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Reachability, observability, minimality, and extendibility for two-point boundary-value descriptor systems

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A deterministic system theory is developed for two-point boundary-value descriptor systems (TPBVDSs). In particular, detailed characterizations of the properties of reachability, observability and minimality are obtained. In addition, *extendibility*, i.e. the concept of considering TPBVDS as being defined on a sequence of intervals of increasing length, is defined and studied. This system theory generalizes our earlier results for the class of *stationary* systems for which the input–output map (weighting pattern) is shift-invariant.

1. Introduction

To model non-causal physical phenomena which usually correspond to processes evolving in space rather than time, the standard state-space models are not appropriate. State-space models are based on the assumption of causality and thus lead to initial-value systems, whereas non-causal models have, in general, boundary specification and thus lead to boundary-value systems.

In recent years there has been considerable interest in characterizing the properties of classes of non-causal models specified by differential and difference equations together with boundary conditions (see, for example, Luenberger 1977, 1978, Krener 1980, 1987, Lewis 1984, Gohberg and Kaashoek 1986 a, b, Gohberg *et al.* 1986, Nikoukhah *et al.* 1987, 1989 a, b). In particular, in our previous work (Nikoukhah *et al.* 1987, 1989 a, b) we have investigated the class of two-point boundary-value descriptor systems (TPBVDSs). As discussed by Nikoukhah *et al.* (1987) input-state and input–output maps for a TPBVDS need *not* be shift-invariant even if the system matrices are constant, and in the analysis this fact presented some difficulties which limited the development given by Nikoukhah *et al.* (1989 a) to shift-invariant systems. In this paper we overcome those difficulties and extend the results of Nikoukhah *et al.* (1989 a) to the full class of constant-coefficient TPBVDSs.

As originally introduced by Krener (1980, 1987), boundary-value systems naturally lead to two notions of recursion: an *outward process*, for which the direction of recursion is outward toward the boundaries, and an *inward process*, which propagates the boundary condition inward from the boundaries. As shown by Nikoukhah *et al.* (1987), the outward process has an easily computed representa-

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tion for any TPBVDS. On the other hand, a closed-form expression for the inward process is only given for the so-called class of *displacement systems*, i.e. those systems with a shift-invariant input-state map. The key to the results presented in this paper is the development of a more general closed-form expression for the inward process. In particular, this expression allows us to follow an approach similar to that of Gohberg *et al.* 1986, in order to derive a general minimality result.

In the next section we review some of the concepts and results from Nikoukhah *et al.* (1987, 1989 a) and in § 3 we review the notions of inward and outward processes and develop the new closed form expression for the inward process. In addition in this section we introduce and characterize the concept of '*extendibility*', i.e. the notion of imbedding a TPBVDS in a set of such systems defined over intervals of increasing length. In § 4, we analyse the properties of reachability and observability, while in § 5 we present results on minimality. We conclude with a few remarks in § 6.

2. Two-point boundary-value descriptor systems

A TPBVDS is described by the dynamic equation

$$Ex(k+1) = Ax(k) + Bu(k), \quad 0 \leqslant k \leqslant N-1 \tag{2.1}$$

with boundary condition

$$V_i x(0) + V_f x(N) = v \tag{2.2}$$

and output

$$y(k) = Cx(k), \quad k = 0, 1, \ldots, N \tag{2.3}$$

Here x and v are n-dimensionai, u is m-dimensional, and E, A, B, V_i, V_f, and C are constant matrices. We also assume that the interval of definition of the system is sufficiently large to observe all the system modes. Specifically, we assume that $N \geqslant 2n$, unless explicitly stated otherwise.

Nikoukhah *et al.* (1987) showed that if (2.1), (2.2) is well-posed (i.e., it yields a well-defined map from $\{u, v\}$ to x), we can assume without loss of generality that (2.1), (2.2) is in *normalized form*, i.e. that there exist scalars α and β such that

$$\alpha E + \beta A = I \tag{2.4}$$

(this is referred to as the *standard form* for the pencil $\{E, A\}$), and in addition

$$V_i E^N + V_f A^N = I \tag{2.5}$$

Note that (2.4) implies that E and A commute and that $\{E^k, A^k\}$ is a regular pencil for all $k > 0$ (see Nikoukhah *et al.* 1987). But most importantly, (2.4) implies that the space of matrices $A^K E^L$, K, $L \geqslant 0$ is spanned by the n matrices $\{A^k E^{n-1-k} \mid k = 0, \ldots, n-1\}$; this property has been introduced by Nikoukhah *et al.* (1987), as the generalized Cayley–Hamilton theorem. We assume throughout this paper that (2.4) and (2.5) hold.

As derived by Nikoukhah *et al.* (1987), the map from $\{u, v\}$ to x has the form

$$x(k) = A^k E^{N-k} v + \sum_{j=0}^{N-1} G(k, j) Bu(j) \tag{2.6}$$

where the Green's function $G(k, j)$ is given by

$$G(k, j) = \begin{cases} A^k(A - E^{N-k}(V_i A + \omega V_f E)E^k)E^{j-k}A^{N-j-1}\Gamma^{-1} & j \geqslant k \\ E^{N-k}(\omega E - A^k(V_i A + \omega V_f E)A^{N-k})E^j A^{k-j-1}\Gamma^{-1} & j < k \end{cases} \tag{2.7}$$

and ω is any number such that

$$\Gamma \equiv \omega E^{N+1} - A^{N+1} \tag{2.8}$$

is invertible.

In marked contrast to the case for causal systems ($E = I$, $V_f = 0$), $G(k, j)$ does *not*, in general, depend on the difference of its arguments. Borrowing some terminology from Gohberg and Kaashoek (1986 a, b), we have the following definition of our first notion of shift-invariance.

Definition 2.1

The TPBVDS (2.1), (2.2) is a *displacement system* if (with the usual abuse of notation) for $0 \leqslant k \leqslant N$, $0 \leqslant j \leqslant N-1$

$$G(k, j) = G(k - j) \tag{2.9}$$

□

A second notion of shift-invariance is the one associated with the input–output map. Specifically, with $v = 0$ in (2.2), the system (2.1)–(2.3) defines the input–output map

$$y(k) = \sum_{j=0}^{N-1} W(k, j)u(j) \tag{2.10}$$

with

$$W(k, j) = CG(k, j)B \tag{2.11}$$

Definition 2.2

The TPBVDS (2.1)–(2.3) is *stationary* if (again with the usual abuse of notation)

$$W(k, j) = W(k - j) \tag{2.12}$$

for $0 \leqslant k \leqslant N$, $0 \leqslant j \leqslant N-1$. □

The following results from Nikoukhah *et al.* (1989 a), characterize the conditions under which a TPBVDS is displacement and stationary.

Theorem 2.1

The TPBVDS (2.1)–(2.3) is stationary if and only if

$$O_s[V_i, E]R_s = O_s[V_i, A]R_s = 0 \tag{2.13}$$

$$O_s[V_f, E]R_s = O_s[V_f, A]R_s = 0 \tag{2.14}$$

where $[X, Y]$ denotes the commutator product of X and Y

$$[X, Y] = XY - YX \tag{2.15}$$

and

$$R_s = [A^{n-1}B \mid EA^{n-2}B \mid \dots \mid E^{n-1}B] \tag{2.16}$$

$$O_s = \begin{bmatrix} CA^{n-1} \\ CEA^{n-2} \\ \vdots \\ CE^{n-1} \end{bmatrix} \tag{2.17}$$

□

Corollary 2.1

The TPBVDS (2.1), (2.2) is a displacement system if and only if

$$[V_i, E] = [V_i, A] = 0 \tag{2.18}$$

$$[V_f, E] = [V_f, A] = 0 \tag{2.19}$$

□

The matrices R_s and O_s in (2.16), (2.17) are, respectively, the *strong reachability* and *strong observability* matrices of the TPBVDS as discussed by Nikoukhah *et al.* (1987), (see also § 4). The results of causal system theory might suggest that the distinction between displacement and stationary systems is a trivial artifact caused by the use of non-minimal realizations. However, as in the case of continuous-time boundary-value systems (Krener 1987), the story is different for TPBVDSs. Specifically, as shown by Nikoukhah *et al.* (1989 a), and in § 5, a TPBVDS can be minimal without being strongly reachable and strongly observable.

Before closing this section we consider another problem, namely that of the degree of freedom in the choice of boundary matrices V_i and V_f. Using (2.7), (2.8), (2.11), the fact that $N > 2n$, and the generalized Cayley–Hamilton theorem, we can readily verify the following theorem.

Theorem 2.2

Consider two TPBVDSs with the same matrices C, E, A and B and identical weighting patterns. Then if one has boundary matrices V_i and V_f, and the other $\hat{V}_i$ and $\hat{V}_f$, we must have

$$O_s V_i R_s = O_s \hat{V}_i R_s \tag{2.20}$$

$$O_s V_f R_s = O_s \hat{V}_f R_s \tag{2.21}$$

Conversely if (2.20), (2.21) holds for two TPBVDSs with identical C, E, A and B system matrices, their weighting patterns must be identical. □

3. Inward processes, outward processes, and extendibility

As discussed by Nikoukhah *et al.* (1987), (with motivation from Krener 1987) the process x in a TPBVDS can be recovered from two processes that each have interpretations as state processes. The outward process has a direction of recursion outward toward the boundaries, summarizing all one needs to know about the input inside any interval in order to determine x outside this interval. For an interval $[k, j]$ with $k < j$, it is given by

$$z_o(k, j) = E^{j-k}x(j) - A^{j-k}x(k) \tag{3.1}$$

As shown by Nikoukhah *et al.* (1987), it can also be expressed in terms of the intervening inputs as

$$z_o(k, j) = \sum_{r=k}^{j-1} E^{r-k}A^{j-r-1}Bu(r) \tag{3.2}$$

Note that $z_o(k, j)$ does not involve the boundary matrices V_i and V_f.

