Internal Models and Recursive Estimation for 2-D Isotropic Random Fields

Ahmed H. Tewfik, *Member, IEEE*, Bernard C. Levy, and Alan S. Willsky, *Fellow, IEEE*

***Abstract*—Efficient recursive smoothing algorithms are developed for isotropic random fields that can be obtained by passing white noise through rational filters. The 2-D estimation problem is shown to be equivalent to a countably infinite set of 1-D separable two-point boundary value smoothing problems. The 1-D *smoothing problems* are solved using a Markovianization approach followed by a standard 1-D smoothing algorithm. The desired field estimate is then obtained as a properly weighted sum of the 1-D smoothed estimates. The 1-D two-point boundary value problems are also shown to have the same asymptotic properties and yield a stable spectral factorization of the power spectrum of the isotropic random fields.**

***Index Terms*—Stochastic processes, Markov processes, innovations methods, Fourier series, recursive estimation, random fields, stochastic differential equations, multidimensional stochastic processes, multidimensional signal processing, filtering, smoothing methods, modeling.**

I. Introduction

PROBLEMS involving spatially-distributed data and phenomena arise in various fields including image processing, meteorology, geophysical signal processing, oceanography and optical processing. A major challenge in any such problem is to develop algorithms capable of dealing effectively with the increased computational complexity of multidimensional problems and that can be implemented in a recursive fashion. In one dimension the ways in which data can be organized for efficient processing are extremely limited and causality typically provides a natural choice. Furthermore, in one dimension, internal differential realizations of random processes were exploited to develop an efficient estimation algorithm, namely the Kalman filtering technique. This has led researchers in estimation theory to investigate the extension of 1-D Kalman filtering and smoothing methods to noncausal 2-D random fields. The work of Woods and Radewan [1], Habibi [2], Attassi [3], Jain and Angel [4], Wong [5], Ogier and Wong [6] to name a few, has shown that such extensions do exist. However, the methods developed by these researchers are either approximate or can be applied only to a limited class of 2-D fields, namely to fields that can be described by hyperbolic partial differential equations, and which therefore are causal in some sense.

Note that, unlike one dimension, the most natural estimation problem in higher dimensions is the smoothing problem, rather than the causal filtering problem. This is because in higher dimensions, the filtering problem requires an artificial partition of the data between past and future, whereas the smoothing problem does not assume any causal ordering of the data. The smoothing problem for 2-D random fields has been studied from an input–output point of view by Ramm [7] and by Levy and Tsitsiklis [8] among others. In particular, Ramm studied the integral equation governing the optimal linear filter for estimating a general random field given some observations, while Levy and Tsitsiklis developed efficient Levinson-like recursions for computing the optimal smoothing filter for the case where both the field of interest and the observations are jointly isotropic.

The objective of this paper is to study the smoothing problem for a class of random fields that have noncausal internal differential realizations but which also have enough structure to allow the development of efficient recursive smoothing algorithms. Specifically, in this paper we investigate efficient recursive smoothing techniques for *isotropic* random fields $z(\vec{r})$[1] that can be represented as the output of rational 2-D filters driven by white noise, and which admit therefore simple internal differential models. Isotropic fields are characterized by the fact that their mean value is a constant independent of position and their autocovariance function is invariant under all rigid body motions, i.e., under translations and rotations. In some sense, isotropy is the natural extension of the notion of stationarity in one dimension. Furthermore, isotropic random fields arise in a number of practical problems such as the black body radiation problem [9], the study of underwater ambient noise in horizontal planes parallel to the surface of the ocean [10], and the investigation of temperature and pressure distributions at constant altitude in the atmosphere [11].

[1]Throughout this paper we use $\vec{r}$ to denote a point in 2-D Cartesian space. The polar coordinates of this point are denoted by r and θ.

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A. H. Tewfik is with the Department of Electrical Engineering, University of Minnesota, Room 4-178 EE/CSci Building, Minneapolis, MN 55455.

B. C. Levy is with the Department of Electrical Engineering and Computer Science, University of California, Davis, Davis, CA 95616.

A. S. Willsky is with the Laboratory for Information and Decision Systems, and the Department of Electrical Engineering and Computer Science, Room 35-437, M.I.T., Cambridge, MA 02139.

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We show that the class of random fields that can be represented as the output of rational 2-D filters driven by white noise may be described in an input-output sense in terms of a 2-D filter whose impulse response is a modified matrix Bessel function of the second kind and order zero. Furthermore, they admit an internal realization that involves the Laplacian operator. The motivation for studying this class of fields is that it can be used to describe a variety of physical phenomena such as the variation of the electric potential created by a random charge distribution.

An important property of 2-D isotropic fields is that when they are expanded in a Fourier series in terms of the polar coordinate angle θ, the Fourier coefficient processes of different orders are uncorrelated [12, p. 5]. Using the input–output model of the process $z(\vec{r})$, 1-D state space two-point boundary value (TPBV) models are constructed for the Fourier coefficient processes. Those models are then used to derive a 2-D space-invariant differential model with appropriate boundary condition for $z(\vec{r})$ over a finite disk of radius R. Given noisy observations of the isotropic random field $z(\vec{r})$ over the disk of radius R, our approach is to reduce the 2-D smoothing problem to a countable set of decoupled 1-D smoothing problems for the uncorrelated Fourier coefficient processes $z_k(r)$ corresponding to the process $z(\vec{r})$. The resulting 1-D TPBV smoothing problems are then solved using a Markovianization technique that transforms the noncausal model to a causal one to which standard 1-D smoothing techniques can be applied. Finally, the best linear least squares estimate of $z(\vec{r})$ given the observations is obtained as a properly weighted combination of the 1-D smoothed estimates of all the Fourier coefficient processes $z_k(r)$. Observe that by properly exploiting the structure of isotropic random fields, a *recursive* solution to the smoothing problem for a noncausal isotropic process has thus been constructed. The recursions here are with respect to the radius r in a polar coordinate representation of the fields.

We also study the *asymptotic* properties of the 1-D causal state space Markovian models of the Fourier coefficient processes. Specifically, we show that all models tend asymptotically to the *same* space-invariant stable model *regardless* of the particular order of the Fourier coefficient which they describe. Furthermore, each model yields asymptotically a *stable spectral factorization* for the original isotropic field $z(\vec{r})$.

This paper is organized as follows. In Section II, we introduce an input–output model and an equivalent differential model for the class of 2-D isotropic fields to be studied. In Section III, two-point boundary value models are developed to describe the 1-D Fourier coefficient processes corresponding to fields in the class that we study. Those models are then used to show that the fields that we consider can be described over a finite disk by a space-invariant differential model with appropriate boundary conditions. The smoothing problem for the isotropic random field $z(\vec{r})$ given noisy measurements over a disk of radius R is defined and reduced to a countably infinite set of decoupled 1-D estimation problem using Fourier series expansions in Section IV. The 1-D smoothing problems are then solved using a Markovianization approach. Finally, Section V studies the asymptotic properties of the Markovian models corresponding to the Fourier coefficients of the signal. The Markovian models are shown to have the same stable space-invariant form and yield a stable spectral factorization for the signal process.

