Stability, Stochastic Stationarity, and Generalized Lyapunov Equations for Two-Point Boundary-Value Descriptor Systems

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Abstract—In this paper, we introduce the concept of internal stability for two-point boundary-value descriptor systems (TPBVDS's). Since TPBVDS's are defined only over a finite interval, the concept of stability is not easy to formulate for these systems. The definition which is used here consists of requiring that as the length of the interval of definition increases, the effect of boundary conditions on states located close to the center of the interval should go to zero. Stochastic TPBVDS's are studied, and the property of stochastic stationarity is characterized in terms of a generalized Lyapunov equation satisfied by the variance of the boundary vector. A second generalized Lyapunov equation satisfied by the state variance of a stochastically stationary TPBVDS is also introduced, and the existence and uniqueness of positive definite solutions to this equation is then used to characterize the property of internal stability.

I. INTRODUCTION

NONCAUSAL physical phenomena arise in many fields of science and engineering. These phenomena correspond usually to processes evolving in space, instead of time. To model such processes, the usual state-space models familiar to system theorists are not appropriate, since these models were developed primarily to describe causality, in the sense that the "state" of a system at a given time is a summary of the past inputs sufficient to compute future outputs. One is then led to ask: what is a natural class of models to describe noncausal phenomena in one dimension? It is the goal of this paper, as well as of earlier papers and reports [1]–[4], to suggest that perhaps the most natural class of discrete-time noncausal models in one dimension is the class of two-point boundary-value descriptor systems (TPBVDS's). This conclusion is drawn from the observation that the impulse response of a time-invariant descriptor system is *noncausal*, and that the dynamics of these systems are symmetric with respect to forwards and backwards propagation. In addition, for systems defined over a finite interval, two-point boundary-value conditions enforce noncausality in the sense that both ends of the interval play a symmetric role in the expression of the boundary conditions.

The noncausality of discrete-time descriptor systems is a well-known feature of these systems. It is, for example, much in evidence in the early work of Luenberger [5]–[7], where it is

pointed out that two-point boundary-value conditions are usually needed to guarantee well-posedness. In Lewis [8], it was shown that these systems could be decomposed into forwards and backwards propagating subsystems, so that their solution involves recursions in both time directions. However, in spite of these useful observations, it is fair to say that most of the literature on descriptor systems has focused mainly on issues of structure [9], [10], and their implication for the control of descriptor systems [11]–[14]. This is primarily due to the fact that in continuous time, descriptor systems display an impulsive behavior, which until recently has been the focus of attention.

The study presented here has been influenced significantly by the work of Krener [15]–[18] on the system-theoretic properties of the standard (i.e., nondescriptor) continuous-time boundary-value systems, and on the analysis of stochastic boundary-value systems. The results of Krener, as well as related work by Gohberg, Kaashoek, and Lerer [19]–[21], have pointed out that boundary-value linear systems have a rich internal structure. The results of this paper, and of [1]–[4] combine in some sense the degree of noncausality attributable to the boundary contributions, which was already present in [15]–[21], with an additional source of noncausality, namely the noncausal dynamics of discrete-time descriptor systems.

Another motivation for this paper is our desire to analyze the properties of optimal estimators for noncausal models developed in [22]–[25], [4]. A new class of Riccati equations for TPBVDS's is obtained in [25], [4], and it is of interest to determine conditions under which positive-definite solutions exist, and the implication of these conditions concerning system stability. One purpose of this paper is to define and study the property of stability for TPBVDS's. The notion of stability is not easy to formulate for these systems, since they are defined over a finite interval. However, a relatively natural concept is that of *internal stability*, whereby as the length of the interval of definition of a TPBVDS grows, the effect of the boundary conditions on states located close to the center of the interval goes to zero. A theory of stability that parallels the standard causal theory is developed by considering stochastically stationary TPBVDS's, and by showing that stochastic stationarity can be characterized in terms of generalized Lyapunov equations. The existence and uniqueness of positive-definite solutions to these equations is then characterized in terms of the property of internal stability.

This paper is organized as follows. Section II provides some background information on the properties of *displacement* two-point boundary-value descriptor systems [2], [4], which is the class of systems considered in this paper. These systems are such that their Green's function is invariant under time-shifts, and they play, therefore, the same role for TPBVDS's as time-invariant systems for causal nondescriptor systems. In Section III, the notion of internal stability is introduced and characterized. In Section IV, we examine stochastic TPBVDS's, and study in particular *stochastically stationary* systems. Two generalized Lyapunov equations which must be satisfied, respectively, by the state variance and the boundary variance of the boundary vector

are introduced, and the property of stochastic stationarity is characterized in terms of the second of these equations. It is shown in Section V that the covariance function of a stochastically stationary TPBVDS satisfies a second-order descriptor equation, with appropriate boundary conditions. Finally, in Section VI the existence and uniqueness of solutions to the generalized Lyapunov equation satisfied by the state variance is characterized in terms of the property of internal stability. The concluding Section VII describes the impact of the results of this paper on the study of the TPBVDS estimators and generalized Riccati equations introduced in [25] and [4].

II. DISPLACEMENT SYSTEMS AND REACHABILITY CONCEPTS

The two-point boundary-value descriptor systems (TPBVDS) considered in this paper satisfy the difference equation

$$Ex(k+1) = Ax(k) + Bu(k), \qquad 0 \le k \le N-1 \qquad (2.1)$$

with the two-point boundary-value condition

$$V_i x(0) + V_f x(N) = v. \qquad (2.2)$$

Here E, A, and B are constant matrices, x and v are n-dimensional vectors, and u is an m-dimensional vector. We refer the reader to [1]–[4] for studies of some of the basic system-theoretic properties of this class of systems.

It was shown in [1] that, without loss of generality, it can be assumed[1] that the system (2.1), (2.2) is in *normalized-form*, i.e., it satisfies the following two properties: i) there exists some scalars α and β such that

$$\alpha E + \beta A = I \qquad (2.3)$$

so that E and A commute; and ii) the boundary matrices V_i and V_f satisfy

$$V_i E^N + V_f A^N = I. \qquad (2.4)$$

A slight generalization of the above normalized form was introduced in [2]. Specifically, (2.1), (2.2) is said to be in *block-normalized form* if (2.4) holds and

$$E = \text{diag } (E_1, \cdots, E_M), \qquad A = \text{diag } (A_1, \cdots, A_M) \qquad (2.5)$$

where: i) the block sizes of E and A are compatible; ii) for each j, there exists (α_j, β_j), possibly varying with j, such that

$$\alpha_j E_j + \beta_j A_j = I; \qquad (2.6)$$

and iii) the eigenmodes of distinct blocks of the system are different, i.e., for any $(s, t) \ne (0, 0)$, $|sE_j - tA_j| = 0$ for at most one value of j. Any well-posed TPBVDS can always be put in normalized or block-normalized form, and we will frequently assume that our system is in one of these two forms.

A special class of two-point boundary-value descriptor systems which is of great interest is the class of displacement TPBVDS's [2], [4].

Definition 2.1: A TPBVDS (2.1), (2.2) is a *displacement system* if the Green's function $G(k, l)$ appearing in the solution

$$x(k) = A^k E^{N-k} v + \sum_{l=0}^{N-1} G(k, l) Bu(l) \qquad (2.7)$$

depends only on the difference between arguments k and l, so that

$$G(k, l) = G(k-l). \qquad (2.8)$$

[1] A necessary and sufficient condition [1] for (2.1), (2.2) to be well-posed—i.e., to yield a well-defined map from $\{v, u\}$ to x—is that, by multiplication on the left only, E, A, V_i, and V_f can be brought to a form satisfying (2.3), (2.4).

Note that the above terminology is consistent with that of Gohberg, Kaashoek, and Lerer [20] (see also [21]) in their study of boundary value systems with standard nondescriptor dynamics. Unlike for causal systems, the fact that the matrices E and A are constant is not sufficient to guarantee that the TPBVDS (2.1), (2.2) is a displacement system. The matrices E and A must also satisfy some properties in relation to the boundary matrices V_i and V_f. The following characterization of displacement systems was established in [2].