The situation is different, however, for the inward process which uses input values near the boundaries to propagate the boundary condition inward. As developed by Nikoukhah *et al.* (1987), for any $K \leqslant L$ the inward process $z_i(K, L)$ is a function of the boundary value v and the inputs $\{u(0), \dots, u(K-1)\}$ and $\{u(L), \dots, u(N-1)\}$ such that the TPBVDS

$$Ex(k+1) = Ax(k) + Bu(k) \tag{3.3}$$

$$V_i(K, L)x(K) + V_f(K, L)x(L) = z_i(K, L) \tag{3.4}$$

yields the same solution as (2.1), (2.2) for $K \leqslant k \leqslant L$. Here $V_i(K, L)$ and $V_f(K, L)$ are assumed to satisfy the normalization condition

$$V_i(K, L)E^{L-K} + V_f(K, L)A^{L-K} = I \tag{3.5}$$

The following is a key new result.

Theorem 3.1

The inwardly-propagated boundary matrices and the inward process can be expressed as

$$V_i(K, L) = E^{N-L}(\omega E - A^K(\omega V_f E + V_i A)A^{N-k})\Gamma^{-1}E^K \tag{3.6}$$

$$V_f(K, L) = -A^K(A - E^{N-L}(\omega V_f E + V_i A)E^L)\Gamma^{-1}A^{N-L} \tag{3.7}$$

and

$$\begin{aligned} z_i(K, L) = {} & E^{N-L}A^K v + E^{N-L}(\omega E - A^K(\omega V_f E + V_i A)A^{N-k})\Gamma^{-1}z_o(0, K) \\ & + A^K(A - E^{N-L}(\omega V_f E + V_i A)E^L)\Gamma^{-1}z_o(L, N) \end{aligned} \tag{3.8}$$

□

Note in particular the starting values

$$z_i(0, N) = v, \quad V_i(0, N) = V_i, \quad V_f(0, N) = V_f \tag{3.9}$$

and the 'final values'

$$z_i(k, k) = x(k) \quad \text{for all } k \tag{3.10}$$

Proof

Let S_h be the $h \times (h+1)$ block matrix

$$S_h = \begin{bmatrix} -A & E & 0 & \dots & 0 \\ 0 & -A & E & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & & -A & E \end{bmatrix} \tag{3.11}$$

Then (2.1), (2.2) can be expressed as

$$\begin{bmatrix} & & S_N & & \\ V_i & 0 & \dots & 0 & V_f \end{bmatrix} \begin{bmatrix} x(0) \\ \vdots \\ x(N) \end{bmatrix} = \begin{bmatrix} Bu(0) \\ \vdots \\ Bu(N-1) \\ v \end{bmatrix} \tag{3.12}$$

It is also easy to see that

$$\begin{bmatrix} -A^K & E^K & 0 & \dots & 0 & 0 \\ 0 & & & & & 0 \\ \vdots & & & S_{L-K} & & \vdots \\ 0 & & & & & 0 \\ 0 & 0 & \dots & 0 & -A^{N-L} & E^{N-L} \\ V_i & 0 & & \dots & 0 & V_f \end{bmatrix} \begin{bmatrix} x(0) \\ x(K) \\ x(K+1) \\ \vdots \\ x(L) \\ x(N) \end{bmatrix} = \begin{bmatrix} z_o(0,K) \\ Bu(K) \\ \vdots \\ Bu(L-1) \\ z_o(L,N) \\ v \end{bmatrix} \tag{3.13}$$

To find $V_i(K, L)$ and $V_f(K, L)$, we need first to construct a full-rank matrix

$$[T_i(K,L) \quad T_f(K,L) \quad P(K,L)]$$

such that

$$[T_i(K,L) \quad T_f(K,L) \quad P(K,L)] \begin{bmatrix} -A^K & 0 \\ 0 & E^{N-L} \\ V_i & V_f \end{bmatrix} = 0 \tag{3.14}$$

Given such a matrix, if we pre-multiply both sides of (3.13) by

$$\Omega(K,L) = \begin{bmatrix} 0 & I & 0 & 0 \\ T_i(K,L) & 0 & T_f(K,L) & P(K,L) \end{bmatrix} \tag{3.15}$$

we obtain

$$\begin{bmatrix} & & S_{L-K} & \\ T_i(K,L)E^K & 0 & \dots & 0 & -T_f(K,L)A^{N-L} \end{bmatrix} \begin{bmatrix} x(K) \\ x(K+1) \\ \vdots \\ x(L) \end{bmatrix}$$

$$= \begin{bmatrix} Bu(K) \\ \vdots \\ Bu(L-1) \\ T_i(K,L)z_o(0,K) + T_f(K,L)z_o(L,N) + P(K,L)v \end{bmatrix} \tag{3.16}$$

This shows that

$$V_i(K, L) = T_i(K, L)E^K \tag{3.17}$$

$$V_f(K, L) = -T_f(K, L)A^{N-L} \tag{3.18}$$

and

$$z_i(K, L) = T_i(K, L)z_o(0, K) + T_f(K, L)z_o(L, N) + P(K, L)v \tag{3.19}$$

It is straightforward to check that

$$T_i(K, L) = E^{N-L}(\omega E - A^K(\omega V_f E + V_i A)A^{N-k})\Gamma^{-1} \tag{3.20}$$

$$T_f(K, L) = A^K(A - E^{N-L}(\omega V_f E + V_i A)E^L)\Gamma^{-1} \tag{3.21}$$

and

$$P(K, L) = E^{N-L}A^K \tag{3.22}$$

satisfy (3.14). Then, substituting (3.20)–(3.22) inside (3.17)–(3.19) gives (3.6)–(3.8). □

Theorem 3.1 can be slightly generalized to give a relationship between all inwardly propagated boundary matrices:

$$V_i(K, L) = E^{J-L}(\omega E - A^{K-I}(\omega V_f(I, J)E + V_i(I, J)A)A^{J-K})\Gamma_{J-I}^{-1}E^{K-I} \tag{3.23}$$

$$V_f(K, L) = -A^{K-I}(A - E^{J-L}(\omega V_f(I, J)E + V_i(I, J)A)E^{L-I})\Gamma_{J-I}^{-1}E^{J-L} \tag{3.24}$$

when $[K, L]$ is contained in $[I, J]$, where

$$\Gamma_M = \omega E^{M+1} - A^{M+1}$$

Remark

Note that if (2.1), (2.2) is a displacement system, the fact that E and A commute with V_i and V_f allows us to recover from (3.6)–(38) the expressions derived in Nikoukhah *et al.* (1989 a):

$$V_i(K, L) = V_i E^{N-L+K} \tag{3.25}$$

$$V_f(K, L) = V_f A^{N-L+K} \tag{3.26}$$

$$z_i(K, L) = E^{N-L}A^K v + V_i E^{N-L} z_o(0, K) - V_f A^K z_o(L, N) \tag{3.27}$$

An important interpretation of the inward process, or more specifically of the inwardly-propagated matrices (3.6), (3.7), is that the Green's function for the system (3.3), (3.4) on the small interval $[K, L]$ is the *restriction* of the Green's function of the original system (2.1), (2.2) defined on $[0, N]$. A logical question then is whether we can also move the boundary conditions outward so that the Green's function for the resulting system, when restricted to $[0, N]$, yields the original Green's function. □

Definition 3.1

The TPBVDS (2.1)–(2.3) is *left (right) input–output extendible* if, given any interval $[K, N]$ ($[0, L]$) containing $[0, N]$, there exists a TPBVDS over this larger interval with the same dynamics as in (2.1) but with new boundary matrices $V_i(K, L)$, $V_f(K, L)$ ($V_i(0, L)$, $V_f(0, L)$) such that the weighting pattern $W(k, j)$ of the original system is the restriction of the weighting pattern $W_e(k, j)$ of the new extended system, i.e.

$$W(k, j) = W_e(k, j), \quad 0 \leqslant k \leqslant N, \quad 0 \leqslant j \leqslant N-1 \tag{3.28}$$

The TPBVDS (2.1)–(2.3) is *input–output extendible* if it is both left and right input–output extendible. □

Definition 3.2

The TPBVDS (2.1), (2.2) is *left (right) exendible* if, given any interval $[K, N]$ ($[0, L]$) containing $[0, N]$, there exists a TPBVDS over this larger interval with the same dynamics as in (2.1) but with new boundary matrices $V_i(K, N)$, $V_f(K, N)$ ($V_i(0, L)$, $V_f(0, L)$) such that the Green's function $G(k, j)$ of the original system is the restriction of the Green's function $G_e(k, j)$ of the new extended system, i.e.

$$G(k, j) = G_e(k, j), \quad 0 \leqslant k \leqslant N, \quad 0 \leqslant j \leqslant N-1 \tag{3.29}$$

The TPBVDS (2.1), (2.2) is *extendible* if it is both left and right extendible.

In order to characterize the conditions under which each of these types of extendibility hold, let us first define two matrices that will appear on several occasions. Specifically, to any matrix F we associate the *Drazin inverse* F^{D} and its *invertible modification* $\tilde{F}$. To define these, let T be an invertible matrix such that

$$F = T \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix} T^{-1} \tag{3.30}$$

where M is invertible and N is nilpotent (e.g. the real Jordan form has this structure). Then,

$$F^{\mathrm{D}} = T \begin{bmatrix} M^{-1} & 0 \\ 0 & 0 \end{bmatrix} T^{-1} \tag{3.31}$$

$$\tilde{F} = T \begin{bmatrix} M & 0 \\ 0 & N+I \end{bmatrix} T^{-1} \tag{3.32}$$

These matrices have a number of important properties (Nikoukhah *et al.* 1989 a). Two that we use are as follows. Let μ be the degree of nilpotency of N, i.e. $N^{\mu-1} \neq 0$, $N^{\mu} = 0$. Then for any matrix G,

$$\operatorname{Ker}(F^{\mu}) \subset \operatorname{Ker}(G) \Leftrightarrow GF^{\mathrm{D}}F = G \tag{3.33}$$

Also, if $\{E, A\}$ is a regular pencil in standard-form,

$$EE^{\mathrm{D}} + AA^{\mathrm{D}} - AA^{\mathrm{D}}EE^{\mathrm{D}} = I \tag{3.34}$$

Equation (3.33) is easily verified. To see why (3.34) is true we need first to pre- and post-multiply (3.34) by T and T^{-1} chosen such that

$$TET^{-1} = \begin{bmatrix} E_1 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & N_e \end{bmatrix}, \quad TAT^{-1} = \begin{bmatrix} N_a & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{bmatrix} \tag{3.35}$$

where E_1, E_2, A_2 and A_3 are invertible and, N_e and N_a are nilpotent (see Nikoukhah *et al.* 1989 a), in which case

$$TE^{D}T^{-1} = \begin{bmatrix} E_1^{-1} & 0 & 0 \\ 0 & E_2^{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad TA^{D}T^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_2^{-1} & 0 \\ 0 & 0 & A_3^{-1} \end{bmatrix} \tag{3.36}$$

Then clearly

$$TEE^{D}T^{-1} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \quad TAA^{D}T^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \tag{3.37}$$

$$TAA^{D}EE^{D}T^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3.38}$$

which imply the desired result.

