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REALIZATION OF ACAUSAL WEIGHTING PATTERNS WITH BOUNDARY-VALUE DESCRIPTOR SYSTEMS*

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Abstract. This paper examines the realization of acausal weighting patterns with two-point boundary-value descriptor systems (TPBVDSs). Attention is restricted to the subclass of TPBVDSs that are *stationary*, so that their input-output weighting pattern is shift-invariant, and *extendible*, i.e., their weighting pattern can be extended outwards indefinitely. Then, given an infinite acausal shift-invariant weighting pattern, the realization problem consists of constructing a minimal TPBVDS over a fixed interval, whose extended weighting pattern matches the given pattern. The realization method that is proposed relies on a new transform, the (s, t)-transform, which is better adapted to the analysis of descriptor dynamics than the standard z-transform, since it handles zero and infinite frequencies in a symmetric way. This new transform is used to determine the dimension of a minimal realization, and then to construct a minimal realization by obtaining state-space representations for two homogeneous rational matrices in s and t obtained from the causal and anticausal components of the weighting pattern.

Key words. acausal weighting pattern, boundary-value descriptor system, realization theory, (s, t) transform, McMillan degree

AMS(MOS) subject classifications. 93B15, 93B20

1. Introduction. There exists an extensive literature [1]-[4] on the state-space realization problem for linear time-invariant causal systems, i.e., for systems which admit an input-output description of the form

$$y(k)=\sum_{l=-\infty}^{\infty} W(k-l)u(l), \tag{1.1}$$

where the impulse response (weighting pattern) $W(.)$ satisfies

$$W(k)=0 \quad \text{for } k\leqq 0. \tag{1.2}$$

However, for many physical systems, in particular when the independent variable is space rather than time, the causality condition (1.2) does not hold. For example, if we consider the temperature of a heated rod, there is no reason to assume that the temperature at any point of the rod depends exclusively on the applied heat on one side of that point. Weighting patterns that do not satisfy (1.2) are called acausal. The objective of this paper is to develop a realization theory for acausal weighting patterns in terms of *two-point boundary-value descriptor systems* (TPBVDSs) of the form

$$Ex(k+1)=Ax(k)+Bu(k), \qquad 0\leqq k\leqq N-1, \tag{1.3}$$

with boundary condition

$$V_i x(0)+V_f x(N)=v, \tag{1.4}$$

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and output

$$y(k)=Cx(k), \qquad 0 \leqq k \leqq N. \tag{1.5}$$

The motivation for considering this class of systems is that the discrete-time descriptor dynamics (1.3) are noncausal, in the sense that they contain components which propagate in both time directions [5]. The boundary conditions (1.4) are another source of noncausality, since they are expressed symmetrically in terms of the system variables at both ends of the interval $[0, N]$. Thus, TPBVDSs have a totally acausal structure which is ideally suited to model noncausal systems [6]-[8]. Motivated by the earlier work of Krener [9]-[10], and Gohberg, Kaashoek, and Lerer [11]-[13] for boundary-value systems with standard nondescriptor dynamics, a complete system theory of TPBVDSs has been developed in [14]-[18], including concepts such as reachability, observability, and minimality. In this paper, we restrict our attention to stationary and extendible TPBVDSs, namely TPBVDSs whose weighting pattern is shift-invariant, and where the interval of definition $[0, N]$ of the TPBVDS can be extended outwards indefinitely, without changing the weighting pattern. This extension process yields an extended weighting pattern $W(k)$ defined for all $k \in \mathbf{Z}$, where the weighting pattern of the original TPBVDS and of all its extensions are restrictions of $W(k)$.

The realization problem that we consider can be stated as follows. Given a weighting pattern $W(k)$, construct a minimal TPBVDS over a sufficiently large interval $[0, N]$, which has $W(k)$ as its extended weighting pattern. As for causal time-invariant systems, where the z-transform plays a useful role in transforming the realization problem into a state-space representation problem for proper rational matrix transfer functions, it is shown that the TPBVDS realization problem can be formulated in the frequency domain as a state-space representation problem for rational transfer functions. However, instead of using the z-transform, we introduce a new transform, the (s, t)-transform, which handles zero and infinite frequencies symmetrically, and is therefore well adapted to the analysis of descriptor systems. Specifically, the (s, t)-transform of a matrix sequence $H(k)$ is defined as

$$H(s, t)=\sum_{k=-\infty}^{\infty} H(k) t^{k-1} / s^{k}. \tag{1.6}$$

Because of its special structure, $H(s, t)$ is strictly proper when viewed as a function of both s and t, but not necessarily strictly proper in s and t separately. When $H(s, t)$ is rational, this last observation leads us to construct minimal state-space representations of the form

$$H(s, t)=K(sD-tF)^{-1} G, \tag{1.7}$$

where the descriptor dynamics appearing in (1.7) generalize the causal dynamics that are usually employed for strictly proper rational matrices in z.

The (s, t) transform is used here to characterize the dimension of TPBVDS realizations in terms of the McMillan degree of rational matrices in s and t, and to formulate the TPBVDS realization problem as a state-space realization in the (s, t)-domain. More precisely, if $W_f(s, t)$ and $W_b(s, t)$ denote the (s, t)-transforms of the causal and anticausal parts of the weighting pattern $W(k)$, and if

$$W(s, t)=W_f(s, t)+W_b(s, t), \tag{1.8a}$$

$$H_r(s, t)=[W_f(s, t)\, W_b(s, t)], \qquad H_o(s, t)=\begin{bmatrix} W_f(s, t) \\ W_b(s, t) \end{bmatrix}, \tag{1.8b}$$

it is shown that minimal TPBVDS realizations of the extended weighting pattern $W(k)$ have dimension

$$n = \omega + \rho - \tau, \tag{1.9}$$

where ω, ρ, and τ denote the McMillan degrees of $H_r(s, t)$, $H_o(s, t)$, and $W(s, t)$, respectively. We also develop a minimal realization procedure, which relies on constructing minimal state-space representations of the form (1.7) for *both* $H_r(s, t)$ and $H_o(s, t)$. The reason why it is necessary to construct state-space representations for two rational matrices, instead of one for the causal case, is that the TPBVDS realization problem requires finding descriptor dynamics (1.3) and boundary conditions (1.4), which together realize $W(k)$. It is the search for boundary conditions that makes the TPBVDS realization problem significantly harder than the causal problem.

This paper is organized as follows. In § 2, we review several results concerning the stationarity, minimality, and extendibility of TPBVDSs that will be used later. It is shown in § 3 that the effect of the boundary conditions on the extended weighting pattern of the system can be characterized completely by a single matrix, called the decomposition matrix, which appears as a parameter of both the causal and anticausal parts of $W(k)$. This matrix simplifies significantly the presentation of our realization results. In § 4, we examine a direct but naive TPBVDS realization procedure consisting in constructing separate minimal realizations of the causal and anticausal components of $W(k)$. Although the resulting realization is generally nonminimal, it is minimal when the weighting pattern $W(k)$ is summable. Furthermore, it yields necessary and sufficient conditions for the realizability of acausal weighting patterns. The (s, t)-transform is introduced in § 5 and is used to formulate the TPBVDS realization problem in the frequency domain. A method for constructing minimal state-space representations of the form (1.7) for rational matrices in s and t is also presented. Finally, § 6 contains the two main results of our paper, namely the characterization (1.9) for the dimension of a minimal realization, and a minimal TPBVDS realization procedure in the frequency domain.

2. Two-point boundary-value descriptor systems. In this section, we review several properties of TPBVDSs, such as stationarity, minimality, and extendibility, that will be needed in the development of our TPBVDS realization procedure.

2.1. Model description. Consider a linear time-invariant TPBVDS of the form (1.3)-(1.5), where x and v are n-dimensional, u is m-dimensional, y is p-dimensional, and E, A, B, and C are constant matrices. We assume that the length N of the interval of definition satisfies $N \geqq 2n$, so that all modes can be excited and observed. In [14] it was shown that if the system (1.3)-(1.4) is well posed, by left multiplication of (1.3) and (1.4) with invertible matrices, we can bring this system to the following *normalized form*, where there exists scalars α and β such that

$$\alpha E + \beta A = I \tag{2.1}$$

(this is referred to as the *standard form* for the pencil $\{E, A\}$), and

$$V_i E^N + V_f A^N = I. \tag{2.2}$$

Note that (2.1) implies that E and A commute, that E, A, and the system have a common set of eigenvectors,[1] and that $\{E^k, A^k\}$ is a regular pencil for all $k \geqq 0$. Another

[1] v is an eigenvector of the system if $v \neq 0$ and for some σ, $(\sigma E - A)v = 0$. σ is called an eigenmode of the system; for descriptor systems σ can be ∞.

consequence of (2.1) is that the space of matrices $\{A^K E^L;\ K, L \geqq 0\}$ is spanned by the n matrices $\{A^k E^{n-1-k};\ 0 \leqq k \leqq n-1\}$. This last result, which was derived in [14], is a generalization of the Cayley-Hamilton theorem to matrix pencils in the standard form (2.1). We assume throughout this paper that (2.1) and (2.2) hold.

