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STOCHASTIC CONTROL OF ROTATIONAL PROCESSES WITH ONE DEGREE OF FREEDOM*

JAMES TING-HO LO† AND ALAN S. WILLSKY‡

Abstract. A class of bilinear stochastic control problems involving single-degree-of-freedom rotation is formulated and resolved. Both synchronization control and orientation control are considered. In each case, the measurement data is first processed through a nonlinear transformation. The transformed process then goes through an ordinary estimator, such as a Kalman–Bucy filter. After another nonlinear processing of the output of the ordinary estimator, the desired optimal control is yielded. A generalization of the approach illustrated by these results to control problems on arbitrary Abelian Lie groups is included.

1. Introduction. In this paper we will study several classes of stochastic control problems associated with single-degree-of-freedom rotation. As we shall see, the relevant state and sensor dynamic equations are bilinear in nature.

In the past, such stochastic control problems have been studied strictly in a vector space setting. While such techniques have been most useful in the study of linear systems, these methods have not yielded closed form optimal synthesis techniques for large classes of nonlinear systems, such as the bilinear systems considered here.

It is the purpose of this paper to use an alternative technique to the vector space approach. The motivation for this is to study the bilinear equations of interest with the aid of algebraic and analytical tools that are as natural to these problems as the vector space methods are to the linear problems. In this sense, one should view the present work as being motivated not only by the failure of vector space theory to handle some nonlinear problems adequately, but also by the success of vector space theory in effectively utilizing the structure of linear systems.

Very recently, the theory of Lie groups and Lie algebras has been successfully applied to a number of bilinear systems problems. Specifically, the results of Wei and Norman [11], [12] on differential equations, Brockett [1], Sussman, and Jurdjevic [5] on the structures of bilinear control systems, and Lo and Willsky [8] on estimation of rotational processes with one degree of freedom indicate that, much as in the theory of linear systems, the differential geometric structure of some bilinear systems may be used to obtain simple, explicit solutions. It is in this spirit that this paper is written.

Specifically, we will concern ourselves with the study of stochastic processes on the circle, S^1, and its extensions to higher dimensions. Topics such as FM modulation, frequency stability, single-degree-of-freedom gyroscopic analysis, and satellite attitude control are well-known examples in this framework.

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† Division of Mathematics, University of Maryland, Balitmore, Maryland 21228. The work of this author was supported in part by the U.S. Office of Naval Research under the Joint Services Electronics Program by Contract N00014–67–A–0298–0006, while he was with the Division of Engineering and Applied Physics, Harvard University, Cambridge, Massachusetts.

‡ Department of Electrical Engineering, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139. The work of this author was done in part while he was a Fannie and John Hertz Foundation Fellow in the Department of Aeronautics and Austronautics, M.I.T.

In the next section, a class of stochastic control problems on the unit circle will be formulated. The state and sensor dynamic processes are constructed by taking the projection module 2π of the corresponding typical 1-dimensional processes. The stochastic differential equations which govern their evolution are bilinear in form. The control function and the observational noise can be viewed as entering multiplicatively.

In § 3, we will briefly discuss two kinds of control criteria on the circle, namely synchronization control and orientation control criteria. An effective optimal control procedure for each of these two kinds of control problems will then be deduced with the aid of the optimal estimation schemes derived in Lo and Willsky [8]. In each case, the measurement data is first processed through a nonlinear transformation. The transformed process then goes through an ordinary estimator, such as a Kalman–Bucy filter. After another nonlinear processing of the output of the ordinary estimator, the desired control is yielded. The approach illustrated by these results can be extended to a large class of problems—those involving processes evolving on Abelian Lie groups. This will be discussed at the end of § 3.

Section 2 is relatively abstract, since it describes the mathematical setting of the problems to be considered. The authors wish to point special attention to § 3, in which we explicitly solve several nonlinear stochastic control problems.

The reader is referred to Lo and Willsky [9] for some examples, which illustrate the application of results in this paper to a number of important practical problems. Among them are a control problem of the synchronous rotation of a prime mover in a hydraulic plant, a feedback frequency modulation problem, and a satellite attitude control problem.

2. Stochastic control systems. In this section, we will formulate a stochastic model of a control system for continuous rotational processes with one degree of freedom. This model consists of equations for the state and the sensor dynamics.