Theorem 2.1: A TPBVDS in block-normalized form is a displacement system if and only if the matrices E and A commute with both V_i and V_f, i.e.,

$$[E, V_i] = [E, V_f] = [A, V_i] = [A, V_f] = 0 \qquad (2.9)$$

where

$$[X, Y] = XY - YX. \qquad (2.10)$$

The class of displacement systems is quite large. For example, it includes *cyclic* systems, for which

$$V_i = -V_f = (E^N - A^N)^{-1} \qquad (2.11a)$$

and *anticyclic* systems with

$$V_i = V_f = (E^N + A^N)^{-1}. \qquad (2.11b)$$

Another useful result from [2] is as follows.

Theorem 2.2: Consider a displacement TPBVDS in block-normalized form. Then V_i and V_f are also block diagonal with block sizes compatible with those of E and A.

In the following, we shall restrict our attention to displacement TPBVDS's. For a system of this type, and in block-normalized form, the Green's function $G(k, l)$ can be expressed as (see [2], [3])

$$G(k, l) = G(k-l) = \begin{cases} V_i A^{k-l-1} E^{N-k+l} & k > l \\ -V_f E^{l-k} A^{N-1-l+k} & k \le l. \end{cases} \qquad (2.12)$$

We will also make use of the concepts of *strong* and *weak reachability* for displacement TPBVDS's. These concepts rely on the outward and inward processes $z_o(k, l)$ and $z_i(k, l)$ ([1], [2]; see also [16], [7]) associated with (2.1), (2.2). For a subinterval $[k, l]$ of $[0, N]$, these processes characterize, respectively, the effect of the inputs $u(\cdot)$ inside, and outside, this subinterval on the boundary states $x(k)$ and $x(l)$. For a displacement TPBVDS, they can be expressed as [2]

$$z_o(k, l) = E^{l-k} x(l) - A^{l-k} x(k) = \sum_{s=k}^{l-1} E^{s-k} A^{l-s-1} Bu(s) \qquad (2.13)$$

$$z_i(k, l) = V_i E^{N-l+k} x(k) + V_f A^{N-l+k} x(l) \qquad (2.14a)$$

$$= E^{N-l} A^k v + V_i E^{N-l} z_o(0, k) - V_f A^k z_o(l, N). \qquad (2.14b)$$

Then, we have the following definitions.

Definition 2.2: The system (2.1), (2.2) is *strongly reachable on* $[k, l]$ if the map

$$\{u(s) : s \in [k, l]\} \to z_o(k, l) \qquad (2.15)$$

is onto. It is *weakly reachable off* $[k, l]$ if the map

$$\{u(s) : s \in [0, k-1] \cup [l, N-1]\} \to z_i(k, l) \qquad (2.16)$$

with the boundary vector $v = 0$, is onto. The system is *strongly reachable* if it is strongly reachable over some interval. It is *weakly reachable* if the union of the range spaces of (2.16) for all k, l is R^n.

To characterize the properties of strong and weak reachability,

the following matrices were introduced in [2]:

$$R_s(k) = [A^{k-1}B \quad EA^{k-2}B \cdots E^{k-1}B], \qquad (2.17)$$

$$R_w(k) = [V_i E^{N-k}R_s(k) \quad V_f A^k R_s(N-k)]. \qquad (2.18)$$

The following was then shown.

Theorem 2.3: The system is strongly reachable iff the range Im (R_s) of $R_s = R_s(n)$ is equal to $\mathbf{R}^n$. The system is weakly reachable iff

$$\bigcup_k \text{Im} (R_w(k)) = \mathbf{R}^n. \qquad (2.19)$$

Finally, it was proved in [2] that strong reachability implies weak reachability.

III. STABILITY

The concept of stability, which is relatively easy to define for causal systems, is more difficult to describe for TPBVDS's, since these systems are defined over a finite interval. By analogy with the causal case, we will require that as the length of the interval of definition [0, N] tends to infinity, the effect of the boundary conditions should vanish for points far away from the boundary.

A. Internal Stability

Consider a displacement TPBVDS defined over a finite interval, and for which the boundary condition (2.2) corresponds to a physical constraint of the problem which cannot be modified. Then, when the dynamics (2.1) and boundary condition (2.2) are fixed, we want to study the effect of increasing the size of the domain [0, N] of definition of the TPBVDS on the state variables $x(k)$ located close to the center of this domain. One issue which arises in this context is that if the TPBVDS (2.1), (2.2) is originally in block-normalized form for a length N_0 of the interval of definition, and if we increase the length to N without changing the matrices V_i, V_f and the vector v appearing in (2.2), the system will not remain in block-normalized form, since (2.4) will not be satisfied. However, if we renormalize (2.2) by a left multiplication by $(V_i E^N + V_f A^N)^{-1}$ and change the matrices V_i, V_f and the vector v accordingly, the TPBVDS will be block-normalized. In this context, it is possible to describe internal stability as follows.

Definition 3.1: The displacement TPBVDS (2.1), (2.2) in block-normalized form is *internally stable* if as the length N of the interval of definition tends to infinity, the effect of the boundary value v on any $x(k)$ located near the midsection of interval [0, N] goes to zero, i.e.,

$$\lim_{N \to \infty} E^{N/2} A^{N/2} (V_i E^N + V_f A^N)^{-1} = 0. \qquad (3.1)$$

To interpret condition (3.1), note that according to (2.7), and taking into account the renormalization described above to put the TPBVDS in block-normalized form as the interval length N is increased, the effect of the boundary vector on state $x(k)$ is given by $A^k E^{N-k}(V_i E^N + V_f A^N)^{-1} v$. Thus, for $k = N/2$, which corresponds to a point in the middle of interval [0, N], the effect of v on $x(N/2)$ is $E^{N/2} A^{N/2}(V_i E^N + V_f A^N)^{-1} v$.

There is another interpretation of the above notion of stability, which we will state without proof. Specifically, as we change N without changing V_i and V_f, except for the renormalization, we actually are changing the entire Green's function of the TPBVDS. Thus, what we have is a *sequence* of Green's functions $G_N(k)$, $1 - N \le k \le N$, indexed by N. Internal stability is then equivalent to

$$\lim_{N \to \infty} \sum_{k=1-N}^{N} \|G_N(k)\| < \infty \qquad (3.2)$$

which relates the concept of stability to the summability of the system's Green's function $G_N(k)$ as the interval N of definition tends to infinity.

As an illustration of the above concept of stability, consider a system that describes the heat distribution around a ring. Since the ring is closed, this system has a periodic boundary condition $x(0) = x(N)$, which is independent of the size of the ring. In this case, if a perturbation in heating conditions is applied at one point of the ring, one would expect that as the size of the ring increases, the effect of this perturbation will become smaller and smaller for points which are located on the opposite side of the ring.

B. Decomposition of a Displacement TPBVDS

Our first objective is to characterize the property of internal stability for a TPBVDS in terms of the system dynamics and boundary conditions. This characterization relies on a particular decomposition for displacement TPBVDS's, obtained using the following Weierstrass-type decomposition (see [26, p. 28]) of a regular matrix pencil. Note that (2.6) guarantees that the pencil $zE - A$ is regular.

Lemma 3.1: Given a TPBVDS, there exists invertible matrices F and T such that

$$E_D = FET = \begin{bmatrix} I & 0 & 0 \\ 0 & A_b & 0 \\ 0 & 0 & I \end{bmatrix} \qquad (3.3a)$$

$$A_D = FAT = \begin{bmatrix} A_f & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & U \end{bmatrix} \qquad (3.3b)$$

where A_f and A_b have eigenvalues inside the unit circle, and U has eigenvalues on the unit circle.

The transformation (3.3) can be achieved by left-multiplication of (2.1) by F and by performing the state transformation

$$x(k) = Tx_D(k). \qquad (3.4)$$

Observe that E_D and A_D satisfy (2.5), (2.6) with $(\alpha_1, \beta_1) = (\alpha_3, \beta_3) = (1, 0)$ and $(\alpha_2, \beta_2) = (0, 1)$. Also, by construction the eigenmodes of the three blocks are different.