As derived in [14], the map from $\{\mathbf{u}, v\}$ to $\mathbf{x}$ has the form:

$$x(k) = A^k E^{N-k} v + \sum_{l=0}^{N-1} G(k, l) B u(l), \tag{2.3}$$

where the Green function $G(k, l)$ is given by

$$G(k, l) = \begin{cases} A^k[A - E^{N-k}(V_i A + \omega V_f E) E^k] E^{l-k} A^{N-l-1} \Gamma^{-1}, & l \geqq k, \\ E^{N-k}[\omega E - A^k(V_i A + \omega V_f E) A^{N-k}] E^l A^{k-l-1} \Gamma^{-1}, & l < k, \end{cases} \tag{2.4}$$

and where ω is any number such that

$$\Gamma = \omega E^{N+1} - A^{N+1} \tag{2.5}$$

is invertible.

The map from inputs $\mathbf{u}$ to outputs $\mathbf{y}$ specifies the weighting pattern W of the system. Setting $v = 0$ in (2.3), we obtain

$$y(k) = \sum_{l=0}^{N-1} W(k, l) u(l), \tag{2.6}$$

with

$$W(k, l) = C G(k, l) B. \tag{2.7}$$

2.2. Stationarity. In contrast with the causal case, where time-invariant state-space models have a time-invariant impulse response, the weighting pattern $W(k, l)$ given by (2.7) is not, in general, a function of the difference $k - l$. TPBVDSs that have this property are called *stationary*.

THEOREM 2.1 [15]. *The* TPBVDS (1.3)-(1.5) *is stationary if and only if*

$$O_s[V_i, E] R_s = O_s[V_i, A] R_s = 0, \tag{2.8a}$$

$$O_s[V_f, E] R_s = O_s[V_f, A] R_s = 0, \tag{2.8b}$$

where $[X, Y]$ *denotes the commutator product of* X *and* Y

$$[X, Y] = XY - YX \tag{2.9}$$

and

$$R_s = [E^{n-1} B\ A E^{n-2} B \cdots A^{n-1} B], \tag{2.10a}$$

$$O_s^T = [(E^{n-1})^T C^T\ (A E^{n-2})^T C^T \cdots (A^{n-1})^T C^T]. \tag{2.10b}$$

The matrices R_s and O_s in (2.10) are the *strong reachability* and *strong observability* matrices of the TPBVDS. If they have full rank, the triplets (E, A, B) and (C, E, A) are said, respectively, to be strongly reachable, and strongly observable (see [14]-[15] for a detailed study of the properties of strong and weak reachability and observability). The stationarity conditions (2.8a) and (2.8b) state that V_i and V_f must commute with E and A, except for parts that are either in the left nullspace of R_s or the right nullspace of O_s. Consequently, if R_s and O_s have full rank, i.e., if the TPBVDS is strongly reachable and strongly observable, V_i and V_f must commute with E and A.

It is easily verified that the weighting pattern of a stationary TPBVDS defined over $[0, N]$ is given by

$$W(k)=\begin{cases} CV_iA^{k-1}E^{N-k}B, & 1\leqq k\leqq N, \\ -CV_fE^{-k}A^{N+k-1}B, & 1-N\leqq k\leqq 0. \end{cases} \tag{2.11}$$

2.3. Minimality. Since our goal is to realize shift-invariant acausal weighting patterns with stationary TPBVDSs, we need to be able to determine whether or not a system in this class is minimal. This issue was examined in [15] and [18], leading to the following definition and characterization of minimality.

DEFINITION 2.1. A TPBVDS is *minimal* if its state x has the lowest dimension among all TPBVDSs having the same weighting pattern.

THEOREM 2.2. *The stationary* TPBVDS (1.3)-(1.5) *is minimal if and only if*

(2.12a) $[V_iR_s \quad V_fR_s]$ *has full row rank,*

(2.12b) $\begin{bmatrix} O_sV_i \\ O_sV_f \end{bmatrix}$ *has full column rank,*

(2.12c) $\ker(O_s)\subset \mathrm{im}\,(R_s)$.

It was also shown in Corollary 5.1 of [15] that Theorem 2.2 implies the following corollary.

COROLLARY. *Let* $(C_j, V_i^j, V_f^j, E_j, A_j, N)$ *with* $j=1,2$ *be two minimal and stationary realizations of the same weighting pattern, where* $\{E_j, A_j\}$, $j=1,2$ *are in standard form for the same* α *and* β. *Then, there exists an invertible matrix* T *such that*

$$B_2=TB_1,\qquad C_2=C_1T^{-1}, \tag{2.13a}$$

$$O_s^1(V_i^1-T^{-1}V_i^2T)R_s^1=O_s^1(V_f^1-T^{-1}V_f^2T)R_s^1=0, \tag{2.13b}$$

and

$$(A_1-T^{-1}A_2T)R_s^1=(E_1-T^{-1}E_2T)R_s^1=0, \tag{2.13c}$$

$$O_s^1(A_1-T^{-1}A_2T)=O_s^1(E_1-T^{-1}E_2T)=0, \tag{2.13d}$$

where R_s^1 *and* O_s^1 *are the strong reachability and observability matrices for system* 1.

2.4. Extendibility. The concept of extendibility was introduced in [15] for stationary TPBVDSs. It was later extended to nonstationary TPBVDSs in [18]. In this paper, we shall consider only the stationary case.

DEFINITION 2.2. The stationary TPBVDS (1.3)-(1.5) is *extendible* (or input-output extendible) if given any interval $[K, L]$ containing $[0, N]$, there exists a stationary TPBVDS over this larger interval with the same dynamics as in (1.3), but with new boundary matrices $V_i(K, L)$ and $V_f(K, L)$ such that the weighting pattern $W_N(k)$ of the original system is the restriction of the weighting pattern $W_{L-K}(k)$ of the new extended system, i.e.,

$$W_N(k)=W_{K-L}(k)\quad \text{for } 1-N\leqq k\leqq N. \tag{2.14}$$

Our characterization of the property of extendibility for stationary TPBVDSs relies on the notion of Drazin inverse of a matrix [19, p. 8].

DEFINITION 2.3. Let F be an arbitrary square matrix, and let T be an invertible real transformation such that

$$F=T\begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix}T^{-1}, \tag{2.15a}$$

where M is invertible and N is nilpotent. For example, the real Jordan form of F has the above structure. The *Drazin inverse* of F is defined as

$$F^D = T\begin{bmatrix} M^{-1} & 0 \\ 0 & 0 \end{bmatrix} T^{-1}. \tag{2.15b}$$

It can be shown that the Drazin inverse is unique and possesses the following properties:

(i) F^D can be expressed as a polynomial of F, so that it commutes with F. Thus, if a subspace is F-invariant, it is also F^D-invariant.

(ii) When F is invertible, $F^D = F^{-1}$.

(iii) If μ is the degree of nilpotency of N, i.e., if $N^{\mu-1} \neq 0$ and $N^{\mu} = 0$, then for $k \geqq \mu$

$$F^{k+1}F^D = F^k. \tag{2.16}$$

(iv) Let G be any matrix. Then, the condition

$$\ker(F^{\mu}) \subset \ker(G) \tag{2.17a}$$

is equivalent to

$$GFF^D = G. \tag{2.17b}$$

(v) If $\{E, A\}$ is a regular pencil in standard form, we have [18, pp. 33–34]

$$EE^D + AA^D - EE^D AA^D = I. \tag{2.18}$$

The extendibility property can then be characterized as follows.

THEOREM 2.3 [15]. *A stationary* TPBVDS *is extendible if and only if*

$$O_s(V_i - V_i E^D E)R_s = 0, \tag{2.19a}$$

$$O_s(V_f - V_f A^D A)R_s = 0. \tag{2.19b}$$

From conditions (2.19), by using the E-, A-, E^D-, and A^D-invariance of im (R_s) [15] and the generalized Cayley–Hamilton theorem, it is easy to check that for an extendible stationary TPBVDS, the weighting pattern (2.11) can be rewritten as

$$W(k) = \begin{cases} CV_i E^N E^D (AE^D)^{k-1} B, & 1 \leqq k \leqq N, \\ -CV_f A^N A^D (EA^D)^{-k} B, & 1-N \leqq k \leqq 0. \end{cases} \tag{2.20}$$

Given an extendible stationary TPBVDS over $[0, N]$ with weighting pattern $W_N(k)$, it is of interest to ask whether it is possible to extend this TPBVDS in a consistent way over intervals of increasing lengths, i.e., so that this progressive extension process gives rise to a unique extended weighting pattern $W(k)$ defined for all k. A procedure to achieve this objective is given by Theorem 2.4.

THEOREM 2.4. *An extendible stationary* TPBVDS *admits extendible extensions over any interval. Furthermore, the weighting pattern of these extendible extensions is unique.*

Proof. Given an extendible stationary TPBVDS $(C, V_i, V_f, E, A, B, N)$, consider the TPBVDS $(C, \tilde{V}_i, \tilde{V}_f, E, A, B, M)$ defined over an interval of length $M > N$, with

$$\tilde{V}_i = V_i E^N (E^D)^M, \qquad \tilde{V}_f = V_f A^N (A^D)^M. \tag{2.21}$$

It is easy to check that this new TPBVDS is in normalized form, and by using the E-, A-, E^D-, and A^D-invariance of im (R_s), that it is stationary and extendible. According to (2.20), its weighting pattern can be expressed as

$$\tilde{W}_M(k) = \begin{cases} C\tilde{V}_i E^M E^D (AE^D)^{k-1} B, & 1 \leqq k \leqq M, \\ -C\tilde{V}_f A^M A^D (EA^D)^{-k} B, & 1-M \leqq k \leqq 0. \end{cases} \tag{2.22}$$

Substituting (2.21) inside (2.22), and noting from $N > n$ that we have $E^{M+N}(E^D)^M = E^N$ and $A^{M+N}(A^D)^M = A^N$, we find

$$\tilde{W}_M(k) = \begin{cases} CV_iE^NE^D(AE^D)^{k-1}B, & 1 \leqq k \leqq M, \\ -CV_fA^NA^D(EA^D)^{-k}B, & 1-M \leqq k \leqq 0. \end{cases} \tag{2.23}$$

This implies

$$\tilde{W}_M(k) = W_N(k) \quad \text{for } 1-N \leqq k \leqq N, \tag{2.24}$$

so that the TPBVDS $(C, \tilde{V}_i, \tilde{V}_f, E, A, B, M)$ specified by (2.21) is an extension of $(C, V_i, V_f, E, A, B, N)$.

