonalization is analogous to what we have done previously in Chapters 3, 5 and 6 for differential and difference realizations of the estimator dynamics for 1-D continuous, 1-D discrete and 2-D discrete processes. We start by investigating <u>equivalent operator representations</u> of dynamics described by equations such as (7.2.1).

Consider the general differential operator form of (7.2.1)

$$\Lambda z = Bu \qquad ; \ \Lambda : X \to Y \tag{7.2.2}$$

where Λ is a differential operator whose domain and range X and Y are two inner product spaces. Motivated by our work with 1-D discrete descriptor form systems in Chapter 5, consider invertible operators $T: X \to X$ and $F: Y \to Y$ which give rise to the equivalence transformation

$$\xi = Tz \tag{7.2.3a}$$

and equivalent dynamics

$$F\Lambda z = FBu$$

$$F\Lambda T^{-1}\xi = FBu \quad . \tag{7.2.3b}$$

Defining

$$\tilde{\Lambda} = F\Lambda T^{-1} \tag{7.2.4a}$$

and

$$\tilde{B} = FB \quad , \tag{7.2.4b}$$

we can write the dynamics in (7.2.3b) in the form of (7.2.2) as

$$\tilde{\Lambda}\xi = \tilde{B}u \quad . \tag{7.2.4c}$$

7.2.2 Operator Diagonalization

In decoupling the estimator dynamics, our objective is to find operators T and F which transform (7.2.1) as

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = T \begin{bmatrix} \hat{x} \\ \hat{\lambda} \end{bmatrix} \tag{7.2.5a}$$

and

$$F \left[\begin{array}{c:c} L & -BQB^{*} \\ \hdashline C^{*}R^{-1}C & L^{\dagger} \end{array} \right] T^{-1} = \left[\begin{array}{c:c} L_1 & 0 \\ \hdashline 0 & L_2 \end{array} \right] \tag{7.2.5b}$$

so that L_1 and L_2 represent dynamics that are stable as discussed earlier.

To obtain the most general expression for (7.2.5b) we could define partitions of F and T^{-1} as

$$F = \begin{bmatrix} F_1 & F_2 \\ F_3 & F_4 \end{bmatrix} \quad \text{and} \quad T^{-1} = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \tag{7.2.6}$$

and carry out the indicated product. The first set of conditions for decoupling would be given by setting the upper right and lower left partitions of the product in (7.2.5b) to zero. Next we would add the constraint that L_1 and L_2, given by the upper left and lower right partitions of the product, must have some appropriate stability properties.

Unfortunately, determining the complete class of F and T which are compatible with this most general statement of the problem is decidely nontrivial for arbitrary estimator dynamics (7.2.1). With the benefit of our previous work on diagonalization of 1-D estimator dynamics, we will constrain the problem statement by assuming the following form for the operator T in (7.2.6)

$$T = \begin{bmatrix} \theta_1 & -I \\ \theta_2 & I \end{bmatrix} \tag{7.2.7a}$$

with its inverse written as

$$T^{-1} = \begin{bmatrix} I & I \\ -\theta_2 & \theta_1 \end{bmatrix} \begin{bmatrix} P_s & 0 \\ 0 & P_s \end{bmatrix} \tag{7.2.7b}$$

where

$$P_s = (\theta_1 + \theta_2)^{-1} \quad . \tag{7.2.7c}$$

Although we have not yet determined the form of either θ_1 or θ_2, we will assume for the time being that their sum is invertible.

Substituting (7.2.7b) in (7.2.5b) and carrying out the product for an arbitrary operator F (see (7.2.6)) gives

$$\left[\begin{array}{c:c} F_1L + F_1BQB^*\theta_2 + F_2C^*R^{-1}C - F_2L^\dagger\theta_2 & F_1L - F_1BQB^*\theta_1 + F_2C^*R^{-1}C + F_2L^\dagger\theta_1 \\ \hdashline F_3L + F_3BQB^*\theta_2 + F_4C^*R^{-1}C - F_4L^\dagger\theta_2 & F_3L - F_3BQB^*\theta_1 + F_4C^*R^{-1}C + F_4L^\dagger\theta_1 \end{array}\right]$$

$$\cdot \begin{bmatrix} P_s & 0 \\ & P_s \end{bmatrix} = \begin{bmatrix} L_1 & 0 \\ & L_2 \end{bmatrix} \qquad (7.2.8)$$

To achieve the diagonal form of (7.2.5b), we require:

$$F_1L - F_1BQB^*\theta_1 + F_2C^*R^{-1}C + F_2L^\dagger\theta_1 = 0 \qquad (7.2.9a)$$

and

$$F_3L + F_3BQB^*\theta_2 + F_4C^*R^{-1}C - F_4L^\dagger\theta_2 = 0 \quad . \qquad (7.2.9b)$$

Expressions for L_1 and L_2 are obtained by substituting from (7.2.9a) for $F_1L + F_2C^*R^{-1}C$ into the upper left partition of (7.2.8) to get

$$L_1 = -F_2L^\dagger + F_1BQB^* \qquad (7.2.9c)$$

and by substituting from (7.2.9b) for $F_3L + F_4C^*R^{-1}C$ into the lower left partition of (7.2.8) to get

$$L_2 = F_4L^\dagger - F_3BQB^* \quad . \qquad (7.2.9d)$$

Thus we must find (if possible) θ_1, θ_2, and the four partitions of F such that:

(i) (7.2.9a) and (7.2.9b) are satisfied,
(ii) the sum of θ_1 and θ_2 is invertible,
(iii) the operator F is invertible

and

(iv) the diagonal elements of (7.2.8) are stable causal and stable anticausal differential operators.

When considering continuous parameter problems in either 1-D or 2-D, we will find it especially useful to have the operator L in the form of a diffusion, i.e. $L = L_t - A$ where L_t is differentiation with respect to t and in the 2-D case A contains no partials with respect to t. In particular, given this form for L, we show that the equations for θ_1 and θ_2 are Riccati equations (in the 2-D case, "operator Riccati" equations [58]), and and we will make use of the available theory regarding the existence of solutions for equations of this type to establish diagonal forms.

In Section 7.3 we reconsider the diagonalization of the estimator dynamics for the 1-D continuous and 1-D discrete problems which we have already solved by other methods in Chapters 2 and 5. The reason for looking at the 1-D problems in this light is to gain insight into how we might go about performing the diagonalization (i.e., determining the operators F and T) for 2-D cases.

7.2.3 Operator Triangularizaton

Here we present an oultine of how one could pursue an implementation of the estimator by triangularizing the estimator dynamics rather than by diagonalizing. As discussed in the introduction to this chapter, the principle advantage of the triangularized representation as compared to the diagonalized representation is that the former can be developed in such a way that no operator inverses are required in recovering the estimate of the process x from the values of the transformed processes (in diagonalization the inverse of $\theta_1 + \theta_2$ is required). In addition, we will see that we only need to solve a single operator equation as opposed to the two in (7.2.9a) and (7.2.9b).

To (lower) triangularize the estimator dynamics, we seek transformations T and F (see (7.2.4)) which lead to

$$F\begin{bmatrix} L & -BQB^* \\ C^*R^{-1}C & L^\dagger \end{bmatrix} T^{-1} = \begin{bmatrix} L_1 & 0 \\ L_{21} & L_2 \end{bmatrix} \tag{7.2.30}$$

with a similar form for an upper triangularization. Next we present specific structures for F and T which lead to a triangularization with no operator inverses. In particular, choose T to have the structure

$$T = \begin{bmatrix} I & -P \\ 0 & I \end{bmatrix} \quad \text{with inverse} \quad T^{-1} = \begin{bmatrix} I & P \\ 0 & I \end{bmatrix}, \tag{7.2.31a}$$

and choose F to have the form (which coincidentally is the same as T^{-1})

$$F = \begin{bmatrix} I & P \\ 0 & I \end{bmatrix} \quad . \tag{7.2.31b}$$

Then it can be shown by direct substitution into (7.2.30) that the estimator dynamics become

$$\left[\begin{array}{c:c} L_1 & 0 \\ \hdashline L_{21} & L_2 \end{array}\right] = \left[\begin{array}{c:c} L + PC^* R^{-1} C & 0 \\ \hdashline C^* R^{-1} C & L^\dagger + C^* R^{-1} CP \end{array}\right] \quad . \tag{7.2.32}$$

The condition for this triagualar form is the existence of a solution to the single operator equation

$$LP + PL^\dagger + PC^* R^{-1} CP - BQB^* = 0 \quad . \tag{7.2.33}$$

In Section 7.4.2 we will present a detailed investigation of the triangular representation of the estimator for the 2-D continuous case.

SECTION 7.3

OPERATOR DIAGONALIZATION FOR 1-D ESTIMATORS

In this section we solve the diagonalization problem for the 1-D continuous and 1-D discrete estimators given the assumed form for T in (7.2.7a). In applying the conditions in (7.2.9a) and (7.2.9b) we will use the basic definition of a zero operator. That is, an operator M is 0 if and only if $M\zeta = 0$ for all ζ in the domain of M.

7.3.1 The 1-D Continuous Case

To write the estimator dynamics in the operator form (7.2.1) recall that the operational definitions of L and $L^\dagger$ for the 1-D continuous case are:

$$(L\eta)(t) = \dot{\eta}(t) - A(t)\eta(t) \tag{7.3.1a}$$

and

$$(L^\dagger\eta)(t) = -\dot{\eta}(t) - A'(t)\eta(t) \quad . \tag{7.3.1b}$$

Note that each of these has the form of a diffusion in t as defined earlier. The action of the operators C, B, R and Q is simply multiplication by the matrices C(t), B(t), R(t) and Q(t). The adjoints C^* and B^* are given by the matrix transposes C'(t) and B'(t). To simplify the notation, hereafter we will omit reference to the independent variable t.

Here we assume that F and T are time-varying invertible matrices and our problem is to find dynamics governing their elements so that the conditions in (7.2.9a) and (7.2.9b) are met. The condition in (7.2.9a) is equivalent (for arbitrary ζ) to

$$F_1(d/dt - A)\zeta + F_2(-d/dt - A')(\theta_1\zeta) - F_1BQB'\theta_1\zeta + F_2C'R^{-1}C\zeta = 0$$

or carrying out the differentiation

$$\left(F_1 - F_2\theta_1\right)\dot{\zeta} + \left(-F_1A - F_2A'\theta_1 - F_2\dot{\theta}_1 - F_1BQB'\theta_1 + F_2C'R^{-1}C\right)\zeta = 0 \quad . \tag{7.3.2a}$$

Since this equation must be true for arbitrary ζ in the space of continuously differentiable functions, the coefficients of both ζ and its derivative must be zero. Considering the coefficient of the derivative first, we have

$$F_1 = F_2\theta_1 \quad . \tag{7.3.2b}$$

Substituting this into the coefficient for ζ and setting that coefficient to zero gives

$$F_2\left(\dot{\theta}_1 + \theta_1 A + A'\theta_1 + \theta_1 BQB'\theta_1 - C'R^{-1}C\right) = 0 \quad . \tag{7.3.2c}$$

A similar application of (7.2.9b) results in

$$F_3 = -F_4\theta_2 \tag{7.3.3a}$$

and

$$F_4\left(\dot{\theta}_2 + \theta_2 A + A'\theta_2 - \theta_2 BQB'\theta_2 + C'R^{-1}C\right) = 0 \quad . \tag{7.3.3b}$$

If we choose F as

$$F = \begin{bmatrix} \theta_1 & I \\ \theta_2 & -I \end{bmatrix} \quad , \tag{7.3.4}$$

then (7.3.2c) and (7.3.3b) are the usual Riccati equations for the dynamics of θ_1 and θ_2 and uniform complete controllabiltiy and reconstructability of the triple $\{A,B,C\}$ guarantees the invertiblity of both F and T. Furthermore, it can be shown by substituting from (7.3.4) into (7.2.9c) and (7.2.9d) that L_1 and L_2 are given by

$$L_1 = -L^{\dagger} + \theta_1 BQB^{*} \tag{7.3.5a}$$

and

$$L_2 = -L^{\dagger} - \theta_2 BQB^{*} \quad . \tag{7.3.5b}$$

These result in the same dynamics as those obtained by the Hamiltonian diagonalization in Chapter 3. Note that we have not specified the boundary conditions for either θ_1 or θ_2. As we found in Chapter 3, these boundary conditions should be chosen to simplify the transformed estimator boundary conditions.