To complete the transformation of our system, B is replaced by

$$B_D = FB \qquad (3.5)$$

and the boundary matrices become

$$V_{Di} = LV_i T, \qquad V_{Df} = LV_f T \qquad (3.6)$$

where the normalizing matrix L is selected here such that relation (2.4) is satisfied by the new TPBVDS. Finally, if the original TPBVDS was a displacement system, the new TPBVDS is also a displacement system since its Green's function is related to the original Greens's function through

$$G_D(k-l) = T^{-1}G(k-l)F^{-1}. \qquad (3.7)$$

In this case, since the TPBVDS specified by (3.3), (3.5), and (3.6) is a displacement system in block-normalized form, we can invoke Theorem 2.2 to conclude that the matrices V_{Di} and V_{Df} are block diagonal, i.e.,

$$V_{Di} = \begin{bmatrix} V_{i1} & 0 & 0 \\ 0 & V_{i2} & 0 \\ 0 & 0 & V_{i3} \end{bmatrix} \quad \text{and} \quad V_{Df} = \begin{bmatrix} V_{f1} & 0 & 0 \\ 0 & V_{f2} & 0 \\ 0 & 0 & V_{f3} \end{bmatrix} \qquad (3.8)$$

which yields the main result of this section.

Theorem 3.1 (Decomposition of a Displacement TPBVDS): Through the use of a state transformation T, and by left multiplication of (2.1) and (2.2) by invertible matrices F and L,

an arbitrary displacement TPBVDS can be decomposed into three decoupled subsystems of the form

$$x_f(k+1) = A_f x_f(k) + B_f u(k), \quad V_{i1} x_f(0) + V_{f1} x_f(N) = v_1, \quad (3.9a)$$

$$x_b(k) = A_b x_b(k+1) - B_b u(k), \quad V_{i2} x_b(0) + V_{f2} x_b(N) = v_2, \quad (3.9b)$$

$$x_m(k+1) = U x_m(k) + B_m u(k), \quad V_{i3} x_m(0) + V_{f3} x_m(N) = v_3 \quad (3.9c)$$

where the matrices A_f and A_b have their roots inside the unit circle, and U has its roots on the unit circle. The subsystems (3.9a)-(3.9c) are displacement systems, in normalized form.

In what follows, for convenience only, we will refer to (3.9a)-(3.9c) as the forward, backward, and marginal parts of the system, respectively. Note, for example, that the dynamics of (3.9a) look like forward dynamics and those of (3.9b) look like backward dynamics, but the boundary conditions in each case can make each of these noncausal.

C. Characterization of Internal Stability

The main feature of the decomposition (3.9) of a displacement TPBVDS is that it reduces the study of internal stability for a TPBVDS to the study of these properties for each of its components.

Lemma 3.2: Consider a displacement TPBVDS given by

$$x(k+1) = Ax(k) + Bu(k) \quad (3.10a)$$

$$V_i x(0) + V_f x(N) = v \quad (3.10b)$$

where A has all its roots inside the unit circle. Then, the system (3.10) is internally stable if and only if the matrix V_i is invertible.

Proof: Taking into account the definition (3.1) of internal stability, we see that (3.10) is internally stable if and only if

$$\lim_{N \to \infty} A^{N/2} (V_i + V_f A^N)^{-1} = 0$$

which is clearly equivalent to requiring that V_i should be invertible. □

This yields the following characterization of internal stability.

Theorem 3.2: A displacement TPBVDS is internally stable if and only if the decomposition (3.9) of this system is such that boundary matrices V_{i1} and V_{f2} are invertible, and the system does not have any eigenmode on the unit circle.

Proof: The first part of the above characterization is obtained by applying Lemma 3.2 to the forward and backward components (3.9a) and (3.9b). The condition concerning the eigenmodes on the unit circle is derived by noting that no choice of boundary matrices V_{i3} and V_{f3} satisfying (2.4) will guarantee that

$$\lim_{N \to \infty} U^{N/2} (V_{i3} + V_{f3} U^N)^{-1} = 0. \quad □$$

D. Stable Extendibility

There is another natural way in which one might consider defining stability for a TPBVDS, which we now briefly describe. As this notion does not lead to particularly surprising results, we omit the details and refer the reader to [3]. The basic idea stems from the observation that the inward process $z_i(k, l)$ can be thought of as the *inward propagation* of the boundary conditions so that the solution to (2.1) with boundary condition (2.14a) is the same as the solution of (2.1), (2.2) over the interval $[k, l]$. Thus, the Green's function of (2.1), (2.14a) is the *restriction* of the Green's function of (2.1), (2.2). It is natural then to ask if one can propagate the boundary condition *outward* to obtain an *extension* of the Green's function. The following definition and theorem are from [2].

Definition 3.2: A displacement TPBVDS (2.1), (2.2) is *extendible* if for any $K \leq 0$ and $L \geq N$, there exist boundary

matrices $V_i(K, L)$ and $V_f(K, L)$ so that the TPBVDS specified by (2.1) and the new boundary condition

$$V_i(K, L)x(K) + V_f(K, L)x(L) = v(K, L) \quad (3.11)$$

is such that
 i) the new extended system is a displacement system;
 ii) the Green's function $G(k - l)$ of the original system is the restriction of the Green's function $G_e(k - l)$ of the extended system

$$G(k-l) = G_e(k-l) \quad \text{for } |k-l| \leq N. \quad (3.12)$$

Theorem 3.3: A displacement TPBVDS in block normalized form is extendible if and only if the following two conditions are satisfied:

$$\text{i) Ker } (E^n) \subset \text{Ker } (V_i) \quad (3.13a)$$

$$\text{ii) Ker } (A^n) \subset \text{Ker } (V_f). \quad (3.13b)$$

In this case one choice for the boundary matrices of the extended system (which is also in block-normalized form) is

$$V_i(K, L) = V_i E^N (E^D)^{L-K} \quad (3.14a)$$

$$V_f(K, L) = V_f A^N (A^D)^{L-K} \quad (3.14b)$$

where E^D and A^D are the Drazin inverses [27, p. 8] of E and A.

For an extendible TPBVDS, we have an entire family of extensions over intervals of increasing size, all of which can be thought of as the restriction of a system defined over $(-\infty, \infty)$. This leads directly to the following.

Definition 3.3: An extendible displacement TPBVDS is *stably extendible* if the Green's function $G_e(k)$ obtained by extension to $(-\infty, \infty)$ is absolutely summable, i.e.,

$$\sum_{-\infty}^{+\infty} \|G_e(k)\| < \infty. \quad (3.15)$$

Theorem 3.4: An extendible displacement TPBVDS is stably extendible if and only if the decomposition (3.9) of this system is such that

$$V_{f1} = V_{i2} = 0 \quad (3.16)$$

and the system does not have any eigenvalue of the unit circle, i.e., it does not contain a marginal component of the form (3.9c).

Proof: See [3].

From condition (3.16), we can immediately deduce that stable extendibility implies internal stability, since subsystems (3.9a) and (3.9b) must each satisfy the normalization condition (2.4), so that boundary matrices V_{i1} and V_{f2} are invertible. Theorem 3.4 shows that the class of stably extendible TPBVDS's consists of systems obtained by combining *completely decoupled* forward and backward causal subsystems. Thus, this class does not contain systems that have a truly acausal behavior since we can associate a time direction to each subsystem. The class of internally stable subsystems, on the other hand, is far richer and does include systems with acausal response characteristics.

IV. STOCHASTIC TPBVDS'S AND GENERALIZED LYAPUNOV EQUATIONS

In this section, we study the class of stochastic TPBVDS's given by (2.1), (2.2), where $u(k)$ is a zero-mean white Gaussian noise with unit intensity, and where v is a zero-mean Gaussian random vector independent of $u(k)$, and with covariance Q. Thus, we have

$$M[u(k)u^T(l)] = I\delta(k-l) \quad (4.1)$$

where $M[z]$ denotes the mean of a random variable z, and $\delta(k)$ is the Kronecker delta function. In addition, it is assumed throughout the remainder of this paper that the TPBVDS (2.1), (2.2) is a displacement system in normalized form. The displacement assumption is quite important, and all the results of this paper concerning stochastic TPBVDS's are restricted to this class of systems.