A natural state space for single-degree-of-freedom rotational processes is the circle group, S^1. It has been shown (Ito and McKean [4]) that the circular Brownian motion on S^1 can be constructed by taking the projection modulo 2π of the standard 1-dimensional Brownian motion onto the unit circle S^1. This method will now be used to construct the continuous state and sensor dynamics to be used in this paper.

We will adopt the following notation:

$(\Omega, \mathscr{A}, P)$ = a probability space;
s = a positive real number;
C_1^s = the family of real-valued continuous functions, a, on $[0, s]$ such that $a(0) = 0$;
C_2^s = the family of 2×2 orthogonal-matrix-valued continuous functions, A, on $[0, s]$ such that $A(0) = I$, the identity matrix;
B_i^s = the Borel σ-field of C_i^s with respect to the uniform topology of C_i^s, for $i = 1$ and 2.

Lower case letters denote elements in C_1^s and upper case letters denote elements in C_2^s.

Let $J:C_1^s \to C_2^s$ be defined by

$$
(1)\qquad (J(a))(t) = \exp(a(t)R) = \begin{bmatrix} \cos a(t) & \sin a(t) \\ -\sin a(t) & \cos a(t) \end{bmatrix}, \qquad R = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}
$$

for $a \in C_1^s$ and $t \in [0, s]$. It is easily seen that J is B_1^s-measurable and bijective. This bijective operator will play a key role in this paper. The reader is referred to Lo and Willsky [8] for a physically appealing argument concerning the bijectivity of J.

Thus a continuous stochastic process Y on S^1 corresponds to a continuous stochastic process y on R in the sense that one can be induced by the other via the bijective operator J. In the following we will refer to y as the C_1-representation and Y as the C_2-representation of the continuous stochastic process under consideration.

2.1. State dynamic equations. We first formulate a state dynamic equation on S^1 as the following scalar Ito differential equation viewed as its C_1-representation:

$$
(2)\qquad \begin{aligned} dx &= a(t)\,dt + F(t)x(t)\,dt + G(t)u(t)\,dt + Q^{1/2}(t)\,dw(t), \\ x(0) &= 0, \end{aligned}
$$

where a, F, G, and $Q^{1/2}$ are scalar functions, w is a standard Brownian motion on $(\Omega, \mathscr{A}, P)$, and u is the scalar control function. In considering rotational processes with one degree of freedom, the dynamic state of the process is specified by the 2×2 orthogonal matrix representation of the process. In this sense, the C_1-representation above is *not* a dynamic-state-space representation of the process. Injecting x into S^1 via the operator J, we obtain, with the aid of the Ito differential rule, the following Ito matrix differential equation satisfied by the C_2-representation $X = J(x)$. This *is* a dynamic-state-space representation of the state dynamics.

$$
(3)\qquad \begin{aligned} dX(t) &= [(A_1(t) + A_2(t))\,dt + B(t)u(t)\,dt + C(t)\,dw(t) + D(t)x(t)\,dt]X(t), \\ X(0) &= I, \\ x(t) &= \left[\int_0^t (dX(s))X^{-1}(s)\right]_{12}, \end{aligned}
$$

where

$$(4)\qquad A_2(t) = a(t)R,$$

$$(5)\qquad B(t) = G(t)R,$$

$$(6)\qquad C(t) = Q^{1/2}(t)R,$$

$$(7)\qquad D(t) = F(t)R,$$

$$(8)\qquad A_1(t) = \tfrac{1}{2}C^2(t).$$

We note that in this equation A_1 is introduced to keep the evolution of X on S^1 when the equation is interpreted as an Ito differential equation. In fact the term $A_1X\,dt$ is precisely the second order correction term that arises in Ito differential calculus. $A_2(t)$ as well as $B(t)$, $C(t)$, and $D(t)$ are skew symmetric matrices.

If we set $C = D = A_1 = 0$, equation (3) then becomes

$$\dot{X}(t) = (A_2(t) + B(t)u(t))X(t),$$

which is a well-known deterministic model (Brockett [1]) for control systems on S^1. This indicates that our formulation (3) introduces randomness (in the form of white Gaussian noise) into the above well-known deterministic model in a very natural way. In addition, the terms involving the coefficient D in (3) allow the physical quantity $x(t)$, the total angle that the considered rotational process has swept, to enter the state dynamics directly.