To prove the uniqueness of the extended weighting pattern $\tilde{W}_M(k)$, observe that if $\tilde{W}_M(k)$ is the weighting pattern of an arbitrary extendible extension of TPBVDS (1.3)-(1.5) to an interval of length $M > N$, it can be expressed as (2.22), and satisfies (2.24), so that it is uniquely specified on $[1-N, N]$. Since $N > n$, by applying the standard Cayley-Hamilton theorem to matrices AE^D and EA^D in (2.22), we see that $\tilde{W}_M(k)$ is also uniquely specified on $[N+1, M]$ and $[1-M, -N]$. □

Thus, we can associate to an extendible system a sequence of extendible systems over progressively larger intervals, and with consistent weighting patterns. In this way, we can construct an infinite weighting pattern, called the *extended weighting pattern* of the system, which is such that the weighting pattern of the system and of all its extensions are restrictions of this extended weighting pattern.

From (2.23), the extended weighting pattern of an extendible stationary TPBVDS (1.3)-(1.5) is given by

$$W(k) = \begin{cases} C(V_iE^N)E^D(AE^D)^{k-1}B, & k > 0, \\ -C[I-(V_iE^N)]A^D(EA^D)^{-k}B, & k \leqq 0, \end{cases} \tag{2.25}$$

where we have taken into account the normalization (2.2).

3. Internal description of a weighting pattern. The matrix V_iE^N specifies entirely the effect of the boundary conditions on the extended weighting pattern $W(k)$ given by (2.25). This motivates the introduction of the following concept.

DEFINITION 3.1. Let $(C, V_i, V_f, E, A, B, N)$ be a stationary and extendible TPBVDS. Then P is a *decomposition matrix* of this system if

$$O_sPR_s = O_s(E^NV_i)R_s. \tag{3.1}$$

The motivation for calling P a decomposition matrix is that the extended weighting pattern (2.25) can be expressed as

$$W(k) = \begin{cases} CPE^D(AE^D)^{k-1}B, & k > 0, \\ -C(I-P)A^D(EA^D)^{-k}B, & k \leqq 0. \end{cases} \tag{3.2}$$

Thus, if the identity matrix is decomposed into P and $I-P$, the matrices P and $I-P$ appear as parameters of the causal and anticausal parts of $W(k)$. Also, by using (2.8), (2.19), (3.1), and the fact that $\operatorname{im}(R_s)$ and $\ker(O_s)$ are E- and A-invariant, it is easy to check that a decomposition matrix P satisfies

$$O_s(PA-AP)R_s = O_s(PE-EP)R_s = 0, \tag{3.3a}$$

$$O_s(P-PEE^D)R_s = 0, \tag{3.3b}$$

$$O_s[(I-P)-(I-P)AA^D]R_s = 0. \tag{3.3c}$$

As is clear from Definition 3.1, one particular choice of decomposition matrix is

$$P = V_iE^N. \tag{3.4}$$

This choice is not unique in general. If P is a decomposition matrix, so is $P+Q$, where Q is any matrix such that O_sQR_s equals zero.

The expression (3.2) for the extended weighting pattern $W(k)$ motivates the introduction of the following concept.

DEFINITION 3.2. A five-tuple (C, P, E, A, B) is said to be an *internal description* of the acausal weighting pattern $W(k)$ if it satisfies (3.2) and (3.3), and if $\{E, A\}$ is in standard form. Furthermore, (C, P, E, A, B) is *minimal* if it has the smallest dimension among all internal descriptions of $W(k)$.

Given an acausal weighting pattern $W(k)$, a possible procedure for constructing a minimal, extendible, stationary TPBVDS $(C, V_i, V_f, E, A, B, N)$ that admits $W(k)$ as extended weighting pattern consists therefore in dividing the realization problem into two steps. First, find a minimal internal description (C, P, E, A, B) of $W(k)$. Next, given a finite interval $[0, N]$, find some appropriate boundary matrices V_i and V_f such that the corresponding TPBVDS is extendible and stationary, and such that P is a decomposition matrix associated to these matrices. The following result guarantees the validity of this two-step realization approach.

THEOREM 3.1. *Consider a weighting pattern $W(k)$ with internal description (C, P, E, A, B). Then, for any interval length N, there exists matrices V_i and V_f such that the* TPBVDS $(C, V_i, V_f, E, A, B, N)$ *is normalized, extendible, stationary, and has $W(k)$ as its extended weighting pattern. P is a decomposition matrix of the* TPBVDS $(C, V_i, V_f, E, A, B, N)$. *Furthermore, this* TPBVDS *is minimal if and only if the internal description (C, P, E, A, B) of $W(k)$ is minimal.*

Proof. Let

$$V_i = P(E^D)^N + \sigma X(\sigma E^N + A^N)^{-1}, \tag{3.5a}$$

$$V_f = (I-P)(A^D)^N + X(\sigma E^N + A^N)^{-1}, \tag{3.5b}$$

where

$$X = I - PEE^D - (I-P)AA^D = (I-P)EE^D + PAA^D - EE^DAA^D, \tag{3.6}$$

and where σ is any scalar such that $\sigma E^N + A^N$ is invertible. The second equality in (3.6) is a consequence of identity (2.18). Relations (3.5)-(3.6) specify a TPBVDS $(C, V_i, V_f, E, A, B, N)$. By direct calculation, it is easy to check that V_i and V_f satisfy the normalization (2.2), and that the stationarity and extendibility conditions (2.8) and (2.19) for $(C, V_i, V_f, E, A, B, N)$ are implied, respectively, by the relations (3.3a) and (3.3b)-(3.3c) for (C, P, E, A, B). Noting that

$$O_sXR_s = 0, \tag{3.7}$$

we can also verify that the extended weighting pattern (2.28) associated to $(C, V_i, V_f, E, A, B, N)$ is equal to the weighting pattern $W(k)$ given by (3.2). Finally, taking (3.7) and (3.3b) into account, the matrix V_i given by (3.5a) satisfies

$$O_sV_iE^NR_s = O_sPE^DER_s = O_sPR_s, \tag{3.8}$$

so that P is a decomposition matrix of $(C, V_i, V_f, E, A, B, N)$.

If the internal description (C, P, E, A, B) is not minimal, there exists an internal description (C', P', E', A', B') of smaller dimension, and the above construction yields an extendible stationary TPBVDS realizing $W(k)$, of smaller dimension than $(C, V_i, V_f, E, A, B, N)$, thus showing that this last TPBVDS is not minimal. Conversely, if the TPBVDS $(C, V_i, V_f, E, A, B, N)$ given by (3.5)-(3.6) is not minimal, we can find a lower-dimensional stationary TPBVDS $(C', V_i', V_f', E', A', B', N)$ that is a minimal realization of $W(k)$ over $[0, N]$. According to Corollary 5.2 of [15], this TPBVDS must be extendible and has $W(k)$ for extended weighting pattern. Then $P' = V_i' E'^N$ is a decomposition matrix for this lower-dimensional realization, thus showing that the internal description (C, P, E, A, B) is not minimal. □

Given an internal description (C, P, E, A, B) of the weighting pattern $W(k)$, the following result shows that it is possible to characterize the minimality of this internal description directly, without invoking minimality conditions for an associated TPBVDS.

THEOREM 3.2. *The internal description* (C, P, E, A, B) *of* $W(k)$ *is minimal if and only if*

(3.9a) $R_w = [R_s \quad PR_s]$ *has full row rank,*

(3.9b) $O_w = \begin{bmatrix} O_s \\ O_s P \end{bmatrix}$ *has full column rank,*

(3.9c) $\ker(O_s) \subset \operatorname{im}(R_s)$.

Proof. According to Theorem 3.1, we can associate to (C, P, E, A, B) an extendible stationary TPBVDS $(C, V_i, V_f, E, A, B, N)$, which is minimal if and only if (C, P, E, A, B) is minimal. This TPBVDS is minimal if and only if conditions (2.12) are satisfied. Thus, we need only to show that conditions (2.12) and (3.9) are equivalent. Suppose that conditions (2.12) are satisfied, but (3.9a) is not. Then, there exists $v \neq 0$ such that

$$v^T R_s = 0, \tag{3.10a}$$

$$v^T P R_s = 0. \tag{3.10b}$$

But from (3.10a) and (2.12c) we can conclude that v^T must belong to the row space of O_s. From (3.10b), we find

$$v^T P R_s = v^T V_i E^N R_s = 0. \tag{3.11}$$

Combining (3.10a) and (3.11) yields

$$v^T V_f A^N R_s = 0. \tag{3.12}$$

Since the system is extendible, we have

$$v^T V_i E E^D R_s = v^T V_i R_s, \tag{3.13a}$$

$$v^T V_f A A^D R_s = v^T V_f R_s. \tag{3.13b}$$

But, since R_s is E- and A-invariant, the range spaces of $E^N R_s$ and $A^N R_s$ coincide, respectively, with the ranges of $EE^D R_s$ and $AA^D R_s$. Combining (3.11)–(3.13), we obtain

$$v^T V_i R_s = v^T V_f R_s = 0, \tag{3.14}$$

which contradicts the assumption that (2.12a) is satisfied. Thus, (3.9a) is implied by (2.12). A similar argument can be used to show that (3.9b) is implied by (2.12).