In the continuous-time case, and for the usual nondescriptor state-space dynamics, a related class of stochastic boundary-value systems was examined by Krener [17], [18], who studied the relation existing between this class of systems and reciprocal processes. In particular, Krener considered the problem of realizing reciprocal processes with stochastic boundary-value systems. Our goal here is somewhat different, in the sense that we shall seek to obtain a complete set of conditions under which a stochastic TPBVDS of the form (2.1), (2.2) is stochastically stationary. It turns out that the characterization that will be obtained involves a Lyapunov equation for the boundary variance Q which generalizes the standard Lyapunov equation for stationary Gauss-Markov processes.

Definition 4.1: A TPBVDS is *stochastically stationary* if

$$M[x(k)x^T(l)] = R(k, l) = R(k - l). \qquad (4.2)$$

If the TPBVDS (2.1), (2.2) is stochastically stationary, the *variance matrix* $P(k) = R(k, k)$ of $x(k)$ must be constant, i.e., $P(k) = P$ for all k. Thus, our first step at this point will be to characterize completely the matrix $P(k)$ for a displacement TPBVDS in normalized form. Let

$$\Pi(k) = \sum_{j=0}^{k} A^{k-j} E^j BB^T (A^{k-j} E^j)^T. \qquad (4.3)$$

Then, using the Green's function solution (2.7), (2.11), multiplying by its transpose, and taking expected values, we obtain

$$P(k) = A^k E^{N-k} Q(A^k E^{N-k})^T + (V_i E^{N-k})\Pi(k-1)(V_i E^{N-k})^T$$
$$+ (V_f A^k)\Pi(N-1-k)(V_f A^k)^T. \qquad (4.4)$$

This expression can also be rewritten as

$$P(k) = A^k E^{N-k} Q(A^k E^{N-k})^T + R_w(k)R_w^T(k) \qquad (4.5)$$

where $R_w(k)$ is the weak reachability matrix (2.18). From (4.5), and noting from Theorem 2.3 that if the TPBVDS is weakly reachable, we have

$$z^T R_w(k) = 0 \qquad \text{for all } k \Rightarrow z = 0 \qquad (4.6)$$

we can conclude that if the TPBVDS is weakly reachable *and* has a constant variance P, then P is positive definite.

The expression (4.4) for $P(k)$ is an *explicit description*, and is valid in general for displacement TPBVDS's in normalized form. However, as in the causal case, where $P(k)$ satisfies a time-dependent Lyapunov equation, it is also possible to obtain an *implicit description* for $P(k)$ in the form of a recursion with boundary conditions. Specifically, multiplying both sides of (2.1) and (2.2) by their transposes, using the Green's function solution (2.7), (2.11), then taking expected values, it can be shown that $P(k)$ satisfies the TPBVDS

$$EP(k+1)E^T - AP(k)A^T = (V_i E^N)BB^T(V_i E^N)^T$$
$$- (V_f A^N)BB^T(V_f A^N)^T \qquad (4.7a)$$

$$V_i P(0)V_i^T - V_f P(N)V_f^T = (V_i E^N)Q(V_i E^N)^T$$
$$- (V_f A^N)Q(V_f A^N)^T \qquad (4.7b)$$

which can be viewed as a generalized time-dependent Lyapunov equation for $P(k)$.

Equations (4.7a) and (4.7b) may or may not characterize completely the variance $P(k)$, i.e., they may have several solutions, one of which will be (4.4). This corresponds to situations where (4.7a) and (4.7b) do not completely capture the structure of (4.4), and in this case, additional conditions would have to be imposed to make sure that we obtain a unique solution equal to (4.4). To obtain conditions under which (4.7a) and (4.7b) specify $P(k)$ uniquely, these equations can be rewritten in the form of a TPBVDS of type (2.1), (2.2), and we can then apply the well-posedness test for TPBVDS's presented in [1]. Let $p(k)$, q, and w denote the vectors obtained by scanning the entries of matrices $P(k)$, Q, and $W = BB^T$ columnwise. We can then rewrite (4.7a) and (4.7b) as

$$(E \otimes E)p(k+1) - (A \otimes A)p(k) = (V_i E^N \otimes V_i E^N)w$$
$$- (V_f A^N \otimes V_f A^N)w \qquad (4.8a)$$

$$(V_i \otimes V_i)p(0) - (V_f \otimes V_f)p(N) = (V_i E^N \otimes V_i E^N)q$$
$$- (V_f A^N \otimes V_f A^N)q \qquad (4.8b)$$

where $\otimes$ denotes the Kronecker product [28]. Note that the right-hand sides of the above equations are irrelevant as far as well-posedness is concerned.

The well-posedness condition for the TPBVDS (4.8a), (4.8b) reduces to the invertibility of the matrix

$$F_N = (V_i \otimes V_i)(E \otimes E)^N - (V_f \otimes V_f)(A \otimes A)^N$$
$$= (V_i E^N) \otimes (V_i E^N) - (V_f A^N) \otimes (V_f A^N). \qquad (4.9)$$

We obtain, therefore, the following result.

Theorem 4.1: Equations (4.7a) and (4.7b) characterize uniquely the variance $P(k)$ if and only if

$$\lambda_j \neq \mu_l \qquad \text{for all } j \text{ and } l \qquad (4.10)$$

where λ_j and μ_j are the eigenvalues of $V_i E^N$ and $V_f A^N$, respectively.

Proof: Since matrices $V_i E^N$ and $V_f A^N$ satisfy (2.4), they can be brought simultaneously to Jordan form. Furthermore, the eigenvalues λ_j and μ_j corresponding to the same eigenvector z satisfy

$$\lambda_j + \mu_j = 1. \qquad (4.11)$$

Then, it is easy to check that the eigenvalues of F_N must have the form $\lambda_j \lambda_l - \mu_j \mu_l$, so that F_N is invertible as long as

$$\lambda_j \lambda_l \neq \mu_j \mu_l.$$

Taking into account (4.11), this gives (4.10). $\qquad \square$

Note that in the causal case the eigenvalues λ_j and μ_j are all equal to 1 and 0, respectively. Thus, according to Theorem 4.1, $P(k)$ is uniquely defined. This is expected, since in this case (4.7a) is a forwards recursion for $P(k)$, and (4.7b) is the initial condition $P(0) = Q$.

Theorem 4.1 indicates that, except under very special circumstances, $P(k)$ can be uniquely computed from the generalized time-dependent Lyapunov equation (4.7a) and (4.7b). In addition, when the TPBVDS is stochastically stationary, the matrix $P(k) = P$ is *constant*, and satisfies the two algebraic equations

$$EPE^T - APA^T = (V_i E^N)BB^T(V_i E^N)^T - (V_f A^N)BB^T(V_f A^N)^T$$
$$(4.12)$$

$$V_i PV_i^T - V_f PV_f^T = (V_i E^N)Q(V_i E^N)^T - (V_f A^N)Q(V_f A^N)^T$$
$$(4.13)$$

obtained from (4.7a) and (4.7b). Equation (4.12) is a generalized algebraic Lyapunov equation, and by analogy with the causal

case, it is tempting to think that, if a TPBVDS has a constant positive definite variance matrix P satisfying (4.12), then the TPBVDS is stochastically stationary. Unfortunately, this is not the case, and the correct condition for stochastic stationarity, which is condition (4.14) below, involves the variance Q of the boundary vector v.