7.3.2 The 1-D Discrete Case

In considering the operator form of the estimator dynamics for the 1-D discrete case, we must be careful to recall that these dynamics were originally derived in Chapter 5 in terms of the adjoint process referred to there as γ:

$$\begin{bmatrix} L & \vdots & -BQB^* \\ - & & - \\ C^*R^{-1}C & \vdots & L^\dagger \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{\gamma} \end{bmatrix} = \begin{bmatrix} 0 \\ - \\ C^*R^{-1} \end{bmatrix} y \tag{7.3.6a}$$

where the formal difference operator L and its formal adjoint are given by

$$(L\zeta)_k = \zeta_{k+1} - A_k\zeta_k \qquad \left(L = D^{-1}I - A\right) \tag{7.3.6b}$$

and

$$(L^\dagger\zeta)_k = \zeta_{k-1} - A_k'\zeta_k \qquad \left(L^\dagger = DI - A'\right) \quad . \tag{7.3.6c}$$

Having derived the dynamics in terms of γ, we found that the estimator boundary condition could be simplified if we rewrote the estimator in terms of the shifted process λ:

$$\lambda_k = \gamma_{k-1} \tag{7.3.7a}$$

or in terms of the 1-D delay operator D:

$$\lambda = D\gamma \qquad (\ \gamma = D^{-1}\lambda\ ) \quad . \tag{7.3.7b}$$

By employing (7.3.7b), the estimator dynamics can be rewritten in terms of λ as

$$\begin{bmatrix} L & \vdots & -BQB^*D^{-1} \\ - & & - \\ C^*R^{-1}C & \vdots & L^\dagger D^{-1} \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{\lambda} \end{bmatrix} = \begin{bmatrix} 0 \\ - \\ C^*R^{-1} \end{bmatrix} y \quad . \tag{7.3.8}$$

We will assume the same form for the operator T and its inverse as we did for the 1-D continuous case (see (7.2.7)) and also that the action of both F and T is multiplication by a k-varying martix. The diagonalization conditions for the operator dynamics in (7.3.8) corresponding to the conditions in (7.2.9a) and (7.2.9b) for the continuous case are

$$F_1L - F_1BQB^*D^{-1}\theta_1 + F_2C^*R^{-1}C + F_2L^\dagger D^{-1}\theta_1 = 0 \tag{7.3.9a}$$

and

$$F_3L + F_3BQB^*D^{-1}\theta_2 + F_4C^*R^{-1}C - F_4L^\dagger D^{-1}\theta_2 = 0 \quad . \tag{7.3.9b}$$

The only change has been to include D^{-1} to account for the change in variable form γ to λ. With this addition, the diagonal operators L_1 and L_2 previously given by (7.2.9c) and (7.2.9d) become

$$L_1 = -F_2L^\dagger D^{-1} + F_1BQB^*D^{-1} \tag{7.3.9c}$$

and

$$L_2 = F_4L^\dagger D^{-1} - F_3BQB^*D^{-1} \quad . \tag{7.3.9d}$$

To determine the dynamics of θ_1 and θ_2, first consider the action of (7.3.9a) on an arbitrary sequence ζ with the coefficients of ζ_k and ζ_{k+1} explictly displayed:

$$\left(F_{1,k}(I - B_kQ_kB_k'\theta_{1,k+1}) - F_{2,k}A_k'\theta_{1,k+1}\right)\zeta_{k+1}$$
$$+ \left(F_{2,k}(C_k'R_k^{-1}C_k + \theta_{1,k}) - F_{1,k}A_k\right)\zeta_k = 0 \quad . \tag{7.3.10}$$

For (7.3.10) to hold for an arbitrary sequence, the coefficients of each of ζ_k and ζ_{k+1} must be zero. Solving for F_2 by setting the coefficient for ζ_k to zero gives

$$F_{2,k} = F_{1,k}A_k(C_k'R_k^{-1}C_k + \theta_{1,k})^{-1} \quad . \tag{7.3.11a}$$

Substituting this expression into the coefficient for ζ_{k+1} in (7.3.10) and equating it to zero, we have

$$F_{1,k}\left(I - B_kQ_kB_k'\theta_{1,k+1} - A_k(C_k'R_k^{-1}C_k + \theta_{1,k})^{-1}A_k'\theta_{1,k+1}\right) = 0 \quad . \tag{7.3.11b}$$

This leads to the forward difference equation for θ_1 that we previously derived in Chapter 5, namely:

$$\theta_{1,k+1} = \left(A_k (C_k' R_k^{-1} C_k + \theta_{1,k})^{-1} A_k' + B_k Q_k B_k' \right)^{-1} \quad . \qquad (7.3.11c)$$

In a similar fashion, a backward difference equation for θ_2 is derived from the action of the zero operator in (7.3.9b) on an arbitrary sequence:

$$\left(F_{3,k}(I + B_k Q_k B_k' \theta_{2,k+1}) + F_{4,k} A_k' \theta_{2,k+1} \right) \zeta_{k+1}$$

$$+ \left(F_{4,k}(C_k' R_k^{-1} C_k - \theta_{2,k}) - F_{3,k} A_k \right) \zeta_k \quad . \qquad (7.3.12a)$$

Equating the coefficients to zero gives:

$$F_{3,k} = -F_{4,k} A_k' \theta_{2,k+1} (I + B_k Q_k B_k' \theta_{2,k+1})^{-1} \qquad (7.3.12b)$$

and

$$F_{4,k}\left(\theta_{2,k} - C_k' R_k^{-1} C_k - A_k' \theta_{2,k+1} (I + B_k Q_k B_k' \theta_{2,k+1})^{-1} A_k \right) = 0 \ . \qquad (7.3.12c)$$

This gives the backward difference equation for θ_2:

$$\theta_{2,k} = C_k' R_k^{-1} C_k + A_k' \theta_{2,k+1} (I + B_k Q_k B_k' \theta_{2,k+1})^{-1} A_k \qquad (7.3.13)$$

If we choose both $F_{1,k}$ and $F_{4,k}$ as the identity, then F_k becomes

$$F_k = \left[\begin{array}{c:c} I & A_k (C_k' R_k^{-1} C_k + \theta_{1,k})^{-1} \\ \hdashline -A_k' \theta_{2,k+1} (I + B_k Q_k B_k' \theta_{2,k+1})^{-1} & I \end{array} \right] . \qquad (7.3.14)$$

Substituting from (7.3.14) into (7.3.9c) and (7.3.9d), it can be shown that the resulting operators L_1 and L_2 are the same as the dynamics found earlier in Chapter 5 for the diagonalized system.

It is interesting to note that had we allowed the operator T to contain shift dynamics rather than restricting it to a k-varying matrix, then we could have applied

$$T = \begin{bmatrix} \theta_1 & -I \\ \theta_2 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & D \end{bmatrix} \tag{7.3.15}$$

(where D is the 1-D delay) directly to (x, γ) in the original estimator dynamics (7.3.6a) to obtain the identical decoupled system.

It has been our objective to demonstrate the operator diagonalization procedure for the familiar 1-D cases as a preview to the more complex 2-D cases to be investigated in the next section. We emphasize that these 1-D diagonalizations were made manageable by our educated guess for the form of the operator T in (7.2.7). We will see that this form is also useful for 2-D problems.

SECTION 7.4

OPERATOR DIAGONALIZATION FOR 2-D ESTIMATORS

7.4.1 Introduction

In this section we consider the implementation problem for continuous and discrete parameter 2-D processes. We start with the continuous parameter case, and in particular, we investigate the solution for an initial-boundary-value process governed by a parabolic partial differential equation representing transient 1-D heat flow. By performing an operator diagonalization of the estimator dynamics in a manner similar to that employed for the 1-D cases in the previous section, we will be able to put our estimator dynamics into a form which is similar to that derived for infinite dimensional space-time systems [6,56]. As we will see, a key to the diagonalization for the parabolic example is an existence and representation result for the solution of an operator Riccati equation.

Having found that a specific structure for the operator T was useful in performing the diagonalization in the 1-D case and for the 2-D parabolic example, we investigate the usefulness of that same structure for more general 2-D problems including hyperbolic and elliptic systems. In keeping the 2-D problem description as close to that of the 1-D, we rewrite the 2-D partial differential operators in the form of a diffusion operator that is similar to the ordinary differential operators in the 1-D case. In the continuous parameter 2-D case, that form has been employed by researchers in the field of infinite dimensional or space-time systems (for example, see [6]). The infinite-dimensional terminology is used because with the partial differential dynamics written as a first order diffusion, the state process is function-valued (infinite dimensional) rather than vector-valued (finite dimensional) as it is in the case of ordinary differential equations. That is, the partial differential equation is written as an ordinary differential equation whose solution is an element of a Hilbert space of functions. The difficulty in proceeding to diagonalization for the more general systems is that for those cases the questions of existence and representation of the solution of the operator Riccati equation have not been thoroughly resolved.

Diagonalization is useful in developing implementation schemes for the estimator in 2 and higher dimensional cases, for the same reasons as those we found in the 1-D case. Specifically, if we can rewrite the dynamics as two decoupled subsystems (each in the form of an infinite dimensional diffusion), one whose dynamics are causally stable and the other whose dynamics are anti-causally stable, then we can compute the dynamical contribution to the estimator solution through zero initial condition solutions of the decoupled dynamical systems (an infinite dimensional two-filter solution). The boundary contribution is then added via superposition (as we also did in the 1-D case). As we will see, the ability to compute the boundary contribution is made possible because under certain conditions (which we discuss later) there is an infinte dimensional version of the variation of constants formula which directly parallels the finite dimensional version for the 1-D case [59].

The number and complexity of technical difficulties encountered when working with partial differential equations as compared to ordinary differential equations is considerable and should not be underestimated. The methodology proposed in this section provides a means for specifying Ricatti operator equations which lead to the diagonalization of the estimator dynamics for the general 2-D case. However, we offer no simple solution for the establishing the existence of solutions to these Riccati equations. So although this section does not present a complete solution to diagonalization problems for general 2-D systems, it does give a structured approach for attacking them.

For the 2-D discrete case, we consider a process governed by quadrant-causal dynamics of the type introduced by Roesser [57]. Although there is a straightforward way in which to write the the operator representation of these 2-D dynamics in a form similar to that of the 1-D discrete case, we will find that, unlike the continuous case, the 2-D discrete estimator diagonalization does not follow as a straightforward extension of the 1-D case. In fact our analysis indicates that for the quadrant causal case, no diagonalization of the estimator dynamics into two quadrant causal systems is possible.