To prove the converse, assume that (3.9) is satisfied and (2.12a) is not. Then, there exists $v \neq 0$ such that

$$v^T V_i R_s = v^T V_f R_s = 0, \tag{3.15}$$

which because of the E- and A-invariance of R_s implies

$$v^T V_i E^N R_s = v^T V_f A^N R_s = 0. \tag{3.16}$$

This in turn implies

$$v^T R_s = 0, \tag{3.17}$$

so that according to (3.9c) v^T belongs to the row space of O_s. Thus,

$$v^T V_i E^N R_s = v^T P R_s = 0. \tag{3.18}$$

But (3.17) and (3.18) contradict (3.9a). Consequently, (2.12a) is implied by (3.9). A similar argument shows that (3.9) implies (2.12b), thus proving the theorem. □

In the following, by analogy with the weak reachability and observability matrices that were introduced in [14] and [15], to characterize the concepts of weak reachability and observability for a TPBVDS $(C, V_i, V_f, E, A, B, N)$, the matrices R_w and O_w appearing in (3.9a) and (3.9b) will be called the *weak reachability* and *weak observability* matrices of the internal description (C, P, E, A, B). As will be shown below, these two matrices play a key role in the construction of a minimal internal description of the weighting pattern $W(k)$.

Theorem 3.2 implies that two minimal internal descriptions of a weighting pattern $W(k)$ can be related as follows.

COROLLARY. *Consider two minimal internal descriptions $(C_j, P_j, E_j, A_j, B_j)$, with $j=1,2$, of the same weighting pattern $W(k)$, which are in standard form for the same α and β. Then, there exists an invertible matrix T such that relations* (2.13a), (2.13c)-(2.13d), *and*

$$O_s^1(P_1 - T^{-1}P_2T)R_s^1 = 0 \tag{3.19}$$

are satisfied.

Proof. According to Theorem 3.1, we can construct two minimal TPBVDSs $(C, V_i^j, V_f^j, E, A, B, N)$ associated to the two given internal descriptions of $W(k)$. Then, there exists an invertible matrix T such that relations (2.13) are satisfied. Consequently, the strong reachability and observability matrices of systems 1 and 2 are related through

$$R_s^2 = TR_s^1, \qquad O_s^2 = O_s^1 T^{-1}. \tag{3.20}$$

From (2.13b) and (3.19), we can deduce that

$$O_s^1 V_i^1 R_s^1 = O_s^2 V_i^2 R_s^2, \tag{3.21}$$

which implies

$$O_s^1 V_i^1 E_1^N R_s^1 = O_s^2 V_i^2 E_2^N R_s^2, \tag{3.22}$$

or equivalently,

$$O_s^1 P_1 R_s^1 = O_s^2 P_2 R_s^2, \tag{3.23}$$

which proves (3.19). □

4. Realizability conditions and separable realization. In § 3, we have reduced the minimal TPBVDS realization problem to the following problem. Given an infinite weighting pattern $W(k)$, find a minimal internal description (C, P, E, A, B) of $W(k)$.

4.1. Realizability conditions. As a first step, we characterize the weighting patterns that admit a finite-dimensional internal description.

THEOREM 4.1. *A sequence $W(k)$ admits a finite-dimensional internal description if and only if there exists scalars a_i, $1 \leqq i \leqq n_f$ and b_i, $1 \leqq i \leqq n_b$ such that*

$$W(n_f + l) = \sum_{i=1}^{n_f} a_i W(n_f + l - i) \quad \textit{for all } l > 0, \tag{4.1a}$$

$$W(-n_b + l) = \sum_{i=1}^{n_b} b_i W(-n_b + l + i) \quad \textit{for all } l \leqq 0. \tag{4.1b}$$

Proof. Necessity is shown by applying the standard Cayley–Hamilton theorem to matrices AE^D and EA^D in (3.2). To prove sufficiency, consider the decomposition

$$W(k) = W_f(k) + W_b(k), \tag{4.2a}$$

$$W_f(k) = W(k)1(k-1), \qquad W_b(k) = W(k)1(-k), \tag{4.2b}$$

of $W(k)$ into its causal and anticausal parts, where $1(k)$ denotes the unit step function, i.e.,

$$1(k) = \begin{cases} 1 & \text{for } k \geqq 0, \\ 0 & \text{for } k < 0. \end{cases}$$

Conditions (4.1a) and (4.1b) imply that $W_f(k)$ and $W_b(k)$ can be realized by finite-dimensional causal and anticausal systems, respectively. Let (C_f, A_f, B_f) and (C_b, A_b, B_b) be such realizations, so that

$$W_f(k) = C_f A_f^{k-1} B_f \quad \text{for } k > 0, \tag{4.3a}$$

$$W_b(k) = C_b A_b^{-k} B_b \quad \text{for } k \leqq 0. \tag{4.3b}$$

Then, it is clear that

$$C = [C_f - C_b], \qquad P = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \tag{4.4a}$$

$$E = \begin{bmatrix} I & 0 \\ 0 & A_b \end{bmatrix}, \quad A = \begin{bmatrix} A_f & 0 \\ 0 & I \end{bmatrix}, \quad B = \begin{bmatrix} B_f \\ B_b \end{bmatrix} \tag{4.4b}$$

is an internal description of $W(k)$, so that the theorem is proved. □

4.2. Separable realization. Let us continue to analyze the realization obtained by decomposing the weighting pattern $W(k)$ into causal and anticausal components, and then constructing minimal realizations for each of these components separately. Given a finite interval $[0, N]$, the internal description (4.4) yields the following extendible stationary TPBVDS realization of $W(k)$:

$$\begin{bmatrix} I & 0 \\ 0 & A_b \end{bmatrix} x(k+1) = \begin{bmatrix} A_f & 0 \\ 0 & I \end{bmatrix} x(k) + \begin{bmatrix} B_f \\ B_b \end{bmatrix} u(k), \tag{4.5a}$$

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} x(0) + \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} x(N) = \begin{bmatrix} v_f \\ v_b \end{bmatrix}, \tag{4.5b}$$

$$y(k) = [C_f - C_b] x(k). \tag{4.5c}$$

An interesting feature of this realization is that it consists of two decoupled subsystems, which propagate, respectively, in the forward and backward directions. An extendible stationary TPBVDS with this structure is called *separable.* Also, observe that in (4.5b) the boundary matrices satisfy $V_i = P$ and $V_f = I - P$, regardless of the interval length. Thus, in the separable case, there is no real distinction between internal descriptions and minimal TPBVDS realizations.

Unfortunately, the separable realization (4.5) is not always minimal, as can be seen from the following example, which is an adaptation of an example presented in [20] for boundary-value systems with standard nondescriptor dynamics.

Example 4.1. Consider the weighting pattern

$$W(k) = \begin{cases} 1, & k \geqq 1, \\ \frac{1}{2}, & k \leqq 0. \end{cases} \tag{4.6}$$

Its separable realization takes the form

$$x_f(k+1)=x_f(k)+u(k), \qquad x_f(0)=v_f, \tag{4.7a}$$

$$x_b(k)=x_b(k+1)+\tfrac{1}{2}u(k), \qquad x_b(N)=v_b, \tag{4.7b}$$

$$y(k)=y_f(k)+y_b(k). \tag{4.7c}$$

However, this realization is not minimal, since W admits also the following one-dimensional realization:

$$x(k+1)=x(k)+u(k), \tag{4.8a}$$

$$y(k)=\tfrac{1}{2}x(k), \tag{4.8b}$$

$$2x(0)-x(N)=v. \tag{4.8c}$$

In this case, the reason we can realize both the causal and anticausal parts of W with a single one-dimensional system is that they have both the same mode, $\sigma=1$. □

In general, when the causal and anticausal parts of W share a common mode, the realization approach consisting in constructing separately minimal realizations of the causal and anticausal parts of W does not yield a minimal realization.

4.3. Summable weighting patterns. Nevertheless, the separable realization (4.5) turns out to be minimal for the class of weighting patterns $W(k)$ that are *summable*, i.e., such that

$$\sum_{k=-\infty}^{\infty} \|W(k)\|<\infty, \tag{4.9}$$

where $\|\cdot\|$ denotes an arbitrary matrix norm. This class of weighting patterns is important, since it corresponds to BIBO stable systems.