Theorem 4.2: A stochastic TPBVDS is stochastically stationary if and only if Q satisfies the equation

$$EQE^T - AQA^T = V_iBB^TV_i^T - V_fBB^TV_f^T. \quad (4.14)$$

Proof: To prove sufficiency, we need to show that when (4.14) is satisfied, then $R(k + 1, l + 1) = R(k, l)$ for all $k, l \in [0, N]$. By using the Green's function solution (2.7), (2.11) to evaluate $R(k, l) = M[x(k)x^T(l)]$ for $k \geq l$, we obtain

$$R(k, l) = A^kE^{N-k}Q(A^lE^{N-l})^T$$

$$+ \sum_{j=0}^{l-1} V_iA^{k-j-1}E^{N-k+j}BB^T(V_iA^{l-j-1}E^{N-l+j})^T$$

$$+ \sum_{j=l}^{k-1} V_fA^{k-j-1}E^{N-k+j}BB^T$$

$$\cdot (V_iA^{N-1-j+l}E^{j-l})^T1(k-l-1)$$

$$+ \sum_{j=k}^{N-1} V_fA^{N-1-j+k}E^{j-k}BB^T(V_fA^{N-1-j+l}E^{j-l})^T$$

$$(4.15)$$

where $1(k)$ is the unit step function, i.e.,

$$1(k) = \begin{cases} 1 & \text{for } k \geq 0 \\ 0 & \text{for } k < 0. \end{cases}$$

From (4.15), it is easy to check that

$$R(k+1, l+1) - R(k, l) = A^kE^{N-1-k}[AQA^T - EQE^T$$

$$+ V_iBB^TV_i^T - V_fBB^TV_f^T](A^lE^{N-1-l})^T \quad (4.16)$$

which indicates clearly that when Q satisfies the generalized Lyapunov equation (4.14), then $R(k + 1, l + 1) = R(k, l)$ for all k, l.

Conversely, to prove necessity, assume that $R(k + 1, l + 1) = R(k, l)$ for all k, l. Then, the right-hand side of (4.16) is zero for all k, l. Thus, if

$$\Delta = AQA^T - EQE^T + V_iBB^TV_i^T - V_fBB^TV_f^T \quad (4.17)$$

we have

$$A^kE^{N-1-k}\Delta(A^lE^{N-1-l})^T = 0 \quad (4.18)$$

for all k, l. Taking into account (2.3), as well as (4.18), yields

$$\Delta = (\alpha E + \beta A)^{N-1}\Delta(\alpha E + \beta A)^{N-1}$$

$$= \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \binom{N-1}{k}\binom{N-1}{l} \alpha^{2(N-1)-k-l}\beta^{k+l}$$

$$\cdot A^kE^{N-1-k}\Delta(A^lE^{N-1-l})^T = 0 \quad (4.19)$$

which shows that Q must obey the generalized Lyapunov equation (4.14). □

Note that for causal systems ($E = V_i = I$, $V_f = 0$) the boundary covariance matrix is simply $P(0)$, (4.14) for Q is *identical* to (4.12) for P (which is the usual Lyapunov equation), and (4.13) reduces to $P = Q$. For a general TPBVDS, however, P and Q are different quantities.

When a TPBVDS is stochastically stationary, it must have a constant variance. However, unlike in the causal case, the converse is not always true. To see what happens, set $k = l$ in (4.16) and note that $R(k, k) = P(k)$. This gives

$$P(k+1) - P(k) = A^kE^{N-1-k}[AQA^T - EQE^T + V_iBB^TV_i^T$$

$$- V_fBB^TV_f^T](A^kE^{N-1-k})^T. \quad (4.20)$$

The relation (4.20) shows that when Q satisfies the Lyapunov equation (4.14), then $P(k + 1) = P(k)$ for all k, as expected. Conversely, if $P(k + 1) = P(k)$ for all k, Q must satisfy the equation

$$A^kE^{N-1-k}[AQA^T - EQE^T + V_iBB^TV_i^T$$

$$- V_fBB^TV_f^T](A^kE^{N-1-k})^T = 0 \quad (4.21)$$

for all k. In the special case when either E or A is invertible, this relation implies that Q must satisfy (4.14). In other words, if either E or A is invertible, the TPBVDS (2.1), (2.2) is stochastically stationary if and only if it has a constant variance. However, this is not true in general, i.e., (4.14) is not necessarily implied by (4.21), as can be seen from the following example.

Example 4.1: Consider the TPBVDS

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x(k+1) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} x(k)$$

$$+ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} u(k) \quad (4.22a)$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x(0) + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & N \\ 0 & 0 & 1 \end{bmatrix} x(N) = v \quad (4.22b)$$

where the variance of v is given by

$$Q = \begin{bmatrix} 1 & N & 1 \\ N & N^2+2 & N \\ 1 & N & 1 \end{bmatrix}. \quad (4.23)$$

The system (4.22) is in normalized form and is a displacement system. Then, it is easy to check that Q satisfies (4.21), but not (4.14), and that (4.22) has a constant variance matrix

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which satisfies both (4.12) and (4.13). This shows that a TPBVDS may have a constant variance matrix even if (4.14) is not satisfied, and therefore, the system is not stochastically stationary.

Note that the two generalized algebraic Lyapunov equations (4.12) and (4.14) for P and Q have exactly the same form, and differ only by their right-hand sides. Consequently, they admit unique solutions under the same condition.

Theorem 4.3: The generalized Lyapunov equations (4.12) and (4.14) have unique solutions if and only if the eigenmodes σ_j of the TPBVDS (2.1), i.e., the values for which $\sigma E - A$ is singular, are such that:

i) $\sigma_j\sigma_l \neq 1$ for all j and l; $\qquad\qquad\qquad (4.24)$

ii) the TPBVDS does not have both zero and infinite eigenmodes, i.e., the matrices E and A are not both singular.

Proof: The proof is similar to that of Theorem 4.1. Equations (4.12) and (4.14) admit unique solutions if and only if the matrix $M = E \otimes E - A \otimes A$ is invertible. But since E and A satisfy (2.3), they can be brought to Jordan form simultane-

ously, and we may denote by λ_j and μ_j the eigenvalues of these two matrices appearing in corresponding Jordan blocks. Assuming that E and A are in Jordan form, it is easy to check that the eigenvalues of M are $\lambda_j \lambda_l - \mu_j \mu_l$. Furthermore, the eigenmodes $\sigma_j = \mu_j/\lambda_j$. Combining these two observations, and noting from (2.3) that λ_j and μ_j cannot both be zero, we see therefore that M is invertible if and only if conditions i) and ii) are satisfied.   □.

Theorem 4.3 indicates that the class of TPBVDS's such that the generalized Lyapunov equations (4.12) and (4.14) have a unique solution is somewhat restricted, since either E and A must be invertible.

Thus, it may happen that a TPBVDS has a constant variance matrix P, but yet the generalized Lyapunov equation (4.12) may not specify P completely, i.e., it may have several solutions. In this case, in order to compute P, instead of using the implicit specification of P provided by the Lyapunov equation (4.12), one should use the explicit expression (4.4) for an arbitrary value of k.

V. COVARIANCE CHARACTERIZATION

In the previous section, it was shown that the variance P of a stochastically stationary TPBVDS satisfies the Lyapunov equation (4.12). As long as the conditions of Theorem 4.3 are satisfied, this provides a simpler method for computing P than the explicit evaluation of (4.4). To this point, however, the only characterization that we have of the covariance function $R(s) = R(l + s, l)$ for a stochastically stationary TPBVDS is (4.15), which we would need to evaluate for every individual value of $s = k - l$. Our goal in this section is to obtain a recursive characterization of $R(s)$ that can be used to compute the covariance in a considerably more efficient fashion. An interesting feature of the recursions that we shall obtain is that unlike the causal nondescriptor case, where the covariance satisfies first-order causal recursions, for the TPBVDS case, the covariance satisfies *second-order* boundary value recursions. Note, however, that this result is not totally unexpected, since it was shown by Krener [17] that the covariance of a continuous-time stationary two-point boundary value process with standard dynamics satisfies a second-order differential equation.

The starting point of our derivation is the observation that

$$ER(k+1, l) = M[Ex(k+1)x^T(l)] = M[(Ax(k)+Bu(k))x^T(l)].$$
(5.1)

Using (2.7), (2.11) to compute $M[u(k)x^T(l)]$, we find that (5.1) yields

$$ER(k+1, l) - AR(k, l) = -BB^T(V_f E^{k-l} A^{N-1-(k-l)})^T$$
for $k \geq l$.   (5.2a)

Similarly, it can be shown that

$$R(k, l+1)E^T - R(k, l)A^T = V_i A^{k-l-1} E^{N-(k-l)} BB^T$$
for $k > l$.   (5.2b)

Combining (5.2a) and (5.2b), we obtain therefore

$$[ER(k+1, l+1) - AR(k, l+1)]E^T - [ER(k+1, l)$$
$$- AR(k, l)]A^T = 0$$ (5.3)

for $k > l$, which holds independently of whether the TPBVDS (2.1), (2.2) is stochastically stationary or not.