For convenience we rewrite the form of the operator T proposed earlier in Section 7.2.2:

$$T = \begin{bmatrix} \theta_1 & -I \\ \theta_2 & I \end{bmatrix} \tag{7.4.1a}$$

with inverse

$$T^{-1} = \begin{bmatrix} I & I \\ -\theta_2 & \theta_1 \end{bmatrix} \begin{bmatrix} P_s & 0 \\ 0 & P_s \end{bmatrix} \tag{7.4.1b}$$

where

$$P_s = (\theta_1 + \theta_2)^{-1} \quad . \tag{7.4.1c}$$

The diagonal representation of the estimator dynamics under this transformation is rewritten from (7.2.8):

$$\left[\begin{array}{c:c} F_1L + F_1BQB^*\theta_2 + F_2C^*R^{-1}C - F_2L^\dagger\theta_2 & F_1L - F_1BQB^*\theta_1 + F_2C^*R^{-1}C + F_2L^\dagger\theta_1 \\ \hdashline F_3L + F_3BQB^*\theta_2 + F_4C^*R^{-1}C - F_4L^\dagger\theta_2 & F_3L - F_3BQB^*\theta_1 + F_4C^*R^{-1}C + F_4L^\dagger\theta_1 \end{array}\right]$$

$$\cdot \begin{bmatrix} P_s & 0 \\ 0 & P_s \end{bmatrix} = \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix} \quad . \tag{7.4.2}$$

7.4.2 The 2-D Continuous Case

We start by considering the specific example of estimating a 2-D process representing the transient heat distribution in a 1-D medium [56], i.e. one independent variable represents time and the other distance. After stating the estimation problem in the appropriate operator notation, we are able to write the estimator dynamics and boundary conditions. Next we seek operators θ_1, θ_2 and F which lead to a diagonal form for the transformed estimator dynamics in (7.4.2). In particular, we find that θ_1 and θ_2 are integral operators. Furthermore, we show that the boundary conditions of the transformed dynamically decoupled estimator processes q_1 and q_2 can also be decoupled by an appropriate choice of the boundary conditions for the kernels of these integral operators. Finally, we consider the diagonalization problem for more generel second order processes and discuss some new problems that may arise in the general case.

With $x(t,s)$ representing temperature at time t and position s, the temperature distribution in a homogeneous rod is described by

$$\frac{\partial x(t,s)}{\partial t} = \alpha^2 \frac{\partial^2 x(t,s)}{\partial s^2} + B(t,s)u(t,s) \tag{7.4.3}$$

where u is a 2-D white noise with continuous covariance parameter $Q(t,s)$, $B(t,s)$ a continuous scaling parameter and α is the thermal diffusivity. See Section 6.3.3 for a further discussion of this equation. The region Ω is the open rectangle $(0,T)\times(a,b)$ (that is, we consider the time interval $(0,T)$ and the position along the rod starting at $s=a$ and ending at $s=b$). The boundary conditions are assumed to be given by the initial condition:

$$x(0,s) = v_0(s) \quad ; \ s \in (a,b) \tag{7.4.4a}$$

and spatial boundary conditions:

$$x(t,a) = v_a(t) \quad ; \ t \in (0,T) \tag{7.4.4b}$$

and

$$x(t,b) = v_b(t) \quad ; \ t \in (0,T) \tag{7.4.4c}$$

where v_0, v_a and v_b are 1-D white noises with nonzero continuous covariance parameters $\Pi_0(s)$, $\Pi_a(t)$ and $\Pi_b(t)$ respectively. This model is a popular one in the distributed parameter systems literature (see for instance [6] and [56]) except there the input noise is modelled as only being white in time. The measurements are given by

$$y(t,s) = C(t,s)x(t,s) + r(t,s) \quad \text{on } \Omega \tag{7.4.5}$$

where C is continuous in both variables and r is a 2-D white noise with nonzero continuous covariance parameter $R(t,s)$.

The problem statement can be rewritten in an operator form which is appropriate for specifying the solution derived in Chapter 2 as follows. Define the formal differential operators

$$L_t = \frac{\partial}{\partial t} \tag{7.4.6a}$$

$$A = \alpha^2 \frac{\partial^2}{\partial s^2} \quad , \tag{7.4.6b}$$

so that their difference is a diffusion operator in t:

$$L = L_t - A \quad . \tag{7.4.6c}$$

It is straightforward to see that the dynamics in (7.4.3) can be expressed in terms of L as

$$Lx = Bu \quad , \tag{7.4.7}$$

which is the form we seek.

Recall that the estimator solution derived in Chapter 2 requires that the boundary condition for x be specified in terms of a process x_b (i.e. $v=Vx_b$) where x_b is determined from Green's identity. Specifically, the Green's identity for L in (7.4.6c) operating on functions whose support is the rectangle $\Omega = (0,T)\times(a,b)$ is given by [3]

$$\langle Lx,\lambda\rangle_{L_2(\Omega)} = \langle x,L^\dagger\lambda\rangle_{L_2(\Omega)} + \text{boundary term} \tag{7.4.8a}$$

where the formal adjoint differential operator $L^\dagger$ is also in the form of a diffusion:

$$L^\dagger = -L_t - A^\dagger \tag{7.4.8b}$$

where for this parabolic example A is formally self-adjoint:

$$A^\dagger = A \quad . \tag{7.4.8c}$$

The "boundary term" is a line integral along the contour defined by the boundary of the rectangle. This line integral can be represented by the sum of four integrals, each corresponding to one of the edges of the rectangle as [24]:

$$\text{b.t.} = \alpha^2 \int_0^T [x\lambda_s - \lambda x_s]_{s=b} dt \; - \; \alpha^2 \int_0^T [x\lambda_s - \lambda x_s]_{s=a} dt$$

$$+ \int_a^b [x\lambda]_{t=T} ds \; - \; \int_a^b [x\lambda]_{t=0} ds \qquad (7.4.8d)$$

where $(\;)_s$ represents the partial with respect to s, $\partial(\;)/\partial s$.

Next we rewrite the expression for the boundary term in (7.4.8d) as an inner product of square integrable functions whose support is the boundary $\partial\Omega$ of the rectangle Ω, i.e. as $\langle x_b, E\lambda_b \rangle$. To determine x_b, λ_b and the operator E, define

$$X_b = \begin{bmatrix} x(\cdot,b) \\ x_s(\cdot,b) \end{bmatrix}, \qquad X_a = \begin{bmatrix} x(\cdot,a) \\ x_s(\cdot,a) \end{bmatrix},$$

$$X_T = \begin{bmatrix} x(T,\cdot) \\ x_t(T,\cdot) \end{bmatrix} \quad \text{and} \quad X_0 = \begin{bmatrix} x(0,\cdot) \\ x_t(0,\cdot) \end{bmatrix} \qquad (7.4.9a)$$

and x_b as

$$x_b = \begin{bmatrix} X_b \\ X_a \\ X_T \\ X_0 \end{bmatrix} \qquad (7.4.9b)$$

with similar definitions for Λ_b, Λ_a, Λ_T, Λ_0 and λ_b.
Also define the partitioned operators

$$E_b = \alpha^2 \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}, \qquad E_a = -E_b$$

and

$$E_T = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \qquad E_0 = -E_T \qquad (7.4.10a)$$

whose partitions are either the zero operator or the identity operator with ranges and domains compatible with the elements of X_b, X_a, X_T and X_0 in (7.4.9). Given these definitions we can write the boundary term in (7.4.8d) as the following sum of inner products

$$\text{b.t.} = \langle X_b, E_b \Lambda_b \rangle_{L_2^2[0,T]} + \langle X_a, E_a \Lambda_a \rangle_{L_2^2[0,T]} + \langle X_T, E_T \Lambda_T \rangle_{L_2^2[a,b]} + \langle X_0, E_0 \Lambda_0 \rangle_{L_2^2[a,b]} \qquad (7.4.10b)$$

Finally, if we combine the operators in (7.4.10a) to define E as the diagonal operator:

$$E = \begin{bmatrix} E_b & & & \\ & E_a & & 0 \\ & & E_T & \\ 0 & & & E_0 \end{bmatrix} , \qquad (7.4.11a)$$

then, with x_b and λ_b defined as in (7.4.9b), the boundary term can be written in the desired inner product form

$$\text{b.t.} = \langle x_b, E\lambda_b \rangle_{L_2^2(\partial\Omega)} . \qquad (7.4.11b)$$

The boundary condition in (7.4.4) can be written in terms of x_b in a similar fashion. Combine all of the boundary value functions into one as

$$v = \begin{bmatrix} v_b(t) \\ v_a(t) \\ v_0(s) \end{bmatrix} \qquad (7.4.12a)$$

and define the following partitioned operators

$$V_b = [\ I \ \vdots \ 0\ ] , \quad V_a = [\ I \ \vdots \ 0\ ] \quad \text{and} \quad V_0 = [\ I \ \vdots \ 0\ ] . \qquad (7.4.12b)$$

With V defined as

$$V = \begin{bmatrix} V_b & 0 & 0 & 0 \\ 0 & V_a & 0 & 0 \\ 0 & 0 & 0 & V_0 \end{bmatrix} , \qquad (7.4.12c)$$

the boundary condition can be expressed in the desired form:

$$v = Vx_b \quad . \tag{7.4.12d}$$

This completes the description of the estimation problem in the form prescribed in Chapter 2.

The boundary condition for the estimator is given by (see (2.5.25b))

$$0 = [V^*\Pi_v^{-1}V \;\vdots\; E]\begin{bmatrix}\hat{x}_b \\ \hline \hat{\lambda}_b\end{bmatrix} \quad . \tag{7.4.13}$$

Given the definitions above, this boundary condition can be represented by the following four conditions, one for each edge of the rectangle.

$$\begin{bmatrix}0 \\ 0\end{bmatrix} = \begin{bmatrix}\Pi_b^{-1}(t)\hat{x}(t,b) - \alpha^2\hat{\lambda}_s(t,b) \\ \hline \alpha^2\hat{\lambda}(t,b)\end{bmatrix} , \tag{7.4.14a}$$

$$\begin{bmatrix}0 \\ 0\end{bmatrix} = \begin{bmatrix}\Pi_a^{-1}(t)\hat{x}(t,a) + \alpha^2\hat{\lambda}_s(t,a) \\ \hline -\alpha^2\hat{\lambda}(t,a)\end{bmatrix} , \tag{7.4.14b}$$

$$0 = \hat{\lambda}(T,s) \tag{7.4.14c}$$

and

$$0 = \Pi_0^{-1}(s)\hat{x}(0,s) - \hat{\lambda}(0,s) \quad . \tag{7.4.14d}$$

The boundary conditions in (7.4.14) along with the dynamics

$$\begin{bmatrix}L_t - A & \vdots & -BQB^* \\ \hline C^*R^{-1}C & \vdots & -L_t - A^{\dagger}\end{bmatrix}\begin{bmatrix}\hat{x} \\ \hat{\lambda}\end{bmatrix} = \begin{bmatrix}0 \\ \hline C^*R^{-1}\end{bmatrix} y \tag{7.4.15}$$

(here L and $L^{\dagger}$ have been replaced by (7.4.6c) and (7.4.8b)) completely specify the estimator. Next we show how the dynamics in (7.4.15) can be diagonalized as in (7.4.2). In particular, in formulating the diagonalized dynamics we will need to apply a result on the existence of solutions to operator Riccati equations which was originally obtained in solving the linear qaudratic control problem for parabolic systems (see for instance [58] or [60]).