2.2. Sensor dynamic equations. We will now formulate a sensor dynamic equation on S^1. The C_1-representation of the sensor dynamics is given by the following scalar Ito differential equation:

$$dz(t) = H(t)x(t)\,dt + R^{1/2}(t)\,dv(t), \qquad z(0) = 0, \tag{9}$$

where v is a standard scalar Brownian motion independent of w, $R^{1/2}(t)$ and $H(t)$ are scalar functions. Injecting z into S^1 via J, we obtain the following Ito matrix differential equation satisfied by the C_2-representation $Z = J(z)$ of the sensor dynamic equation:

$$\begin{aligned} dZ(t) &= [\tfrac{1}{2}N(t)\,dt + S(t)x(t)\,dt + E(t)\,dv(t)]Z(t), \\ Z(0) &= I, \end{aligned} \tag{10}$$

where

$$N(t) = E^2(t), \tag{11}$$

$$S(t) = H(t)R, \tag{12}$$

$$E(t) = R^{1/2}(t)R. \tag{13}$$

We note that this is a dynamic-state-space representation of the sensor dynamics. The term $\frac{1}{2}NZ\,dt$ in (10) plays the same role as $\frac{1}{2}A_1X\,dt$ did in (3). The matrices $S(t)$ and $E(t)$ are skew-symmetric.

We note that because J is a bijective operator and $Z = J(z)$, the σ-field in $(\Omega, \mathscr{A}, P)$ generated by $Z^t = \{Z(s), 0 \leqq s \leqq t\}$ is the same as that generated by $z^t = \{z(s), 0 \leqq s \leqq t\}$. In other words, Z^t and z^t carry the same amount of information about X. This enables the C_1-representation (9) to serve as an extremely useful auxiliary equation in the analysis of detection, estimation, and control. While the detection and the estimation problems were treated in Lo and Willsky [8], and Lo [17], the application of the C_1-representation to control problems will be considered in this paper.

Since a sensor cannot take measurement of future state evolution, the observation process Z (the output of the sensor) must be nonanticipative with respect to state evolution. More specifically $Z(t)$ must be a function of $X^t = \{X(s), 0 \leqq s \leqq t\}$

or equivalently x^t (from the mathematical viewpoint) since $X^t = J(x^t)$ and J is bijective. We note that the sensor dynamic equation (10) does, in fact, have this essential feature.

From a physical viewpoint the sensors used to observe single-degree-of-freedom rotational processes can be classified into two kinds by the way in which the measurement is taken. The first type measures the orientation $X(t)$ directly (as in the measurement of a gimbal angle in an inertial navigation system (Wrigley, Hollister, and Denhard [14]). A sensor of the second kind measures the total angle swept $x(t)$ directly (e.g., an integrating gyroscope).

3. Cost criteria and feedback control. A cost criterion for a control system operating over some time period T is usually defined as a real-valued functional η on the direct product of the space of state trajectories and the space U (to be specified later) of admissible control functions over T. As shown in the previous section, there are two representations of the space of state trajectories—C_1 and C_2, which are related by the bijective operator J. Therefore we may define the cost criterion as a real-valued functional on either $C_1 \times U$ or $C_2 \times U$. One form of the criterion can be easily obtained from the other via the operator J. A cost criterion in the form of a function on $C_i \times U$ will be called its C_i-representation for $i = 1$ and 2.

Just as with the classification of sensors from a physical viewpoint, the C_1- and the C_2-representations of the cost criterion have different physical interpretations. When the cost is induced directly by the time history of the deviation of the total swept angle of the controlled rotational process from some desired total swept angle (or, alternatively, when it is induced by the deviation of the angular velocity of the controlled process from some desired rotational rate), it is obviously physically more natural to first write down the C_1-representation of the cost criterion. A notable example of this kind is the control of synchronous rotation such as the control of a rotor in a motor or electric generator, or in the adjustment of a high-accuracy clock or an oscillator used for frequency modulation.

On the other hand, when the cost is induced directly by the time history of the "deviation" (a measure of angular deviation will be specified later) of the orientation of the controlled rotational process from the desired orientation, it is then physically more natural to write down the C_2-representation of the cost criterion. A notable example of this kind is the satellite attitude control problem (Leondes [7]). In the following we will study the control problems for these two kinds of cost criteria. They will be referred to as synchronization control and orientation control respectively.