In the special case when the TPBVDS that we consider is stochastically stationary, but setting $k - l = s + 1$ in (5.3), we obtain the following result.

Theorem 5.1: The covariance $R(s)$ of a stochastically stationary TPBVDS satisfies the second-order descriptor recursions

$$ER(s+1)E^T + AR(s+1)A^T = AR(s)E^T + ER(s+2)A^T$$ (5.4)

with the boundary conditions

$$V_i R(0) + V_f R(N) = Q(E^N)^T$$ (5.5a)

$$L(0)(V_i A)^T + L(N-1)(V_f E)^T = -BB^T V_f^T$$ (5.5b)

where

$$L(s) = ER(s+1) - AR(s), \quad 0 \leq s \leq N-1.$$ (5.6)

Furthermore, the second-order boundary-value system (5.4), (5.5) is *well-posed.*

The recursions (5.4) are similar to the second-order differential equation obtained by Krener [17] for a continuous-time stationary two-point boundary value process with standard dynamics. We still need to derive the boundary conditions (5.5) and to show that, when combined with (5.4), they define a well-posed system. To do so, we use (5.2a), where the TPBVDS is now assumed to be stochastically stationary. Setting $k - l = s$ inside (5.2a) gives (5.6), where

$$L(s) = -BB^T(V_f E^s A^{N-1-s})^T.$$ (5.7)

Then, noting that

$$L(s+1)A^T - L(s)E^T = 0, \quad 0 \leq s \leq N-2$$ (5.8)

it is easy to check that the coupled system of first-order descriptor equations (5.6), (5.8) is equivalent to (5.4). A set of boundary conditions for this system will therefore be also applicable to (5.4).

Suppose for the moment that the function $L(s)$ appearing on the left-hand side of (5.6) has already been computed, with either the analytical expression (5.7), or through recursions (5.8). For the first-order recursions (5.6) for $R(s)$, we can use the boundary condition (5.5a), which is derived by multiplying (2.2) on the right by $x^T(0)$, taking expected values, and using the Green's function expression (2.7). The pair (5.6), (5.5a) defines a well-posed TPBVDS for $R(s)$, since its dynamics and boundary matrices are the same as for system (2.1), (2.2).

This leaves us with the problem of computing $L(s)$ for $0 \leq s \leq N - 1$ from the first-order recursions (5.8). However, we already know that the solution must be given by (5.7). This implies in particular that

$$L(0) = -BB^T(V_f A^{N-1})^T$$ (5.9a)

and

$$L(N-1) = -BB^T V_i (E^{N-1})^T.$$ (5.9b)

Combining (5.9a) and (5.9b) yields the boundary condition (5.5b). Furthermore, the well-posedness of the TPBVDS system (5.8), (5.5b) for $L(s)$ is guaranteed by the well-posedness of (2.1), (2.2).

In the above discussion we have focused our attention on a specific set of boundary conditions, namely (5.5a), (5.5b), for the second-order system (5.4). However, there exist many choices of boundary conditions involving only $R(0)$, $R(1)$, $R(N - 1)$, and $R(N)$, which when combined with (5.4) will define a well-posed boundary-value system. For example, one obvious boundary condition is given by $R(0) = P$, where P can be found either by solving the algebraic Lyapunov equation (4.12) or by using analytic expression (4.4).

Example 5.1: Consider the anticyclic system

$$x(k+1) = x(k) + bu(k)$$ (5.10a)

$$(1/2)(x(0) + x(N)) = v$$ (5.10b)

where the variance of v is q. In this case, both sides of the generalized Lyapunov equation (4.14) are equal to 0, so that the TPBVDS (5.10) is stochastically stationary independently of the

choice of q. The Lyapunov equation (4.12) for the state variance p also reduces to zero on both sides, and cannot therefore be used to compute p. However, by direct evaluation of (4.4), it is easy to verify that

$$p = r(0) = Nb^2/4 + q. \tag{5.11}$$

We now compute the covariance function $r(k)$ of (5.10) for $k \in [0, N]$. We use the second-order recursion (5.4), which here takes the form

$$r(k+2) = 2r(k+1) - r(k). \tag{5.12}$$

Since $r(0)$ is already known, we only need $r(1)$ to be able to solve (5.12) in the forward direction. But according to (5.9a), we have

$$r(1) - r(0) = -b^2/2$$

so that

$$r(1) = (N-2)b^2/4 + q$$

and then using (5.12), we find

$$r(k) = (N - 2k)b^2/4 + q. \tag{5.13}$$

VI. LYAPUNOV STABILITY THEORY

For causal systems, the relationship between the existence of a positive definite solution to the standard Lyapunov equation and stability is well known. Specifically, for a causal and reachable system, the Lyapunov equation has a positive definite solution if and only if the system is strictly stable. In this section, for the class of displacement TPBVDS's, we study the relation existing between the existence and uniqueness of positive definite solutions to the generalized Lyapunov equation (4.12) for the state variance P, and the property of internal stability. Note that, whereas the generalized Lyapunov equation (4.14) for Q was the key to the characterization of stochastic stationarity derived in the previous section, (4.12) for P plays the main role in our study of internal stability. An important feature of this equation, which was not present in the causal case, is that it depends on the interval length N. This dependence on interval length is in fact the key to its usefulness in characterizing internal stability.

Since our study is centered on the generalized Lyapunov equation (4.12), it is useful to observe that this equation may admit a nonnegative definite solution P even when the system cannot be made stationary by any choice of boundary vector variance Q, i.e., there may be a nonnegative solution to (4.12) when there is no nonnegative solution to (4.14). This is illustrated by the following.

Example 6.1: Consider the system

$$x(k+1) = (1/2)x(k) + u(k) \tag{6.1a}$$

$$m(x(0) + 2x(N)) = v \tag{6.1b}$$

where $m = (1 + 2(1/2)^N)^{-1}$, and $u(k)$ is a white noise sequence with unit variance. System (6.1) is in normalized form and internally stable. The generalized Lyapunov equation (4.14) for q takes the form

$$(3/4)q = -3m^2 \tag{6.2}$$

which yields a negative value of q, so that the system cannot be made stationary over any interval $[0, N]$. Yet, the Lyapunov equation (4.12) if given by

$$(3/4)p = m^2(1 - 4(1/4)^N) \tag{6.3}$$

and its solution p is positive provided that N is larger than 1. However, this solution is *not* the state variance of (6.1), which in this case is not even constant. This can be seen by noting from

(4.3), (4.4) that the state variance is given by

$$p(k) = \frac{q}{4^k} + \frac{4}{3} m^2 \left(1 - \frac{4}{4^N} + \frac{3}{4^k}\right) \tag{6.4}$$

which is clearly not constant.

Example 6.1 shows that the generalized Lyapunov equation (4.12) may admit a unique positive definite solution P even when the TPBVDS (2.1), (2.2) cannot be made stochastically stationary for any choice of boundary vector variance Q, but in general this matrix P bears no relation whatsoever to the state variance. However, it will be shown below in Theorem 6.3 that, for an internally stable displacement TPBVDS, independently of the choice of boundary matrix Q, as the interval length $N \to \infty$, the variance matrices $P(k)$ of states near the center of the interval approach a constant matrix P^* which *is* the solution to the generalized Lyapunov equation (4.12) with N set equal to ∞.

The main objective of this section is to characterize the property of internal stability in terms of positive definite solutions of (4.12), regardless of whether such solutions correspond to the variance of a stochastically stationary TPBVDS or not. Specifically, it will be shown that for a displacement TPBVDS with no eigenvalues on the unit circle, if for any N, the generalized Lyapunov equation (4.12) has a nonnegative definite solution P, then the system (2.1), (2.2) is internally stable. The assumption that there are no roots on the unit circle is introduced here to rule out a situation such as that of Example 5.1, where since the Lyapunov equation (4.12) is identically zero, it has positive definite solutions, even though the TPBVDS is unstable since it has an eigenvalue on the unit circle.