Note that without loss of generality, it can always be assumed that the E and A matrices of a TPBVDS in normalized form are in the block form (3.35). This can always be achieved by a coordinate transformation. In this coordinate system, the boundary matrices must have the form

$$V_i = \begin{bmatrix} E_1^{-N} & V_{12}^{i} & V_{13}^{i} \\ 0 & V_{22}^{i} & V_{23}^{i} \\ 0 & V_{32}^{i} & V_{33}^{i} \end{bmatrix}, \quad V_f = \begin{bmatrix} V_{11}^{f} & V_{12}^{f} & 0 \\ V_{21}^{f} & V_{22}^{f} & 0 \\ V_{31}^{f} & V_{32}^{f} & A_3^{-N} \end{bmatrix} \tag{3.39}$$

This is because the TPBVDS is supposed to be in normalized form, which means that V_i and V_f must satisfy (2.5), where E^N and A^N have the block structure

$$\begin{bmatrix} E_1^N & 0 & 0 \\ 0 & E_2^N & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_2^N & 0 \\ 0 & 0 & A_3^N \end{bmatrix}$$

Theorem 3.2

A TPBVDS is left extendible if and only if

$$V_i - V_i E^{D} E = 0 \tag{3.40}$$

$$V_f - A^{D} A V_f = 0 \tag{3.41}$$

It is right extendible if and only if

$$V_i - E^D E V_i = 0 \tag{3.42}$$

$$V_f - V_f A^D A = 0 \tag{3.43}$$

It is extendible if and only if

$$V_i - E^D E V_i E^D E = 0 \tag{3.44}$$

$$V_f - A^D A V_f A^D A = 0 \tag{3.45}$$

□

Since for a displacement system E, E^D, A and A^D commute with V_i and V_f, we have the following slightly strengthened version of a result due to Nikoukhah *et al.* (1989 a).

Corollary 3.1

For a displacement TPBVDS, the following statements about it are equivalent.

(i) It is right extendible.

(ii) It is left extendible.

(iii) It is extendible.

(iv) The following equations hold:

$$V_i - V_i E^D E = 0 \tag{3.46}$$

$$V_f - V_f A^D A = 0 \tag{3.47}$$

□

Proof of Theorem 3.2

First we show necessity. Let the TPBVDS be left extendible. Then it must be obtained by moving in the left boundary of another TPBVDS. From (3.6), (3.7) it can be seen that

$$\ker(V_i) \subset \ker(E^k) \tag{3.48}$$

$$\ker(V_f^T) \subset \ker[(A^k)^T] \tag{3.49}$$

where k is the number of steps that the boundary has moved. If k is larger than the maximum of the nilpotency degrees of E and A, the property (3.33) of the Drazin inverse implies that equations (3.48), (3.49) and (3.40), (3.41) are equivalent. If the system is right extendible, (3.42), (3.43) can be shown to be true similarly. Finally, (3.40), (3.41) and (3.42), (3.43) imply (3.44), (3.45).

To show the sufficiency of (3.40), (3.41), we need to construct matrices $V_i(K, N)$ and $V_f(K, N)$ for each $K < 0$, so that when we move in these boundary matrices to $[0, N]$, we recover V_i and V_f. Assume then that (3.40), (3.41) hold, and let

$$V_i(K, N) = [I - (A^D)^{-K} V_f A^{N-K}](E^D)^{N-K} \tag{3.50}$$

$$V_f(K, N) = (A^D)^{-K} V_f \tag{3.51}$$

First we need to check that the extended system is in normalized form, i.e.

$$V_i(K, N)E^{N-K} + V_f(K, N)A^{N-K} = I \tag{3.52}$$

From (3.50), (3.51), and using (3.34) and the fact that V_i and V_f are in normalized form, we get

$$V_i(K, N)E^{N-K} + V_f(K, N)A^{N-K} = (I - AA^{\mathrm{D}})EE^{\mathrm{D}} + AA^{\mathrm{D}} = I \tag{3.53}$$

Now we have to verify that by moving in $V_i(K, N)$ and $V_f(K, N)$ to $V_i(0, N)$ and $V_f(0, N)$ we recover V_i and V_f. This can be checked by substituting the matrices in (3.50), (3.51) into (3.23), (3.24), with $K = 0$, $L = J = N$ and $I = K$.

The sufficiency of (3.42), (3.43) for right extendibility can be proved in a similar way by considering the right extended matrices

$$V_i(0, L) = (E^{\mathrm{D}})^{L-N}V_i \tag{3.54}$$

$$V_f(0, L) = [I - (E^{\mathrm{D}})^{L-N}V_iE^L](A^{\mathrm{D}})^L \tag{3.55}$$

To show the sufficiency of (3.44), (3.45) for extendibility, simply note that (3.44), (3.45) clearly implies (3.40), (3.41) and (3.42), (3.43). □

Theorem 3.3

Let a TPBVDS be left (right) input–output extendible. Then we can find an equivalent TPBVDS using the freedom in its boundary matrices such that this new TPBVDS is left (right) extendible.

Conversely, every left (right) extendible TPBVDS is left (right) input–output extendible. □

Proof

Let a TPBVDS defined over $[0, N]$ be left input–output extendible. Then there exists a TPBVDS defined over $[-n, N]$ such that when we move in its boundaries to $[0, N]$ we get a TPBVDS with weighting pattern identical with the weighting pattern of the original TPBVDS, possibly with different boundary matrices. This new representation of the TPBVDS is clearly left extendible because it has been obtained by moving in the left boundary of another system n steps. A similar argument can be used for the case of right extendibility.

The proof of the converse of the theorem is trivial. □

Theorem 3.4

A TPBVDS is left input–output extendible if and only if

$$O_s(V_i - V_iE^{\mathrm{D}}E)R_s = 0 \tag{3.56}$$

$$O_s(V_f - A^{\mathrm{D}}AV_f)R_s = 0 \tag{3.57}$$

It is right input–output extendible if and only if

$$O_s(V_i - E^{\mathrm{D}}EV_i)R_s = 0 \tag{3.58}$$

$$O_s(V_f - V_fA^{\mathrm{D}}A)R_s = 0 \tag{3.59}$$

It is input–output extendible if and only if

$$O_s(V_i - E^{\mathrm{D}}EV_iE^{\mathrm{D}}E)R_s = 0 \tag{3.60}$$

$$O_s(V_f - A^{\mathrm{D}}AV_fA^{\mathrm{D}}A)R_s = 0 \tag{3.61}$$

□

Again we have a slight extension of a result due to Nikoukhah *et al.* (1989 a).

Corollary 3.2

For a stationary TPBVDS, the following statements about it are equivalent.

(i) It is right input–output extendible.

(ii) It is left input–output extendible.

(iii) It is input–output extendible.

(iv) The following equations hold:

$$O_s(V_i - V_i E^D E)R_s = 0 \tag{3.62}$$

$$O_s(V_f - V_f A^D A)R_s = 0 \tag{3.63}$$

□

Proof of Theorem 3.4

Suppose that the TPBVDS is left input–output extendible. Then, from Theorem 3.3, there exists a TPBVDS with the same weighting pattern which is left extendible, i.e. there exist matrices V_i^* and V_f^* which satisfy (3.40), (3.41) and such that

$$O_s V_i^* R_s = O_s V_i R_s \tag{3.64}$$

$$O_s V_f^* R_s = O_s V_f R_s \tag{3.65}$$

The relations (3.64), (3.65) imply

$$O_s V_i^* E^D E R_s = O_s V_i E^D E R_s \tag{3.66}$$

$$O_s A^D A V_f^* R_s = O_s A^D A V_f R_s \tag{3.67}$$

because of the invariance properties of the strong reachability and observability matrices (see Nikoukhah *et al.* 1989 a, and § 4). Pre-multiplying and post-multiplying (3.40), (3.41) (with V replaced by V^*) by O_s and R_s, respectively, and using (3.64), (3.65) and (3.66), (3.67), we obtain (3.56), (3.57).

Now suppose that (3.56), (3.57) holds. Let

$$V_i^* = (I - AA^D)(E^D)^N + AA^D V_i E^D E \tag{3.68}$$

$$V_f^* = AA^D V_f \tag{3.69}$$

We have to show that the new system obtained by replacing V_i and V_f with these boundary matrices is just another representation of the original system. First we verify that this new system is in normalized form:

$$\begin{aligned} V_i^* E^N + V_f^* A^N &= [(I - AA^D)(E^D)^N + AA^D V_i E^D E]E^n + AA^D V_f A^N \\ &= (I - AA^D)(EE^D)^N + AA^D = I \end{aligned} \tag{3.70}$$

where the last equality can be checked as in (3.34), (3.38). What remains to be shown is that (3.64), (3.65) holds for these matrices. Clearly (3.65) holds from (3.69) and (3.57). Showing (3.64) is more complicated, and again we suppose that the system is in the block form (3.35), (3.39). The matrix V_i^* in (3.68) is given by

$$V_i^* = \begin{bmatrix} I & 0 & \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} E_1^{-N} & 0 & 0 \\ 0 & E_2^{-N} & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} E_1^{-N} & V_{12}^i & V_{13}^i \\ 0 & V_{22}^i & V_{23}^i \\ 0 & V_{32}^i & V_{33}^i \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} E_1^{-N} & 0 & 0 \\ 0 & V_{22}^i & 0 \\ 0 & V_{32}^i & 0 \end{bmatrix} \tag{3.71}$$

where V_{22}^i and V_{32}^i are (2, 2) and (3, 2) blocks of V_i.