In the following, we will consider control systems defined on the fixed interval $T = [0, t_*]$. The space U of admissible control functions is defined as follows: let the mapping $\pi_t : C_2 \to C_2$ be defined by

$$\begin{aligned}(\pi_t A)(s) &= A(s), \qquad 0 \leqq s \leqq t,\\ &= A(t), \qquad t \leqq s \leqq t_*,\end{aligned}$$

for $A \in C_2$. Let $\|\cdot\|_s$ denote the supremum norm in C_2 defined by

$$\|A\|_s = \sup_{t \in T} (\operatorname{tr} A(t)A'(t))$$

and let $\psi : T \times C_2 \to U$ (a convex subset of R^1) be a mapping with the following properties: $\psi(t, A)$ is Hölder continuous in t for each $A \in C_2$ and satisfies a uniform Lipschitz condition

$$|\psi(t, A_1) - \psi(t, A_2)| < c_3 \|A_1 - A_2\|_s$$

for $t \in T$ and $A_1, A_2 \in C_2$. Let Ψ denote the family of functions ψ. We call a control u admissible, and write $u \in U$, if

$$u(t) = \psi(t, \pi_t Z), \qquad t \in T,$$

for some $\psi \in \Psi$. These conditions ensure the causality of the control.

The control problem to be studied in the following subsections is: given a cost function η, find $u^* \in U$ such that

$$\eta[u^*] = \min \{\eta[u] : u \in U\}.$$

The corresponding function ψ^* will be called an optimal control law.

3.1. Synchronization control. Let the C_1-representation of a desired rotational process be continuous and denoted by $\phi(t)$. Assume that the cost criterion η can be expressed as follows:

$$\eta[u] = E\left[\int_0^{t_*} (x(s) - \phi(s))^2 W(s)\, ds + \int_0^{t_*} u_2(s) V(s)\, ds\right], \tag{14}$$

where $W(t)$ and $V(t)$ are respectively nonnegative and positive-valued functions with $V^{-1}(t)$ bounded on T. We have mentioned that since linear control theory is better established than bilinear control theory, the C_1-representation of the dynamic-state-space representation (2) and (9) serves as a very useful auxiliary equation. This is best shown by the following derivation of the optimal control law for the synchronization control problem.

Given the dynamic-state-representation, (3) and (10), of the control system, we first write down the C_1-representations, (2) and (9), of the system with the coefficients $a(t)$, $F(t)$, $G(t)$, $Q^{1/2}(t)$, $H(t)$, $R^{1/2}(t)$ being determined by (4) $\sim$ (8), (12), and (13).

In addition, we now define the set U_1 of admissible control functions, defined with respect to the C_1 system representation (as opposed to the set U, which was defined earlier with respect to the C_2-representation).

Let π_t, defined earlier, also denote the mapping from C_1 into C_1 defined by

$$(\pi_t(a))(s) = \begin{cases} a(s), & 0 \leqq s \leqq t, \\ a(t), & t \leqq s \leqq t_*, \end{cases}$$

for $a \in C_1$. Let $\|\cdot\|_s$ also denote the supremum norm in C_1 and let $\psi_1 : T \times C_1 \to R$ be a mapping with the properties: $\psi_1(t, a)$ is Hölder continuous in t for each $a \in C_1$ and satisfies a uniform Lipschitz condition

$$|\psi_1(t, a_1) - \psi_1(t, a_2)| < c_4 \|a_1 - a_2\|_s$$

for $t \in T$ and $a_1, a_2 \in C_1$. Let Ψ_1 denote the class of functionals ψ_1. We call the control u admissible and write $u \in U_1$ if

$$u(t) = \psi_1(t, \pi_t z), \qquad t \in T,$$

for some $\psi_1 \in \Psi_1$. An element $\psi_1^0 \in \Psi_1$ is called an optimal control law if

$$\eta[u^0] = \min \{\eta[u] : u \in U_1\},$$

where $u^0(t) = \psi_1^0(t, \pi_t z)$.

By either completing squares or applying Lemma 5.1 (optimality criterion) of Wonham [13], the following lemma can be easily obtained.

LEMMA 1. *Consider the cost criterion* (14) *and the control system described by* (2) *and* (9), *with $x(t)$ regarded as the dynamic state. Then the optimal control law, $u^0(t) = \psi_1^0(t, \pi_t z)$, is given by*

$$u^0(t) = -V^{-1}(t)G(t)(P_1(t)[\hat{x}(t) - \phi(t)] + b(t)), \tag{15}$$

$$\begin{aligned} \dot{b}(t) &= P_1(t)G^2(t)V^{-1}(t)b(t) - F(t)b(t) - P_1(t)(\dot{a}(t) - \phi(t)), \\ b(t_*) &= 0, \end{aligned} \tag{16}$$