II. Random Field Model

A. Input–Output Model

Isotropic random fields that can be obtained by passing white noise through rational rotationally symmetric filters can be described in several ways. Here, our starting point will be an input–output description of such fields in terms of a multidimensional Wiener integral. Specifically, the random fields $z(\vec{r})$ considered in this paper are described over the plane $\boldsymbol{R}^2$ by

$$x(\vec{r}) = -\frac{1}{2\pi}\int_{\boldsymbol{R}^2} K_0(A|\vec{r}-\vec{r}'|)Bu(\vec{r}')\,d\vec{r}', \qquad \vec{r}\in \boldsymbol{R}^2, \tag{2.1}$$

$$z(\vec{r}) = Cx(\vec{r}), \tag{2.2}$$

where $d\vec{r}' = dx'dy'$ denotes an element of area. In (2.1)–(2.2), $x(\vec{r})\in \boldsymbol{R}^n$, $z(\vec{r})\in \boldsymbol{R}^p$, and $u(\vec{r})\in \boldsymbol{R}^m$ is a random zero-mean two-dimensional white Gaussian noise process with

$$E\left[u(\vec{r})u^T(\vec{s})\right] = I_m\delta(\vec{r}-\vec{s}), \tag{2.3}$$

where I_m is the $m\times m$ identity matrix. The matrices $K_0(Ar)$, B, and C are real matrices of appropriate dimensions. In particular, $K_0(Ar)$ denotes a matrix modified Bessel function of the second kind and of order zero [13]. Matrix modified Bessel functions of the first and second kinds arise naturally in the study of rational isotropic random fields. A brief discussion of some of their properties appears in Appendix A. (For more details see [13] and the references therein.) In (2.1) the eigenvalues of the $n\times 1$ real matrix A are assumed to have strictly positive real parts. This insures that the 2-D shaping filter $K_0(Ar)$ is square-integrable. Furthermore, it guarantees that for any measurable and square-integrable n-vector $f(\vec{r})$, $\vec{r}\in\boldsymbol{R}^2$, the Gaussian random variable $\int_{\boldsymbol{R}^2} f^T(\vec{r})x(\vec{r})\,d\vec{r}$ has finite variance. Here, $f^T(\vec{r})$ denotes the transpose of $f(\vec{r})$. The main property of the process $x(\vec{r})$ defined by (2.1) is that it is a 2-D rational isotropic random field as is shown in Theorem 1.

Theorem 1: The process $x(\vec{r})$ defined by (2.1) is an isotropic random field, i.e., its autocorrelation function $R_x(\vec{r},\vec{s}) = E[x(\vec{r})x^T(\vec{s})]$ is invariant under translations and rotations.

Proof: We will first show that $R_x(\vec{r},\vec{s})$ is invariant under translation. From (2.1) we have

$$R_x(\vec{r},\vec{s}) = E\left[x(\vec{r})x^T(\vec{s})\right] \tag{2.4}$$

$$= \frac{1}{4\pi^2}\int_{R^2} K_0(A|\vec{r}-\vec{u}|)BB^TK_0^T(A|\vec{s}-\vec{u}|)\,d\vec{u}. \tag{2.5}$$

Now perform the transformation

$$\vec{v} = \vec{u} + \vec{h} \tag{2.6}$$

to obtain

$$R_x(\vec{r},\vec{s}) = \frac{1}{4\pi^2}\int_{R^2} K_0^T\left(A|\vec{r}+\vec{h}-\vec{v}|\right) \cdot BB^TK_0\left(A|\vec{s}+\vec{h}-\vec{v}|\right)d\vec{v}. \tag{2.7}$$

This shows that $R_x(\vec{r},\vec{s})$ is invariant under translation. Using this fact, we can write

$$R_x(\vec{r},\vec{s}) = R_x(\vec{v},0), \tag{2.8}$$

where $\vec{v} = \vec{r} - \vec{s}$. Hence,

$$R_x(\vec{r},\vec{s}) = \frac{1}{4\pi^2}\int_{R^2} K_0(A|\vec{v}-\vec{u}|)BB^TK_0^T(A|\vec{u}|)\,d\vec{u} \tag{2.9}$$

$$= \frac{1}{4\pi^2}\int_{R^2} K_0\left(A\left(v^2+u^2-2uv\cos(\theta-\phi)\right)^{1/2}\right) \cdot BB^TK_0^T(Au)\,d\vec{u}, \tag{2.10}$$

where $\vec{v}=(v,\phi)$ and $\vec{u}=(u,\theta)$. Letting $\alpha=\phi-\theta$, we conclude from (2.10) and the periodicity of $\cos\alpha$ that

$$R_x(\vec{r},\vec{s}) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{\infty} K_0\left(A\left(v^2+u^2-2uv\cos\alpha\right)^{1/2}\right) \cdot BB^TK_0^T(Au)u\,du\,d\alpha. \quad \square \tag{2.11}$$

Theorem 1 implies also that the output process $z(\vec{r})$ is isotropic with autocorrelation function

$$R_z(\vec{r},\vec{s}) = CR_x(\vec{r},\vec{s})C^T. \tag{2.12}$$

Since $R_x(\cdot)$ is translation-invariant we can define its spectral density matrix $S_x(\vec{\lambda})$, which is the 2-D Fourier transform of $R_x(\vec{r})$:

$$S_x(\vec{\lambda}) = \int_{R^2} R_x(\vec{r})e^{-j\vec{\lambda}\cdot\vec{r}}d\vec{r} \tag{2.13}$$

$$= 2\pi\int_0^{\infty} R_x(r)J_0(\lambda r)r\,dr \tag{2.14}$$

$$= \left(\lambda^2 I_n + M\right)^{-1}BB^T\left(\lambda^2 I_n + M^T\right)^{-1} \tag{2.15}$$

$$= S_x(\lambda), \tag{2.16}$$

where we have taken advantage of the circular symmetry of $R_x(\vec{r})$ and $M = A^2$. Observe that $S_x(\lambda)$ is *rational* in λ, the magnitude of $\vec{\lambda}$. Furthermore, the poles of the spectrum $S_x(\lambda)$, obtained by setting $p = j\lambda$ in (2.15), have a quadrantal symmetry property when plotted in the complex p-plane.

B. Differential Model

To develop an internal realization for the field $z(\vec{r})$ we shall need the notion of a *generalized random field*. We define a generalized random field as follows. Let $(\Omega, \mathscr{A}, P)$ be a probability space and $\mathscr{K}(\boldsymbol{R}^d)$ be the Schwartz space of n-vector functions on $\boldsymbol{R}^d$, $d \geq 1$, with square-integrable derivatives of all orders. Furthermore, let $\mathscr{K}'(\boldsymbol{R}^d)$ be the family of generalized functions on $\mathscr{K}(\boldsymbol{R}^d)$, i.e., the family of all linear functionals continuous in the topology of $\mathscr{K}$.

By a generalized $n\times 1$ vector random field $z(\vec{r},\omega)$, $\vec{r}\in\boldsymbol{R}^d$, $\omega\in\Omega$, we mean that the mapping $z(f,\omega) = \langle f, z\rangle = \int_{\boldsymbol{R}^d} f^T(\vec{r})z(\vec{r},\omega)\,d\vec{r}$ from $\mathscr{K}(\boldsymbol{R}^d)\times\Omega$ to $\boldsymbol{R}^1$ is such that

1) $z(f,\omega)$ is a random variable with a finite variance for every $f\in\mathscr{K}(\boldsymbol{R}^d)$,
2) $z(\cdot)\in\mathscr{K}'(\boldsymbol{R}^d)$ with probability one.

The correlation functional of $z(f,\omega)$ is defined as the bilinear functional

$$B(f,g) = \int_{\boldsymbol{R}^d}\int_{\boldsymbol{R}^d} f^T(\vec{r})K_z(\vec{r},\vec{s})g(\vec{s})\,d\vec{r}d\vec{s}, \qquad f,g\in\mathscr{K}.$$

In particular, for an isotropic generalized random field

$$K_z(\vec{r},\vec{s}) = K_z(|\vec{r}-\vec{s}|),$$

i.e., $K_z(\cdot)$ is invariant under rotation and translation. Note that any mean-square continuous isotropic random field with a finite variance is also a generalized isotropic random field in this sense.

For $z(f,\omega) = \langle f, z\rangle$ we define the operation of differentiation as it is usually defined for ordinary generalized functions:

$$D^{\vec{k}}z(f,\omega) = (-1)^{|\vec{k}|}\langle D^{\vec{k}}f, z\rangle,$$

where

$$D^{\vec{k}}f(\vec{r}) = \frac{\partial^{|\vec{k}|}f(\vec{r})}{\partial r_1^{k_1}\cdots\partial r_d^{k_d}}$$

$$\vec{r} = (r_1,\cdots,r_d)\in\boldsymbol{R}^d$$

$$\vec{k} = (k_1,\cdots,k_d)$$

$$|\vec{k}| = k_1+\cdots+k_d.$$

By abuse of notation we use $D^{\vec{k}}z(\vec{r},\omega)$ to denote the generalized derivative of $z(\vec{r},\omega)$ that may not exist in any usual sense. In particular, $D^{\vec{k}}z(\vec{r},\omega)$ has to be interpreted as meaning that the mapping

$$D^{\vec{k}}z(f,\omega) = \langle f, D^{\vec{k}}z\rangle = (-1)^{|\vec{k}|}\langle D^{\vec{k}}f, z\rangle$$

is a random variable with a finite variance for every $f\in\mathscr{K}(\boldsymbol{R}^d)$. Note that in the previous equation the first equality only makes sense when interpreted according to the second equality.