Our results will require the following lemma.

Lemma 6.1: Let A and V be two square matrices which commute, i.e.,

$$AV = VA. \tag{6.5}$$

Then, if V is singular, there exists a right (left) eigenvector of A in the right (left) null space of V.

Proof: We focus on the right eigenvector case. Let $x \in \text{Ker}$ (V). Then,

$$VAx = AVx = 0$$

and consequently $Ax \in \text{Ker}(V)$. Thus, $\text{Ker}(V)$ *is* A invariant, which implies that A has at least one eigenvector in the null space of V.   □

We can now prove the following result.

Theorem 6.1: Assume that the TPBVDS (2.1), (2.2) is a weakly reachable displacement system with no eigenvalues on the unit circle. Then, if for some N, the generalized Lyapunov equation (4.12) has a nonnegative definite solution P, the TPBVDS is internally stable.

Proof: Since the TPBVDS that we consider has no eigenmodes on the unit circle, the decomposition of Theorem 3.1 takes the form

$$E = \begin{bmatrix} I & 0 \\ 0 & A_b \end{bmatrix}, A = \begin{bmatrix} A_f & 0 \\ 0 & I \end{bmatrix}, B = \begin{bmatrix} B_f \\ B_b \end{bmatrix} \tag{6.6a}$$

where the eigenvalues of A_f and A_b are inside the unit circle, and

$$V_i = \begin{bmatrix} V_{i1} & 0 \\ 0 & V_{i2} \end{bmatrix}, V_f = \begin{bmatrix} V_{f1} & 0 \\ 0 & V_{f2} \end{bmatrix}. \tag{6.6b}$$

To prove stability, we need to show that V_{i1} and V_{f2} are invertible. Using the above decomposition, the generalized Lyapunov equation (4.12) becomes

$$P_f - A_f P_f A_f^T = V_{i1} B_f B_f^T V_{i1}^T - (V_{f1} A_f^N) B_f B_f^T (V_{f1} A_f^N)^T \tag{6.7a}$$

$$A_b P_b A_b^T - P_b = (V_{i2} A_b^N) B_b B_b^T (V_{i2} A_b^N)^T - V_{f2} B_b B_b^T V_{f2}^T \tag{6.7b}$$

$$P_{fb}A_b^T - A_f P_{fb} = V_{i1}B_f B_b^T(V_{i2}A_b^N)^T - (V_{f1}A_f^N)B_f B_b^T V_{f2}^T \quad (6.7c)$$

where

$$P = \begin{bmatrix} P_f & P_{fb} \\ P_{fb}^T & P_b \end{bmatrix}. \quad (6.8)$$

Clearly, if P is nonnegative definite, so is P_f. Since we also know that A_f is strictly stable, from (6.7a) we can conclude that if x^T is an arbitrary left eigenvector of A_f then

$$x^T(V_{i1}B_f B_f^T V_{i1}^T - (V_{f1}A_f^N)B_f B_f^T(V_{f1}A_f^N)^T)x \geq 0. \quad (6.9)$$

We would like to show that V_{i1} is invertible. To do so, assume the contrary. Then, according to Lemma 6.1, there exists a vector x such that

$$x^T A_f = \lambda x^T, \quad (6.10a)$$

$$x^T V_{i1} = 0. \quad (6.10b)$$

We also know that the system is weakly reachable, and from the characterization of weak reachability presented in [2, Proposition 4.4], we have

$$x^T[V_{i1}B_f \quad V_{f1}B_f] \neq 0$$

so that

$$x^T V_{f1}B_f \neq 0. \quad (6.11)$$

Now, taking (6.10b) into account in (6.9), and observing that A_f and V_{f1} commute, we find that

$$0 = x^T V_{f1}A_f^N B_f = \lambda^N x^T V_{f1}B_f \quad (6.12)$$

where λ is the eigenvalue appearing in (6.10a). But (6.12) is compatible with (6.11) only if we have $\lambda = 0$, so that x^T must be in the left null space of both A_f and V_{i1}. However, in this case the matrix

$$V_{i1} + V_{f1}A_f^N$$

characterizing the well-posedness of the forward stable subsystem is not invertible, which contradicts our assumptions. Thus V_{i1} must be invertible. Similarly, it can be proved that V_{f2} is invertible. $\quad \square$.

As in the causal case, the above result has also a converse, i.e., given an internally stable TPBVDS, there exists a positive definite solution to the Lyapunov equation (4.12). However, this result is only valid for large N, and it requires stronger conditions than those of Theorem 6.1. First, the conditions of Theorem 4.3 on the eigenmodes of the TPBVDS must be satisfied, so that (4.12) will be guaranteed to have a solution independently of the choice of input matrix B and of boundary matrices V_i and V_f, in which case this solution will in fact be unique. The second condition is that the TPBVDS must be strongly reachable, instead of weakly reachable as in Theorem 6.1. This is due to the fact that we need to make sure that as $N \to \infty$, the solution of (4.12) is positive definite, instead of merely nonnegative definite.

Theorem 6.2: Consider a displacement TPBVDS which is internally stable, strongly reachable, and whose eigenmodes σ_j satisfy the conditions of Theorem 4.3 for the existence of a unique solution P_N to the generalized Lyapunov equation (4.12). Here the interval length N is allowed to vary, and the dependence of P on N is denoted by the subscript N of P_N. Then, there exists $N^* > 0$ such that P_N is positive definite for all $N > N^*$. Furthermore, as $N \to \infty$,

$$P_N \to P^* = \begin{bmatrix} P_f^* & 0 \\ 0 & P_b^* \end{bmatrix} \quad (6.13)$$

where P_f^* and P_b^* are, respectively, the solutions of the usual

algebraic Lyapunov equations for the forward and backward stable subsystems, i.e.,

$$P_f^* - A_f P_f^* A_f^T = B_f B_f^T, \quad (6.14a)$$

$$P_b^* - A_b P_b^* A_b^T = B_b B_b^T. \quad (6.14b)$$

Proof: First, observe that since the interval length N varies, the boundary matrices V_{i1}, V_{f1}, and V_{i2}, V_{f2} associated, respectively, to the forward and backward stable subsystems need to be rescaled in order to satisfy the normalized form identity (2.4) for all N. The rescaled boundary matrices are given by

$$V_{i1}(N) = (V_{i1} + V_{f1}A_f^N)^{-1}V_{i1},$$

$$V_{f1}(N) = (V_{i1} + V_{f1}A_f^N)^{-1}V_{f1} \quad (6.15a)$$

$$V_{i2}(N) = (V_{i2}A_b^N + V_{f2})^{-1}V_{i2},$$

$$V_{f2}(N) = (V_{i2}A_b^N + V_{f2})^{-1}V_{f2} \quad (6.15b)$$

and since the TPBVDS is internally stable, the matrices V_{i1} and V_{f2} are invertible, so that as $N \to \infty$,

$$V_{i1}(N) \to I, \quad V_{f1}(N) \to V_{i1}^{-1}V_{f1}, \quad V_{i2}(N) \to V_{f2}^{-1}V_{i2}, \quad V_{f2}(N) \to I. \quad (6.16)$$

Consider now the matrix P_N given by (6.8), whose entries satisfy (6.7a)–(6.7c), where the boundary matrices on the right-hand side are replaced by the scaled matrices (6.15). We want to show that for N large enough, the solutions $P_{f,N}$ and $P_{b,N}$ of (6.7a) and (6.7b) are positive definite and tend to P_f^* and P_b^* given by (6.14), and that the solution $P_{fb,N}$ of (6.7c) goes to zero as $N \to \infty$.

The first step is to observe that, as $N \to \infty$, since the scaled boundary matrices tend to finite limits given by (6.16), the right-hand side of (6.7c) tends to zero. But the eigenmodes of the system are such that the solution P_N is unique, and therefore the solution $P_{fb,N}$ of (6.7c) is unique and tends to zero as N goes to infinity.