$$\begin{aligned} \dot{P}_1(t) &= -F(t)P_1(t) - A(t)P_1(t) + G^2(t)V^{-1}(t)P_1^2(t) - W(t), \\ P_1(t_*) &= 0, \end{aligned} \tag{17}$$

$$\begin{aligned} \hat{x}(t) &= E(x(t)|z^t), \\ z^t &= \{z(s), 0 \leqq s \leqq t\}, \end{aligned} \tag{18}$$

$$\begin{aligned} d\hat{x}(t) &= a(t)\,dt + F(t)\hat{x}(t)\,dt + G(t)u^0(t)\,dt \\ &\quad + P_2(t)H(t)R^{-1}(t)(dz(t) - H(t)\hat{x}(t)\,dt), \\ \hat{x}(0) &= 0, \end{aligned} \tag{19}$$

$$\begin{aligned} \dot{P}_2(t) &= 2F(t)P_2(t) + Q(t) - H^2(t)R^{-1}(t)P_2^2(t), \\ P_2(0) &= 0. \end{aligned} \tag{20}$$

Using Lemma 1, we can now determine the optimal synchronization control law. We observe that the σ-subfield of $\mathscr{A}$ generated by z^t is the same as that generated by $Z^t = \{Z(s), 0 \leqq s \leqq t\}$, because $Z^t = J(z^t)$ and J is bijective. In other words, z and Z are causally equivalent. Let this σ-subfield be denoted by $\mathscr{A}_z^t$. Then the conditional expectation $E(x(t)|\mathscr{A}_z^t)$ is both a B_1-measurable functional f_1 of z^t and a B_2-measurable functional f_2 of Z^t, and

$$f_2(Z^t) = f_1(J^{-1}(Z^t)).$$

Let $\hat{x}(t)$ and $\hat{x}(t|t)$ denote $f_1(z^t) \triangleq E(x(t)|z^t)$ and $f_2(Z^t) \triangleq E(x(t)|Z^t)$, respectively. Note that this notation is consistent with (18). Referring to Lo and Willsky [8], it is easily seen that

$$\begin{aligned} d\hat{x}(t|t) &= a(t)\,dt + F(t)\hat{x}(t|t)\,dt + G(t)\psi_1^0(t, \pi_t(J^{-1}(Z^t)))\,dt \\ &\quad + P^2(t)H(t)R^{-1}(t)\{[(dZ(t))Z'(t) - \tfrac{1}{2}N(t)\,dt]_{12} \\ &\qquad - H(t)\hat{x}(t|t)\,dt\}, \\ \hat{x}(0|0) &= 0. \end{aligned} \tag{21}$$

Again because z and Z are causally equivalent, we may define a $\psi_1 \in \Psi_1$ for each $\psi \in \Psi$ by

$$\psi_1(t, \pi_t z) = \psi[t, \pi_t(J(z))],$$

and we may define $\psi^0 \in \Psi$ by

$$\psi^0(t, \pi_t Z) = \psi_1^0[t, \pi_t(J^{-1}(Z))].$$

We note here that the properties of Ψ and Ψ_1 do not give rise to trouble in the above argument. Since ψ_1^0 isoptimal, we have

$$\eta[\psi^0(t, \pi_t Z)] = \eta[\psi_1^0(t, \pi_t(J^{-1}(Z))] \leqq \eta[\psi_1(t, \pi_t(z))] = \eta[\psi(t, \pi_t Z)]$$

for all $\psi \in \Psi$. Hence ψ^0 is optimal. Summarizing what has been shown, we obtain the following theorem.

THEOREM 2. *Consider the control system of rotational processes described by the bilinear matrix Ito differential equations* (3) *and* (10) *and consider the cost criterion* (14). *The optimal control law* ψ^* *is given by*

$$u^*(t) = \psi^*(t, \pi_t Z) = -V^{-1}(t)G(t)(P_1(t)(\hat{x}(t|t) - \phi(t)) + b(t))$$

and

$$\begin{aligned} d\hat{x}(t|t) &= a(t)\,dt + F(t)\hat{x}(t|t)\,dt + G(t)u^*(t)\,dt + P_2(t)H(t)R^{-1}(t) \\ &\quad \cdot \{[dZ(t)Z'(t) - \tfrac{1}{2}N(t)\,dt]_{12} - H(t)\hat{x}(t|t)\,dt\} \\ \hat{x}(o|o) &= 0, \end{aligned}$$

where $a, F, G, R, H, P_1, b, P_2$ *are determined by* (4), (7), (5), (13), (12), (17), (16), (20), *respectively.*