Now recall that in 1-D stationary random processes that can be obtained by passing white noise through rational 1-D stable filters have time-invariant state space models over the real axis. An analog result holds for the class of isotropic random fields that we are studying. Specifically, the random field $x(\vec{r})$ in (2.1) is a generalized solution [14] of the stochastic equation

$$\left(I_n\nabla^2 - A^2\right)x(\vec{r}) = Bu(\vec{r}) \tag{2.17}$$

$$z(\vec{r}) = Cx(\vec{r}). \tag{2.18}$$

In (2.17), $\nabla^2 x(\vec{r})$ has to be interpreted as previously indicated.

More precisely, let $L^\dagger = (I_n\nabla^2 - A^{2T})$ be the adjoint of the operator $L = (I_n\nabla^2 - A^2)$. $x(\vec{r})$ is a solution of the stochastic differential equation (2.17) in the sense that for any vector $f(\vec{r}) \in \mathscr{K}(\boldsymbol{R}^2)$ the Gaussian random variables $v_1 = \langle L^\dagger f, x\rangle$, and $v_2 = \langle f, Bu\rangle$ are equal with probability one. Indeed, using the facts that

$$(L^\dagger f, g) = (f, Lg), \qquad f, g \in L^2(\boldsymbol{R}^2), \tag{2.19}$$

$$\left(I_n\nabla^2 - A^2\right)\frac{1}{2\pi}K_0(A|\vec{r} - \vec{s}|) = I_n\delta(\vec{r} - \vec{s}), \tag{2.20}$$

we find by direct substitution that

$$E\left[(v_1 - v_2)\right] = 0, \tag{2.21}$$

$$E\left[(v_1 - v_2)^2\right] = 0. \tag{2.22}$$

Note that $x(\vec{r})$ is not the unique weak solution of (2.17). In fact, the isotropic process

$$y(\vec{r}) = -\frac{1}{2\pi}\int_{\boldsymbol{R}^2} I_0(A|\vec{r} - \vec{r}'|)Bu(\vec{r}')\,d\vec{r}', \qquad \vec{r} \in \boldsymbol{R}^2, \tag{2.23}$$

where $I_0(Ar)$ denotes a matrix modified Bessel function of the first kind and of order zero [13], is also a weak solution in the above sense. However the covariance of $y(\vec{r})$ is not well behaved at infinity. In particular, $y(\vec{r})$ does not define a valid generalized isotropic random field since there exists $f(\cdot) \in \mathscr{K}(\boldsymbol{R}^2)$ such that the random variable $\langle f, y\rangle$ has an infinite variance (e.g., for $f(\vec{r}) = e^{-r}$, $A = \alpha^2 > 1$, $\alpha \in \boldsymbol{R}$, and $B = 1$). Note also that while in 2-D the weak solution $x(\vec{r})$ of (2.17) is an ordinary random field that is not mean-square differentiable (i.e., all the derivatives of $x(\vec{r})$ are generalized random fields), an examination of the power spectrum of each weak solution of (2.17) in M-D reveals that all the weak solution of (2.17) in dimension higher than 2 (i.e., in $\boldsymbol{R}^m$, $m > 2$) are generalized random fields.

C. Motivation

The motivation for considering models (2.1)–(2.3) or (2.17)–(2.18) is that they can be used to describe a large class of physical phenomena such as the variation of the electric potential created by a uniformly distributed random sources in a lossy medium, where the loss is described here by A^2. Another important motivation for considering such a model is as follows.

Theorem 2: Any generalized isotropic random process that is obtained by passing 2-D white noise through a rational stable and proper 2-D circularly symmetric linear filter has a realization of the form (2.1)–(2.2) or (2.17)–(2.18).

Proof: Consider the scalar generalized 2-D random field $z(\vec{r})$ that satisfies the partial differential equation

$$P\left(\frac{\partial}{\partial r_1}, \frac{\partial}{\partial r_2}\right)z(\vec{r}) = Q\left(\frac{\partial}{\partial r_1}, \frac{\partial}{\partial r_2}\right)u(\vec{r}), \tag{2.24}$$

where $u(\vec{r})$ is a 2-D white noise process of intensity I_m. Here, $P(s_1, s_2)$ and $Q(s_1, s_2)$ are 2-D polynomials in the variables s_1 and s_2. Equation (2.24) implies that $z(\vec{r})$ is the output of a rational 2-D filter $H(\vec{\lambda})$ driven by the noise process $u(\vec{r})$, where

$$H(\vec{\lambda}) = \frac{Q(j\lambda_1, j\lambda_2)}{P(j\lambda_1, j\lambda_2)}. \tag{2.25}$$

The spectrum of $z(\vec{r})$ is given by

$$S_z(\vec{\lambda}) = \left|H(\vec{\lambda})\right|^2. \tag{2.26}$$

In [12, p. 23], Yadrenko shows that the process $z(\vec{r})$ is isotropic if and only if the 2-D polynomials $P(\cdot,\cdot)$ and $Q(\cdot,\cdot)$ are functions of $\lambda = (\lambda_1^2 + \lambda_2^2)^{1/2}$ only, i.e., if $P(\cdot,\cdot)$ and $Q(\cdot,\cdot)$ are of the form

$$P(j\lambda_1, j\lambda_2) = \sum_{k=0}^{n} p_k(-\lambda^2)^k = P(-\lambda^2), \tag{2.27}$$

$$Q(j\lambda_1, j\lambda_2) = \sum_{k=0}^{q} q_k(-\lambda^2)^k = Q(-\lambda^2). \tag{2.28}$$

In this case, the model (2.24) reduces to

$$P(\nabla^2)z(\vec{r}) = Q(\nabla^2)u(\vec{r}). \tag{2.29}$$

Furthermore, if $n > q$, we can compute a stable spectral factorization of $S_z(-\lambda^2) = |H(-\lambda^2)|^2$

$$H(-\lambda^2) = \frac{Q(-\lambda^2)}{P(-\lambda^2)} \tag{2.30}$$

$$= C\left(-\lambda^2 I_n - A^2\right)^{-1}B, \tag{2.31}$$

where the eigenvalues of A have strictly positive real parts. This condition is necessary to insure that for any measurable and square integrable scalar function $f(\vec{r})$ the Gaussian random variable $\langle f, z\rangle$ has finite variance, where

$$z(\vec{r}) = -\frac{C}{2\pi}\int_{\boldsymbol{R}^2} h(\vec{r} - \vec{r}')Bu(\vec{r}')\,d\vec{r}', \qquad \vec{r} \in \boldsymbol{R}^2, \tag{2.32}$$

and $h(\vec{r})$ is the inverse Fourier transform of $H(\vec{\lambda})$. Using any of the standard 1-D state-space realization techniques with the variable s replaced by λ^2 and the operator d/dt by the operator ∇^2, we can obtain a state space realization of $z(\vec{r})$ in the form (2.17)–(2.18) or an input-output representation of the form (2.1)–(2.2). □

We see therefore that the class of random fields with the representation (2.1)–(2.2) or (2.17)–(2.18) is quite large.