THEOREM 4.2. *When the extended weighting pattern $W(k)$ is summable, the separable* TPBVDS *realization* (4.4)-(4.5) *is strongly reachable and observable, and is therefore minimal.*

Proof. Since $W(k)$ is summable, its causal and anticausal parts $W_f(k)$ and $W_b(k)$ are also summable. This implies that the matrices A_f and A_b appearing in the minimal realizations of W_f and W_b are stable, i.e., their eigenvalues are located inside the unit circle. Consider now the matrix

$$[sE-tA\,|\,B]=\begin{bmatrix} sI-tA_f & 0 & B_f \\ 0 & sA_b-tI & B_b \end{bmatrix}. \tag{4.10}$$

It is shown in Theorem 4.1 of [14] that if this matrix has full rank for $(s,t)\neq(0,0)$, the system is strongly reachable. But since A_f and A_b are stable, the eigenmodes of $sI-tA_f$ and sA_b-tI are such that $s/t<1$ and $s/t>1$, respectively, so that these two matrix pencils do not have any common eigenmode. Furthermore, since the state-space realizations (C_f, A_f, B_f) and (C_b, A_b, B_b) are minimal, the submatrices

$$[sI-tA_f\,|\,B_f] \quad \text{and} \quad [sA_b-tI\,|\,B_b]$$

have full rank. This implies that the matrix (4.10) has full rank, so that TPBVDS (4.5) is strongly reachable. By a similar argument, we can show that

$$\begin{bmatrix} sE-tA \\ C \end{bmatrix}=\begin{bmatrix} sI-tA_f & 0 \\ 0 & sA_b-tI \\ C_f & C_b \end{bmatrix} \tag{4.11}$$

has full rank and that the TPBVDS (4.5) is strongly observable. According to Theorem 3.2, the TPBVDS (4.5) is therefore minimal. □

In the remainder of the paper, we will focus our attention on the general case where $W(k)$ is not summable. In this case, minimal realizations are usually not separable. To obtain a minimal realization, two approaches are possible. One method consists in starting from a nonminimal TPBVDS realization, say the separable realization (4.5), and then using the procedure described in [15] for removing the components of this TPBVDS that are not weakly reachable, not weakly observable, or simultaneously not strongly reachable and observable. An alternative realization approach, that we shall follow here, relies on the introduction of a new transform, the (s, t)-transform, and on formulating the realization problem as a state-space representation problem in the (s, t) domain.

5. The (s, t)-transform and state-space representation of rational matrices. One difficulty associated with the use of the z-transform for analyzing discrete-time descriptor systems is that since the dynamics of such systems are singular, infinite frequencies cannot be handled in the same way as other frequencies [21]. This motivates the introduction of the transform

$$H(s, t) = \sum_{k=-\infty}^{\infty} H(k) t^{k-1}/s^{k}. \tag{5.1}$$

It can be expressed in terms of the standard z-transform $H(z)$ as

$$H(s, t) = H(s/t)/t. \tag{5.2}$$

Relation (5.2) shows that the z-transform can be obtained from the (s, t)-transform simply by replacing (s, t) by $(z, 1)$, and conversely, the (s, t)-transform is obtained from the z-transform by replacing z with s/t, and dividing the result by t. From (5.2), we see also that when $H(s, t)$ exists, it has a particular type of homogeneity and is strictly proper in (s, t) in the sense that

$$\lim_{c\to\infty} H(cs, ct) = \lim_{c\to\infty} H(s, t)/c = 0. \tag{5.3}$$

Note, however, that it is not necessarily strictly proper in s and t separately, so that the corresponding z-transform may not be proper.

In the following, we shall restrict our attention to the case when $H(z)$ and $H(s, t)$ are *rational*. Then, from (5.2), we see that the numerator and denominator polynomials of all entries of $H(s, t)$ are *homogeneous*, i.e., each such polynomial has the form

$$p(s, t) = \sum_{i=0}^{d} p_i s^{d-i} t^{i},$$

where d is the degree of p. Furthermore, from (5.3), we see also that the relative degree in s and t of all entries of $H(s, t)$, i.e., the difference between the denominator and numerator degrees is exactly one. Thus, the transformation (5.2) has the effect of transforming rational matrices $H(z)$, proper or not, into strictly proper homogeneous rational matrices in the two variables s and t with relative degree one. The analysis of this paper will focus exclusively on this specific class of rational matrices. Note that the idea of studying the structure at infinity of rational matrices in z through the introduction of a homogenizing transform is not totally new. It has been considered, for example, in [22, pp. 158–162, 182–187] and [23].

5.1. Formulation of the realization problem. In the causal case, the z-transform plays an important role in the solution of the minimal realization problem. Specifically, given a causal weighting pattern $W(k)$, the minimal realization problem is equivalent to finding matrices (C, A, B) of minimal dimension such that the z-transform $W(z)$ admits the *state-space representation*

$$W(z) = C(zI - A)^{-1}B. \tag{5.4}$$

For the case of acausal weighting patterns, the situation is more complex. If (C, P, E, A, B) is an internal description of the weighting pattern $W(k)$, and if $W_f(k)$ and $W_b(k)$ are the causal and anticausal parts of $W(k)$, the (s, t)-transforms of $W_f(k)$ and $W_b(k)$ can be expressed as

$$W_f(s, t) = \sum_{k=1}^{\infty} CPE^D(AE^D)^{k-1}Bt^{k-1}/s^k = CPE^D(sI - tAE^D)^{-1}B, \tag{5.5a}$$

$$\begin{aligned} W_b(s, t) &= \sum_{k=-\infty}^{0} -C(I-P)A^D(EA^D)^kBt^{k-1}/s^k \\ &= C(I-P)A^D(sEA^D - tI)^{-1}B. \end{aligned} \tag{5.5b}$$

Then, we use the matrix identities [19, p. 80]

$$(sE - tA)^{-1} = E^D(sI - tAE^D)^{-1} - A^D(I - EE^D) \sum_{k=0}^{\mu_E - 1} (sEA^D)^k/t^{k+1}, \tag{5.6a}$$

$$(sE - tA)^{-1} = A^D(sEA^D - tI)^{-1} + E^D(I - AA^D) \sum_{k=0}^{\mu_A - 1} (tAE^D)^k/s^{k+1}, \tag{5.6b}$$

where μ_E and μ_A denote the indices of the nilpotent parts of E and A, respectively. Taking into account the properties (3.3a)–(3.3c) of the decomposition matrix P, we obtain

$$W_f(s, t) = CP(sE - tA)^{-1}B = C(sE - tA)^{-1}PB, \tag{5.7a}$$

$$W_b(s, t) = C(I-P)(sE - tA)^{-1}B = C(sE - tA)^{-1}(I-P)B. \tag{5.7b}$$

Note that $W_f(s, t)$ and $W_b(s, t)$ do not have, in general, the same regions of convergence. However, by analytic continuation, it is possible to extend their domains of definition to the whole plane while using the same notation. This yields the three representations:

$$W(s, t) = W_f(s, t) + W_b(s, t) = C(sE - tA)^{-1}B, \tag{5.8}$$

$$H_r(s, t) = [W_f(s, t)\; W_b(s, t)] = C(sE - tA)^{-1}[PB\; (I-P)B], \tag{5.9}$$

$$H_o(s, t) = \begin{bmatrix} W_f(s, t) \\ W_b(s, t) \end{bmatrix} = \begin{bmatrix} CP \\ C(I-P) \end{bmatrix} (sE - tA)^{-1}B. \tag{5.10}$$

Since the specification of an acausal weighting pattern $W(k)$ is equivalent to the specification of $W_f(s, t)$ and $W_b(s, t)$, we see from (5.8)–(5.10) that the construction of an internal description (C, P, E, A, B) of $W(k)$ can be expressed as a state-space realization problem for rational matrices in s and t. However, in contrast to the causal case, the need to specify P and to achieve minimality implies that we must, in general, obtain state-space representations for the *three* rational matrices $W(s, t)$, $H_r(s, t)$, and $H_b(s, t)$, instead of a single rational matrix for causal systems. Furthermore, since we are considering acausal systems, the computation of any of these state-space representations requires an extension of known state-space realization techniques. We consider this problem first in the next section.

5.2. State-space representations of homogeneous rational matrices in s and t. The above discussion motivates the following *minimal state-space representation problem.* Given an homogeneous rational matrix function $H(s, t)$ of relative degree one, find matrices (K, D, F, G) of lowest possible dimension such that

$$H(s, t) = K(sD - tF)^{-1}G. \tag{5.11}$$

This problem is the counterpart of the minimal state-space representation problem for a strictly proper rational matrix $H(z)$, where we seek to find matrices (K, F, G) of smallest size such that

$$H(z) = K(zI - F)^{-1}G. \tag{5.12}$$

The difference between (5.11) and (5.12) is that, as was noted earlier, the one-dimensional rational transfer function $H(z) = H(z, 1)$ associated to (5.11) is not necessarily proper, so that the representation (5.12) is not applicable to this case.

An important feature of the minimal representation (5.12) is that it is unique up to a similarity transform. For the minimal representation (5.11), even if we impose the additional requirement that $\{D, F\}$ should be in standard form, i.e., that there exists α and β such that

$$\alpha D + \beta F = I, \tag{5.13}$$

the matrices (K, D, F, G) are not unique. To ensure uniqueness, α and β must be chosen a priori. In the causal case, i.e., when $H(z)$ is strictly proper, this was done implicitly in (5.12) by forcing D to be equal to I, which corresponds to selecting $\alpha = 1$ and $\beta = 0$. For the more general case that we consider here, any pair (α, β) is acceptable as long as

$$H(\alpha, -\beta) < \infty. \tag{5.14}$$

This last condition can be viewed as an extension of the condition $H(\infty) < \infty$ for proper transfer functions.

