Estimation for Rotational Processes with One Degree of Freedom—Part III: Implementation

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***Abstract*—In this paper we interpret and comment on the practical significance of the estimation problems considered in Part I [1] of this series. The problem of implementation of our results is addressed, and the results of a numerical simulation are reported.**

I. Introduction

In the first two parts of this series [1], [2], we have considered a variety of estimation problems on the circle S^1. These analytical results indicate how the structure of stochastic processes on S^1 can be utilized in designing easily implemented, high performance, optimal, or suboptimal estimation systems. In the next section we illustrate the basic technique described in Part I for a particular complex signal problem and comment on several of the practical aspects of this problem. In Section III we present simulation results for a demodulation problem in the presence of phase drift noise.

II. A Complex-Signal Estimation Problem

In this section we illustrate the type of estimation problem that can be analyzed using the techniques developed in [1]. Let x and y be 2-dimensional random processes satisfying

$$dx(t)=Ax(t)\,dt+B\,dw(t) \tag{1}$$

$$dy(t)=\alpha dt+x(t)\,dt+dv(t), \qquad y(0)=0 \tag{2}$$

where α is a known constant vector and w and v are independent Brownian motion processes with $E[dw(t)\,dw'(t)]=R\,dt, E[dv(t)dv'(t)]=Q\,dt,\ Q>0$.

We wish to estimate the process x given the received signal

$$z(t)=e^{y_1(t)+iy_2(t)}. \tag{3}$$

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This problem can be reduced to a linear filtering problem via a nonlinear processing of z.

$$dy_1(t)=\mathrm{Re}\left[\frac{dz(t)}{z(t)}\right]-\frac{Q_{11}-Q_{22}}{2}dt \tag{4}$$

$$dy_2(t)=\mathrm{Im}\left[\frac{dz(t)}{z(t)}\right]-Q_{12}dt. \tag{5}$$

There are several important observations that can be made concerning this problem.

1) The recovery of the y process requires the use of the *differential* signal $dz(t)$. For actual implementation it may be more advisable to work with the integrated process $y(t)$ obtained by the integration of (4) and (5).

2) We have that

$$\mathrm{Re}[z(t)]=e^{y_1(t)}\cos y_2(t) \tag{6}$$

$$\mathrm{Im}[z(t)]=e^{y_1(t)}\sin y_2(t) \tag{7}$$

where

$$y_i(t)=\alpha_i t+\int_0^t x_i(s)\,ds+v_i(t), \qquad i=1,2. \tag{8}$$

Thus, our signal process is both amplitude and phase modulated—α represents a "carrier" signal, x the modulating information to be recovered, and v the effects of random fluctuations and drift.

For the sake of specificity, consider the case when $y_1=\alpha_1=x_1=v_1=0$. In this case our signal is

$$\mathrm{Re}[z(t)]=\cos\left(\alpha_2 t+\int_0^t x_2(s)\,ds+v(t)\right) \tag{9}$$

$$\mathrm{Im}[z(t)]=\sin\left(\alpha_2 t+\int_0^t x_2(s)\,ds+v(t)\right). \tag{10}$$

Thus, our problem is one of frequency demodulation in the face of phase drift noise, and the present formulation does not allow additive channel noise; however, it should be noted that FM signal processing methods that involve limiter-discriminators transform additive noise into equivalent frequency and phase variations [4], [5].

Also note that in the usual FM model we receive a signal of the form of (9) *or* (10), but not both. The other component can, in principle, be recovered by passing the received signal through a filter that has 90° of phase shift and unity gain at the carrier frequency α_2. In the next section we indicate how to implement the desired filtering system without direct computation of the second component.

3) The received signal $z(t)$ can also be written in the form

$$z(t)=[\exp(v_1(t)+iv_2(t))]\left[\exp\int_0^t(x_1(s)+ix_2(s))ds\right] \tag{11}$$

which emphasizes the fact that the observation noise process is a multiplicative lognormal process [6]. Noise processes of this type arise in such problems as optical communication through a turbulent atmosphere [6].