III. Fourier Series Coefficients

An important property of 2-D isotropic fields is that when they are expanded in a Fourier series in terms of the polar coordinate angle θ, the Fourier coefficient processes of different orders are *uncorrelated* [12, p. 5]. Specifically, any isotropic random field $f(\cdot)$ can be expanded in a series of the form

$$f(r,\theta) = \sum_{k=-\infty}^{\infty} f_k(r) e^{jk\theta}, \tag{3.1}$$

where the equality holds in the mean square sense. In (3.1) the kth-order Fourier coefficients process $f_k(r)$ is a 1-D nonstationary process given by

$$f_k(r) = \frac{1}{2\pi}\int_0^{2\pi} f(r,\theta) e^{-jk\theta}\, d\theta. \tag{3.2}$$

Furthermore, [12, p. 5]

$$E\left[f_k(r) f_l^H(s)\right] = \left(\int_{\boldsymbol{R}^d} S_f(\lambda) J_k(\lambda r) J_k(\lambda s)\, d\lambda\right)\delta_{k,l}, \quad \forall r,s. \tag{3.3}$$

Here, $f_l^H(r)$ denotes the complex conjugate transpose of $f_l(r)$, $S_f(\lambda)$ is the power spectral density matrix of $f(\cdot)$, $J_k(\cdot)$ is a Bessel function of the first kind and order k and $\delta_{k,l}$ is a Kronecker delta function, i.e., $\delta_{k,l} = 1$ if $k = l$ and $\delta_{k,l} = 0$, otherwise. Thus, by using Fourier series expansions of isotropic random fields it is possible to reduce the study of any problem involving such fields into the study of a countably infinite number of 1-D equivalent problems for the Fourier coefficient processes.

A. State-Space Models for the Fourier Processes

Let us now use the model (2.1)–(2.2) for the process $z(\vec{r})$ to construct 1-D state-space two-point boundary value models for the Fourier coefficient processes $z_k(r)$ over a finite interval $[0, R]$. Those models will be used in the next section to develop recursive solutions to a smoothing problem for $z(\cdot)$.

Theorem 3: A two-point boundary value (TPBV) model describing $z_k(r)$ and $y_k(r)$ over the interval $[0, R]$ is given by

$$\frac{d}{dr}\begin{bmatrix}\xi_k(r)\\ \eta_k(r)\end{bmatrix} = \begin{bmatrix}-rI_k(Ar)B\\ rK_k(Ar)B\end{bmatrix} u_k(r) \tag{3.4}$$

$$x_k(r) = \begin{bmatrix}K_k(Ar) & I_k(Ar)\end{bmatrix}\begin{bmatrix}\xi_k(r)\\ \eta_k(r)\end{bmatrix} \tag{3.5}$$

$$z_k(r) = Cx_k(r) \tag{3.6}$$

with the boundary conditions

$$\xi_k(0) = 0, \qquad \text{with probability 1} \tag{3.7}$$

and

$$\eta_k(R) \sim N\left(0, \Pi_{\eta_k}(R)\right) \tag{3.8}$$

with

$$\Pi_{\eta_k}(R) = \frac{1}{2\pi}\int_R^\infty K_k(As) BB^T K_k^T(As)\, s\, ds. \tag{3.9}$$

Here, $u_k(r)$ and $v_k(r)$ are two one-dimensional zero-mean white Gaussian noise processes with covariance

$$E\left[\begin{bmatrix}u_k(r)\\ v_k(r)\end{bmatrix}\begin{bmatrix}u_k^T(s) & v_k^T(s)\end{bmatrix}\right] = \begin{bmatrix}I & 0\\ 0 & V\end{bmatrix}\frac{\delta(r-s)}{2\pi r}. \tag{3.10}$$

In Theorem 3 $I_k(Ar)$ and $K_k(Ar)$ are matrix modified Bessel functions of the first and second kind respectively, and of order k. (See Appendix A and [13].)

Note that the TPBV model dynamics (3.4) are extremely simple, consisting of a gain matrix multiplying the input noise process $u_k(r)$. This is to be contrasted with the more complicated dynamics of an equivalent Markovian model for $z_k(r)$ that we shall develop in the next section.

Proof: To derive (3.4)–(3.6), we shall use the following identity [13]:

$$K_0(A|\vec{r} - \vec{s}|) = \sum_k I_k(Ar_<) K_k(Ar_>) e^{jk(\theta-\phi)}, \tag{3.11}$$

where $\vec{r} = (r,\theta)$, $\vec{s} = (s,\phi)$, $r_< = \min(r,s)$, and $r_> = \max(r,s)$. Substituting (3.11) into (2.1) we obtain

$$x(\vec{r}) = -\frac{1}{2\pi}\int_{\boldsymbol{R}^2}\left(\sum_k I_k(Ar_<) K_k(Ar_>) e^{jk(\theta-\phi)}\right) \cdot Bu(\vec{r}')\, d\vec{r}', \tag{3.12}$$

$$x(\vec{r}) = \sum_k \left(-\frac{1}{2\pi}\int_{\boldsymbol{R}^2}\left(I_k(Ar_<) K_k(Ar_> e^{jk(\theta-\phi)}\right) \cdot Bu(\vec{r}')\, d\vec{r}'\right), \tag{3.13}$$

where the second equality holds in the mean-square sense. Evaluating the integral with respect to the angular variable for each term on the right-hand side of (3.13) we obtain

$$x(\vec{r}) = \sum_{k=-\infty}^{\infty} x_k(r) e^{jk\theta}, \tag{3.14}$$

where

$$x_k(r) = -K_k(Ar)\int_0^r I_k(As) Bu_k(s)\, s\, ds - I_k(Ar)\int_r^\infty K_k(As) Bu_k(s)\, s\, ds. \tag{3.15}$$

Thus the kth Fourier coefficient of $x(\vec{r})$ is $x_k(r)$.

Furthermore, upon multiplying both sides of (2.2) by $e^{-jk\theta}/2\pi$ and integrating from 0 to 2π, we obtain

$$z_k(r) = Cx_k(r). \tag{3.16}$$

Define the state variables $\xi_k(r)$ and $\eta_k(r)$ by

$$\xi_k(r) = -\int_0^r I_k(As) Bu_k(s)\, s\, ds \tag{3.17}$$

and

$$\eta_k(r) = -\int_r^\infty K_k(As) Bu_k(s)\, s\, ds. \tag{3.18}$$

Then, it follows from (3.15)–(3.18) that a TPBV model describing $z_k(r)$ over the interval $[0, R]$ is given by the system (3.4)–(3.6). □

B. Differential Random Field Model over a Finite Disk

In 1-D stationary random processes that may be obtained by passing white noise through rational filters can be described over a finite or semi-infinite interval by time-invariant state space models with *appropriate* initial conditions. In particular, the initial conditions are chosen to guarantee that the covariance of the state space model is initially equal to its steady state value. Theorem 3 can be used to show that an analog modeling result holds for the class (2.1)–(2.2). Specifically, this class admits an internal description of the form (2.17) with appropriate boundary conditions as the following theorem indicates.

Theorem 4: The process $x(\vec{r})$ given by (2.1) is the unique solution to (2.17) over the disk $D_R = \{\vec{r}: r < R\}$ with the boundary conditions

1)
$$E\left[\left(c^T x(\vec{r})\right)^2\right] < \infty \tag{3.19}$$

for any finite norm n-vector c,

2)
$$\int_\Gamma \left[G(\vec{R},\vec{s}) \frac{\partial x}{\partial n}(\vec{s}) - \left(\frac{\partial G}{\partial n}(\vec{R},\vec{s}) \right) x(\vec{s}) \right] dl = \beta(R,\theta), \quad 0 \le \theta < 2\pi, \tag{3.20}$$

where Γ is the circle of radius R, $G(\vec{r},\vec{s}) = 1/2\pi K_0(A|\vec{r}-\vec{s}|)$, and

$$E[\beta(R,\theta)] = 0 \tag{3.21}$$

$$E[\beta(R,\theta)\beta^T(R,\phi)] = \Pi_\beta(R;\theta-\phi) = \sum_{k=-\infty}^{\infty} I_k(AR)\Pi_{\eta_k}(R) I_k^T(AR) e^{jk(\theta-\phi)} \tag{3.22}$$

with

$$\Pi_{\eta_k}(R) = \frac{1}{2\pi}\int_R^\infty K_k(As) BB^T K_k^T(As) s\,ds. \tag{3.23}$$

Furthermore,

$$E[\beta(R,\theta)u^T(r,\phi)] = 0, \qquad \text{for } r < R. \tag{3.24}$$

Here, $\partial/\partial n$ and dl denote respectively the normal derivative with respect to Γ and an infinitesimal element of arc length along Γ. Furthermore, the notation $\partial x/\partial n(\vec{s})$ has to be interpreted as in Section II-B.