THEOREM 5.1. *A matrix function $H(s, t)$ admits a state-space representation* (5.11) *if and only if it is rational, homogeneous in s and t, and with relative degree one. Under these conditions, if (α, β) is a pair of scalars such that $H(\alpha, -\beta)$ exists, $H(s, t)$ admits a unique minimal representation, up to a similarity transform, satisfying* (5.11) *and* (5.13). *The dimension r of this minimal realization, i.e., the size of D and F, is given by*

$$r = d(H(\alpha z, 1 - \beta z)), \tag{5.15}$$

where $d(\cdot)$ denotes the usual McMillan degree, and where $H(\alpha z, 1 - \beta z)$ is a strictly proper rational matrix in z.

Proof. If $H(s, t)$ admits a representation of the form (5.11), it is clear that it must be rational, homogeneous in s and t, and of relative degree one. To prove sufficiency, we need to construct such a representation. Let α and β be such that $H(\alpha, -\beta)$ exists. Then, consider the rational matrix $H(\alpha z, 1 - \beta z)$. This matrix is strictly proper in z because

$$\lim_{z \to \infty} H(\alpha z, 1 - \beta z) = \lim_{z \to \infty} H(\alpha, -\beta)/z = 0. \tag{5.16}$$

It can therefore be realized as

$$H(\alpha z, 1 - \beta z) = K(zI - F)^{-1}G. \tag{5.17}$$

Now, assume that $\alpha \neq 0$ (otherwise, reverse the roles of D and F), and let

$$w = \frac{\alpha}{\alpha t + \beta s}, \qquad z = \frac{s}{\alpha t + \beta s}. \tag{5.18}$$

In this case

$$s = \frac{\alpha z}{w}, \qquad t = \frac{1 - \beta z}{w}, \tag{5.19}$$

which implies that

$$H(s, t) = wH(\alpha z, 1 - \beta z) = wK(zI - F)^{-1}G = K(sD - tF)^{-1}G, \tag{5.20}$$

with

$$D = \frac{I - \beta F}{\alpha}. \tag{5.21}$$

Since there is a one-to-one correspondence between the representation (5.17) of $H(\alpha z, 1-\beta z)$ and the representation (5.20) of $H(s, t)$ with D given by (5.21), the dimension and uniqueness properties of these two representations are the same. This implies that minimal state-space representations of $H(s, t)$ satisfying (5.20) and (5.21) are related by a similarity transform, and have a dimension r equal to the McMillan degree of $H(\alpha z, 1-\beta z)$. □

COROLLARY. *The state-space representation* (5.11), (5.13) *is minimal if and only if* (D, F, G) *is strongly reachable and* (K, D, F) *is strongly observable. Furthermore, the dimension of a minimal state-space representation is equal to the rank of the Hankel matrix* O_sR_s, *where* O_s *and* R_s *are the strong observability and reachability matrices associated, respectively, to* (K, D, F) *and* (D, F, G).

Proof. It can be assumed without loss of generality that $\alpha \neq 0$ in (5.13). Then, the representation (5.11), (5.13) of $H(s, t)$ is minimal if and only if the representation (5.17) of $H(\alpha z, 1-\beta z)$ is minimal, or equivalently if and only if (K, F) is observable and (F, G) is reachable, where observability and reachability are defined here in the sense of causal systems. Since $\alpha \neq 0$, this is equivalent to requiring that (K, D, F) and (D, F, G) are strongly observable, and strongly reachable, respectively (see [14, Thm. 4.1]).

It was also shown in Theorem 5.1 that the dimension r of a minimal state-space representation is equal to the McMillan degree of $H(\alpha z, 1-\beta z)$. But according to the realization theory of causal systems, this McMillan degree is equal to the rank of the Hankel matrix

$$\bar{H} = \overline{OR}, \tag{5.22}$$

where $\bar{O}$ and $\bar{R}$ are the observability and reachability matrices associated to the pairs (K, F) and (F, G), respectively. But with $\alpha \neq 0$, the nullspace of $\bar{O}$ coincides with that of O_s, and the range of $\bar{R}$ with that of R_s. This implies that the rank of $\bar{H}$ is equal to that of O_sR_s, thus proving the corollary. □

One relatively unsatisfactory aspect of Theorem 5.1 is that the dimension r of a minimal state-space representation of $H(s, t)$ is characterized in terms of the McMillan degree of the one-dimensional rational matrix $H(\alpha z, 1-\beta z)$, and not directly in terms of $H(s, t)$. It turns out that it is possible to characterize r directly from $H(s, t)$ by extending the concept of McMillan degree as follows.

DEFINITION 5.1. Given a homogeneous and strictly proper rational matrix $H(s, t)$ in s and t, the *McMillan degree* of $H(s, t)$ is defined as the degree of the least common multiple of the denominators of all minors of $H(s, t)$.

Then, we have Theorem 5.2.

THEOREM 5.2. *If $H(s, t)$ is realizable, i.e., if it is homogeneous of relative degree one, the dimension of a minimal state-space representation of $H(s, t)$ is equal to its McMillan degree.*

Proof. Consider the minimal representation

$$H(s, t) = K(sD - tF)^{-1}G. \tag{5.23}$$

Without loss of generality, it can be assumed that the pencil $sD - tF$ is in Weierstrass canonical form (see [24, p. 28]), so that

$$K = [K_1 \quad K_2], \quad D = \begin{bmatrix} D_1 & 0 \\ 0 & N \end{bmatrix}, \quad F = \begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix}, \quad G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}, \tag{5.24}$$

where N is nilpotent and D_1 and F_2 are invertible. The rational matrix

$$H_1(s, t) = K_1(sD_1 - tF_1)^{-1}G_1 \tag{5.25}$$

can then be expressed as

$$H_1(s, t) = K_1(zD_1 - F_1)^{-1}G_1/t = \bar{H}_1(z)/t, \tag{5.26}$$

where $z = s/t$. Since there is a one-to-one correspondence between $H_1(s, t)$ and $\bar{H}_1(z)$, the dimensions of minimal representations of these two rational matrices must be equal. But, $\bar{H}_1(z)$ is a strictly proper rational matrix in z, so that the dimension of its minimal representation is equal to its McMillan degree, i.e., to the degree of the least common multiple $a_1(z)$ of the denominators of all minors of $\bar{H}_1(z)$. Also, since D_1 is invertible, t is not a factor of the denominator of any of the entries, and thus of any of the minors of $H_1(t, s)$. Let $p_1(s, t)$ denote the least common multiple of the denominators of the minors of $H_1(s, t)$. Since t is not a factor of $p_1(s, t)$, the degree of $p_1(s, t)$ is just the degree in z of $p_1(z, 1) = a_1(z)$. This shows that the degree of $p_1(s, t)$ equals the McMillan degree of $\bar{H}_1(z)$, which is in turn equal to the dimension of D_1 and F_1.

For the second block of the representation (5.24), we proceed similarly. Let

$$H_2(s, t) = K_2(sN - tF_2)^{-1}G_2, \tag{5.27}$$

and denote by $p_2(s, t)$ the least common multiple of the denominators of the minors of $H_2(s, t)$. Since F_2 is invertible, s is not a factor of $p_2(s, t)$. This implies that the degree of $p_2(s, t)$ is just the degree in t of $p_2(1, t)$, which, by analogy with the previous case, is just the dimension of N and F_2. Also, since N is nilpotent and (N, F_2) is in standard form, we have

$$p_2(s, t) = \det(sN - tF_2) = at^{n_2}, \tag{5.28}$$

where a is a constant, and n_2 the dimension of N and F_2.

Noting that

$$H(s, t) = H_1(s, t) + H_2(s, t) \tag{5.29}$$

and the fact that $p_1(s, t)$ and $p_2(s, t)$ have no common factors, we can easily deduce that the least common multiple $p(s, t)$ of the denominators of the minors of H satisfies

$$p(s, t) = p_1(s, t)p_2(s, t). \tag{5.30}$$

The degree of $p(s, t)$ is therefore equal to the sum of the dimensions of the blocks of (5.24), which is the dimension of D and F. □

Example 5.1. Consider the sequence

$$H(k)=\begin{cases}-1, & k=0,\\ 1, & k=1,\\ 0 & \text{otherwise}.\end{cases} \tag{5.31}$$

Its (s, t)- and z-transforms are, respectively,

$$H(s,t)=\frac{1}{s}-\frac{1}{t}, \tag{5.32a}$$

$$H(z)=-1+\frac{1}{z}. \tag{5.32b}$$

Already we can see the advantage of using the (s, t)-transform: $H(s, t)$ has one mode at zero and one at infinity, where $H(z)$ has only a pole at $z=0$.

From Theorem 5.2, we see that the dimension of a minimal representation, simply select $\alpha=\beta=1$, and perform the realization

$$H(z,1-z)=\frac{1}{z}-\frac{1}{1-z}=[1 \quad 1]\left(zI-\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right)^{-1}\begin{bmatrix}1\\ 1\end{bmatrix}, \tag{5.33}$$

which implies that

$$K=[1 \quad], \quad D=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \quad F=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}, \quad G=\begin{bmatrix}1\\ 1\end{bmatrix}. \tag{5.34}$$

6. Minimal realization. In § 5, it was shown that the specification of an internal description (C, P, E, A, B) of a weighting pattern $W(k)$ yields the three state-space representations (5.8)–(5.10) for the rational matrices $W(s, t)$, $H_r(s, t)$, and $H_o(s, t)$. This suggests that the construction of a minimal internal description of $W(k)$ can be formulted as a state-space representation problem in the (s, t)-domain. It turns out that the link existing between minimal state-space representations of rational matrices and minimal internal descriptions is less direct than for causal systems, since an internal description (C, P, E, A, B) can be minimal, even though *none* of the state-space representations (5.8)–(5.10) is minimal.