3.2. Orientation control. The standard distance function (Riemannian metric) on the circle—i.e., the distance, ρ, between two points on the circle is the arc length of the shortest path (geodesic line) joining them. Any valid mathematical expression for the "distance" between two orientations must be a positive-valued function $\lambda: S^1 \times S^1 \to R^1$, which is nondecreasing with respect to ρ, i.e.,

$$\rho(\Theta_1, \Theta_2) > \rho(\Theta_1, \Theta_2) \Leftrightarrow \lambda(\Theta_1, \Theta_2) > \lambda(\Theta_1, \Theta_3)$$

for $\Theta_i \in S^1$, $i = 1, 2$. In this subsection, we will consider only

$$\lambda(\Theta_1, \Theta_2) = \tfrac{1}{2}(2 - \operatorname{tr} \Theta_1 \Theta_2')$$

to avoid complexity in illustrating the approach.

Let $\Phi \in C_2$ be the desired evolution of the orientation. Then a cost criterion η for orientation control can be expressed as follows:

$$\eta[u] = E\left[\int_0^{t_*} \tfrac{1}{2}(2 - \operatorname{tr} X(s)\Phi'(s))\,ds + \int_0^{t_*} \gamma(s)u^2(s)\,ds\right], \tag{22}$$

where ϕ is a nonnegative scalar function over T.

Let y be the C_1-representation of Φ. It is easily seen that the C_1-representation of η can be written as follows:

$$\eta[u] = E\left[\int_0^{t_*} (1 - \cos(x(s) - \phi(s)))\,ds + \int_0^{t_*} \gamma(s)u^2(s)\,ds\right]. \tag{23}$$

We note that the function $1 - \cos x$ was used in estimation problems in Bucy and Mallinckrodt [16].

In view of the C_1-representation (2) and (9), setting $y(t) = x(t) - \phi(t)$, we have

$$\begin{aligned} dy &= (a - \dot{\phi} + F\phi)\,dt + Fy\,dt + Gu\,dt + Q^{1/2}\,dw, \\ y(0) &= 0, \\ dz &= H\phi\,dt + Hy\,dt + R^{1/2}\,dv, \\ z(0) &= 0, \\ \eta[u] &= E\left[\int_0^{t_*} (1 - \cos y(s))\,ds + \int_0^{t_*} \gamma(s)u^2(s)\,ds\right]. \end{aligned}$$

Thus the Bellman functional equation (Kushner [6] and Wonham [13]) is

$$\min_{u\in U} [V_t(t, \xi) + \tfrac{1}{2}P^2H^2R^{-1}V_{\xi\xi}(t, \xi) + (F\xi + a - \dot{\phi} + F\phi + Gu)V_\xi(t, \xi) + 1 + \gamma u^2 - \exp(-\tfrac{1}{2}P)\cos \xi] = 0,$$

$$V(t_*, \xi) = 0,$$

where

(24) $$\dot{P} = 2FP - H^2R^{-1}P^2 + Q, \qquad P(0) = 0.$$

We set

$$u = -\gamma^{-1}GV_\xi(t, \xi).$$

Then the control law $-\gamma^{-1}(t)G(t)V_\xi(t, \hat{y}(t))$ is optimal in U, if there exists a solution to the following partial differential equation (see Lemma 5.1 of Wonham [13]):

(25) $$\begin{aligned} &V_t(t, \xi) + \tfrac{1}{2}P^2H^2R^{-1}V_{\xi\xi}(t, \xi) + (F\xi + a - \dot{\phi} + F\phi)V_\xi(t, \xi) + 1 \\ &\qquad - \exp(-\tfrac{1}{2}P)\cos \xi = 0, \\ &V(t_*, \xi) = 0. \end{aligned}$$

Let

$$L(\cdot) \triangleq \frac{\partial}{\partial t}(\cdot) + \frac{1}{2}P^2H^2R^{-1}\frac{\partial^2}{\partial \xi^2}(\cdot) + (F\xi + a - \dot{\phi} + F\phi)\frac{\partial}{\partial \xi}(\cdot).$$

We note that L is a Kolmogorov backward operator (Doob [3, p. 275]). It is well known that there exists one and only one solution $V(t, \xi)$ to (25) and it can be written as