We select an operator F for the 2-D case which has the same structure as the matrix F in the 1-D continuous case, i.e.

$$F = \begin{bmatrix} \theta_1 & I \\ \theta_2 & -I \end{bmatrix} \quad . \tag{7.4.16}$$

Due to the similarity between the diffusion representation of the 2-D formal differential operators L and $L^\dagger$ and their 1-D counterparts in (7.3.4a) and (7.3.4b), it can be shown that the transformed dynamics in (7.4.2) are diagonalized with a similar representation for the diagonal operators L_1 and L_2 as that of the 1-D case in (7.3.8), namely:

$$\begin{aligned} L_1 &= -L^\dagger + \theta_1 BQB^* \\ &= L_t + (A + \theta_1 BQB^*) \end{aligned} \tag{7.4.17a}$$

and

$$\begin{aligned} L_2 &= -L^\dagger - \theta_2 BQB^* \\ &= L_t + (A^\dagger - \theta_2 BQB^*) \quad . \end{aligned} \tag{7.4.17b}$$

Thus, the dynamics of the transformed estimator processes

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = T \begin{bmatrix} \hat{x} \\ \hat{\lambda} \end{bmatrix} \tag{7.4.17c}$$

are decoupled as

$$L_1 q_1 = C^* R^{-1} y \qquad \text{and} \qquad L_2 q_2 = -C^* R^{-1} y \quad . \tag{7.4.17d}$$

Of course, as in the 1-D case, our ability to perform the diagonalization (to force the off-diagonal elements of (7.4.2) to zero) is dependent upon the existence of operators θ_1 and θ_2 which satisfy

$$(\theta_1 L_t - L_t\theta_1 - A^{\dagger}\theta_1 - \theta_1 A - \theta_1 BQB^{*}\theta_1 + C^{*}R^{-1}C)\xi = 0 \qquad (7.4.18a)$$

and

$$(\theta_2 L_t - L_t\theta_2 - A^{\dagger}\theta_2 - \theta_2 A + \theta_2 BQB^{*}\theta_2 - C^{*}R^{-1}C)\xi = 0 \qquad (7.4.18b)$$

and whose sum $(\theta_1 + \theta_2)$ is invertible. As mentioned earlier, the question of existence has been resolved for the parabolic case [60]. By assuming the following integral representation for θ_1

$$(\theta_1\xi)(t,s) = \int_a^b \theta_1(t,s,\sigma)\xi(t,\sigma)d\sigma \qquad (7.4.19a)$$

and substituting this representation into (7.4.18a), it can be shown that the integration kernel satisfies the Riccati integro-differential equation:

$$-\frac{\partial}{\partial t}\theta_1(t,s,\sigma) = \frac{\partial^2}{\partial s^2}\theta_1(t,s,\sigma) + \frac{\partial^2}{\partial\sigma^2}\theta_1(t,s,\sigma) - C(t,s)R^{-1}(t,s)\delta(s-\sigma)C(t,\sigma)$$

$$+ \int_a^b \theta_1(t,s,\rho)B(t,\rho)Q(t,\rho)B(t,\rho)\theta_1(t,\rho,\sigma)d\rho \quad . \qquad (7.4.19b)$$

In deriving (7.4.19b) we have replaced $L_t\theta_1\xi$ in (7.4.18a) by

$$L_t\theta_1\xi = \theta_1 L_t\xi + \int_a^b \left[\frac{\partial}{\partial t}\theta_1(t,s,\sigma)\right]\xi(t,\sigma)d\sigma \qquad (7.4.19c)$$

$$= \theta_1 L_t\xi + \left[L_t\theta_1\right]\xi \quad . \qquad (7.4.19d)$$

In addition, by integrating by parts twice we can express $\theta_1 A\xi$ as follows

$$\theta_1 A\xi = \int_a^b \frac{\partial^2}{\partial\sigma^2}\theta_1(t,s,\sigma)\xi(t,\sigma)d\sigma + \frac{\partial\xi(t,\sigma)}{\partial\sigma}\theta_1(t,s,\sigma)\Big|_{\sigma=a}^{\sigma=b}$$

$$- \frac{\partial\theta_1(t,s,\sigma)}{\partial\sigma}\xi(t,\sigma)\Big|_{\sigma=a}^{\sigma=b} \quad . \qquad (7.4.19e)$$

Then choosing the following boundary conditions for θ_1 so that (7.4.18b) is satisfied for arbitrary ξ:

$$\frac{\partial\theta_1(t,s,\sigma)}{\partial\sigma}\Big|_{\sigma=a} = \frac{\partial\theta_1(t,s,\sigma)}{\partial\sigma}\Big|_{\sigma=b} = 0 \tag{7.4.19f}$$

and

$$\theta_1(t,s,a) = \theta_1(t,s,b) = 0 \tag{7.4.19g}$$

we can replace $\theta_1 A\xi$ in (7.4.18a) by

$$\theta_1 A\xi = A_\sigma^\dagger\theta_1\xi \tag{7.4.19h}$$

where the subscript σ indicates partial differentiation with respect to σ. A similar procedure establishes a partial differential equation governing the kernel for an integral representation of the operator θ_2.

We remark that the Riccati integro-differential equation in (7.4.19b) differs slightly from that typically encountered in the distributed parameter systems literature because we have assumed that the input process u and the observation noise r are white in both space and time rather than simply white in time. As a consequence of this assumption the covariance operators R^{-1} and Q having kernels which contain the product of two delta functions $(\delta(t-\tau)\delta(s-\sigma))$ rather than a single delta function. We also note that the existence of the inverse of the sum of θ_1 and θ_2 remains an open question. At the end of this section we outline an alternative approach in which the estimator dynamics are triangularized and for which no operator inversion is required.

As yet, we have not specified the boundary conditions for the integration kernels $\theta_1(t,s,\sigma)$ and $\theta_2(t,s,\sigma)$ whose dynamics are given by Riccati equations of the type (7.4.19b). As in the 1-D case, we exercise the freedom in choosing these boundary conditions to simplify as much as possible the boundary conditions for the transformed processes q_1 and q_2. Substituting (7.4.1) into (7.4.17c) gives the explicit expressions for q_1 and q_2:

$$q_1 = \theta_1\hat{x} - \hat{\lambda} \tag{7.4.20a}$$

$$q_2 = \theta_2\hat{x} + \hat{\lambda} \tag{7.4.20b}$$

with inverse

$$\hat{x} = P_s(q_1 + q_2) \tag{7.4.20c}$$

$$\hat{\lambda} = -\theta_2 P_s q_1 + \theta_1 P_s q_2 \tag{7.4.20d}$$

where P_s is the inverse operator in (7.4.1c). By choosing initial and final values for the kernels as

$$\theta_1(0,s,\sigma) = \Pi_0^{-1}(s)\delta(s-\sigma) \qquad ; \; s,\sigma \in (a,b) \tag{7.4.21a}$$

and

$$\theta_2(T,s,\sigma) = 0 \qquad ; \; s,\sigma \in (a,b) \quad , \tag{7.4.21b}$$

it can be shown by direct substitution from (7.4.20) that the initial and final estimator boundary conditions in (7.4.14d) and (7.4.14c) when expressed in terms of q_1 and q_2 become decoupled:

$$q_1(0,s) = 0 \tag{7.4.21c}$$

and

$$q_2(T,s) = 0 \quad . \tag{7.4.21d}$$

Unfortunately, a similar decoupling of the spatial boundary conditions (7.4.14a) and (7.4.14b) is not so straightforward due to the presence of the partial derivatives λ_s. In particular, to write these two spatial boundary conditions in terms of the transformed variables q_1 and q_2, we must take the partial with respect to s of the expression for λ in (7.4.20d) which contains the inverse operator P_s and evaluate that expression at s=a and s=b. Hopefully, further study (beyond this thesis) will lead to a method for choosing spatial boundary conditions for θ_1 and θ_2 which result in a decoupling of the transformed spatial boundary conditions for q_1 and q_2. This spatial decoupling along with the decoupling of the initial and final conditions in (7.4.21) and the dynamics in (7.4.17) would leave q_1 and q_2 completely decoupled.

Clearly, there is a great deal to be done. For our particular parabolic example alone we have left unanswered:

(1) The invertibility of $(\theta_1 + \theta_2)$ and a representation of that inverse.

(2) The stability of the operator L_1 as a forward operator and L_2 as a backward operator.

(3) The choice of spatial boundary conditions for the kernels of θ_1 and θ_2 which result in the maximum decoupling of the boundary conditions for the transformed processes q_1 and q_2.

Assuming that each of these has been favorably resolved, the estimator could be implemented efficiently as follows. Since L_1 is forward (in time) stable and L_2 is backward stable, q_1 and q_2 could be solved by numerically stable forward and backward finite difference approximations of L_1 and L_2, respectively. Given a representation for the operator P_s, the inverse of the sum of θ_1 and θ_2, the estimate of x could be formed from a finite difference approximation of the realization of P_s acting on the discretized solutions for q_1 and q_2 (see (7.4.20c) above).

Diagonalization for the General Case

Here we briefly discuss some additional complexities which arise when diagonalizing the estimator dynamics for general 2-D processes including space-time processes governed by hyperbolic as well as parabolic equations and purely spatial processes satisfying elliptic partial differential equations. That is, we consider the estimator for the more general class of scalar processes which are governed by

$$Lx = Bu \tag{7.4.22a}$$

(along with an appropriate boundary condition to make the problem well-posed) where L is a formal second order partial differential operator of the form

$$(Lx)(t,s) = \alpha_1(t,s)\frac{\partial^2}{\partial t^2}x(t,s) + 2\alpha_2(t,s)\frac{\partial^2}{\partial t\,\partial s}x(t,s) + \alpha_3(t,s)\frac{\partial^2}{\partial s^2}x(t,s)$$
$$+ \alpha_4\frac{\partial}{\partial t}x(t,s) + \alpha_5\frac{\partial}{\partial s}x(t,s) + \alpha_6(t,s)x(t,s) \tag{7.4.22b}$$

Although not necessary for our developments, we will for simplicity of notation asssume that each of the α_i is constant. In this case, the formal adjoint of this operator is shown in [24] to be given by

$$L^{\dagger}\lambda = \alpha_1\lambda_{tt} + 2\alpha_2\lambda_{ts} + \alpha_3\lambda_{ss} - \alpha_4\lambda_t - \alpha_5\lambda_s + \alpha_6\lambda \tag{7.4.23}$$

where the subscripts s and t represent partial differentiation with respect to that variable. Since as in the parabolic example our intent is to extend the 1-D results, we will rewrite the formal partial differential operators in (7.4.22b) and (7.4.23) as diffusion operators and then express the estimator dynamics in terms of these diffusion operators.