Proof: By substituting the Fourier series expansions of $x(\cdot)$ and $u(\cdot)$ into (2.17), we find that (2.17) is equivalent to the countably infinite set of 1-D stochastic differential equations

$$\frac{d^2 x_k(r)}{dr^2} + \frac{1}{r}\frac{dx_k(r)}{dr} - \left(A^2 + \frac{k^2}{r^2} I\right) x_k(r) = u_k(r), \quad -\infty < k < \infty. \tag{3.25}$$

Furthermore, by choosing the state variables

$$\begin{bmatrix} \xi_k(r) \\ \eta_k(r) \end{bmatrix} = r \begin{bmatrix} AI_{k+1}(Ar) & -AI_k(Ar) \\ AK_{k+1}(Ar) & AK_k(Ar) \end{bmatrix} \cdot \begin{bmatrix} x_k(r) \\ A^{-1}\left(\frac{d}{dr} x_k(r) - \frac{k}{r} x_k(r) \right) \end{bmatrix}, \tag{3.26}$$

we find that for each k (3.25) is equivalent to the state space model (3.4)–(3.5).

To derive an initial condition for $\xi_k(r)$ and a final condition for $\eta_k(r)$ we proceed as follows. First observe that for $k \neq 0$ (3.3) implies that $x_k(0) = 0$ with probability one. Combining this fact with the asymptotic behavior of $I_k(Ar)$ and $K_k(Ar)$ as r tends to zero (cf. Appendix A) we conclude that $\xi_k(0) = 0$ with probability one for $k \neq 0$. Furthermore, (3.19) indicates that each component of $x(\cdot)$ has a finite variance. In particular, this property holds at and near the origin. Now $x(0) = x_0(0)$ since for $k \neq 0$ $x_k(0) = 0$ with probability one. But

$$x_0(0) = \lim_{r \to 0} [K_k(Ar) I_k(Ar)] \begin{bmatrix} \xi_k(r) \\ \eta_k(r) \end{bmatrix}. \tag{3.27}$$

By combining the previous discussion with the asymptotic behavior of $I_k(Ar)$ and $K_k(Ar)$ as r tends to zero we conclude that $\xi_0(0) = 0$ with probability one. Hence, $\xi_k(0) = 0$ with probability one for all k.

Next observe that for any $f(\cdot) \in \mathscr{K}(\boldsymbol{R}^2)$ the mapping $\frac{\partial}{\partial r} x(f,\omega)$ can be expressed as

$$\frac{\partial}{\partial r} x(f,\omega) = \sum_{k=-\infty}^{\infty} \frac{d}{dr} x_k(f_k,\omega), \tag{3.28}$$

where the equality holds in the mean-square sense, $f_k(r)$ is the kth-order Fourier coefficient of $f(\cdot)$ and

$$\frac{d}{dr} x_k(f_k,\omega) = (-1)\int_{\boldsymbol{R}} \frac{d}{dr} f_k^H(r) x_k(r)\,dr. \tag{3.29}$$

Using this fact, (3.11), and (3.26), it follows that

$$\beta(R,\theta) = \sum_{k=-\infty}^{\infty} I_k(AR)\eta_k(R) e^{jk\theta}, \tag{3.30}$$

where the equality holds in the mean-square sense. Thus (2.17) together with the boundary conditions (3.19)–(3.20) over the disk D_R is equivalent to the countably infinite set of TPBV models (3.4)–(3.5) with the boundary conditions (3.7)–(3.8). Since for each k the system (3.4)–(3.5) with the boundary conditions (3.7)–(3.8) has a unique solution $x_k(r)$, we conclude that (2.17) together with the boundary conditions (3.19)–(3.20) over the disk has the unique solution $x(\vec{r})$ defined in (2.1).

Finally, observe that since $\eta_k(R)$ depends only on the values of $u_k(r)$ with $r > R$, (3.30) implies that

$$E[\beta(R,\theta)u^T(r,\phi)] = 0, \qquad \text{for } r < R. \quad \square \tag{3.31}$$

IV. The Smoothing Problem

A. Problem Statement

Let

$$y(\vec{r}) = z(\vec{r}) + v(\vec{r}), \qquad \vec{r} \in D_R \tag{4.1}$$

with $D_R = \{\vec{r}: r \le R\}$, be noisy observations of the isotropic field $z(\vec{r})$ defined by the model (2.1)–(2.2). Here, $v(\vec{r})$ is a two-dimensional white Gaussian noise field of dimension p uncorrelated with $u(\vec{r})$ and $\beta(R,\theta)$, and with intensity V, where V is a positive definite matrix. Thus,

$$E[v(\vec{r})u^T(\vec{s})] = 0, \tag{4.2}$$

$$E[v(\vec{r})\beta^T(R,\theta)] = 0, \tag{4.3}$$

$$E[v(\vec{r})v^T(\vec{s})] = V\delta(\vec{r}-\vec{s}), \tag{4.4}$$

where $\delta(\vec{r})$ denotes a two-dimensional delta function. The estimation problem that we consider here consists in computing the conditional mean

$$\hat{z}(\vec{r}|R) = E[z(\vec{r})|y(\vec{s}): 0 \le s \le R], \qquad \text{for all } \vec{r} \in D_R. \tag{4.5}$$

Following [8], we shall solve this smoothing problem using Fourier series expansions of the observation, signal and observation and process noise processes. Substituting the Fourier series expansions of $y(\cdot)$, $z(\cdot)$, and $v(\cdot)$ into (3.1) yields

$$y_k(r) = z_k(r) + v_k(r), \qquad 0 \le r \le R. \tag{4.6}$$

Since the Fourier coefficient process of different orders are uncorrelated our original two-dimensional estimation problem requires only the solution of a countable set of decoupled 1-D smoothing problems for the Fourier coefficient process $z_k(r)$ given the observations $y_k(s)$ over the interval $0 \le s \le R$. Once the smoothed estimates $\hat{z}_k(r|R) = E[z_k(r)|y_k(s):\ 0 \le s \le R]$ are found, $\hat{z}(\vec{r}|R)$ may be computed as

$$\hat{z}(\vec{r}|R) = \sum_{k=-\infty}^{\infty} \hat{z}_k(r|R)e^{jk\theta}, \tag{4.7}$$

where the equality in (4.7) is to be understood in the mean-square sense. In practice, of course, one would consider only a finite number N of the previous one-dimensional estimation problems.

B. 1-D Smoothers

Let us now develop a solution to the 1-D TPBV smoothing problems for the Fourier coefficient processes. Our solution is based on a Markovianization procedure followed by standard 1-D smoothing techniques.