(26) $$V(t, \xi) = \int_t^{t_*}\left\{\int_{-\infty}^{\infty} g(t, \xi; s, \zeta)\left[1 - \exp\left(-\frac{P}{2}\right)\cos \zeta\right]d\zeta\right\}ds,$$

where g is a Green's function which satisfies

(27) $$L[g(t, \xi; s, \zeta)] = 0, \qquad g(s, \xi; s, \zeta) = \delta(\xi - \zeta),$$

δ being the Dirac delta function. It can be checked by simple calculation that the solution to (27) is

(28) $$g(t, \xi; s, \eta) = \frac{\beta(t; s)}{\sqrt{2\pi\alpha(t; s)}}\exp\left[-\frac{(\xi - \mu(t; s, \eta))^2}{2\alpha(t; s)}\right],$$

where

$$\beta(t; s) = \exp\left[-\int_t^s F(\tau)\,d\tau\right], \tag{29}$$

$$\mu(t; s, \eta) = \beta(t; s)\eta + \int_t^s \beta(t; \tau)(a(\tau) - \dot{\phi}(\tau) + F(\tau)\phi(\tau))\,d\tau, \tag{30}$$

$$\alpha(t; s) = \int_t^s \beta^2(t; \tau)P^2(\tau)H^2(\tau)R^{-1}(\tau)\,d\tau. \tag{31}$$

Substituting (28) into (26) yields

$$V(t, \xi) = (t_* - t) - \int_t^{t_*} \exp\left\{-\frac{1}{2}\left[\frac{\alpha(t; s)}{\beta^2(t; s)} + P(s)\right]\right\} \cos\left[\frac{\xi - \mu(t; s, 0)}{\beta(t; s)}\right] ds.$$

Thus,

$$V_\xi(t, \xi) = \int_t^{t_*} \frac{1}{\beta(t; s)} \exp\left\{-\frac{1}{2}\left[\frac{\alpha(t; s)}{\beta^2(t; s)} + P(s)\right]\right\} \sin\left[\frac{\xi - \mu(t; s, 0)}{\beta(t; s)}\right] ds.$$

Summarizing what has been shown, we obtain the following theorem for optimal orientation control.

THEOREM 3. *Consider the control system of rotational processes described by* (3), (10) *and consider the cost criterion* (14). *The optimal control law* ψ^* *is given by*

$$u^*(t) = \psi^*(t, \pi_t Z) = \int_y^{t_*} K(t, s) \sin\left[\frac{\hat{x}(t|t) - \phi(t) - \mu(t; s, 0)}{\beta(t; s)}\right] ds,$$

$$K(t, s) = -\frac{G(t)}{\gamma(t)\beta(t; s)} \exp\left\{-\frac{1}{2}\left[\frac{\alpha(t; s)}{\beta^2(t; s)} + P(s)\right]\right\},$$

where β, μ, α *are determined by* (29) $\sim$ (31), *and*

$$\begin{aligned} d\hat{x}(t|t) &= a(t)\,dt + F(t)\hat{x}(t|t)\,dt + G(t)u^*(t)\,dt \\ &\quad + P(t)H(t)R^{-1}(t)([(dZ(t))Z'(t)]_{12} - H(t)\hat{x}(t|t)\,dt), \\ \hat{x}(0|0) &= 0, \end{aligned}$$

where a, F, G, R, H, P *are determined by* (4), (7), (5), (13), (12), (24), *respectively.*

We remark that $K(t, s)$, $\beta(t; s)$, and $\mu(t; s, 0)$ can be precomputed and stored in the feedback controller. Hence it is believed that the optimal control scheme of the previous theorem can easily be implemented.

When x does not directly enter the state dynamics (3), i.e., when $D \equiv 0$, the optimal orientation control law takes a very simple and interesting form. We state it in the following corollary.

COROLLARY. *Consider the control problem in the previous theorem. If* $D \equiv 0$, *the optimal control law* ψ^* *is given by*

$$\begin{aligned} u^*(t) &= \psi^*(t, \pi_t Z) \\ &= c_1(t)\cos(x(t|t) - \phi(t)) + c_2(t)\sin(\hat{x}(t|t) - \phi(t)), \end{aligned}$$

where

$$c_1(t) = -\int_t^{t_*} \exp\left\{-\frac{1}{2}\left[\int_t^{\tau} P^2(s)H^2(s)R^{-1}(s)\,ds + P(\tau)\right]\right\} \sin\left[\int_t^{\tau} (a(s) - \dot{\phi}(s) + F(s)\phi(s))\,ds\right] d\tau.$$

$$c_2(t) = \int_t^{t_*} \exp\left\{-\frac{1}{2}\left[\int_t^{\tau} P^2(s)H^2(s)R^{-1}(s)\,ds + P(\tau)\right]\right\} \cos\left[\int_t^{\tau} (a(s) - \dot{\phi}(s) + F(s)\phi(s))\,ds\right] d\tau,$$

and $\hat{x}(t|t)$, a, F, G, R, H, P *are determined as in the previous theorem.*