The strong reachability and observability matrices have a block structure as well, i.e.

$$O_s = W \begin{bmatrix} O_s^1 & 0 & 0 \\ 0 & O_s^2 & 0 \\ 0 & 0 & O_s^3 \end{bmatrix}, \quad R_s = \begin{bmatrix} R_s^1 & 0 & 0 \\ 0 & R_s^2 & 0 \\ 0 & 0 & R_s^3 \end{bmatrix} Z \tag{3.72}$$

for some invertible matrices Z and W (this is due to the fact that the three blocks of the system have distinct eigenvalues, see Nikoukhah *et al.* 1989 a). Also observe that

$$O_s(V_iE^N + V_fA^N)R_s = O_sV_iE^NR_s + O_sAA^DV_fA^NR_s = O_sR_s \tag{3.73}$$

Pre- and post-multiplying (3.73) by W^{-1} and Z^{-1}, respectively, and inspecting the (1, 2) block yields

$$O_s^1V_{12}^iE_2^NR_s^2 = 0 \tag{3.74}$$

Since R_s^2 is E_2-invariant (see § 4) and E_2 is invertible, this implies

$$O_s^1V_{12}^iR_s^2 = 0 \tag{3.75}$$

Also note that (3.56) implies

$$O_s^kV_{k3}^iR_s^3 = 0, \quad k = 1, 2, 3 \tag{3.76}$$

Finally, by noting the expression for V_i^* in (3.71) and (3.75) and (3.76), we find

$$O_sV_iR_s = O_sV_i^*R_s \tag{3.77}$$

which is the desired result. The other cases can be argued similarly. □

The following example demonstrates the fact that left and right extendibility are indeed distinct notions.

Example 3.1

Consider the TPBVDS

$$x(k+1) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x(k) + u(k) \tag{3.78}$$

$$x(0) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(N) = v \tag{3.79}$$

This TPBVDS is well-posed and in normalized form. It is easy to check that for this system (3.43) is violated, but (3.40) and (3.41) hold. Thus this system is left extendible but not right extendible. □

The input–output extendibility feature is a property of the weighting pattern of the system, and not any specific representation, so that it is possible to refer to this property as extendibility of the weighting pattern. The following theorem justifies this.

Theorem 3.5

Let 2 TPBVDSs (of possibly different dimension) defined over $[0, N]$ have identical weighting patterns. Then if one is input–output extendible so is the other. □

Proof

It is readily seen that the proof due to Nikoukhah *et al.* (1989 a), although stated for stationary systems, is actually valid for general TPBVDSs. □

The extendibility property is a very important property because it allows us to associate with each system a sequence of systems defined over any desired interval. We present a way of constructing this sequence. But first we give the following characterization of extendible systems which can be derived from the extendibility condition and the fact that the system is in normalized form.

Theorem 3.6

Let a TPBVDS be extendible and in block form (3.35), (3.39). Then the boundary matrices must have the structure

$$V_{\rm i} = \begin{bmatrix} E_1^{-N} & 0 & 0 \\ 0 & V_{22}^{\rm i} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad V_{\rm f} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & V_{22}^{\rm f} & 0 \\ 0 & 0 & A_3^{-N} \end{bmatrix} \tag{3.80}$$

which means that the TPBVDS is separated into a purely causal part and a purely anticausal part, each having nilpotent dynamics, and a non-descriptor acausal part. □

Note that if a system is input–output extendible, then it has a representation of the form (3.80).

Theorem 3.6 allows us to simplify the expression for the Green's function solution of an extendible system. By replacing the $V_{\rm i}$ and $V_{\rm f}$ in the general Green's function solution by $V_{\rm i}$ and $V_{\rm f}$ in (3.80), we obtain the following expression for the Green's function of an extendible system:

$$\begin{aligned} G(k, j) &= \begin{cases} -\tilde{A}^k \tilde{E}^{N-k} V_{\rm f} E E^{\rm D} \tilde{E}^{j-N} \tilde{A}^{N-j-1} - (I - EE^{\rm D}) E^{j-k} (A^{\rm D})^{j-k+1} & j \geqslant k \\ \tilde{A}^k \tilde{E}^{N-k} V_{\rm i} A A^{\rm D} \tilde{A}^{-j-1} \tilde{E}^j + (I - AA^{\rm D})(E^{\rm D})^{k-j} A^{k-j-1} & j < k \end{cases} \\ &= \begin{cases} -\tilde{A}^k \tilde{E}^{-k} [I - (E^N V_{\rm i})] E E^{\rm D} \tilde{A}^{-j-1} \tilde{E}^j - (I - EE^{\rm D}) E^{j-k} (A^{\rm D})^{j-k+1} & j \geqslant k \\ \tilde{A}^k \tilde{E}^{-k} (E^N V_{\rm i}) A A^{\rm D} \tilde{A}^{-j-1} \tilde{E}^j + (I - AA^{\rm D})(E^{\rm D})^{k-j} A^{k-j-1} & j < k \end{cases} \end{aligned} \tag{3.81}$$

Accordingly, the weighting pattern of an input–output extendible system can be expressed as

$$W(k, j) =$$
$$\begin{cases} -C\{\tilde{A}^k\tilde{E}^{-k}[I-(E^N V_\mathrm{i})]EE^\mathrm{D}\tilde{A}^{-j-1}\tilde{E}^j-(I-EE^\mathrm{D})E^{j-k}(A^\mathrm{D})^{j-k+1}\}B & j \geqslant k \\ C\{\tilde{A}^k\tilde{E}^{-k}(E^N V_\mathrm{i})AA^\mathrm{D}\tilde{A}^{-j-1}\tilde{E}^j+(I-AA^\mathrm{D})(E^\mathrm{D})^{k-j}A^{k-j-1}\}B & j < k \end{cases} \tag{3.82}$$

Note that (3.81) expresses the Green's function of an extendible TPBVDS and all of its extensions. Similarly, (3.82) expresses the weighting pattern of an input–output extendible TPBVDS and all of its extensions. This observation deserves further comment. Specifically, what we have done is the following. We begin with a specific extendible TPBVDS defined on $[0, N]$, with boundary matrices V_i, V_f so that the system is in standard form over this specific interval. Equations (3.81) and (3.82) then provide us with the Green's function and weighting pattern for *all* extensions of the TPBVDS. Thus we use the parameters associated with any one of the family of extensions to obtain G and W for the whole family. These expressions must of course, not depend on the particular member of the family used in the computation. In particular (3.81) and (3.82) do not depend on N. Rather $E^N V_\mathrm{i}$ is, in a sense, an invariant for the entire family (remember that V_i also depends on N, as it is chosen so that the system is in standard form over $[0, N]$). In the simpler stationary case this point can be made much more explicit. In particular, the weighting pattern of an input–output extendible stationary TPBVDS can be expressed as follows (Nikoukhah *et al.* 1989 a):

$$W(k) = \begin{cases} C(E^N V_\mathrm{i})E^\mathrm{D}(AE^\mathrm{D})^{k-1}B & k > 0 \\ -C(A^N V_\mathrm{f})A^\mathrm{D}(EA^\mathrm{D})^{-k}B & k \leqslant 0 \end{cases} \tag{3.83}$$

Note also that if we are in the basis (3.35), by partitioning

$$C = [C_1 \quad C_2 \quad C_3], \quad B = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}$$

accordingly, $W(k, j)$ can be expressed as

$$W(k, j) = \begin{cases} -C_2A_2^kE_2^{N-k}V_{22}^\mathrm{f}E_2^{j-N}A_2^{N-j-1}B_2 - C_3A_3^{k-j-1}N_\mathrm{e}^{j-k}B_3 & j \geqslant k \\ C_2A_2^kE_2^{N-k}V_{22}^\mathrm{i}E_2^jA_2^{-j-1}B_2 + C_1E_1^{j-k}N_a^{k-j-1}B_1 & j < k \end{cases} \tag{3.84}$$

We can construct the sequence of (inward and outward) extensions (in standard form) of our extendible or input–output extendible TPBVDS as follows:

$$V_\mathrm{i}(I, J) = \tilde{E}^{-J}\tilde{A}^I(E^N V_\mathrm{i})AA^\mathrm{D}\tilde{E}^I\tilde{A}^I + (I - AA^\mathrm{D})\tilde{E}^{I-J} \tag{3.85}$$

$$V_\mathrm{f}(I, J) = \tilde{E}^{-J}\tilde{A}^I[I-(E^N V_\mathrm{i})]EE^\mathrm{D}\tilde{E}^J\tilde{A}^{-J} + (I - EE^\mathrm{D})\tilde{A}^{I-J} \tag{3.86}$$

In the basis (3.35), (3.85), (3.86) becomes

$$V_\mathrm{i}(I, J) = \begin{bmatrix} E_1^{-(J-I)} & 0 & 0 \\ 0 & E_2^{N-J}A_2^IV_{22}^\mathrm{i}E_2^IA_2^{-I} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3.87}$$

$$V_\mathrm{f}(I, J) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_2^IE_2^{N-J}V_{22}^\mathrm{f}E_2^{J-N}A_2^{N-J} & 0 \\ 0 & 0 & A_3^{-(J-I)} \end{bmatrix} \tag{3.88}$$

4. Reachability and observability

As discussed by Nikoukhah *et al.* (1987), there are two notions for both reachability and observability for TPBVDSs. In this section we provide brief reviews of these definitions and present additional results.