The main feature of the TPBV model (3.4)–(3.8) describing the kth-order Fourier coefficient is that it is separable, i.e., the boundary conditions $\xi_k(0)$ and $\eta_k(R)$ are decoupled (cf. [15]). Hence, a Markovian model of the same order as the model (3.4)–(3.8) can be constructed for $x_k(r)$ by reversing the direction of propagation of $\eta_k(r)$ using a technique introduced by Verghese and Kailath [16] for constructing backwards Markovian models. Specifically, if we extract the part $\hat{u}_k(r)$ of $u_k(r)$ that may be estimated from $\{\eta_k(s),\ 0 \le s \le r\}$, we find that

$$\begin{aligned}\hat{u}_k(r) &= E[u_k(r)|\eta_k(s),\quad 0 \le s \le r] \\ &= E[u_k(r)\eta_k^H(r)]E[\eta_k(r)\eta_k^H(r)]^{-1}\eta_k(r) \\ &= -\frac{1}{2\pi}B^TK_k(Ar)\Pi_{\eta_k}^{-1}(r)\eta_k(r),\end{aligned} \tag{4.8}$$

where

$$\Pi_{\eta_k}(r) = \frac{1}{2\pi}\int_r^\infty K_k(As)BB^TK_k^T(As)s\,ds. \tag{4.9}$$

It may then be shown that the process $\tilde{u}_k(r)$ defined as

$$\tilde{u}_k(r) = u_k(r) - \hat{u}_k(r) \tag{4.10}$$

is a white process with the *same* intensity $I_m/2\pi r$ as $u_k(r)$. Substituting (4.10) and (4.8) into (3.4)–(3.8) yields the forwards propagating model:

$$\frac{d}{dr}\begin{bmatrix}\xi_k(r)\\ \eta_k(r)\end{bmatrix} = \begin{bmatrix}0 & G_k(r)\\ 0 & F_k(r)\end{bmatrix}\begin{bmatrix}\xi_k(r)\\ \eta_k(r)\end{bmatrix} + \begin{bmatrix}-rI_k(Ar)B\\ rK_k(Ar)B\end{bmatrix}\tilde{u}_k(r), \tag{4.11}$$

$$y_k = [CK_k(Ar) \quad CI_k(Ar)]\begin{bmatrix}\xi_k(r)\\ \eta_k(r)\end{bmatrix} + v_k(r), \tag{4.12}$$

with

$$E[\tilde{u}_k(r)] = 0, \tag{4.13}$$

$$E[v_k(r)] = 0, \tag{4.14}$$

$$E\left[\begin{bmatrix}\tilde{u}_k(r)\\ v_k(r)\end{bmatrix}[\tilde{u}_k^H(s) \quad v_k^H(s)]\right] = \begin{bmatrix}I & 0\\ 0 & V\end{bmatrix}\frac{\delta(r-s)}{2\pi r}, \tag{4.15}$$

and where

$$G_k(r) = \frac{r}{2\pi}I_k(Ar)BB^TK_k^T(Ar)\Pi_{\eta_k}^{-1}(r) \tag{4.16}$$

and

$$F_k(r) = -\frac{r}{2\pi}K_k(Ar)BB^TK_k^T(Ar)\Pi_{\eta_k}^{-1}(r). \tag{4.17}$$

The initial conditions for the state-space model (4.11) at $r = 0$ are given by

$$\begin{bmatrix}\xi_k(r)\\ \eta_k(r)\end{bmatrix} \sim N(0, \Pi_k(0)) \tag{4.18}$$

with

$$\Pi_k(0) = \begin{bmatrix}0 & 0\\ 0 & \Pi_{\eta_k}(0)\end{bmatrix}, \tag{4.19}$$

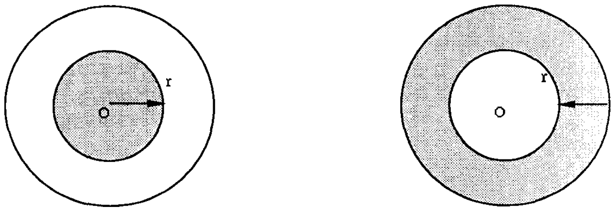

Fig. 1. Outgoing and incoming radial recursions.

where we have used the fact that

$$E[\xi_k(0)\eta_k^H(0)] = 0. \tag{4.20}$$

Here, $\eta_k^H(r)$ denotes the complex conjugate transpose of $\eta_k(r)$. The smoothing problem associated with the system (4.11)–(4.12) over $[0, R]$ is a standard causal smoothing problem and can be solved using any of the 1-D smoothing techniques such as the Mayne–Fraser two-filter formula [17]–[18], or the Rauch–Tung–Striebel formula [19], among others.

Note that the TPBV model (3.4)–(3.8) is well posed in the sense of [20], since $z_k(r)$ can be expressed uniquely in terms of $u_k(r)$ and $\eta_k(R)$. Furthermore, observe that $\eta_k(R)$ is independent of $u_k(r)$ for $r \leq R$. Thus, we could have directly applied the results of [20] to the TPBV model (3.4)–(3.8) to obtain the Hamiltonian TPBV system that governs the smoothed estimates of $\xi_k(r)$ and $\eta_k(r)$, $\hat{\xi}_k(r)$ and $\hat{\eta}_k(r)$. Conceptually, the difference between the approach that we presented and that of [20] lies in the way they deal with the boundary conditions for the smoother. In the method of Adams *et al.*, the boundary conditions are replaced initially by zero boundary conditions and a two-filter smoothing formula with simple dynamics is used. Once all the measurements $y_k(r)$ have been processed, a second step is required to take the true boundary conditions into account. On the other hand, the Markovianization approach deals with the boundary conditions directly as the measurements are processed. It does so by properly incorporating the boundary conditions into the dynamics of the estimator, a step that results in a more complicated smoother implementation.

Once the smoothed estimates $\hat{\xi}_k(r)$ and $\hat{\eta}_k(r)$ have been computed for all k, the smoothed estimate $\hat{z}(\vec{r}|R)$ of $z(\vec{r})$ can be found as

$$\hat{z}(\vec{r}|R) = \sum_{k=-\infty}^{\infty} C\big(K_k(Ar)\hat{\xi}_k(r) + I_k(Ar)\hat{\eta}_k(r)\big)e^{jk\theta}. \tag{4.21}$$

Finally, as noted earlier, the two efficient processing schemes that we have developed for estimating isotropic random fields of the form (2.1)–(2.2) or (2.17)–(2.18) are based on a concept of causality where the data is processed outwards or inwards with respect to a disk of observation as shown in Fig. 1. Observe that this concept of causality follows naturally from the special geometrical structure of isotropic random fields.

V. Asymptotic Behavior of the Differential Models at Infinity

The Fourier coefficient processes $x_n(r)$ have a finite variance for all $r \in \boldsymbol{R}$ since by definition $x(\vec{r})$ has finite variance over the whole plane (see Section II-A.) Hence, the optimal estimator for the Fourier coefficient process $x_n(r)$ written in integral form, must have a well-behaved kernel for all r. However, the matrices appearing in (3.4)–(3.6) and (4.11)–(4.12) are *not* well behaved as r tends to zero or infinity. This ill-behavior is due to the singularity of $K_k(Ar)$ and $I_k(Ar)$ as r tends to zero and infinity respectively [13]. Furthermore, model (4.11)–(4.12) defines a singular estimation problem as r tends to infinity. This follows from the fact that the intensity of the noise processes $u_k(r)$, $\tilde{u}_k(\cdot)$ and $v_k(\cdot)$ varies as r^{-1}. The singularity of the model (4.11)–(4.12) as r tends to zero is of no practical consequence and a strategy for dealing with it is briefly discussed in [21]. Here, we introduce differential models for the Fourier coefficient processes that are well behaved as r tends to infinity.

A. Models

The models that we develop are obtained by applying the state transformation

$$\chi_k(r) = T_k(r)\begin{bmatrix} \xi_k(r) \\ \eta_k(r) \end{bmatrix} \tag{5.1}$$

$$T_k(r) = \begin{bmatrix} K_k(Ar) & I_k(Ar) \\ -K_{k+1}(Ar) & I_{k+1}(Ar) \end{bmatrix} \tag{5.2}$$

to model (4.11)–(4.12), followed by a normalization of all the processes. The normalization consists in multiplying all processes by $r^{1/2}$, which forces the intensity of the noise processes to be a constant.