In Lo and Willsky [8], orientation estimation of rotational processes with one degree of freedom was studied. It was shown that the optimal orientation estimate $\hat{X}(t)|t)$ of $X(t)$ given observation Z^t is

$$\hat{X}(t|t) = \exp(R\hat{x}(t|t)).$$

Hence the optimal control law in the previous corollary is in fact linear:

$$u^*(t) = [c_1(t), c_2(t)]\Phi'(t)\hat{X}(t|t)[1, 0]',$$

where $\Phi(t) = \exp(R\phi(t))$ is the C_2-representation of $\phi(t)$.

3.3. Control on Abelian Lie groups. The results of the previous subsections can be extended to a large class of problems—those involving processes evolving on Abelian Lie groups. It is well known (Warner [10]) that a given Abelian Lie group G is isomorphic to the direct product of a number of copies of the real line and a number of copies of the unit circle, i.e.,

$$G \approx R^n \times (S^1)^m.$$

The diffusion processes on this type of space have been used to model some interesting satellite and pendulum systems in Ku and Sheporaitis [15]. Following Lo and Willsky [8], a bijective mapping $J_{nm}:(C_1^s)^{n+m} \to (C_1^s)^n \times (C_2^s)^m$ is defined by

$$(J_{nm}(a))(t) = [a_1(t), \cdots, a_n(t), (J(a_{n+1}))(t), \cdots, (J(a_{n+m}))(t)]$$

for $a \in (C_1^s)^{n+m}$, a_i being the ith component of a. Thus a continuous random signal process on G which is described by an $\mathscr{A}$-measurable function $X:\Omega \to (C_1^s)^n \times (C_2^s)^m$ corresponds to a unique continuous random signal process on R^{n+m} which is described by an $\mathscr{A}$-measurable function $x:\Omega \to (C_1^s)^{m+n}$ such that

$$X(t) = (J_{nm}(x))(t), \qquad t \in [0, s].$$

The mathematical model for a control system on G can be obtained by first using J_{nm} to inject the following $(n + m)$-*vector* random differential equation into

$R^n \times (S^1)^m$:

$$dx(t) = a(t)\,dt + F(t)x(t)\,dt + C(t)u(t)\,dt + Q^{1/2}(t)\,dw(t),$$
$$x(o) = 0,$$

and using J_{pq} to inject the following $p + q$-*vector* random differential equation into $R^p \times (S^1)^q$:

$$dz(t) = H(t)x(t)\,dt + R^{1/2}(t)\,dv(t),$$
$$z(0) = 0,$$

where the coefficient functions are of appropriate dimension and w and v are independent vector Brownian motions. Differentiating $X(t) = (J_{mn}(x))(t)$ and $Z(t) = (J_{pq}(z))(t)$ by the stochastic differentiation rule, we obtain a set of joint linear and bilinear stochastic differential equations. This calculation is straightforward and thus we will not display those equations. Let $X(t) = [x_1(t), \cdots, x_n(t), X_{n+1}(t), \cdots, X_{n+m}(t)]$, where $X_{n+i}(t) = (J(x_{n+i}))(t)$. A joint synchronization and orientation cost criterion can be written as follows: for $0 \leqq l \leqq m$,

$$n[u] = \sum_{i=1}^{n+l} \int_0^{t_*} \gamma_i(s)(x_i(s) - \phi_i(s))^2\,ds + \sum_{i=n+l+1}^{n+m} \int_0^{t_*} \gamma_i(s)(2 - \operatorname{tr} X_i(s)\Phi_i'(s))\,ds$$
$$+ \int_0^{t_*} u'(s)V(s)u(s)\,ds,$$

where γ_i are nonnegative functions over T, V is nonnegative definite over T, and $\phi_i(s)$ and $\Phi_i(s)$ are the desired total swept angles and the desired orientations at $t = s$. Because of the bijective property of J_{nm} and J_{pq}, it is clear that the optimal control analysis in the previous subsections can be easily generalized to this general Abelian case with little modification. The reader is referred to Lo and Willsky [9] for some examples, which illustrate the approach.

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