Definition 4.1

The system (2.1), (2.2) is *strongly reachable* on $[K, L]$ if the map

$$\{u(k) : k \in [K, L]\} \to z_o(K, L) \tag{4.1}$$

is onto. The system is *strongly reachable* if it is strongly reachable on some interval. □

From (3.2) we can see that the range of the map (4.1) is just the range of $R_s(L-K)$ where

$$R_s(j) = [A^{j-1}B \mid EA^{j-2}B \mid \dots \mid E^{j-1}B] \tag{4.2}$$

Note that $R_s = R_s(n)$. Furthermore a TPBVDS is strongly reachable if and only if R_s has full rank (this is a consequence of the generalized Cayley–Hamilton theorem). In addition, the strongly reachable spaces have the usual nesting property, i.e.

$$\mathscr{R}_s(k) = \operatorname{Im}(R_s(k)) \subset \operatorname{Im}(R_s(k+1)) = \mathscr{R}_s(k+1) \tag{4.3}$$

We refer the reader to Nikoukhah *et al.* (1987), for proofs of these and other results related to strong reachability. For future reference, we define the strongly reachable subspace

$$\mathscr{R}_s = \operatorname{Im}(R_s) \tag{4.4}$$

Definition 4.2

The system (2.1)–(2.3) is *strongly observable* on $[K, L]$ if the map

$$z_i(K, L) \to \{y(k) : k \in [K, L]\} \tag{4.5}$$

defined by (3.3), (3.4), and (2.3) with $u \equiv 0$ on $[K, L]$ is one to one. The system is *strongly observable* if it is strongly observable on some $[K, L]$. □

It is easily checked (Nikoukhah *et al.* 1987), that the kernel of the mapping (4.5) is the kernel of $O_s(L-K)$, where

$$O_s(j) = \begin{bmatrix} CE^j \\ CAE^{j-1} \\ \vdots \\ CA^j \end{bmatrix} \tag{4.6}$$

Note that $O_s = O_s(n-1)$. Furthermore, a TPBVDS is strongly observable if and only if O_s has full rank. In addition, the strong unobservability subspaces have the usual nesting property

$$\mathscr{O}_s(k+1) = \ker(O_s(k+1)) \subset \mathscr{O}_s(k) = \ker(O_s(k)) \tag{4.7}$$

Again for future reference we define the strongly unobservable suspace

$$\mathcal{O}_s = \ker(O_s) \tag{4.8}$$

Note that the properties of strong reachability and observability involve only the matrices C, E, A and B. As we shall see, the other weaker set of notions of reachability and observability involve the boundary matrices as well.

Definition 4.3

The system (2.1), (2.2) is *weakly reachable* off $[K, L]$ if the map

$$\{u(k) : k \in [0, K-1] \cup [L, N-1]\} \to z_i(K, L) \tag{4.9}$$

with $v = 0$ is onto. The weakly reachable subspace $\mathcal{R}_w(K, L)$ is the range of this map. The system is called *weakly reachable* if

$$\mathcal{R}_w \equiv \bigvee_{K,L} \mathcal{R}_w(K, L) = \mathbb{R}^n \tag{4.10}$$

The space $\mathcal{R}_w$ is called the weak reachability space. □

While it is shown by Nikoukhah *et al.* (1987), that for K and L far from the boundaries the dimension of $\mathcal{R}_w(K, L)$ is constant, it is *not* generally true that this space is fixed or that any nesting of weak reachability spaces occurs as K and L move inward from the boundaries. That is why we may very well have a system which is weakly reachable, but where $\mathcal{R}_w(K, L)$ is not the whole space for any K and L. The authors (Nikoukhah *et al.* 1987) defined weak reachability differently; specifically we called a system weakly reachable if $\mathcal{R}_w(K, L)$ equaled $\mathbb{R}^n$ for K and L far from the boundaries. We see later that Definition 4.3 is more appropriate.

Theorem 4.1

The weak reachability space $\mathcal{R}_w$ can be expressed as

$$\mathcal{R}_w = \bigvee_{0 \leqslant k < n} A^k E^{n-1-k} \operatorname{Im}[V_i R_s \mid V_f R_s] \tag{4.11}$$

□

Corollary 4.1

For an extendible system, the weak reachability space $\mathcal{R}_w$ can be expressed as

$$\mathcal{R}_w = \bigvee_{0 \leqslant k < n} A^k E^{n-1-k} V_i \mathcal{R}_s + \mathcal{R}_s \tag{4.12}$$

□

Proof

First we prove the following Lemma, which justifies the use of the terms 'strong' and 'weak'. □

Lemma 4.1

For any TPBVDS

$$\mathcal{R}_s \subset \mathcal{R}_w \tag{4.13}$$

□

Proof

We show the stronger result

$$\mathcal{R}_s \subset \mathcal{R}_w(K, L) \quad \text{for } K, L \in [n, N-n] \tag{4.14}$$

From expression (3.8) for $z_i(K, L)$ with $v = 0$, and the fact that the space reached by $z_o(0, K)$ and $z_o(L, N)$ is exactly $\mathcal{R}_s$, we can easily deduce that

$$\begin{aligned}\mathcal{R}_w(K, L) = {} & E^{N-L}(\omega E - A^K(\omega V_f E + V_i A)A^{N-K})\Gamma^{-1}\mathcal{R}_s \\ & + A^K(A - E^{N-L}(\omega V_f E + V_i A)E^L)\Gamma^{-1}\mathcal{R}_s\end{aligned} \tag{4.15}$$

By noting that $A^{N-K}\mathcal{R}_s \subset \mathcal{R}_s$ and $E^L\mathcal{R}_s \subset \mathcal{R}_s$, (4.15) implies that

$$\begin{aligned}\mathcal{R}_w(K, L) \supset {} & E^{N-L}(\omega E - A^K(\omega V_f E + V_i A)A^{N-K})E^L\Gamma^{-1}\mathcal{R}_s \\ & + A^K(A - E^{N-L}(\omega V_f E + V_i A)E^L)A^{N-K}\Gamma^{-1}\mathcal{R}_s \\ = {} & \mathcal{R}_s + E^{N-L}A^K(\omega V_f E + V_i A)A^{N-K}E^L\Gamma^{-1}\mathcal{R}_s\end{aligned} \tag{4.16}$$

which in turn implies (4.14). Clearly then (4.13) is a consequence of (4.14). □

To prove Theorem 4.1, observe that

$$\mathcal{R}_w \supset \mathcal{R}_s + \bigvee_K \mathcal{R}_w(K, K) \tag{4.17}$$

Using (2.7) and the fact that $\Gamma^{-1}\mathcal{R}_s = \mathcal{R}_s$, this implies

$$\mathcal{R}_w \supset \bigvee_K \{A^K E^{N-K}(\omega V_f E + V_i A)\mathcal{R}_s\} + \mathcal{R}_s, \quad \text{for all } \omega \tag{4.18}$$

But thanks to (2.5) and the E- and A-invariance of $\mathcal{R}_s$,

$$\mathcal{R}_s + (\omega V_f E + V_i A)\mathcal{R}_s = V_i\mathcal{R}_s + V_f\mathcal{R}_s \tag{4.19}$$

which along with (4.17) and the Cayley–Hamilton theorem proves that

$$\mathcal{R}_w \supset \bigvee_{0 \leq k < n} A^k E^{n-1-k} \operatorname{Im}[V_i R_s \mid V_f R_s] \tag{4.20}$$

The other inclusion is trivial since in expression (3.8) for $z_i(K, L)$, the range of the map $u \to z_i$ is essentially the range of matrices $A^r E^s V_i A^t E^l$ and $A^r E^s V_f A^t E^l$.

To prove the corollary, simply note that we can decompose the system into three subsystems as in (3.35), in which case V_i and V_f are expressed as in (3.80). Now using the fact that for an extendible system $V_f = (I - V_i E^N)(A^D)^N$, we can show that

$$\operatorname{Im}[V_i R_s \mid R_s] = \operatorname{Im}[V_i R_s \mid V_f R_s] \tag{4.21}$$

which yields the desired result. □

In the case of displacement systems, expression (4.11) simplifies and we obtain the result given by Nikoukhah *et al.* (1989 a).

$$\mathcal{R}_w = \operatorname{Im}[V_i R_s \mid V_f R_s] \tag{4.22}$$

Definition 4.4

The system (2.1)–(2.3) is *weakly observable* off $[K, L]$ if the map

$$z_o(K, L) \to \{y(k) : k \in [0, K] \cup [L, N]\} \tag{4.23}$$

with $v = 0$ and $u(j) = 0$, $j \in [0, K-1] \cup [L, N-1]$, is one to one. The weakly unobservable subspace $\mathcal{O}_w(K, L)$ is the null space of this map. The system is called weakly observable if

$$\mathcal{O}_w \equiv \bigcap_{K,L} \mathcal{O}_w(K, L) = \{0\} \tag{4.24}$$

The space $\mathcal{O}_w$ is the weak unobservability space. □

By analogy with the weak reachability case, we simply present the dual set of results concerning weak observability.

Theorem 4.2

The weakly unobservable space can be expressed as

$$\mathcal{O}_w = \bigcap_{0 \leqslant k < n} \ker\left(\begin{bmatrix} O_s V_i \\ O_s V_f \end{bmatrix}\right) E^{n-1-k} A^k \tag{4.25}$$

□

Corollary 4.2

For an extendible system the weakly unobservable space can be expressed as follows

$$\mathcal{O}_w = \mathcal{O}_s \cap \left\{ \bigcap_{0 \leqslant k < n} \ker\,(O_s V_i E^{n-1-k} A^k) \right\} \tag{4.26}$$

□

Lemma 4.2

For any TPBVDS

$$\mathcal{O}_w \subset \mathcal{O}_s \tag{4.27}$$

□

This lemma shows that weak observability is a weaker condition than strong observability.

If the TPBVDS is displacement, (4.25) simplifies and yields the result given by Nikoukhah *et al.* (1989 a):

$$\mathcal{O}_w = \ker \begin{bmatrix} O_s V_i \\ O_s V_f \end{bmatrix} \tag{4.28}$$

The following example illustrates the difference between the concepts of strong and weak reachability.