Note that by using (3.5) we can identify

$$\chi_k(r) = \begin{bmatrix} x_k(r) \\ A^{-1}\left(\dfrac{d}{dr}x_k(r) - \dfrac{k}{r}x_k(r)\right) \end{bmatrix}. \tag{5.3}$$

Note also that the transformation $T_k(r)$ has the properties that

$$\frac{d}{dr}T_k(r) = \begin{bmatrix} \frac{k}{r}I & A \\ A & -\frac{(k+1)}{r}I \end{bmatrix} T_k(r), \tag{5.4}$$

$$T_k^{-1}(r) = r\begin{bmatrix} AI_{k+1}(Ar) & -AI_k(Ar) \\ AK_{k+1}(Ar) & AK_k(Ar) \end{bmatrix}. \tag{5.5}$$

Identities (5.4)–(5.5) can be derived by using the recurrence relations for modified Bessel functions [13] and the Wronskian identity [13]

$$I_{k+1}(Ar)K_k(Ar) + I_k(Ar)K_{k+1}(Ar) = A^{-1}r^{-1}. \tag{5.6}$$

If we apply the state transformation $T_k(r)$ to the model (4.11)–(4.12) and if we introduce the *normalized processes*

$$\bar{\alpha}_k(r) = \sqrt{r}\,\alpha_k(r), \tag{5.7}$$

where $\alpha_k(r)$ stands for $\chi_k(r)$, $u_k(r)$, $y_k(r)$ or $v_k(r)$, we obtain

$$\frac{d}{dr}\bar{\chi}_k(r) = \left(A_k'(r) + \frac{I}{2r}\right)\bar{\chi}_k(r) + \bar{B}\bar{u}_k(r), \tag{5.8}$$

$$\bar{y}_k(r) = \bar{C}\bar{\chi}_k(r) + \bar{v}_k(r), \tag{5.9}$$

where

$$A_k'(r) = \begin{bmatrix} \frac{k}{r}I & A \\ A + D_k(r) & -\frac{(k+1)}{r}I + E_k(r) \end{bmatrix}, \tag{5.10}$$

$$D_k(r) = -\frac{r}{2\pi}A^{-1}BB^TK_k^T(Ar)\Pi_{\eta_k}^{-1}(r)K_{k+1}(Ar)A, \tag{5.11}$$

$$E_k(r) = -\frac{r}{2\pi}A^{-1}BB^TK_k^T(Ar)\Pi_{\eta_k}^{-1}(r)K_k(Ar)A, \tag{5.12}$$

and where $\bar{u}_k(r)$ and $\bar{v}_k(r)$ are two uncorrelated zero-mean Gaussian noise processes with intensities $I/2\pi$ and $V/2\pi$ respectively. Note that this implies that (5.8)–(5.9) defines a nonsingular estimation problem.

Let us now make two comments. First note that the transformation $T_k(r)$, its inverse $T_k^{-1}(r)$ and the normalization gain $r^{1/2}$ blow up as r tends to infinity. (The transformation $T_k(r)$ and its inverse $T_k^{-1}(r)$ blow up as r tends to infinity because of the singularity of the matrix functions $I_k(Ar)$ as r tends to infinity.) However, the normalized processes that appear in (5.8)–(5.9) are well behaved and have a finite variance as r tends to infinity. In fact, by using the asymptotic forms of $K_k(Ar)$ and $I_k(Ar)$ as r tends to infinity (cf. Appendix A) and (5.1), it can be shown that the process $\chi_k(r)$ has a variance that tends to zero as r^{-1} as r tends to infinity. Furthermore, recall that the intensity of the noise processes $u_k(r)$ and $v_k(r)$ is also proportional to r^{-1}. Hence, the variance of all the Fourier coefficient processes tends to zero as r^{-1} as r tends to infinity. This is precisely the reason why we have to keep a very large number of terms in (4.21) to obtain meaningful results as r tends to infinity. Note that this also implies that all the normalized processes are well behaved with variances and noise intensities that tend to a finite constant as r tends to infinity.

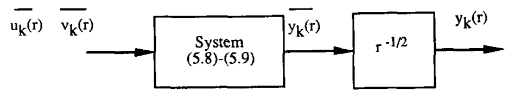

Fig. 2. Model for $y_k(r)$ for large values of r.

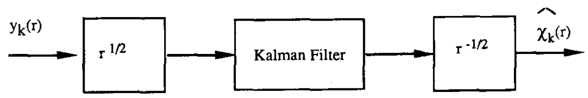

Fig. 3. Filtering procedure for large values of r.

Second, observe that the model (5.8)–(5.9) shows that we can interpret the Fourier coefficient process $y_k(r)$ as being the output of a cascaded system which is driven by the nonsingular noise processes $\bar{u}_k(r)$ and $\bar{v}_k(r)$. The cascaded system consists of a system that is well behaved as r tends to infinity followed by a gain stage with a gain of $r^{-1/2}$, as shown in Figs. 2 and 3.

B. Asymptotic Behavior

To study the asymptotic behavior of model (5.8)–(5.9) as r tends to infinity, we note that, as r tends to infinity, the modified Bessel functions $K_k(Ar)$ and $I_k(Ar)$ have the asymptotic forms [13]

$$I_k(Ar) \sim (2\pi Ar)^{-1/2}e^{Ar}, \tag{5.13}$$

$$K_k(Ar) \sim (2Ar/\pi)^{-1/2}e^{-Ar}. \tag{5.14}$$

Hence, if we assume that the pair (A, B) is controllable, we obtain

$$\begin{aligned} \lim_{r\to\infty} D_k(r) &= \lim_{r\to\infty} -A^{-1}BB^Te^{-A^Tr} \\ &\quad\cdot\left(\int_r^{\infty} e^{-As}BB^Te^{-A^Ts}\,ds\right)^{-1} e^{-Ar}A \\ &= -A^{-1}BB^TQ^{-1}A \\ &= D, \end{aligned} \tag{5.15}$$

where Q is the matrix

$$Q = \int_0^{\infty} e^{-As}BB^Te^{-A^Ts}\,ds. \tag{5.16}$$

Note that since $-A$ is a stable matrix and since the pair (A, B) is controllable then Q is the unique positive definite solution of the matrix equation [22]

$$-AQ - QA^T + BB^T = 0. \tag{5.17}$$

Similarly, we have

$$\lim_{r\to\infty} E_k(r) = D. \tag{5.18}$$

Thus, as r tends to infinity the Markovian model (5.8)–(5.9) takes the form

$$\frac{d}{dr}\bar{\chi}_k(r) = \bar{A}'\bar{\chi}_k(r) + \bar{B}\bar{u}_k(r), \tag{5.19}$$

$$\bar{y}_k(r) = \bar{C}\bar{\chi}_k(r) + \bar{v}_k(r), \tag{5.20}$$

where

$$\bar{A}' = \begin{bmatrix} 0 & A \\ A+D & D \end{bmatrix}. \tag{5.21}$$

Note that the asymptotic model (5.19)–(5.20) implies that the model (5.8)–(5.9) is well behaved as r tends to infinity. Note also that the asymptotic model (5.19)–(5.20) is space invariant and does *not* depend on the order k of the Fourier coefficient process under consideration. This reflects the fact that as r tends to infinity all the Fourier coefficient processes have an equal importance in the sense that we would have to retain a very large number of terms in (4.21) to obtain meaningful results, as was already observed. This also implies that for large values of r we can use the *same* filter to obtain smoothed estimates of all Fourier coefficients.

C. Stable Spectral Factorizations

Model (5.19)–(5.20) provides a *stable spectral factorization* of $S_x(\lambda)$. In particular, observe that the transfer function associated with (5.19) is

$$\begin{aligned} W_f(s) &= A(sI+A)^{-1}(sI - A + A^{-1}BB^TQ^{-1}A)^{-1}A^{-1}B \\ &= (sI+A)^{-1}(sI - A + BB^TQ^{-1})^{-1}B. \end{aligned} \tag{5.22}$$