Define (here we assume α_1 is nonzero) the formal differential operators

$$A_1 = -\frac{1}{\alpha_1}\left[\alpha_3 \frac{\partial^2}{\partial s^2} + \alpha_5 \frac{\partial}{\partial s} + \alpha_6\right] \quad , \tag{7.4.24a}$$

$$A_2 = -\frac{1}{\alpha_1}\left[2\alpha_2 \frac{\partial}{\partial s} + \alpha_4 \right] \quad , \tag{7.4.24b}$$

$$A = \begin{bmatrix} 0 & I \\ A_1 & A_2 \end{bmatrix} \quad , \tag{7.4.24c}$$

and

$$M = L_t I - A \quad . \tag{7.4.24d}$$

Then with

$$X = \begin{bmatrix} x \\ L_t x \end{bmatrix} = \begin{bmatrix} x \\ x_t \end{bmatrix} \tag{7.4.25a}$$

the dynamics for x in (7.4.22a) ($Lx = Bu$) is equivalent to the second row of

$$MX = \begin{bmatrix} 0 \\ B \end{bmatrix} u \quad . \tag{7.4.25b}$$

<u>Example: Poisson's Equation</u>

The methodology for writing space-time type systems (i.e. parabolic or hyperbolic) in the diffusion form (infinite dimensional state-space form) above is extensively addressed in the infinite dimensional systems literature

[6]. We include this simple example to emphasize that elliptic problems as well can easily be expressed in this form. Consider the case where L represents the Laplacian (the independent variables s and t are spatial variables)

$$L = \nabla^2 = \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial s^2} \quad . \tag{7.4.26a}$$

Then the dynamics can be written as

$$\begin{bmatrix} L_t & 0 \\ 0 & L_t \end{bmatrix} \begin{bmatrix} x \\ x_t \end{bmatrix} = \begin{bmatrix} 0 & I \\ -\frac{\partial^2}{\partial s^2} & 0 \end{bmatrix} \begin{bmatrix} x \\ x_t \end{bmatrix} + \begin{bmatrix} 0 \\ B \end{bmatrix} u \tag{7.4.26b}$$

so that for this case the operator M is

$$M = L_t I - \begin{bmatrix} 0 & I \\ -\frac{\partial^2}{\partial s^2} & 0 \end{bmatrix} \tag{7.4.26c}$$

which is the diffusion form we seek. $\equiv$

Returning to the general case, the formal adjoint for the operator M in (7.4.24d) is found by taking the adjoint of its constituent elements:

$$M^\dagger = (L_t)^\dagger I - A^\dagger \tag{7.4.27a}$$

where from (7.4.23)

$$(L_t)^\dagger = -L_t \tag{7.4.27b}$$

and the formal adjoint of A in (7.4.24c) is

$$A^\dagger = \begin{bmatrix} 0 & A_1^\dagger \\ I & A_2^\dagger \end{bmatrix} \tag{7.4.27c}$$

with the partitions given by (again from (7.4.23))

$$A_1^\dagger = \frac{-1}{\alpha_1} \left[\alpha_3 \frac{\partial^2}{\partial s^2} - \alpha_5 \frac{\partial}{\partial s} + \alpha_6 \right] \tag{7.4.27d}$$

$$A_2^{\dagger} = \frac{-1}{\alpha_1}\left[-2\alpha_2 \frac{\partial}{\partial s} + \alpha_4\right] \quad . \tag{7.4.27e}$$

To rewrite the estimator dynamics in terms of $M^{\dagger}$ we need to express the action of $L^{\dagger}$ in terms of that of $M^{\dagger}$. It can be shown that $L^{\dagger}\lambda$ is given by the first row of

$$M^{\dagger}\Lambda = \begin{bmatrix} L^{\dagger}\lambda \\ 0 \end{bmatrix} \tag{7.4.28a}$$

if we specify the partitions of Λ by

$$\Lambda = \begin{bmatrix} -\lambda_t + (1/\alpha_1)[-2\alpha_2\lambda_s + \alpha_4\lambda] \\ \hline \lambda \end{bmatrix} \quad . \tag{7.4.28b}$$

This form for the partitions of Λ is found by applying the constraint that the second row of (7.4.28a) is zero.

Assuming that the observations are of the form

$$y = Cx + r \tag{7.4.29}$$

where r is white in both independent variables, the estimator dynamics can be rewritten in terms of M, X, $M^{\dagger}$ and Λ as

$$\left[\begin{array}{ccc|cc} & M & & 0 & 0 \\ & & & 0 & -BQB^* \\ \hline C^*R^{-1}C & \vdots & 0 & & M^{\dagger} \\ 0 & \vdots & 0 & & \end{array}\right] \begin{bmatrix} \hat{X} \\ \hline \hat{\Lambda} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \hline C^*R^{-1} \\ 0 \end{bmatrix} \quad . \tag{7.4.30}$$

As in the unaugmented case, the transformed estimator process is defined as

$$\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = \begin{bmatrix} z_{11} \\ z_{12} \\ z_{21} \\ z_{22} \end{bmatrix} = \begin{bmatrix} \theta_1 & -I \\ \theta_2 & I \end{bmatrix} \begin{bmatrix} \hat{X} \\ \hat{\Lambda} \end{bmatrix} \tag{7.4.29a}$$

where now θ_1 and θ_2 are partitioned as

$$\theta_1 = \begin{bmatrix} \theta_1^{11} & \theta_1^{12} \\ \theta_1^{21} & \theta_1^{22} \end{bmatrix} . \tag{7.4.29b}$$

Let the operator F be defined in terms of θ_1 and θ_2 just as before in (7.4.16). Then if we define

$$\tilde{Q} = \begin{bmatrix} 0 & 0 \\ 0 & BQB^* \end{bmatrix} \quad \text{and} \quad \tilde{R} = \begin{bmatrix} C^*R^{-1}C & 0 \\ 0 & 0 \end{bmatrix} \tag{7.4.30}$$

and if we assume the existence of solutions for the two operator Riccati equations (for arbitrary Ξ)

$$\left(\theta_1 L_t - L_t\theta_1 - A^\dagger\theta_1 - \theta_1 A - \theta_1\tilde{Q}\theta_1 + \tilde{R}\right)\Xi = 0 \tag{7.4.31a}$$

and

$$\left(\theta_2 L_t - L_t\theta_2 - A^\dagger\theta_2 - \theta_2 A + \theta_1\tilde{Q}\theta_1 - \tilde{R}\right)\Xi = 0 \quad , \tag{7.4.31b}$$

then the transformed processes Z_1 and Z_2 will have decoupled dynamics

$$L_1 Z_1 = \begin{bmatrix} C^*R^{-1} \\ 0 \end{bmatrix} y \tag{7 4.32a}$$

and

$$L_2 Z_2 = \begin{bmatrix} -C^*R^{-1} \\ 0 \end{bmatrix} y \tag{7.4.32b}$$

where L_1 and L_2 are the operators

$$L_1 = L_t + (A + \theta_1\tilde{Q}) \tag{7.4.32c}$$

and

$$L_2 = L_t + (A - \theta_2\tilde{Q}) \quad . \tag{7.4.32d}$$

Existence of solutions to the operator Riccati equations (7.4.31a) and (7.4.31b) remains the first of several open issues. If the differential operator A were the infinitesimal generator of a strongly continuous semigroup (something which in general is difficult to establish, e.g. see the Hille-Yosida theorem in [56]), then it has been shown [56] that there exists a solution to these equations. This is a sufficient but not a necessary condition. Given existence, there still remains the question of realizations for θ_1 and θ_2 for the general case (recall for the parabolic case that we found that θ_1 and θ_2 had integral representations).

If we assume that the existence and representation questions regarding the operator Riccati equations have been resolved, then, as in the parabolic example, we would choose boundary conditions for these operators in such a way that the boundary conditions for Z_1 and Z_2 were simplified. Drawing from our experience with nonseparable 1-D TPBVPs in Chapter 3, we speculate that for a purely spatial noncausal process, e.g. the solution to Poisson's equation on a rectangle, that we will be unable to completely decouple the initial and final conditions for Z_1 and Z_2. Thus, we would have an infinite dimensional two-point boundary-value problem for Z_1 and Z_2. In the following we indicate how, under certain conditions, this TPBVP can be solved. In particular, this solution is nearly identical in form to the two-filter solution derived in Chapter 3 for the 1-D TPBVP.

Employing notation similar to that used for the 1-D TPBVP, we represent the coupled initial and final boundary conditions by

$$v_{0T}(s) = [\, V_1^0 \,\vdots\, V_2^0 \,] \begin{bmatrix} Z_1(0,s) \\ Z_2(0,s) \end{bmatrix} + [\, V_1^T \,\vdots\, V_2^T \,] \begin{bmatrix} Z_1(T,s) \\ Z_2(T,s) \end{bmatrix} \quad ; \; s \in (a,b). \qquad (7.4.33)$$

Next define two differential operators A_1 and A_2 by using the spatial boundary conditions (at s=a and s=b) for Z_1 and Z_2 to restrict the domains of the formal differential operators (see (7.4.32) above)

$$-(A + \theta_1\tilde{Q}) \qquad (7.4.34a)$$

and

$$-(A - \theta_2\tilde{Q}) \qquad (7.4.34b)$$

respectively. When each of A_1 and A_2 so-defined is an infinitesimal generator of strongly continuous semigroup, then we can invoke a fundamental result from the theory of infinte dimensional systems to obtain an expression for the solutions of the following infinite dimensional diffusions

$$\dot{q}_1(t) = A_1 q_1(t) + u_1(t) \tag{7.4.35a}$$

and

$$\dot{q}_2(t) = A_2 q_2(t) + u_2(t) \tag{7.4.35b}$$

with intitial and final conditions

$$q_1(0) = q_0 \tag{7.4.35c}$$

and

$$q_2(T) = q_T \tag{7.4.35d}$$

In particular, the solutions $q_1(t)$ and $q_2(t)$ can be written in a form which is analogous to the variation of constants formula for finite dimensional systems [59][1]:

$$q_1(t) = \Phi_1(t,0)q_0 + \int_0^t \Phi_1(t,\tau)u_1(\tau)d\tau \tag{7.4.36a}$$

and

$$q_2(t) = \Phi_2(t,T)q_T + \int_T^t \Phi_2(t,\tau)u_2(\tau)d\tau \tag{7.4.36b}$$

where $\Phi_1(t,\tau)$ is a foward evolution operator in the semigroup of operators generated by A_1 and $\Phi_2(t,\tau)$ is a backward evolution operator in the semigroup of operators generated by A_2. Given this result, a derivation identical to that used to obtain the two-filter solution for the 1-D case in

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[1] Some additional technical restrictions on the u_i are requied to ensure the integrability of $\Phi_i u_i$ (see [60]).

Chapter 3 gives the solution to (7.4.32a) and (7.4.32b) with coupled boundary condition (7.4.33) as

$$\begin{bmatrix} Z_1(t) \\ Z_2(t) \end{bmatrix} = \Phi_Z(t)F_Z^{-1}\{v_{0T} - V_1^T Z_1^0(T) - V_2^0 Z_2^0(0)\} + \begin{bmatrix} Z_1^0(t) \\ Z_2^0(t) \end{bmatrix} \qquad (7.4.37a)$$

where

$$\Phi_Q(t) = \begin{bmatrix} \Phi_1(t,0) & 0 \\ 0 & \Phi_2(t,T) \end{bmatrix} , \qquad (7.4.37b)$$

$$F_Z = \left[V_1^0 + V_1^T \Phi_1(T,0) \,\vdots\, V_2^T + V_2^0 \Phi_2(0,T) \right] , \qquad (7.4.37c)$$

Z_1^0 is the solution to (7.4.32a) with zero initial condition $Z_1^0(0) = 0$ and Z_2^0 is the solution to (7.4.32b) with zero final condition $Z_2^0(T) = 0$. The implementation of the solution of (7.4.37a) could be accomplished by solving for Z_1^0 and Z_2^0 by forward and backward finite difference approximations of (7.4.32a) and (7.4.32b) respectively. The boundary value contribution to the solution:

$$\Phi_Z(t)F_Z^{-1}\{v_{0T} - V_1^T Z_1^0(T) - V_2^0 Z_2^0(0)\} \qquad (7.4.37d)$$

would require approximations of the evolution operators Φ_1 and Φ_2 and the inverse of F in (7.4.37c). Conceivably, these approximations could also be accomplished by similar discretization methods.

Finally, to recover the estimate of the process x, we must invert the transformation (7.4.29a), i.e.:

$$\begin{bmatrix} \hat{x} \\ \hat{x}_t \end{bmatrix} = \hat{X} = (\theta_1 + \theta_2)^{-1}(Z_1 + Z_2) \quad . \qquad (7.4.38a)$$

In the case for which θ_1 and θ_2 are known to be integral operators, the estimate could be recovered by an approximation of the solution to the integral equation

$$(\theta_1 + \theta_2)\hat{x} = z_1 + z_2 \quad . \tag{7.4.38b}$$

Clearly, each case must be considered individualy.

Here we summarize some of the major issues pertaining to the diagonalized implementation of the estimator for second order processes.

(1) In order to diagonalize the estimator dynamics we must establish existence of solutions to the operator Riccati equations (7.4.31a) and (7.4.31b).