Example 4.1

Consider the following displacement TPBVDS

$$x(k+1) = x(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k) \tag{4.29}$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} x(0) + \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} x(N) = 0 \tag{4.30}$$

where

$$x(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \tag{4.31}$$

This system is well-posed and in normalized form. The strong reachability space for this system is just

$$\operatorname{Im}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

so that the system is not strongly reachable. In fact, we can easily see that only x_1 is strongly reachable and x_2 is not. However, using (4.22), we can check that the system is weakly reachable. In fact, we can check that this system is weakly reachable off any interval $[K, L]$, $0 < K$, $L < N$. To understand this fact, note that boundary condition (4.30) can be rewritten as

$$x_1(0) = 0, \quad x_2(0) = x_1(N) \tag{4.32}$$

It is clear that $x_1(k)$ can be made arbitrary by proper choice of inputs $u(j)$, $j < k$. On the other hand, $x_1(N)$, and thus $x_2(0)$, can also be independently made arbitrary by proper choice of $u(j)$, $k \leqslant j < N$. But (4.29) implies that $x_2(k)$ is constant for all k, so that it must equal $x_2(0)$ and $x_1(N)$. The result is that $x_1(k)$ and $x_2(k)$, which form $x(k)$, can be made arbitrary by proper choice of the input u. □

5. Minimality

In this section we present minimality results for TPBVDSs, extending the results for stationary systems by Nikoukhah *et al.* (1989 a), and using an approach analogous to that of Krener 1987, and Gohberg *et al.* (1986), with differences due to possible singularity of E and A.

Definition 5.1

A TPBVDS is *minimal* if x has the lowest dimension among all TPBVDSs having the same weighting pattern. □

Theorem 5.1

A TPBVDS with $N \geqslant 4n$ is minimal if and only if

(*a*) $$\mathscr{R}_w = \mathbb{R}^n \tag{5.1}$$

(*b*) $$\mathscr{O}_w = \{0\} \tag{5.2}$$

(*c*) $$\mathscr{O}_s \subset \mathscr{R}_s \tag{5.3}$$

(i.e. if it is weakly reachable and observable, and any strongly unobserved mode is strongly reached). □

As did Nikoukhah *et al.* (1989 a), we need to introduce three different Hankel matrices and also, as did Krener (1987) and Gohberg *et al.* (1986), we may have certain level of non-uniqueness in minimal realizations that is not present in the causal case. The length of the interval here is assumed to be larger than four times the dimension of the system so that all the modes on both sides of a state in the middle of the interval can be reached and observed (see the proof for details on where this assumption is needed). If N is not large enough, the conditions of Theorem 5.1 become necessary but not sufficient.

Proof

The approach that we use is the same as that of Nikoukhah *et al.* (1989 a) for the stationary case. We focus therefore on the new aspects of the non-stationary case, and refer the reader to Nikoukhah *et al.* (1989 a) where appropriate. We begin with the reduction procedures if any of the conditions (5.1)–(5.3) are not satisfied. Consider first the case in which $\mathscr{R}_{\mathrm{w}} \neq \mathbb{R}^n$. Let $\mathscr{R}_2$ be any subspace such that

$$\mathscr{R}_{\mathrm{w}} \oplus \mathscr{R}_2 = \mathbb{R}^n \tag{5.4}$$

By performing a similarity transformation on x to represent it in a basis compatible with (5.4), we can assume that

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad E = \begin{bmatrix} E_{11} & E_{12} \\ 0 & E_{22} \end{bmatrix}, \quad V_{\mathrm{i}} = \begin{bmatrix} V^{\mathrm{i}}_{11} & V^{\mathrm{i}}_{12} \\ V^{\mathrm{i}}_{21} & V^{\mathrm{i}}_{22} \end{bmatrix} \tag{5.5}$$

$$V_{\mathrm{f}} = \begin{bmatrix} V^{\mathrm{f}}_{11} & V^{\mathrm{f}}_{12} \\ V^{\mathrm{f}}_{21} & V^{\mathrm{f}}_{22} \end{bmatrix}, \quad C = [C_1 \quad C_2], \quad B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \tag{5.6}$$

The 0-blocks in A and E follow from the A- and E-invariance of $\mathscr{R}_{\mathrm{w}}$. The 0-block in B is due to the fact that $\operatorname{Im}[B] \subset \mathscr{R}_{\mathrm{s}}$ (Cayley–Hamilton) and $\mathscr{R}_{\mathrm{s}} \subset \mathscr{R}_{\mathrm{w}}$. In addition, since

$$\mathscr{R}_{\mathrm{w}} \supset V_{\mathrm{i}} \mathscr{R}_{\mathrm{s}} + V_{\mathrm{f}} R_{\mathrm{s}} \tag{5.7}$$

we must have

$$V^{\mathrm{i}}_{21} A^k_{11} E^j_{11} B_1 = V^{\mathrm{f}}_{21} A^k_{11} E^j_{11} B_1 = 0 \tag{5.8}$$

From the form (2.7) for the weighting pattern of a TPBVDS we can then conclude that the weighting pattern of the system is determined by the system $(C_1, V^{\mathrm{i}}_{11}, V^{\mathrm{f}}_{11}, E_{11}, A_{11}, B_{11})$. Note since E and A are in standard form, so are E_{11} and A_{11}. However, the boundary matrices V^{i}_{11} and V^{f}_{11} need not be normalized and indeed there is no guarantee that this TPBVDS is well-posed. However, thanks to the following result from Nikoukhah *et al.* (1989a) and Theorem 2.2, we can modify the boundary matrices in order to make $(C_1, V^{\mathrm{i}}_{11}, V^{\mathrm{f}}_{11}, E_{11}, A_{11}, B_1)$ well-posed while leaving the weighting pattern unchanged. □

Lemma 5.1

Consider a (possibly not well-posed) TPBVDS (2.1), (2.2), with E and A in standard form and for which the following holds:

$$O_{\mathrm{s}}(V_{\mathrm{i}} E^N + V_{\mathrm{f}} A^N) R_{\mathrm{s}} = O_{\mathrm{s}} R_{\mathrm{s}} \tag{5.9}$$

Then we can find $\tilde{V}_{\mathrm{i}}$, $\tilde{V}_{\mathrm{f}}$ such that

$$\tilde{V}_{\mathrm{i}} E^N + \tilde{V}_{\mathrm{f}} A^N = I \tag{5.10}$$

and

$$O_{\mathrm{s}} V_{\mathrm{i}} R_{\mathrm{s}} = O_{\mathrm{s}} \tilde{V}_{\mathrm{i}} R_{\mathrm{s}}, \quad O_{\mathrm{s}} V_{f} R_{\mathrm{s}} = O_{\mathrm{s}} \tilde{V}_{\mathrm{f}} R_{\mathrm{s}} \tag{5.11}$$

□

We refer the reader to the work of Nikoukhah *et al.* (1989 a), where it is shown that (5.9) holds for our reduced system, so that we may apply the lemma.

To continue with the proof of the theorem, note that the problem of reducing the dimension of the realization if (5.2) is violated is merely the dual of the problem that we have just considered. Consequently, we omit the details. Also the analysis for the case in which condition (5.3) is violated is the same for stationary and non-stationary systems, and we refer to Nikoukhah *et al.* (1989 a), for details.

What remains to be shown is that two TPBVDSs with the same weighting pattern and both satisfying (5.1)–(5.3) must have the same dimension, and consequently are minimal. To proceed with the proof we need the following Lemma due to Nikoukhah *et al.* (1989 a), which is based on Gohberg *et al.* (1987).

Lemma 5.2

Let $\{E_i, A_i\}$, $i = 1, 2$, be two regular pencils so that $\alpha E_i + \beta A_i = I$, $i = 1, 2$, where $\dim(E_i) = \dim(A_i) = n_i$. Suppose that $N \geqslant 2\max(n_1, n_2)$. Also suppose that for some matrices $\{M_i, N_i\}$, $i = 1, 2$,

$$M_1 A_1^k E_1^{N-1-k} N_1 = M_2 A_2^k E_2^{N-1-k} N_2, \quad 0 \leqslant k \leqslant N-1 \tag{5.12}$$

Then for *all* K, L

$$M_1 A_1^K E_1^L N_1 = M_2 A_2^K E_2^L N_2 \tag{5.13}$$

□

Proceeding with the proof, consider two systems $(C_j, E_j, A_j, V_j^{\mathrm{i}}, V_j^{\mathrm{f}}, B_j)$, $j = 1, 2$, satisfying minimality conditions (5.1)–(5.3), and, without loss of generality, assume that both are in normalized form with the same α and β. What we know is that

$$\begin{aligned} &C_1 A_1^k (A_1 - E_1^{N-k}(V_1^{\mathrm{i}} A_1 + \omega V_1^{\mathrm{f}} E_1) E_1^k) E_1^{j-k} A_1^{N-j-1} \Gamma_1^{-1} B_1 \\ &\quad = C_2 A_2^k (A_2 - E_2^{N-k}(V_2^{\mathrm{i}} A_2 + \omega V_2^{\mathrm{f}} E_2) E_2^k) E_2^{j-k} A_2^{N-j-1} \Gamma_2^{-1} B_2, \quad j \geqslant k \end{aligned} \tag{5.14}$$

$$\begin{aligned} &C_1 E_1^{N-k} (\omega E_1 - A_1^k (V_1^{\mathrm{i}} A_1 + \omega V_1^{\mathrm{f}} E_1) A_1^{N-k}) E_1^j A_1^{k-j-1} \Gamma_1^{-1} B_1 \\ &\quad = C_2 E_2^{N-k} (\omega E_2 - A_2^k (V_2^{\mathrm{i}} A_2 + \omega V_2^{\mathrm{f}} E_2) A_2^{N-k}) E_2^j A_2^{k-j-1} \Gamma_2^{-1} B_2, \quad j < k \end{aligned} \tag{5.15}$$