The formula

$$-A + BB^TQ^{-1} = QA^TQ^{-1} \tag{5.23}$$

(which is easily derived from (5.17)) now shows that $-A + BB^TQ^{-1}$ and A have the same eigenvalues. Therefore, $W_f(s)$ will have its poles in the left half-plane since all the eigenvalues of A have a positive real part by assumption. Note that this also implies that the matrix $\bar{A}'$ is a stable matrix. Furthermore, observe that

$$W_f(s)U(s) = W_b(s), \tag{5.24}$$

where

$$\begin{aligned} W_b(s) &= (sI+A)^{-1}(sI-A)^{-1}B \\ &= (s^2I - A^2)^{-1}B \\ &= (s^2I - M)^{-1}B, \end{aligned} \tag{5.25}$$

and

$$U(s) = I + B^TQ^{-1}(sI-A)^{-1}B. \tag{5.26}$$

It is easy to verify that $U(s)$ is a paraunitary or allpass transfer function in the sense that

$$U(s)U^T(-s) = U^T(-s)U(s) = I. \tag{5.27}$$

Hence, we have

$$\begin{aligned} W_f(s)W_f^T(-s) &= W_b(s)W_b^T(-s) \\ &= (sI+A)^{-1}(sI-A)^{-1} \\ &\quad \cdot BB^T(-sI-A^T)^{-1}(-sI+A^T)^{-1} \\ &= S_x(\lambda)|_{\lambda=-js}, \end{aligned} \tag{5.28}$$

which proves that the asymptotic model (5.19)–(5.20) does lead to a stable spectral factorization of $S_x(\lambda)$. In particular, note that (5.28) means that even though the Markovian models (5.8)–(5.9) of different order Fourier coefficients converge at different rates (which are functions of k) to the asymptotic model (5.19)–(5.20), *each* does asymptotically lead to a stable factorization of $S_x(\lambda)$. In other words, it suffices to consider the asymptotics of the Markovian model of *any* Fourier coefficient to obtain a stable spectral factorization of $S_x(\lambda)$. Note also that the results of [16] imply that $(sI - A + BB^TQ^{-1})^{-1}B$ is the transfer function of a stable *forward* Markovian model corresponding to the stable *backwards* Markovian model with transfer function $(sI-A)^{-1}B$.

D. Stability Analysis

Finally, observe that according to the previous subsection all the eigenvalues of the matrix $\bar{A}'$ lie in the left half s-plane. Thus, the asymptotic model (5.19)–(5.20) is exponentially stable. Hence, by direct application of Theorem 4.11 of [23], we obtain the following asymptotic stability result for the Kalman filter associated with (5.19)–(5.20).

Theorem 5: The Kalman filter associated with the model (5.19)–(5.20) is asymptotically stable. Furthermore, the error covariance associated with the normalized process $\bar{\chi}_k(r)$ converges to a non-negative definite matrix $\bar{P}$ as r tends to infinity, where $\bar{P}$ is the solution of the algebraic Riccati equation

$$0 = \bar{A}'\bar{P} + \bar{P}\bar{A}'^T + \bar{B}\bar{B}^T - \bar{P}\bar{C}^TV^{-1}\bar{C}\bar{P}, \tag{5.29}$$

where matrix $\bar{A}'$ is defined in (5.21).

VI. Conclusion

In this paper we have obtained efficient recursive estimation techniques for isotropic random fields described by noncausal internal differential realizations. By exploiting the properties of isotropic random fields, we showed that the problem of estimating an isotropic random field given noisy observations over a finite disk of radius R is equivalent to a countably infinite set of decoupled one-dimensional two-point boundary value system (TPBV) estimation problems for the Fourier coefficient processes of the random field. We then solved the 1-D TPBV estimation problems using a Markovianization approach followed by standard 1-D smoothing techniques. The smoothing schemes that we have developed result in a processing structure that is recursive with respect to the radius r in a polar coordinate representation of the field. A brief discussion of the numerical implementation of the smoother derived in Section IV may be found in [21]. We have also studied the asymptotic behavior of the Markovian models that we developed as the radius R of the disk of observation tends to infinity. In particular, we have shown that the asymptotic form of all models is the same and is stable and spatially invariant. Furthermore, the asymptotic model led to a stable spectral factorization for the original 2-D signal.

Observe that the approach that we have used to solve the smoothing problem for isotropic random fields has elements of the approaches that use a full Karhunen–Loeve expansion of the 2-D field and those that use the values of the field directly. In 1-D and 2-D one can certainly use full Karhunen–Loeve expansions of 1-D processes or 2-D fields to solve estimation problems. The disadvantage of such an approach is that it leads to a nonrecursive scheme for estimating a set of random variables. On the other hand, we expanded a 2-D field in a Karhunen–Loeve expansion in terms of the coordinate angle θ only. This mixed approach lead to a set of 1-D random processes (rather than random variables) that have recursive internal representations. By exploiting those recursive representations we were able to develop the computationally efficient recursive estimation schemes of Section IV.

Note also that the approach that we have used in this paper carries over to the case where the source term $u(\cdot)$ appearing on the right-hand side of (2.1) is not spatially white but has a covariance function that is invariant under rotations only. In particular, it applies to the case where the field $u(\cdot)$ has a covariance function of the form

$$\begin{aligned} E\left[u(\vec{r})u^T(\vec{s})\right] &= K_1(r,s)K_2(\theta-\phi) \\ &= K_1(r,s)\sum_k a_k e^{jk(\theta-\phi)}, \end{aligned} \tag{6.1}$$

where $K_1(r,s)$ is a positive definite function of the variables r and s that is assumed to have a *finite*-dimensional state-space realization. Our approach can also be used in the case where the matrices, A, B, and C of (2.1)–(2.2) are functions of the polar coordinate variable r only. In both of these cases the Fourier coefficients of the processes $x(\cdot)$, $u(\cdot)$, $v(\cdot)$ and $y(\cdot)$ are uncorrelated. However, alternative estimation approaches have to be developed to deal with the case where the source term $u(\cdot)$ has a covariance function that is not invariant under rotations. This latter case is of importance in a number of applications, e.g., in ocean acoustics where the source term $u(\cdot)$ is often homogeneous with a 2-D power spectrum that has an angular dependence only in the wave-number plane.

Appendix

In this paper, we make frequent use of the matrix modified Bessel functions of the first and second kinds, $I_k(Ar)$ and $K_k(Ar)$. These functions are a generalization of the corresponding scalar modified Bessel functions, and they satisfy the matrix differential equation

$$\left(I_n\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{k^2}{r^2}\right)-A^2\right)F(r)=0 \tag{A.1}$$

with the limiting forms

$$I_k(Ar) \sim (k!)^{-1}\left(\frac{Ar}{2}\right)^k, \tag{A.2}$$

$$K_0(Ar) \sim \ln(Ar), \tag{A.3}$$

$$K_k(Ar) \sim \frac{(k-1)!}{2}\left(\frac{Ar}{2}\right)^{-k}, \quad k \geq 1, \tag{A.4}$$

as r tends to zero, and with the asymptotic forms

$$I_k(Ar) \sim (2\pi Ar)^{1/2}e^{Ar}, \tag{A.5}$$

$$K_k(Ar) \sim \left(\frac{2Ar}{\pi}\right)^{-1/2} e^{-Ar}, \tag{A.6}$$

as r tends to infinity. Thus $I_k(Ar)$ and $K_k(Ar)$ are regular at $r=0$, and as r tends to infinity, respectively.

$I_k(Ar)$ and $K_k(Ar)$ have the series expansions

$$I_k(Ar)=\left(\frac{Ar}{2}\right)^k \sum_{n=0}^{\infty} \frac{\left(\frac{Ar}{2}\right)^{2n}}{\Gamma(n+1)\Gamma(n+k+1)}, \tag{A.7}$$

$$\begin{aligned} K_k(Ar) = {} & \frac{1}{2}\left(\frac{Ar}{2}\right)^{-k}\sum_{n=0}^{k-1}(-1)^n\frac{\Gamma(k-n)}{\Gamma(n+1)}\left(\frac{Ar}{2}\right)^{2n} \\ & +(-1)^{k+1}\ln\left(\tfrac{1}{2}Ar\right)I_k(Ar) \\ & +(-1)^k\frac{1}{2}\left(\frac{Ar}{2}\right)^k\sum_{n=0}^{\infty}\{\psi(n+1) \\ & +\psi(n+k+1)\}\frac{\left(\frac{Ar}{2}\right)^{2n}}{\Gamma(n+1)\Gamma(n+k+1)}, \end{aligned} \tag{A.8}$$

where $\Gamma(\cdot)$ is a Gamma function and $\psi(x)=d(\ln(x))/dx$ is the Psi or Digamma function.

Bessel functions have a number of useful properties that are listed in [13].

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