6.1. Dimension of a minimal realization.

THEOREM 6.1. *The dimension n of a minimal internal description of* $W(k)$ *is given by*

$$n=\omega+\rho-\tau, \tag{6.1}$$

where if $d(\cdot)$ *denotes the generalized McMillan degree introduced in Definition* 5.1,

$$\omega=d(H_r(s,t)), \quad \rho=d(H_o(s,t)), \quad \tau=d(W(s,t)). \tag{6.2}$$

Proof. Let (C, P, E, A, B) be a minimal internal description of $W(k)$. Then, $W(s, t)$, $H_r(s, t)$, and $H_o(s, t)$ admit state-space representations of the form (5.8)–(5.10), and from the corollary of Theorem 5.1, ω, ρ, and τ are the ranks of the Hankel matrices O_sR_w, O_wR_s, and O_sR_s, respectively. But, according to the minimality conditions (3.9a)–(3.9b), R_w and O_w have full rank, which implies that ω and ρ are the ranks of the strong observability and reachability matrices O_s and R_s, respectively. From condition (3.9c), we can also deduce that the rank of O_sR_s equals the rank of O_s plus that of R_s minus n, so that

$$\tau = \rho + \omega - n, \tag{6.3}$$

which implies (6.1). □

Example 6.1. Consider the weighting pattern

$$W(k) = \begin{cases} a^k & k \geqq 1, \\ ba^k, & k < 1, \end{cases} \tag{6.4}$$

where a and b are scalar parameters with $a < 1$. Using Theorem 4.1, it is straightforward to check that $W(k)$ is realizable. From Theorem 6.1, we find that the dimension of a minimal internal description of $W(k)$ is given by

$$\begin{aligned} n &= d\left(\left[\frac{a}{s-at}\ \frac{ab}{s-at}\right]\right) + d\left(\begin{bmatrix} \dfrac{a}{s-at} \\ \dfrac{-ab}{s-at} \end{bmatrix}\right) - d\left(\frac{(1-b)a}{s-at}\right) \\ &= \begin{cases} 1+1-1=1 & \text{for } b \neq 1, \\ 1+1-0=2 & \text{for } b=1. \end{cases} \end{aligned} \tag{6.5}$$

When $b \neq 1$, a minimal internal description of $W(k)$ is

$$C = \frac{a}{1-b}, \quad P = \frac{1}{1-b}, \quad E = 1, \quad A = a, \quad B = 1. \tag{6.6}$$

The causal and anticausal parts $W_f(s, t)$ and $W_b(s, t)$ of W have the same pole, namely $s/t = a$, which explains why they can be realized with a single eigenmode. The resulting TPBVDS realization is strongly reachable, strongly observable, and nonseparable.

When $b = 1$, a minimal internal description of $W(k)$ is

$$C = [a \quad a], \quad P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad E = I, \quad A = aI, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{6.7}$$

This separable realization is not strongly reachable, and is not strongly observable. Note that in the realization (6.6), the system matrices tend to ∞ as $b \to 1$. Thus, $b = 1$ can be viewed as a singularity in the sense that the dimension of a minimal internal description of W is two only when b is exactly equal to one. □

6.2. Minimal realization procedure. One interesting aspect of Theorem 6.1 is that as an intermediate step in the evaluation of the dimension n of a minimal internal description of $W(k)$, we obtain ω and ρ, which are, respectively, the ranks of the strong observability and reachability matrices of a minimal internal description. This observation leads to the following procedure for constructing a minimal internal description of W.

Step 1. Construct the minimal state-space representations

$$H_r(s, t) = [W_f(s, t)\, W_b(s, t)] = \bar{C}(s\bar{E} - t\bar{A})^{-1}[\bar{B}_f \quad \bar{B}_b], \tag{6.8}$$

$$H_o(s, t) = \begin{bmatrix} W_f(s, t) \\ W_b(s, t) \end{bmatrix} = \begin{bmatrix} \tilde{C}_f \\ \tilde{C}_b \end{bmatrix}(s\tilde{E} - t\tilde{A})^{-1}\tilde{B}, \tag{6.9}$$

where if α and β are such that $W_f(\alpha, -\beta)$ and $W_b(\alpha, -\beta)$ are defined, the pairs $\{\bar{E}, \bar{A}\}$ and $\{\tilde{E}, \tilde{A}\}$ satisfy the normalization condition (2.1) for the same α and β. Since the representations (6.8) and (6.9) are both minimal, the sizes of the matrices $\{\bar{E}, \bar{A}\}$ and $\{\tilde{E}, \tilde{A}\}$ are equal, respectively, to ω and ρ.

Step 2. Let

$$\bar{B} = \bar{B}_f + \bar{B}_b, \qquad \tilde{C} = \tilde{C}_f + \tilde{C}_b. \tag{6.10}$$

From (6.8)-(6.9), we find

$$\begin{aligned} W(s,t) &= W_f(s,t) + W_b(s,t) \\ &= \bar{C}(s\bar{E} - t\bar{A})^{-1}\bar{B} = \tilde{C}(s\tilde{E} - t\tilde{A})^{-1}\tilde{B}, \end{aligned} \tag{6.11}$$

so that $(\bar{C}, \bar{E}, \bar{A}, \bar{B})$ and $(\tilde{C}, \tilde{E}, \tilde{A}, \tilde{B})$ are two state-space representations, in general nonminimal, of $W(s, t)$. The minimality of representations (6.8) and (6.9) implies that $(\bar{C}, \bar{E}, \bar{A}, \bar{B})$ and $(\tilde{C}, \tilde{E}, \tilde{A}, \tilde{B})$ are, respectively, strongly observable and strongly reachable. By decomposing these two representations into strongly reachable/unreachable, and strongly observable/unobservable components, respectively, we obtain

$$\bar{C} = [\bar{C}_1 \quad \bar{C}_2], \quad \bar{E} = \begin{bmatrix} \bar{E}_1 & \bar{E}_2 \\ 0 & \bar{E}_4 \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} \bar{A}_1 & \bar{A}_2 \\ 0 & \bar{A}_4 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix}, \tag{6.12}$$

and

$$\tilde{C} = [0 \quad \tilde{C}_2], \quad \tilde{E} = \begin{bmatrix} \tilde{E}_1 & \tilde{E}_2 \\ 0 & \tilde{E}_4 \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} \tilde{A}_1 & \tilde{A}_2 \\ 0 & \tilde{A}_4 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix}. \tag{6.13}$$

In the following, it will be assumed that the representations (6.8) and (6.9) are in the coordinate systems corresponding to (6.12) and (6.13), respectively.

Step 3. From (6.11), we find that

$$W(s,t) = \bar{C}_1(s\bar{E}_1 - t\bar{A}_1)^{-1}\bar{B}_1 = \tilde{C}_2(s\tilde{E}_4 - t\tilde{A}_4)^{-1}\tilde{B}_2, \tag{6.14}$$

where the representations $(\bar{C}_1, \bar{E}_1, \bar{A}_1, \bar{B}_1)$ and $(\tilde{C}_2, \tilde{E}_4, \tilde{A}_4, \tilde{B}_2)$ are both strongly reachable and observable. This implies that they must be related by a similarity transformation, i.e., there exists a matrix T such that

$$\bar{C}_1 = \tilde{C}_2 T^{-1}, \quad \bar{E}_1 = T\tilde{E}_4 T^{-1}, \quad \bar{A}_1 = T\tilde{A}_4 T^{-1}, \quad \bar{B}_1 = T\tilde{B}_2. \tag{6.15}$$

The matrix T is given by

$$T = \bar{M}_s \tilde{M}_s^T (\tilde{M}_s \tilde{M}_s^T)^{-1}, \tag{6.16}$$

where $\bar{M}_s$ and $\tilde{M}_s$ denote, respectively, the strong reachability matrices of $(\bar{E}_1, \bar{A}_1, \bar{B}_1)$ and $(\tilde{E}_4, \tilde{A}_4, \tilde{B}_2)$. Furthermore, since the representations (6.14) are minimal, the matrices $\bar{E}_1$, $\bar{A}_1$, $\tilde{E}_4$ and $\tilde{A}_4$ have dimension τ, where τ is given by (6.2), and consequently, the blocks $\{\bar{E}_4, \bar{A}_4\}$, and $\{\tilde{E}_1, \tilde{A}_1\}$ in the decompositions (6.12) and (6.13) have respective dimensions $\omega - \tau$ and $\rho - \tau$.