Let $k \in [2n, N-2n]$ (remember that $N \geqslant 4n$). Then we can apply Lemma 5.2 to get

$$\begin{aligned} &C_1 A_1^k (A_1 - E_1^{N-k}(V_1^{\mathrm{i}} A_1 + \omega V_1^{\mathrm{f}} E_1) E_1^k) E_1^K A_1^L \Gamma_1^{-1} B_1 \\ &\quad = C_2 A_2^k (A_2 - E_2^{N-k}(V_2^{\mathrm{i}} A_2 + \omega V_2^{\mathrm{f}} E_2) E_2^k) E_2^K A_2^L \Gamma_2^{-1} B_2, \quad \text{for all } K, L \end{aligned} \tag{5.16}$$

$$\begin{aligned} &C_1 E_1^{N-k} (\omega E_1 - A_1^k (V_1^{\mathrm{i}} A_1 + \omega V_1^{\mathrm{f}} E_1) A_1^{N-k}) E_1^K A_1^L \Gamma_1^{-1} B_1 \\ &\quad = C_2 E_2^{N-k} (\omega E_2 - A_2^k (V_2^{\mathrm{i}} A_2 + \omega V_2^{\mathrm{f}} E_2) A_2^{N-k}) E_2^K A_2^L \Gamma_2^{-1} B_2, \quad \text{for all } K, L \end{aligned} \tag{5.17}$$

By taking $K = r$, $L = N - k + s$ in (5.16) and $K = k + r$ and $L = s$ in (5.17) and subtracting the two sides of (5.16) from (5.17), we obtain

$$C_1 E_1^r A_1^s B_1 = C_2 E_2^r A_2^s B_2, \quad \text{for all } r, s \geqslant 0 \tag{5.18}$$

Using (5.14), (5.15) and (5.18), we can show that

$$\begin{aligned} &C_1 A_1^k E_1^{N-k} (V_1^{\mathrm{i}} A_1 + \omega V_1^{\mathrm{f}} E_1) E_1^j A_1^{N-j-1} \Gamma_1^{-1} B_1 \\ &\qquad = C_2 A_2^k E_2^{N-k} (V_2^{\mathrm{i}} A_2 + \omega V_2^{\mathrm{f}} E_2) E_2^j A_2^{N-j-1} \Gamma_2^{-1} B_2 \end{aligned} \tag{5.19}$$

and taking into account Lemma 5.2, this implies

$$C_1 A_1^r E_1^s (V_1^{\mathrm{i}} A_1 + \omega V_1^{\mathrm{f}} E_1) E_1^t A_1^u \Gamma_1^{-1} B_1 = C_2 A_1^r E_2^s (V_2^{\mathrm{i}} A_2 + \omega V_2^{\mathrm{f}} E_2) E_2^t A_2^u \Gamma_2^{-1} B_2 \tag{5.20}$$

for all $r, s, t, u \geqslant 0$. Then, using the fact that both systems are in normalized form, we obtain

$$C_1 A_1^r E_1^s V_1^{\mathrm{i}} E_1^t A_1^u B_1 = C_2 A_2^r E_2^s V_2^{\mathrm{i}} E_2^t A_2^u B_2 \tag{5.21}$$

$$C_1 A_1^r E_1^s V_1^{\mathrm{f}} E_1^t A_1^u B_1 = C_2 A_2^r E_2^s V_2^{\mathrm{f}} E_2^t A_2^u B_2 \tag{5.22}$$

for all $r, s, t, u \geqslant 0$. To proceed, we now introduce three Hankel matrices:

$$H_{\mathrm{in}} = O_{\mathrm{s}}^1 R_{\mathrm{w}}^1 = O_{\mathrm{s}}^2 R_{\mathrm{w}}^2 \tag{5.23}$$

$$H_{\mathrm{out}} = O_{\mathrm{w}}^1 R_{\mathrm{s}}^1 = O_{\mathrm{w}}^2 R_{\mathrm{s}}^2 \tag{5.24}$$

$$H_{\mathrm{s}} = O_{\mathrm{s}}^1 R_{\mathrm{s}}^1 = O_{\mathrm{s}}^2 R_{\mathrm{s}}^2 \tag{5.25}$$

where R_{s}^j and O_{s}^j are the strong reachability and observability matrices of system j, respectively, and

$$R_{\mathrm{w}}^j = [A_j^{n-1}(V_{\mathrm{i}}^j R_{\mathrm{s}}^j \mid V_{\mathrm{f}}^j R_{\mathrm{s}}^j) \mid E_j A_j^{n-2}(V_{\mathrm{i}}^j R_{\mathrm{s}}^j \mid V_{\mathrm{f}}^j R_{\mathrm{s}}^j) \mid \ldots \mid E_j^{n-1}(V_{\mathrm{i}}^j R_{\mathrm{s}}^j \mid V_{\mathrm{f}}^j R_{\mathrm{s}}^j)]$$

$$O_{\mathrm{w}}^j = \begin{bmatrix} \begin{bmatrix} O_{\mathrm{s}}^j V_{\mathrm{i}}^j \\ O_{\mathrm{s}}^j V_{\mathrm{f}}^j \end{bmatrix} A_j^{n-1} \\ \vdots \\ \begin{bmatrix} O_{\mathrm{s}}^j V_{\mathrm{i}}^j \\ O_{\mathrm{s}}^j V_{\mathrm{f}}^j \end{bmatrix} E_j^{n-1} \end{bmatrix}, \quad j = 1, 2$$

where R_{w}^j and O_{w}^j are respectively the *weak reachability* and *weak observability matrices* of system j. Equations (5.23)–(5.25) are direct consequences of (5.18) and (5.21), (5.22). We note that these definitions differ somewhat from those of Nikoukhah *et al.* (1989 a), due to the differences in characterizing weak reachability and observability in the non-stationary case.

Clearly

$$\mathscr{R}_{\mathrm{w}}^j = \operatorname{Im}(R_{\mathrm{w}}^j) \tag{5.26}$$

$$\mathscr{O}_{\mathrm{w}}^j = \ker(O_{\mathrm{w}}^j) \tag{5.27}$$

for $j = 1, 2$. Thus $\mathscr{R}_{\mathrm{w}}^j$ and $\mathscr{O}_{\mathrm{w}}^j$ have full rank since both systems satisfy (5.1)–(5.3). Consequently we can find a matrix U so that

$$R_{\mathrm{s}}^2 = U R_{\mathrm{s}}^1 \tag{5.28}$$

Similarly we can obtain an analogous expression for R_{s}^1 in terms of R_{s}^2. These allow us to conclude that

$$\operatorname{rank}(R_{\mathrm{s}}^1) = \operatorname{rank}(R_{\mathrm{s}}^2) = \rho \tag{5.29}$$

and in an analogous way we can show that

$$\operatorname{rank}(O_{\mathrm{s}}^1) = \operatorname{rank}(O_{\mathrm{s}}^2) = \omega \tag{5.30}$$

Finally, condition (5.3) together with (5.25) imply that

$$\rho - (n_1 - \omega) = \operatorname{rank}(H_{\mathrm{s}}) = \rho - (n_2 - \omega) \tag{5.31}$$

from which we see that

$$n_1 = n_2 \tag{5.32}$$

completing the proof of the Theorem. □

The following result, proven in Nikoukhah *et al.* (1989 a), is now easily seen to be valid for *all* minimal TPBVDSs.

Corollary 5.1

Let $(C_j, V_j^{\mathrm{i}}, V_j^{\mathrm{f}}, E_j, A_j, B_j, N)$, $j = 1, 2$, be two minimal realizations of the same weighting pattern, where $\{E_j, A_j\}$, $j = 1, 2$, are in standard form for the same α and β. Then there exists an invertible matrix T such that

$$B_2 = TB_1 \tag{5.33}$$

$$C_2 = C_1 T^{-1} \tag{5.34}$$

$$O_s^1(V_1^{\mathrm{i}} - T^{-1}V_2^{\mathrm{i}}T)R_s^1 = 0 \tag{5.35}$$

$$O_s^1(V_1^{\mathrm{f}} - T^{-1}V_2^{\mathrm{f}}T)R_s^1 = 0 \tag{5.36}$$

and

$$(A_1 - T^{-1}A_2T)R_s^1 = 0 \tag{5.37}$$

$$(E_1 - T^{-1}E_2T)R_s^1 = 0 \tag{5.38}$$

$$O_s^1(A_1 - T^{-1}A_2T) = 0 \tag{5.39}$$

$$O_s^1(E_1 - T^{-1}E_2T) = 0 \tag{5.40}$$

where R_s^1 and O_s^1 are the strong reachability and observability matrices for system 1. □

Corollary 5.2

(*a*) Every left (right) input–output extendible TPBVDS has a minimal realization that is also left (right) input–output extendible.

(*b*) Every left (right) extension of a minimal left (right) input–output extendible TPBVDS is minimal. □

Proof

Part (*a*) follows from Theorem 3.5. To show (*b*), suppose that an extension of a minimal system defined on the interval $[0, N]$ is not minimal and thus can be reduced. Reduce the extension and move in its boundaries to the interval $[0, N]$. The system obtained has clearly the same weighting pattern as the original system defined on $[0, N]$, but has lower dimension, which is a contradiction. □

We now have an alternative method for proving the main result of Nikoukhah *et al.* (1989 a).

Theorem 5.2

A stationary TPBVDS, with $N \geqslant 2n$, is minimal if and only if

$$(a) \qquad \operatorname{Im}[V_i R_s \mid V_f R_s] = \mathbb{R}^n \tag{5.41}$$

$$(b) \qquad \ker \begin{bmatrix} O_s V_i \\ O_s V_f \end{bmatrix} = \{0\} \tag{5.42}$$

$$(c) \qquad \mathcal{O}_s \subset \mathcal{R}_s \tag{5.43}$$

□

Proof

First, note that the minimality conditions of Theorem 5.1 are necessary and sufficient for this case as well, even though we have a weaker condition on the length of the interval. This is because the only place that the assumption $N \geqslant 4n$ was used in the proof of Theorem 5.1 was in the derivation of (5.18) and (5.21), (5.22). But in the stationary case, as long as $N \geqslant 2n$, (5.21), (5.22) immediately follow from Lemma 5.2 and the assumption that the weighting patterns of the two systems are identical. In addition, (5.18) follows from (5.21), (5.22) and the assumption that the two systems are in normalized form. So all we need to show is that conditions (5.1)–(5.3) and (5.41)–(5.43) are equivalent in the stationary case.