(2) Given existence of solutions, we must determine a representation or realization of these operators and then choose boundary conditions in such a way that the boundary conditions for the transformed estimator processes z_1 and z_2 are maximally decoupled.

(3) With the boundary conditions and dynamics for z_1 and z_2 established, we will in general have an infinite dimensional TPBVP. As discussed above, a solution for this type of problem can be written in a two-filter form if the system operators A_1 and A_2 in (7.4.43a) and (7.4.34b) can be shown to be infinitesimal generators of strongly continuous semigroups.

(4) Finally, with (1) through (3) resolved, we still need to determine efficient numerical methods for (a) solving the operator Riccati equations, (b) solving the TPBVP for the transformed estimator processes and (c) inverting the transformation to obtain values of the estimate of the original process x.

Next we comment on how this method can be extended to estimation of higher order (greater than 2-D) processes. Finally, in closing this section we discuss implementation of the estimator via triangularization of the estimator dynmaics. This approach was introduced earlier in Section 7.2.3 as an alternative to diagonalization which does not require an operator inverse (as in (7.4.38)) to recover the estimates of the original process x from the transformed processes.

Higher Dimensional Processes

As we have seen, the method of diagonalizing the estimator dynamics which is based on the solution of an operator Riccati is dependent upon writing the dynamics of the process to be estimated as a diffusion in one variable, call it t. In the 1-D case the system matrix A(t) is simply a t-varying matrix, and in the 2-D case the system operator $A(t)$ can vary with t but must contain partials only with respect to the other independent variable s. This has a natural extension to 3 and higher dimensions. For example, consider a 4-D process with independent variables (t,s,s',s''). We simply require that the dynamics can be written as

$$\left(L_t - A(t)\right)x = Bu$$

where A contains partials with respect to s, s' and s'' only. We note that the results regarding the existence of solutions to Riccati equations cited for the 2-D case when A is an infinitesimal generator of a strongly continuous semigroup have extensions to higher dimensions [60]. In addition, the variation of constants formula also applies in higher dimensions.

Triangularization of the Estimator Dynamics

Here we apply the transformation T proposed earlier in (7.2.31a):

$$T = \begin{bmatrix} I & -P \\ 0 & I \end{bmatrix} \quad \text{with inverse} \quad T^{-1} = \begin{bmatrix} I & P \\ 0 & I \end{bmatrix}, \tag{7.4.39a}$$

where the transformed process is denoted as before by

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = T\begin{bmatrix} \hat{x} \\ \hat{\lambda} \end{bmatrix}, \tag{7.4.39b}$$

and F is from (7.2.31b)

$$F = \begin{bmatrix} I & P \\ 0 & I \end{bmatrix}. \tag{7.4.39c}$$

Assuming L in the form of a diffusion, $L = L_t - A$, it can be shown by direct substitution into (7.2.32) that the estimator dynamics become

$$\left[\begin{array}{c:c} L_t - A + PC^*R^{-1}C & 0 \\ \hdashline C^*R^{-1}C & -L_t - A^\dagger + C^*R^{-1}CP \end{array}\right]\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \left[\begin{array}{c} PC^*R^{-1} \\ \hdashline C^*R^{-1} \end{array}\right] y \,. \tag{7.4.39d}$$

In addition, with L in this form, the operator equation in (7.2.23) becomes the Riccati equation:

$$\left(L_t P - AP - PL_t - PA^{\dagger} + PC^{*}R^{-1}CP - BQB^{*}\right)\Xi = 0 \quad . \tag{7.4.40}$$

Applying T^{-1} in (7.4.39a), to invert from the transformed processes to the original processes gives

$$\hat{\lambda} = Z_2 \tag{7.4.41a}$$

and

$$\hat{x} = Z_1 + PZ_2 \quad . \tag{7.4.41b}$$

For ease of reference to previous work, we will replace Z_2 with $\hat{\lambda}$ and denote Z_1 by

$$\hat{x}_f = Z_1 \tag{7.4.41c}$$

so that from (7.4.41b)

$$\hat{x} = \hat{x}_f + P\hat{\lambda} \quad . \tag{7.4.41d}$$

We will assume as before that the spatial boundary conditions have been absorbed into the domains of the system operators defined by[2]

$$A_f = A - PC^{*}R^{-1}C \tag{7.4.42a}$$

and

$$A_{\lambda} = A^{\dagger} - C^{*}R^{-1}CP \quad . \tag{7.4.42b}$$

Then the transformed process will in general have a coupled two-point boundary condition of the form

$$v_{x\lambda} = [V_x^0 \ \vdots \ V_{\lambda}^0]\begin{bmatrix}\hat{x}_f(0) \\ \hat{\lambda}(0)\end{bmatrix} + [V_x^T \ \vdots \ V_{\lambda}^T]\begin{bmatrix}\hat{x}_f(T) \\ \hat{\lambda}(T)\end{bmatrix} \tag{7.4.43}$$

---

[2] We will also need to assume that A_f and A_{λ} are infinitesimal generators of strongly continuous semigroups so that we can apply the variation of constants formula.

and the dynamics of $\hat{x}_f$ will be decoupled from those of $\hat{\lambda}$ (but not vice-versa):

$$L_t\hat{x}_f = A_f\hat{x}_f + PC^*R^{-1}y$$

$$= A\hat{x}_f + PC^*R^{-1}(y - C\hat{x}_f) \qquad (7.4.49a)$$

and

$$-L_t\hat{\lambda} = A_\lambda\hat{\lambda} + C^*R^{-1}(y - C\hat{x}_f) \quad . \qquad (7.4.49b)$$

Note that these dynamics are identical to those of the innovations form of the smoother for both causal finite dimensional [55] and infinite dimensional processes [6].

A form of the the general solution for the TPBVP defined in (7.4.43) and (7.4.44) can be derived by an application of the infinte dimensional variation of constants formula[3] as follows. First note that an application of the variation of constants formula allows us to write

$$\hat{x}_f(t) = \Phi_f(t,0)\hat{x}_f(0) + \hat{x}_f^0(t) \qquad (7.4.45a)$$

and

$$\hat{\lambda}(t) = \Phi_\lambda(t,T)\hat{\lambda}(T) + \hat{\lambda}^0(t) + \Phi_{\lambda x}(t)\hat{x}_f(0) \qquad (7.4.45b)$$

where the zero initial and final condition solutions are given by

$$\hat{x}_f^0(t) = \int_0^t \Phi_f(t,s)P(s)C^*(s)R^{-1}(s)y(s)ds \quad , \qquad (7.4.45c)$$

and

$$\hat{\lambda}^0(t) = \int_T^t \Phi_\lambda(t,s)C^*(s)R^{-1}(s)\left[y(s) - C(s)\hat{x}_f^0(s)\right]ds \qquad (7.4.45d)$$

---

[3] Because the variation of constants formula is expressed in the same notation for both finite and infinte dimensional systems, all of the this can be directly translated into results for finite dimensional systems.

respectively, and the coefficient of the initial value of $\hat{x}_f$ is given by

$$\Phi_{\lambda x}(t) = \int_T^t \Phi_\lambda(t,s)C^*(s)R^{-1}(s)C(s)\Phi_f(s,0)ds \quad . \tag{7.4.45e}$$

Of course, the initial condition $\hat{x}_f(0)$ and final condition $\hat{\lambda}(T)$ are unknown. Substituting into the boundary condition (7.4.43) from (7.4.45a) and (7.4.45b) evaluated at t=T and t=0 respectively, it can be shown that these initial and final values are given by

$$\begin{bmatrix} \hat{x}_f(0) \\ \hat{\lambda}(T) \end{bmatrix} = F_{x\lambda}^{-1}\left\{ v_{x\lambda} - V_x^T \hat{x}_f^0(T) - V_\lambda^0 \hat{\lambda}^0(0) \right\} \tag{7.4.46a}$$

where

$$F_{x\lambda} = \left[V_x^0 + \Phi_x(T,0)V_x^T + V_\lambda^0 \Phi_{\lambda x}(0) \,\vdots\, V_\lambda^0 \Phi_\lambda(0,T) + V_\lambda^T \right] \quad . \tag{7.4.46b}$$

Therefore, solving the zero initial condition problem for $\hat{x}_f^0(t)$ and using that solution as an input to the zero final condition problem for $\hat{\lambda}^0(t)$, then using these two to compute the initial and final conditions in (7.4.46a), we can construct the entire solution for $\hat{x}_f$ and $\hat{\lambda}$ from (7.4.45). Finally, these are combined via (7.4.41d) to give the estimate $\hat{x}$. Of course, as was also true for the diagonalization method, finding an efficient numerical method to solve for for these processes may be nontrivial.

As a final remark, we note that if we had chosen F as

$$F = \begin{bmatrix} I & P \\ P^{-1} & 0 \end{bmatrix} \tag{7.4.47}$$

rather than as in (7.4.44b), then we would have obtained a form for the estimator dynamics which corresponds to the Rauch-Tung-Striebel solution for the 1-D smoothing problem. However, we see that this choice would require the operator inverse P^{-1}, which is something we had endeavored to avoid.

7.4.3 A Quadrant-Causal 2-D Discrete Process

A recursive model for discrete parameter 2-D processes with a special quadrant causal form has been introduced by Roesser[57]. One form of quadrant causality is depicted in Figure 7.4.1 where the value of the process x at the point (i,j) is a function of the inputs in the quadrant to its southwest. Thus the recursion is a northeast recursion (its "past" is in the southwest).

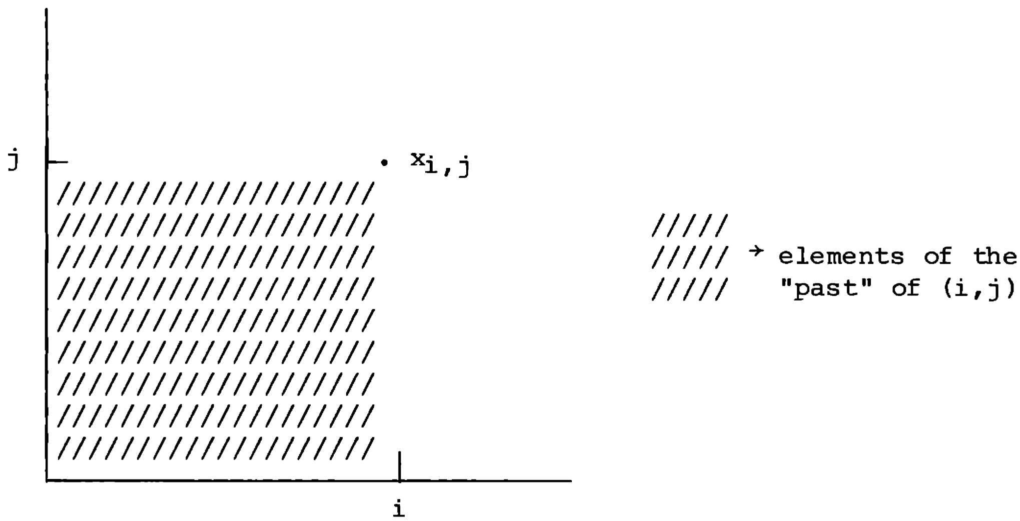

Figure 7.4.1 Northeast Causality of Roesser's Model

With the process at the point i,j defined as

$$x_{i,j} = \begin{bmatrix} v_{i,j} \\ h_{i,j} \end{bmatrix} , \tag{7.4.48a}$$

Roesser's difference equation is given by

$$\begin{bmatrix} v_{i+1,j} \\ h_{i,j+1} \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} v_{i,j} \\ h_{i,j} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_{i,j} \tag{7.4.48b}$$

on $[0,I-1]\times[0,J-1]$. The northeast recursion in (7.4.48) will be written in operator form as follows. Recall from Appendix 6D that D_1 is a delay in the first index, i, and D_2 is a delay in the second index, j. Define

$$\Delta = \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix} , \tag{7.4.49a}$$

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} , \qquad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \tag{7.4.49b}$$

then the dynamics in (7.4.48b) can be written as

$$Lx = Bu \tag{7.4.50a}$$

where

$$L = \Delta^{-1} - A \quad . \tag{7.4.50b}$$

By following a similar developement as that used in Appendix 6D to form the Green's identity for the nearest neighbor model, it can be shown that the formal adjoint of the northeast diffusion operator L in (7.4.50b) is given by

$$L^{\dagger} = \Delta - A' \tag{7.4.51}$$

which represents a southwest recursion.