The signal-noise model described in this section arises in a number of practical problems including radio navigation and tracking systems based on phase and frequency comparisons (e.g., Doppler radar, Omega, Loran), frequency stability, standards, and the measurement of frequency drifts, AM, FM, and joint AM–FM problems, and the processing of data from an integrating gyroscope. The reader is referred to [3] for a discussion of these problems.

III. A Frequency Demodulation Example

Let v and w be independent Brownian motions, and define x to be the scalar process satisfying

$$dx(t)=ax(t)dt+b^{\frac{1}{2}}dv(t) \tag{12}$$

($a<0$), and suppose we observe

$$z_1(t)=\sin\left(\omega_c t+c\int_0^t x(s)ds+q^{\frac{1}{2}}w(t)\right) \tag{13}$$

where ω_c is a known carrier frequency (there is no difficulty in taking x to be the output of a more complicated linear diffusion process). Suppose we wish to demodulate z_1 to obtain an estimate of x. In order to apply the results of [1] directly, we need to compute

$$z_2(t)=\cos\left(\omega_c t+c\int_0^t x(s)ds+q^{\frac{1}{2}}w(t)\right) \tag{14}$$

and then we must compute

$$y(t)=\int_0^t[z_2(\tau)dz_1(\tau)-z_1(\tau)dz_2(\tau)]$$

$$=\omega_c t+c\int_0^t x(s)ds+q^{\frac{1}{2}}w(t). \tag{15}$$

Given (15), we can produce the optimal steady-state filter for x by standard Kalman or Wiener filter methods.

One method for producing (14) was described in Section II, and several others are proposed in [3]. We include here a method, based on the use of cycle counters, that was used in the simulations described later in this section. Assuming that ω_c is large and positive, we can be assured that $\dot{y}>0$. We can also write

$$y(t)=2\pi\left[\frac{y(t)}{2\pi}\right]+\psi(t) \tag{16}$$

where $[x]$ is the largest integer that is $\leqslant x$, and $[y/2\pi]$ represents the output of a cycle counter. Using the facts that $\dot{y}>0$ and $\dot{z}_1=\dot{y}z_2$, we compute $\psi(t)$ and $y(t)$ mod 2π as follows:

$$\psi(t)=[\sin^{-1}(z_1(t))]\bmod 2\pi, \qquad \text{if } \operatorname{sgn}(\dot{z}_1(t))\geqslant 0 \tag{17}$$

$$\psi(t)=[\pi-\sin^{-1}(z_1(t))]\bmod 2\pi, \qquad \text{if } (\dot{z}_1(t))<0. \tag{18}$$

Suppose we have implemented a system that produces $dy(t)$ (or $\dot{y}(t)$) given the input z_1. In this case, the optimal steady-state filter equations are

$$d\hat{x}(t|t)=\hat{a}x(t|t)dt+\frac{P_\infty c}{q}(dy(t)-c\hat{x}(t|t)dt-\omega_c dt) \tag{19}$$

$$P_\infty=\frac{aq+\sqrt{a^2q^2+bc^2q}}{c^2}. \tag{20}$$

Here P_∞ is the optimum steady-state error variance.

If we can produce $y(t)$ as opposed to $dy(t)$, we obtain the optimal steady-state filtering equations

$$\dot{r}(t)=-\alpha r(t)+y(t)-\omega_c t \tag{21}$$

$$\hat{x}(t|t)=M(-\alpha r(t)+y(t)-\omega_c t) \tag{22}$$

where

$$\alpha=\sqrt{a^2+bc^2/q} \qquad M=\frac{bc}{q(\alpha+|a|)}=\frac{a+\alpha}{c}. \tag{23}$$

The steady-state error variances are identical for the two filters.