Note that since

$$\operatorname{Im}[V_i R_s \mid V_f R_s] \subset R_w \tag{5.44}$$

$$\ker \begin{bmatrix} O_s V_i \\ O_s V_f \end{bmatrix} \supset \mathcal{O}_s \tag{5.45}$$

the conditions (5.41)–(5.43) are sufficient for minimality. To show necessity, let us assume that (5.1)–(5.3) hold. Suppose that (5.41) fails. Then there exists a vector $q \neq 0$ such that

$$q^T[V_i R_s \quad V_f R_s] = 0 \tag{5.46}$$

and, consequently,

$$q^T R_s = 0 \tag{5.47}$$

Noting that condition (5.3) is equivalent to

$$\text{Left-ker}(R_s) = \mathcal{R}_s^{\perp} \subset \mathcal{O}_s^{\perp} = \text{Row-Im}(O_s) \tag{5.48}$$

(5.47) implies that

$$q^T \in \text{Row-Im}(O_s) \tag{5.49}$$

which, thanks to the stationarity conditions (2.13) and (2.14), implies

$$q^T(V_i E^r A^s - E^r A^s V_i) R_s = 0 \tag{5.50}$$

$$q^T(V_f E^r A^s - E^r A^s V_f) R_s = 0 \tag{5.51}$$

for all r and s. Because of the E- and A-invariance of R_s, there exists a matrix D such that

$$E^{n-1-k} A^k R_s = R_s D \tag{5.52}$$

Then (5.50), (5.51) imply

$$q^{\mathrm{T}} E^{n-1-k} A^k [V_{\mathrm{i}} R_{\mathrm{s}} \mid V_{\mathrm{f}} R_{\mathrm{s}}] = q^{\mathrm{T}} [V_{\mathrm{i}} R_{\mathrm{s}} D \mid V_{\mathrm{f}} R_{\mathrm{s}} D] = 0 \tag{5.53}$$

Since (5.53) holds for all $k \in [0, n-1]$ we obtain

$$q^{\mathrm{T}} R_{\mathrm{w}} = 0 \tag{5.54}$$

which violates (5.1). Similarly, we can show that if (5.42) fails, then (5.2) is violated. □

We have shown above that conditions (5.1), (5.3) are equivalent to conditions (5.41)–(5.43) for stationary systems. However, note that this does not imply that (5.1) is equivalent to (5.41), and (5.2) to (5.42). As can be seen from the proof of Theorem 5.2, condition (5.3) must be true for (5.1) to be equivalent to (5.41), and for (5.2) to be equivalent to (5.42). The following example illustrates this point.

Example 5.1

Consider the following stationary TPBVDS in normalized form defined over an interval of length N

$$C = [0 \quad 0 \quad 1], \quad V_{\mathrm{i}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad V_{\mathrm{f}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & N & -N^2/2 \end{bmatrix}$$

$$E = I, \quad A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{5.55}$$

For this system, the strong reachability space $\mathscr{R}_{\mathrm{s}}$ is

$$\mathrm{Im}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right)$$

and so $\mathrm{Im}\,[V_{\mathrm{i}} R_{\mathrm{s}} \quad V_{\mathrm{f}} R_{\mathrm{s}}]$ is equal to

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & -1 \end{bmatrix}$$

and thus is not $\mathbb{R}^3$. This implies that condition (5.41) is not satisfied. On the other hand, we have

$$\mathscr{R}_{\mathrm{w}} = \bigvee_{k=0,\ldots,n-1} \mathrm{Im}\,(E^{n-k-1} A^k [V_{\mathrm{i}} R_{\mathrm{s}} \mid V_{\mathrm{f}} R_{\mathrm{s}}])$$

$$= \bigvee_{k=0,1,2} \mathrm{Im}\left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^k \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & -1 \end{bmatrix} \right) = \mathbb{R}^3 \tag{5.56}$$

which means that condition (5.1) is satisfied. This example illustrates that, if (5.3) does not hold, (5.1) and (5.41) are not equivalent. In this example, (5.3) does not hold since the strong unobservability space $\mathcal{O}_s$ is equal to ker [0 0 1] which implies that

$$\mathcal{O}_s = \operatorname{Im}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\right)$$

which is clearly not included in the strongly reachable space

$$\operatorname{Im}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$$

Finally, let us state an extension of another result due to Nikoukhah *et al.* (1989 a). In that work, we considered the generalization of standard and normalized forms to block forms. Specifically, a *block standard form* for a regular pencil $\{E, A\}$ is given by

$$TET^{-1} = \operatorname{diag}(E_1, E_2, \ldots, E_M) \tag{5.57}$$

$$TAT^{-1} = \operatorname{diag}(A_1, A_2, \ldots, A_M) \tag{5.58}$$

where T is an invertible matrix, where each $\{E_i, A_i\}$ pair is in standard form, i.e. there exist α_i, β_i such that

$$\alpha_i E_i + \beta_i A_i = I, \quad 1 \leqslant i \leqslant M \tag{5.59}$$

and furthermore where $\{E_i, A_i\}$ and $\{E_j, A_j\}$, $i \neq j$, have no eigenmode in common. That is, for any pair $(s, t) \neq (0, 0)$, $|sE_i - tA_i| = 0$ for at most one value of i (the value (s/t) for which this occurs for $\{E_i, A_i\}$ is the eigenmode of this pair). It has been shown by Nikoukhah *et al.* (1989 a) that if a TPBVDS is stationary, then V_{i} and V_{f} can also be chosen to have the same block diagonal form as E and A, i.e.

$$TV_{\mathrm{i}}T^{-1} = \operatorname{diag}(V_1^{\mathrm{i}}, \ldots, V_M^{\mathrm{i}}) \tag{5.60}$$

$$TV_{\mathrm{f}}T^{-1} = \operatorname{diag}(V_1^{\mathrm{f}}, \ldots, V_M^{\mathrm{f}}) \tag{5.61}$$

and moreover each of the subsystems $(C_k, V_k^{\mathrm{i}}, V_k^{\mathrm{f}}, E_k, A_k, B_k, N)$ is stationary.

In this case the reachability, observability and minimality of the overall system can be examined by looking at each subsystem in turn. In particular it allows us to study the properties of individual system eigenmodes. To see this, consider a TPBVDS transformed into the following normalized or block normalized form with

$$E = \operatorname{diag}(E_1, \ldots, E_M) \tag{5.62}$$

$$A = \operatorname{diag}(A_1, \ldots, A_M) \tag{5.63}$$

where $\{E_i, A_i\}$ has a unique eigenmode σ_i, with $\sigma_i \neq \sigma_j$ for $i \neq j$. Then we say that the eigenmode σ_j is strongly reachable if (E_j, A_j, B_j) is strongly reachable (i.e. R_s^j has full rank). It can easily be verified that σ_j is strongly reachable if and only if

$$[\sigma_j E - A \mid B]$$

has full row rank ($\sigma_j = \infty$ is strongly reachable if and only if $[E \mid B]$ has full row rank). Similarly, we say that an eigenmode σ_j is strongly observable if (C_j, E_j, A_j)

is strongly observable (i.e. O_s^j has full rank). Eigenmode σ_j is strongly observable if and only if

$$\begin{bmatrix} \sigma_j E - A \\ C \end{bmatrix}$$

has full column rank ($\sigma_j = \infty$ is strongly observable if and only if

$$\begin{bmatrix} E \\ C \end{bmatrix}$$

has full column rank). Since the boundary matrices can also be taken in block diagonal form,

$$V_i = \operatorname{diag}(V_1^i, \ldots, V_M^i) \tag{5.64}$$

$$V_f = \operatorname{diag}(V_1^f, \ldots, V_M^f) \tag{5.65}$$

we can also consider weak reachability and observability of individual eigenmodes. An eigenmode σ_j is called weakly reachable (observable) if subsystem j is weakly reachable (observable). Also σ_j is weakly reachable if and only if

$$[\sigma_j E - A \mid V_i B \mid V_f B]$$

has full row rank; it is weakly observable if and only if

$$\begin{bmatrix} \sigma_j E - A \\ C V_i \\ C V_f \end{bmatrix}$$

has full column rank. □

We now have the following result.

Theorem 5.3

Consider a minimal, stationary TPBVDS; then any eigenmode of the strongly unreachable (unobservable) part of system is also an eigenmode of the strongly reachable (observable) part of the system. □

Proof

Suppose that σ_k is an eigenmode of the strongly unreachable part of the system. As just described, let us break down the system into subsystems, each one of which has a distinct eigenmode. In particular, let $\Sigma_k = (C_k, V_k^i, V_k^f, E_k, A_k, B_k, N)$ denote the subsystem associated with eigenmode σ_k. Then, since Σ_k is minimal, it has a strongly reachable part (otherwise, B_k must be zero, the subsystem has weighting pattern 0 and the minimal realization has dimension 0). Thus, σ_k is an eigenmode of the strongly reachable part of Σ_k and of the original system. □

6. Conclusions

In this paper we have developed some of the system-theoretic properties of two-point boundary-value descriptor systems. We have derived detailed characterizations of reachability, observability, and minimality, extending previous results for the shift-invariant case of Nikoukhah *et al.* (1989 a). As had already been noted for continuous-time, non-descriptor boundary-value systems, minimality for TPBVDSs is a bit more complicated than for causal systems. Indeed there is a certain degree of non-uniqueness in minimal realizations.

Another concept that we have introduced and studied in this paper is extendibility, i.e. the idea of thinking of a TPBVDS as being defined on a sequence of intervals of increasing length. Once one introduces such a notion, it becomes possible to talk about the realization problem (Nikoukhah *et al.* 1988), and asymptotic properties such as stability (Nikoukhah *et al.* 1989 c).

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