With observations of the form

$$y_{i,j} = C_{ij}x_{ij} + r_{ij} \tag{7.4.52}$$

the estimator dynamics are given by

$$\begin{bmatrix} L & -BQB^{*} \\ C^{*}R^{-1}C & L^{\dagger} \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{\gamma} \end{bmatrix} = \begin{bmatrix} 0 \\ C^{*}R^{-1} \end{bmatrix} y \quad . \tag{7.4.53a}$$

As in the 1-D case we will rewrite the estimator dynamics in terms of the transformed variable

$$\hat{\lambda} = \Delta\hat{\gamma} \qquad (\hat{\gamma} = \Delta^{-1}\hat{\lambda}) \tag{7.4.53b}$$

as

$$\begin{bmatrix} \Delta^{-1} - A & -BQB^{*}\Delta^{-1} \\ C^{*}R^{-1}C & I - A'\Delta^{-1} \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{\lambda} \end{bmatrix} = \begin{bmatrix} 0 \\ C^{*}R^{-1} \end{bmatrix} y \ . \tag{7.4.53c}$$

The estimator boundary conditions will not be addressed here. We will only investigate the dynamical diagonalization. In particular we will consider the special case for which T and F are matrices, i.e. they contain no dynamics.

Using the form for T and its inverse in (7.4.1) the transformed dynamics become

$$\begin{bmatrix} F_1 & F_2 \\ F_3 & F_4 \end{bmatrix} \left[\begin{array}{c:c} \Delta^{-1} - A & -BQB^{*}\Delta^{-1} \\ \hdashline C^{*}R^{-1}C & I - A'\Delta^{-1} \end{array}\right] \begin{bmatrix} I & I \\ -\theta_2 & \theta_1 \end{bmatrix} \begin{bmatrix} P_s & 0 \\ 0 & P_s \end{bmatrix} = \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix} \quad (7.4.54)$$

where the condition for diagonalization is the existence of solutions to

$$\Big(F_1(\Delta^{-1} - BQB'\Delta^{-1}\theta_1) - F_2A'\Delta^{-1}\theta_1 - F_1A + F_2(\theta_1 + C'R^{-1}C) \Big)\xi = 0 \quad (7.4.55a)$$

and

$$\Big(F_3(\Delta^{-1} - BQB'\Delta^{-1}\theta_2) + F_4A'\Delta^{-1}\theta_2 - F_3A + F_4(-\theta_2 + C'R^{-1}C) \Big)\xi = 0 \quad (7.4.55b)$$

for arbitrary ξ. From (7.3.9c) of the 1-D discrete case, the diagonal operator L_1 is of the form

$$L_1 = (F_1BQB' + F_2A')\Delta^{-1} - F_2 \quad . \quad (7.4.55c)$$

Thus, with the coefficient of Δ^{-1} an invertible matrix, L_1 will have a quadrant causal form. Otherwise L_1 is in the descriptor representation of quadrant causal dynamics.

First consider the existence of a sequence $\theta_{1_{ij}}$ satisfying (7.4.55a). Equating the coefficients of terms containing no shift operator Δ^{-1} to zero (see the 1-D example) gives

$$F_2 = F_1A(\theta_1 + C'R^{-1}C)^{-1} \quad . \quad (7.4.56a)$$

Substituting this into the terms containing a shift gives

$$F_1\left[\Delta^{-1} - \Big(BQB' + A(\theta_1 + C'R^{-1}C)A'\Big)\Delta^{-1}\theta_1\right]\xi = 0 \quad (7.4.56b)$$

Thus we require that

$$\Big(BQB' + A(\theta_1 + C'R^{-1}C)A'\Big)\Delta^{-1}\theta_1 = \Delta^{-1} \quad (7.4.57a)$$

or equivalently that

$$\Delta^{-1}\theta_1\Delta = \Big(BQB' + A(\theta_1 + C'R^{-1}C)A'\Big)^{-1} \quad . \quad (7.4.57b)$$

Consider the left hand side of (7.4.57b):

$$(\Delta^{-1}\theta_1\Delta)_{ij} = \left[\begin{array}{c:c} \theta^{11}_{1\,i+1,j} & \theta^{12}_{1\,i+1,j}\,D_1^{-1}D_2 \\ \hdashline \theta^{21}_{1\,i,j+1}\,D_2^{-1}D_1 & \theta^{22}_{1\,i,j+1} \end{array}\right] \qquad (7.4.58)$$

For the partitions of the right hand side to be compatible with those in (7.4.58) for nonzero θ^{12} and θ^{21}, the right hand side must also contain shift operators. Since B, Q, C and A as defined contain no shifts, θ_1 must. However, our initial assumption was that it had none. Therefore, only in the special case that both the right and left hand sides are diagonal will there exist a decoupling transformation with no shift operations. A similar argument shows that no θ_2 exists which satisfies (7.4.55b). Thus, in general, no two-filter implementation exists for this estimator.

An alternative approach for which we also restrict F and T to contain no dynamics might be to find a four-filter coupled solution of the lower triangular form

$$F\left[\begin{array}{c:c} \Delta^{-1} - A & -BQB^*\Delta^{-1} \\ \hdashline C^*R^{-1}C & I - A'\Delta^{-1} \end{array}\right]T^{-1} = \left[\begin{array}{c:c:c:c} L_1 & 0 & 0 & 0 \\ \hdashline L_{12} & L_2 & 0 & 0 \\ \hdashline L_{13} & L_{23} & L_3 & 0 \\ \hdashline L_{14} & L_{24} & L_{34} & L_4 \end{array}\right] \qquad (7.4.59)$$

where, for instance, L_1 is an eastward recursion (i.e. L_1 is of the form $D_1 - A_1$), L_2 northward, L_3 westward and L_4 southward. In this way, L_1 could be solved first with its solution fed back to L_2 via L_{12} etc. Of course, determining F and T, if they exist at all, will be a nontrivial task. Finally, we could also consider the case where θ_1 and θ_2 contain dynamics which would be analogous to the 2-D parablic example where they were shown to be integral operators.

SECTION 7.5

CONCLUSIONS

As a preliminary for and to provide insight into the operator diagonalization of the estimator dynamics for 2-D processes we have reformulated the two-filter solutions for the 1-D discrete and 1-D continuous cases by an operator diagonalization approach. By writing 2-D partial differential operators as infinite dimensional diffusion operators, we have been able to directly extend the operator diagonalization applied in the 1-D continuous case. Furthermore, by employing an infinite dimensional version of the variation of constants formula we have hypothesized a (infinite dimensional) two-filter implementation of the estimator for the general 2-D boundary value processes. It should be emphasized that there remain many unanswered questions with respect to the operator diagonalization problem for the general 2-D case. Indeed, more research is required before this problem is completely resolved. In particular, many of the questions are related to issues of existence and representation of solutions to operator Riccati equations (see (7.4.31)).

As mentioned above, the 2-D diagonalization was obtained in large part by a direct extension of the 1-D solution. In each case, the estimate is computed via transformed estimator processes and ultimately the estimates must be recovered from these processes by an inversion of that transformation. In the 1-D case, the inverse transformation is simply a matrix inverse. However, in the 2-D case we find that an operator inverse is required. In general, this operator inversion will represent a nontrivial problem. Therefore, we have investigated an alternative estimator implementation (which is also applicable in 1-D) which requires no inversions. This alternative approach is based on a specific form of triangularization of the estimator dynamics rather than on diagonalization. For purely causal processes (both finite and infinite dimensional), this form of the estimator corresponds to the smoother derived from the innovations approach.

We also have applied our diagonalization methodolgy to the estimator for a discrete 2-D process satisfying Roesser's quadrant causal dynamical model. We found for the special case that the diagonalizing operators were restricted to matrices (i.e. they contained no dynamics) that no two-filter solution is attainable through diagonalization. Under this same restriction for the transformation operator, we have suggested an approach for developing a four-filter implementation via triangularization of the estimator dynamics.

CHAPTER 8: SUMMARY OF CONTRIBUTIONS AND SUGGESTIONS FOR FURTHER RESEARCH

SECTION 8.1
CONTRIBUTIONS

The principal contribution of this thesis is the derivation of the differential (difference) operator form of the estimator equations in Chapter 2. This single representation of the estimator is applicable to boundary value stochastic processes satisfying linear ordinary differential and difference equations as well as linear partial differential and difference equations. Thus, the fundamental structure of linear estimators for boundary value stochastic processes is embodied in these equations. An attractive feature of our estimator equations is that they are completely specified in terms of a) the differential operator and boundary condition used to describe the process to be estimated, b) the operators used to describe the observations and c) the Green's identity for the differential (difference) operator governing the process to be estimated. In addition, the differential (difference) form of the estimation equations provides an excellent starting point for formulating methods of implementing the estimator. Indeed, the derivation of efficient and numerically stable algorithms for implementing the estimator is the subject of the second half of this thesis.

Due to the nature of our derivation of the estimator equations by the method of complementary models, we have been able, with very little additional effort, to write the estimation error equations in a differential operator form similar to that of the estimator. These error equations also represent a significant contribution. As illustrated by our study of 1-D continuous parameter processes in Chapter 3, because the estimate and estimation error satisfy similar equations, many of the computations required in calculating the estimate are also applicable to the calculation of the estimation error covariance.

Our studies of the implementation of the estimator begin in Chapter 3 with the consideration of 1-D n^{th} order continuous parameter two-point boundary value stochastic processes. An application of the differential operator representation of the estimator derived in Chapter 2 shows that the estimator for these processes is a $2n^{th}$ order two-point boundary value process. A significant result is the formulation of a stable (forward/backward) two-filter solution for the estimator by appropriate block-diagonalization of its $2n^{th}$ order dynamics. Consequently, the estimator can be implemented by numerically stable recursive solutions of each of the forward and backward differential equations. In particular, the derivation of this two-filter solution as a linear combination of 1) a zero initial condition foward recursion, 2) a zero final condition backward recursion and 3) the boundary value should be considered another contribution of this thesis. In addition to the two-filter solution, we have also established matrix differential equations by which the estimation error covariance is computed. As mentioned above, many of these matrices are also employed in the two-filter computation of the estimator.

The second class of processes to which our estimator solution is applied is that of 1-D discrete parameter two-point boundary value processes (Chapter 5). As in the continuous parameter case, the estimator for an n^{th} order discrete process is a $2n^{th}$ order two-point boundary value process. Although the estimator dynamics in the discrete case are in descriptor form, we have shown that they also can be diagonalized into stable forward and stable backward recursions. So again we have derived a numerically stable and efficient method for implementing the estimator, as well as some matrix difference equations for computing the error covariance.