Step 4. The matrices C, E, A, and B of a minimal internal description are now selected as

$$E = \begin{bmatrix} \tilde{E}_1 & \tilde{E}_2 T^{-1} & * \\ 0 & \bar{E}_1 & \bar{E}_2 \\ 0 & 0 & \bar{E}_4 \end{bmatrix}, \qquad A = \begin{bmatrix} \tilde{A}_1 & \tilde{A}_2 T^{-1} & * \\ 0 & \bar{A}_1 & \bar{A}_2 \\ 0 & 0 & \bar{A}_4 \end{bmatrix}, \tag{6.17a}$$

$$C = [0 \quad \bar{C}_1 \quad \bar{C}_2], \qquad B = \begin{bmatrix} \tilde{B}_1 \\ \bar{B}_1 \\ 0 \end{bmatrix}, \tag{6.17b}$$

where * indicates an arbitrary block entry. The role of the similarity transformation T is to guarantee that the component which is common to state-space representations (6.8) and (6.9) is expressed in the same coordinate system. Note that (5.17) corresponds to a four part Kalman decomposition of (C, E, A, B) into strongly reachable/unreachable and observable/unobservable parts, where according to (3.9c), there is no strongly unreachable and unobservable component, since the internal description that we are constructing must be minimal.

By using this last observation, we can immediately conclude from (6.11) that

$$W(s, t) = C(sE - tA)^{-1}B. \tag{6.18}$$

If we denote

$$B_f = \begin{bmatrix} * \\ \bar{B}_f \end{bmatrix}, \qquad C_f = [\tilde{C}_f \quad *]\begin{bmatrix} I & 0 & 0 \\ 0 & T^{-1} & 0 \\ 0 & 0 & I \end{bmatrix}, \tag{6.19}$$

from (6.8) and (6.9), it is also easy to check that

$$W_f(s, t) = C(sE - tA)^{-1}B_f = C_f(sE - tA)^{-1}B. \tag{6.20}$$

Expanding $W_f(s, t)$ in power series of $s-\alpha$ and $t+\beta$ in the vicinity of $(s, t) = (\alpha, -\beta)$, noting that $\alpha E + \beta A = I$, and matching the coefficients of $(s-\alpha)^i(t+\beta)^j$ for all i, j in (6.20) yields

$$O_s R_s^f = O_s^f R_s, \tag{6.21}$$

where R_s^f and O_s^f denote the strong reachability and observability matrices associated respectively to (E, A, B_f) and (C_f, E, A).

Step 5. The matrix P is then obtained by solving the equation

$$O_s P R_s = O_s R_s^f. \tag{6.22}$$

The existence of a solution is guaranteed by identity (6.21), which shows that the row and column spaces of the matrix on the right side of (6.22) are spanned by O_s and R_s, respectively. The solution of (6.22) is generally not unique, since we can add to any solution P a matrix Q such that $O_s Q R_s = 0$, i.e., a matrix of the form

$$Q = \begin{bmatrix} * & * & * \\ 0 & 0 & * \\ 0 & 0 & * \end{bmatrix}. \tag{6.23}$$

We must now prove that the matrices (C, P, E, A, B) given by (6.17) and (6.22) specify an internal description of $W(k)$. This requires showing that the state-space representation identities (5.9)-(5.10) are satisfied, as well as properties (3.3). The relation

$$O_s P R_s = O_s R_s^f = O_s^f R_s, \tag{6.24}$$

implies

$$CPR_s = CR_s^f, \qquad O_s PB = O_s^f B, \tag{6.25}$$

so that from the Cayley-Hamilton theorem and (6.20), we have

$$CP(sE - tA)^{-1}B = C(sE - tA)^{-1}PB = W_f(s, t). \tag{6.26}$$

When combined with (6.18), this yields the representations (5.9)-(5.10). To prove relations (3.3a), we use (6.24) and the fact that the reachability matrices R_s and R_s^f, and observability matrices O_s and O_s^f are constructed from the same matrices E and A. Then, from the Cayley-Hamilton theorem, there exist matrices K_E and K_A which satisfy

$$ER_s = R_s K_E, \qquad AR_s = R_s K_A, \tag{6.27a}$$

$$ER_s^f = R_s^f K_E, \qquad AR_s = R_s^f K_A, \tag{6.27b}$$

i.e., the same matrices K_E and K_A can be used to characterize the E- and A-invariance of the range spaces of both R_s and R_s^f. Similarly, the E- and A-invariance of the nullspaces of O_s and O_s^f can be characterized by a single pair of matrices. Taking this feature into account in (6.24), it can be checked easily that the constraints (3.3a) are satisfied. To prove relations (3.3b) and (3.3c), we use identities (5.6a)-(5.6b). Substituting (5.6a) inside (6.26), and noting that the weighting pattern $W_f(k)$ is causal, we find

$$CPA^D(I-EE^D)(EA^D)^k B = 0 \tag{6.28}$$

for $0 \leqq k \leqq \mu_E - 1$. Expressing the pencil $\{E, A\}$ in Weierstrass canonical form, it is then easy to check that (6.28) is equivalent to (3.3b). Similarly, to derive (3.3c), we substitute (5.6b) inside the state-space representation

$$W_b(s,t) = C(I-P)(sE-tA)^{-1}B \tag{6.29}$$

and use the fact that $W_b(k)$ is an anticausal weighting pattern. Thus, (C, P, E, A, B) is an internal description of $W(k)$. Since its dimension n obeys (6.1), it is *minimal.*

Example 6.2. Let

$$W(k) = \begin{cases} 0, & k=1, \\ -1, & k \neq 1. \end{cases} \tag{6.30}$$

Then

$$W_f(s,t) = \frac{-t}{s(s-t)}, \qquad W_b(s,t) = \frac{1}{s-t}, \tag{6.31}$$

and according to Theorem 6.1, the dimension of a minimal internal description of $W(k)$ is

$$n = 2+2-1 = 3. \tag{6.32}$$

Since $\omega = \rho = 2$, we can also conclude that the minimal internal description is neither strongly reachable nor strongly observable. To obtain a minimal description, the first step is to construct the minimal state-space representations

$$[W_f \quad W_b] = \begin{bmatrix} \dfrac{-t}{s(s-t)} & \dfrac{1}{s-t} \end{bmatrix} = [1 \quad 1]\left(sI - t\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)^{-1} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \tag{6.33a}$$

$$\begin{bmatrix} W_f \\ W_b \end{bmatrix} = \begin{bmatrix} \dfrac{-t}{s(s-t)} \\ \dfrac{1}{s-t} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}\left(sI - t\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)^{-1} \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \tag{6.33b}$$

which satisfy the normalization condition (2.1) with $\alpha = 1$, $\beta = 0$. This yields

$$\bar{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \tilde{C} = [1 \quad 0]. \tag{6.34}$$

In this case, we can select $T = 1$, and

(6.35) $$C = [0 \quad 1 \quad 1], \quad E = I, \quad A = \begin{bmatrix} 1 & 0 & * \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix},$$

where * denotes an arbitrary entry. Finally, by solving (6.22) we find

(6.36) $$P = \begin{bmatrix} * & * & * \\ 0 & 1 & * \\ 1 & 0 & * \end{bmatrix}.$$

The above realization procedure can be simplified significantly if the minimal internal description is either strongly observable or strongly reachable, i.e., if the integers ω and ρ in (6.2) satisfy either $\omega = n$ or $\rho = n$.

Strongly observable case ($\omega = n$). In this case, only the state-space representation (6.8) is needed, and we can select $(C, E, A, B) = (\bar{C}, \bar{E}, \bar{A}, \bar{B})$. Also, since O_s has full rank, (6.24) for P reduces to

(6.37) $$PR_s = R_s^f.$$

Strongly reachable case ($\rho = n$). Then, only the representation (6.9) is needed, and we can select $(C, E, A, B) = (\tilde{C}, \tilde{E}, \tilde{A}, \tilde{B})$. Furthermore, (6.24) for P becomes

(6.38) $$O_s P = O_s^f.$$

The previous realization procedure, or its simplification for the strongly observable and reachable cases, is of interest only when it yields a minimal internal description which is not separable, since in the separable case, the realization of § 4 is minimal. the following result provides a test for determining whether a weighting pattern admits a separable minimal description.

THEOREM 6.2. *$W(k)$ has a separable minimal realization if and only if the minimal dimension n given by* (6.1) *satisfies*

(6.39) $$n = d(W_f(s, t)) + d(W_b(s, t)).$$

Proof. If we construct two minimal realizations of W_f and W_b, and combine them to realize $W(k)$ as shown in (4.4)-(4.5), we obtain a description of dimension $d(W_f(s, t)) + d(W_b(s, t))$. This description will therefore be minimal if and only if (6.39) is satisfied, where n is given by (6.1). □

7. Conclusions. In this paper, the minimal TPBVDS realization problem for acausal shift-invariant weighting patterns has been examined. By restricting our attention to extendible stationary TPBVDSs, it was shown that the minimal TPBVDS realization problem is equivalent to the problem of finding a minimal internal description for the weighting pattern $W(k)$ of interest. Introducing the (s, t) transform and characterizing minimal state-space representations of homogeneous rational matrices in (s, t), a frequency-domain approach was developed for finding the dimension of a minimal internal description, and for constructing such a description.

Since the assumption that the weighting pattern $W(k)$ is shift-invariant is restrictive, particularly for acausal systems, it would be of interest to extend the above theory to the nonstationary case. Also, we have limited our attention here to deterministic systems. Since there exists a complete and elegant stochastic realization theory for causal systems [25]-[27], it is natural to ask whether a similar theory can be developed for acausal stochastic systems. For the Gaussian case, some preliminary stochastic realization results have been presented in [28] for boundary value systems with standard nondescriptor dynamics, and in [18] for TPBVDSs.

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