In addition to simulating systems of the type just discussed, we have also obtained results for a system involving a first-order phase-lock loop (PLL). The reader is referred to [7] for a detailed description of PLL systems. The baseband equations of the PLL are

$$\dot{r}(t)=Ke(t) \tag{24}$$

$$e(t)=\frac{\sqrt{2}}{2}\sin(y(t)-\omega_c t-r(t)) \tag{25}$$

and the transfer function $G(s)$ from the error signal e to the estimate $\hat{x}$ is computed via Wiener filtering methods

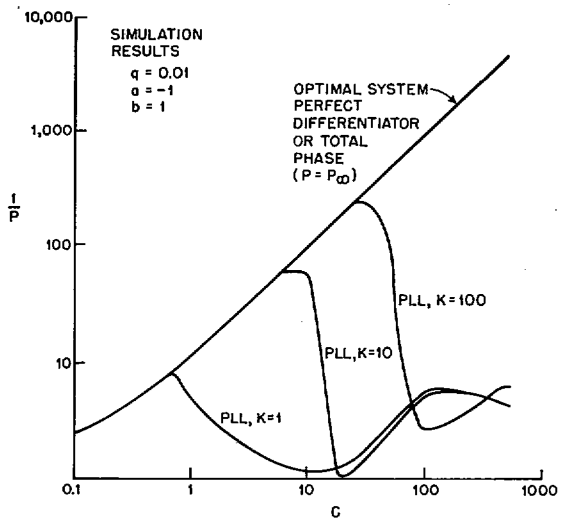

Fig. 1. A graph of some of the FM simulation results.

assuming that $\sin(y-\omega_c t-r)\approx y-\omega_c t-r$. This yields

$$G(s)=M\sqrt{2}\,\frac{s+\frac{\sqrt{2}}{2}K}{s+\alpha} \tag{26}$$

where α and M are given in (23). If the linearizing assumption holds (i.e., if we are in the so-called "above threshold" range) the PLL performs in a manner identical to the filter (21)–(23).

Nominal values $a=-1, b=1$ were chosen for the simulation and the values of c and q were varied to test system performance. Note that c is often called the *frequency deviation* [7] and the larger c is, the better we would expect our system to perform. Also $1/q$ is sometimes called the *oscillator coherence time* [7], and the quantity

$$\Lambda=c^2/q \tag{27}$$

plays the role of a signal to noise ratio. The figure of merit we use to compare various system performances is P, the inverse of the empirically computed steady-state sample error variance.

A number of different values of c and q were used in the simulations, and runs were made using mostly the baseband system models given in (19)–(26). Also, in order to test out the cycle-counter-total phase system followed by the filter (21)–(23), a number of runs were made using a carrier frequency $f_c=\omega_c/2\pi=10000\,\text{Hz}$. The full set of simulation results are reported and discussed in [3], and the results for $q=0.01$ and for a range of values of c are graphically displayed in Fig. 1. We note that the simulation results for the various optimal systems—the perfect differentiator system (19)–(20), the perfect total phase detector system (21)–(23), and the cycle-counter-total phase system—match quite well with the analytically determined performance measure, $1/P_\infty$.

Note that Fig. 1 indicates that for small values of c the PLL results are quite close to the optimal values (this is the "above threshold" region), but for large values of c—the region in which the optimal system performance is quite good—the performance of the PLL falls off drastically. The cause of this is the violation of the linearization assumption $\sin\epsilon\approx\epsilon$ on which the PLL design was based (we fall "below threshold"). We note that as K increases, the range of values of c over which PLL performance is near optimal also increases. In fact, for the baseband PLL model, one can show that as $K\to\infty$, baseband PLL performance approaches the optimal. However, increasing K tends to invalidate the use of the baseband model to approximate the actual PLL (we violate the so-called "bandwidth constraint"). Thus, taking into account both the threshold and bandwidth constraints on the PLL, we see that the performance of the PLL is limited relative to the optimal.

IV. Conclusions

In this paper we have considered some of the practical implications of the results derived in Part I [1] of this series. We have illustrated the optimal Abelian Lie group estimation result derived in [1] and have discussed the significance of this formulation. In addition, we have discussed some of the implementation problems related to the techniques we have developed, and have presented the results of numerical simulations. These results indicate that systems designed using these new techniques compare favorably to standard phase-lock loop systems.

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Alan S. Willsky (S'70–M'73), for a photograph and biography, see this issue, page 21.

James Ting-Ho Lo (M'73), for a photograph and biography, see this issue, page 21.