Expanding our investigation to include 2-D discrete parameter processes, we have shown that the concept of splitting the dynamics into forward stable and backward stable parts is also useful in implementing the estimator for a large class of 2-D discrete processes. In particular, by unifying the description of 2-D dynamics into a single model (that we have called the nearest neighbor model, NNM), we have determined the conditions for rewriting the 2-D dynamics in a 1-D form of high vector dimension. Diagonalization is then performed on this 1-D model to obtain a two-filter solution. Under certain symmetry conditions, the estimator implementation can be further simplified into a system of decoupled 1-D processes each of which can be diagonalized separately.

Our success in developing two-filter forms for the estimators of 1-D discrete and continuous and 2-D discrete processes suggested that this approach could be extended to the estimator for 2-D continuous parameter processes. By studying two-filter (diagonal) representations in the context of the original differential operator description of the estimator, we have presented a unified methodology for diagonalizing this differential operator. Applying this methodology to the 2-D continuous case we have shown that the existence of diagonal representations for the estimator dynamics is conditioned on the existence of solutions to operator Riccati equations. By relying on the existing theory regarding solutions of operator Riccati equations, we were able to obtain a diagonal form for the dynamics of the estimator of a parabolic process. However, the complete specification of a two-filter solution has not yet been achieved due to some unanswered questions regarding coupling of spatial boundary conditions. For the more general cases of elliptic and hyperbolic systems, existence of solutions of the operator Riccati equations must be established on an individual basis.

SECTION 8.2

SUGGESTIONS FOR FUTURE RESEARCH

In deriving the differential operator form of the estimator we have made several assumptions which, with additional study, might be relaxed:

1) The estimator as presented in equation (2.5.25) has been derived under the assumption that the correlation operator for the boundary value v, Π_v, is invertible. In Section 2.5.4 we have derived an alternative expression for the estimator for the case when this operator might be singular and have applied that form of the estimator solution to discrete 1-D periodic process. However, in specifying the estimator using this alternative expression, we have not been able to completely eliminate its dependence on operators associated with the definition of the internal differential realization for the associated complementary process. It should be possible, as we have shown for the case of Π_v invertible, to write this estimator in a form which is <u>independent</u> of the realization for the complementary process.

2) To simplify the expression for the complementary process and ultimately to simplify the proof of the complementation property, we assumed that the elements of the underlying process $\zeta = \{u, v, r, r_b\}$ were uncorrelated (i.e. that the correlation operator Σ_ζ is block-diagonal). It should be possible to extend our results to the case where elements of ζ are correlated. A good starting point for this extension would probably be with the static example in Chapter 2. In particular, the derivation in Appendix 2A of the representation for the complementary process for the static example should be broadened to include correlations between x and r.

3) In applying Green's identity in the derivation of the internal realization of the complementary process, we found it convenient to restrict the boundary condition for the process to be estimated to be a linear function of what we have referred to as the boundary process

x_b (i.e. $v = Vx_b$). In Appendix 3B we have considered the implications of relaxing this assumption and have derived the estimator equations for a 1-D continuous parameter process with a boundary condition of the form:

$$v = \int_0^T V(s)x(s)ds \quad .$$

It would be of interest to determine the form of the estimator when this restriction is relaxed in the general case.

4) We have only considered measurements, $y = Cx + r$, for continuous parameter processes which are themselves continuous parameter processes. There are many important cases for which measurements are available only at discrete points. Thus, the estimator should be extended to include measurements of this type.

With regard to application of the estimator and its implementation, the following questions have been raised in the course of our research:

1) It has been shown that the well-posedness of a 1-D two-point boundary value problem is dependent on the invertibility of a matrix we have denoted as F (see (3.2.3b) for the continous case and (5.2.8b) for the discrete case). Since the estimator for this type of process is also a two-point boundary value problem, there is a corresponding matrix which must be invertible (see e.g. (3.4.12)). Thus far we have not determined the general conditions for invertibiltiy of this matrix, i.e. the well-posedness conditions for the 1-D estimator. One can conjecture that well-posedness of the smoother will depend on both the well-posedness of the original TPBVP governing the process to be estimated as well as controllability and reconstructibility conditions. This important point deserves further consideration.

2) In each of our 2-D continuous examples (Poisson's equation in Chapter 2 and the heat equation in Chapter 7) we have assumed that the boundary value v is simply a white noise process. This assumption has been made because the specification of the estimator boundary

condition has required the inverse of the covariance operator Π_v. For v a 2-D white noise, this inversion is trivial. Other than white noise boundary conditions might be considered by applying the estimator solution, discussed above, for which the inverse of Π_v is not required.

3) Our study of Markov models for 1-D TPBVPs in Chapter 4 raises some interesting questions. We found that the Markov model for an n^{th} order TPBVP was of order greater than or equal to n (only in special "separable" conditions is it equal to n). Thus, a Mayne-Fraser two-filter smoother for this Markov model must be of that same order. In Chapter 3 we showed that the estimator obtained directly from the n^{th} order two-point boundary value model could be implemented by an n^{th} order two-filter solution. Although the estimator in each of these two cases combines the results from its forward and backward filters in a different manner, it is intriguing that the order of the differential equations that must be solved over the interval of interest is <u>different</u> for the two approaches. Of course, both must ultimately give the same minimum variance solution. Therefore, the two are necessarily related and furthering our understanding of this relationship deserves more study. For instance, each of the filters in the Mayne-Fraser solution for causal processes has an interpretation as either a foward or backward Kalman filter. Is there a similar interpretation of each of the two filters in our lower order solution?

4) In studying the estimator for a mixed causal/noncausal 2-D discrete process in Chapter 6, we found that although the invertibility condition for the existence of a 1-D marching method form did not hold, we were able to write a 1-D descriptor form representation for the estimator to which we could apply the 1-D diagonalization results developed in Chapter 5. It would be of interest to determine the general conditions under which the 2-D equations could be written in a 1-D descriptor form that is amenable to diagonalization by the 1-D methods of Chapter 5.

5) In considering Roesser's 2-D "state-space" model in Chapter 7, we presented estimator dynamics that were written in terms of a difference operator for this model and its adjoint that were not in NNM form. That is, we did not first write the dynamics in NNM form as we have proposed in Chapter 6. It may be the case that by forcing every 2-D discrete problem into NNM form, we may be losing sight of (and thus the advantages of) the natural structure of the estimator dynamics, and as discussed in 7) below, this structure may suggest other than two-filter forms for implementation of the estimator. It is likely that this question can be answered only by considering a variety of problems in both their natural form and NNM form. A good place to start would be the space time example in Chapter 6 for which we have already found the solution for the NNM model.

6) As mentioned in its introduction, Chapter 7 can itself be considered, in part, a suggestion for future research. Specifically, some of the major questions raised there are:

i) The approach taken in Chapter 7 is formal. Ultimately, to resolve many of the technical issues raised a more rigourous approach must be taken.

ii) Existence of solutions of the operator Riccati equation resulting from triangularization or diagonalization of the estimator dynamics for the general elliptic and hyperbolic cases.

iii) Given existence of the solution to the Riccati equation for a specific case, we must determine a realization for the operator which gives a representation of that solution (e.g. the integral realization for θ_1 and θ_2 in the parabolic example) in order to perform numerical computations.

iv) In equation (7.4.37d) we have shown that if there are coupled temporal boundary conditions for the transformed estimator processes, these can be expressed as an infinite dimensional two-point boundary condition . However, there remain unanswered questions with respect to the spatial boundary conditions for these processes. In particular, to what extent can they be decoupled through appropriate choice of boundary conditions for the decoupling operators θ_1 and θ_2? What are the conditions for which the spatial boundary can be totally decoupled? In the event that they cannot be decoupled, how should they be factored into the two-filter solution we have proposed?

v) To gain further insight into diagonalization of 2-D estimators, additional simple examples should be studied, such as Poisson's equation on a rectangle or disc.

7) In Chapter 7 we have limited our investigation of algorithms for implementing the estimator to two-filter forms which are the result of either diagonalization or triangularization of the estimator dynamics. Of course, other decoupled forms may also lead to efficient algorithms. As suggested in Chapter 7 when discussing Roesser's model, four-filter forms are probably more natural for 2-D problems than are two-filter forms. Furthermore, for 2-D problems when the region of interest is other than a rectangle, other decoupling schemes may be more appropriate. For instance, if the region is a disc, starting from the center and filtering outward on circles of constant radius or filtering on a radial line as it sweeps out an angle from 0 to 2π may be the best approaches. As in our diagonalization studies, a good starting point for studying alternative dynamically decoupled solutions would be investigating transformations of the differential operator representation of the estimator dynamics which lead to various equivalent forms. As a final point, recall that motivated by 1-D solutions, we have concentrated on developing decoupled solutions for 2-D problems which are in the form of a recursion in one variable which we have referred to as t. Other forms deserve consideration especially for 2-D problems where none of the independent variables need play the role of "time".

Finally, we mention that the operator decoupling method we have proposed in Chapter 7 has a wider applicability. As discussed below, there is potential for employing the triangularization and diagonalization methodology developed in Chapter 7 to transform arbitrary partial differential equations into forms which are amenable to numerical solution. In particular, consider the representation (7.4.26b) of Poisson's equation:

$$\begin{bmatrix} L_t & -I \\ A & L_t \end{bmatrix} \begin{bmatrix} x \\ x_t \end{bmatrix} = \begin{bmatrix} 0 \\ B \end{bmatrix} u \quad . \tag{8.2.1}$$

Applying transformations similar to those suggested in the discussion of triangularization of the estimator dynamics in Section 7.2.3, we can transform

(8.2.1) as

$$\begin{bmatrix} I & -P \\ 0 & I \end{bmatrix} \begin{bmatrix} L_t & -I \\ A & L_t \end{bmatrix} \begin{bmatrix} I & P \\ 0 & I \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = \begin{bmatrix} I & -P \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 \\ B \end{bmatrix} u \tag{8.2.2a}$$

and carrying out the products, gives:

$$\left[\begin{array}{c:c} L_t - PA & L_tP - PL_t - PAP - I \\ \hdashline A & L_t + AP \end{array}\right] \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = \begin{bmatrix} -PB \\ B \end{bmatrix} u \quad . \tag{8.2.2b}$$

Given existence of a solution to the operator Riccati equation:

$$\left(L_tP - PL_t - PAP - I\right)\xi = 0 \quad , \tag{8.2.3}$$

the dynamics of Z_1 in (8.2.2b) become decoupled from those of Z_2 as:

$$(L_t - PA)Z_1 = -PBu \tag{8.2.4a}$$

and

$$(L_t + AP)Z_2 = Bu - AZ_1 \tag{8.2.4b}$$

where x is recovered by

$$x = Z_1 - PZ_2 \quad . \tag{8.2.4c}$$

Of course, this is only a sketch of what needs to be done. As we have seen, an appropriate choice of the boundary conditions for the Riccati equation solution which leads to decoupled boundary conditions for Z_1 and Z_2 is crucial. We note that the decoupled form (8.2.4) of writing Poisson's equation has previously been proposed by Angel and Jain [61] although not explicitly as a triangularization problem as we have presented it. It is straightforward to see how this diagonalization/triangularization methodology could be extended to decoupling of general partial differential equations when they are written in the form of (8.2.1).

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BIOGRAPHY

Milton B. Adams was born in Teheran, Iran on June 2, 1948. As a military dependent, he has lived throughout the U.S.. After graduating from Phillips Andover Academy in June of 1967, he attended Brown University. At Brown he received both the Sc. B. and Sc. M. in Electrical Engineering in 1971 and 1972, respectively. In the fall of 1972 he become a staff member of the Charles Stark Draper Laboratory (then the MIT Instrumentation Laboratory). In September 1977 Mr. Adams returned to graduate school and entered a Sc.D. program in the Department of Aeronautics and Astronautics at MIT. His education at MIT has been supported by a Draper Fellowship. He is married to the former Marilyn Louise Jager, and they are expecting their first child in June 1983.