Next, consider Lyapunov equation (6.7a), and observe that since the TPBVDS is strongly reachable, the matrix pair (A_f, B_f) is reachable in the usual sense for causal systems. But since the system is internally stable, $V_{i1}(N)$ given by (6.15a) is invertible, and noting that it commutes with A_f, we can conclude that the pair $(A_f, V_{i1}(N)B_f)$ is also reachable in the usual sense. Then, the solution $P_{f,N}$ of (6.7a) can be expressed as

$$P_{f,N} = P_{f,N}^+ - P_{f,N}^- \quad (6.17)$$

where $P_{f,N}^+$ and $P_{f,N}^-$ are, respectively, the solutions of

$$P_{f,N}^+ - A_f P_{f,N}^+ A_f^T = V_{i1}(N)B_f B_f^T V_{i1}(N)^T \quad (6.18a)$$

$$P_{f,N}^- - A_f P_{f,N}^- A_f^T = V_{f1}(N)A_f^N B_f B_f^T(V_{f1}(N)A_f^N)^T. \quad (6.18b)$$

Since $(A_f, V_{i1}(N)B_f)$ is reachable, $P_{f,N}^+$ is positive definite for all N, and since $V_{i1}(N) \to I$ as $N \to \infty$, $P_{f,N}^+ \to P_f^*$, where P_f^* is the unique positive definite solution of (6.14a). Furthermore, as $N \to \infty$, the right-hand side of (6.18b) tends to zero, so that $P_{f,N}^-$ tends to zero. From (6.17), we can therefore conclude that there exists an integer N^* such that $P_{f,N}$ is positive definite for all $N \geq N^*$. Similarly, it can be shown that the solution $P_{b,N}$ of (6.7b) is positive definite for large enough N and tends to P_b^*, which is the unique positive definite solution of (6.14b).

We have therefore shown that as $N \to \infty$, $P_{f,N}$ and $P_{b,N}$ approach positive definite matrices P_f^* and P_b^*, and the $P_{fb,N}$ tends to zero. Thus, the matrix P_N is positive definite for sufficiently large N and has the limit P^* in (6.13). $\quad \square$

Example 6.2: Consider system (6.1), which is both internally stable and strongly reachable. Then, the solution of the general-

ized Lyapunov equation (6.3) is

$$p_N = \frac{4}{3} m^2 \left(1 - \frac{4}{4^N}\right)$$

which is positive definite for $N \geq 2$. Furthermore, as $N \to \infty$,

$$p_N \to p^* = 4m^2/3 \tag{6.19}$$

where p^* is the solution of the generalized Lyapunov equation (6.3) with $N = \infty$.

It is worth noting that when $N = \infty$, if the TPBVDS is internally stable, in the coordinate system corresponding to decomposition (6.6), the generalized Lyapunov equation (4.12) takes the form

$$EPE^T - APA^T = W, \tag{6.20}$$

$$W = \begin{bmatrix} B_f B_f^T & 0 \\ 0 & -B_b B_b^T \end{bmatrix}. \tag{6.21}$$

Then, independent of whether eigenmodes σ_j satisfy the conditions of Theorem 4.3, one solution of (6.20) is P^* given by (6.13), (6.14), which is nonnegative definite regardless of the reachability properties of the TPBVDS (2.1), (2.2). In other words, for $N = \infty$, the conditions of Theorem 6.2 can be weakened, thus giving the following result.

Corollary 6.1: Let the displacement TPBVDS (2.1), (2.2) be internally stable. Then the generalized Lyapunov equation (4.12) with $N = \infty$ has a nonnegative definite solution P^*. This solution is positive definite if the system is strongly reachable.

For an internally stable TPBVDS, the solution P^* of (4.12) with $N = \infty$ has the following stochastic interpretation.

Theorem 6.3: Let displacement system (2.1), (2.2) be internally stable. Then, for *any* choice of boundary variance Q, as N goes to infinity, the variance matrix of states located close to the center of interval $[0, N]$ converges to the solution P^* of the generalized Lyapunov equation with $N = \infty$.

Proof: Let $P_N(k)$ be the variance matrix of the state $x(k)$ of (2.1), (2.2) defined over $[0, N]$. If l is an arbitrary but fixed integer, we must show that

$$\lim_{N \to \infty} P_N((N/2) + l) = P^* \tag{6.22}$$

where for simplicity it has been assumed that N is even. Our starting point is expression (4.4) for the state variance, i.e.,

$$P_N((N/2) + l) = A^{(N/2)+l} E^{(N/2)-l} Q (A^{(N/2)+l} E^{(N/2)-l})^T$$

$$+ (V_i(N)E^{(N/2)-l})\Pi((N/2)+l-1)$$

$$\cdot (V_i(N)E^{(N/2)-l})^T$$

$$+ (V_f(N)A^{(N/2)+l})\Pi((N/2)-l-1)$$

$$\cdot (V_f(N)A^{(N/2)+l})^T$$

where $\Pi(k)$ is given by (4.3), and boundary matrices $V_i(N)$ and $V_f(N)$ are obtained by rescaling V_i and V_f so that the normalized form identity (2.4) is satisfied for all N. Then, in the coordinate system corresponding to decomposition (6.6) of the TPBVDS in its forward and backward stable components, by using expressions (6.16) for the limit of $V_i(N)$ and $V_f(N)$ as $N \to \infty$, and taking into account the fact that A_f and A_b are stable matrices, we find that

$$\lim_{N \to \infty} P_N((N/2) + l) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \Pi(\infty) \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \Pi(\infty) \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}. \tag{6.23}$$

But since

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}$$

commute with both E and A, (6.23) can be rewritten as

$$\lim_{N \to \infty} P_N((N/2) + l) = \lim_{k \to \infty} \sum_{j=0}^{k} A^{k-j} E^j$$

$$\cdot \begin{bmatrix} B_f B_f^T & 0 \\ 0 & B_b B_b^T \end{bmatrix} (A^{k-j} E^j)^T. \tag{6.24}$$

Thus,

$$\lim_{N \to \infty} P_N((N/2) + l)$$

$$= \begin{bmatrix} \sum_{j=0}^{\infty} A_f^j B_f B_f^T (A_f^j)^T & 0 \\ 0 & \sum_{j=0}^{\infty} A_b^j B_b B_b^T (A_b^j)^T \end{bmatrix}$$

$$= \begin{bmatrix} P_f^* & 0 \\ 0 & P_b^* \end{bmatrix} = P^*. \tag{6.25}$$

This completes the proof of Theorem 6.3. □

Example 6.3: Consider the TPBVDS (6.1). According to (6.19), the solution of (4.12) with $N = \infty$ is $p^* = 4m^2/3$. Then, setting $k = (N/2) + l$ in expression (6.4) for the state variance, we obtain, as expected

$$\lim_{N \to \infty} p_N((N/2) + l) = 4m^2/3 = p^*.$$

Theorem 6.3 shows that, regardless of the boundary variance Q, the state variance of an internally stable displacement TPBVDS converges to the constraint matrix P^* given by (6.13), (6.14). However, an even more interesting observation is that under the above assumptions the TPBVDS will converge to a *stochastically stationary* system as $N \to \infty$. More precisely, if we denote by

$$R_N((N/2) + k, (N/2) + l) = M[x((N/2)+k)x^T((N/2)+l)] \tag{6.26}$$

the correlation matrix of states $x((N/2) + k)$ and $x((N/2) + l)$, where k and l fixed integers, by using the analytic expression (4.15) for the correlation matrix and following steps similar to those used in the proof of Theorem 6.3, it can be shown that in the coordinate system corresponding to the forward and backward stable decomposition (6.6), we have

$$\lim_{N \to \infty} R_N((N/2) + k, (N/2) + l)$$

$$= R^*(k - l)$$

$$= \begin{bmatrix} A_f^{k-l} P_f^* - \sum_{j=0}^{k-l-1} A_f^{k-l-j-1} B_f B_f^T (A_b^j)^T \\ 0 \quad\quad\quad P_b^* (A_b^{k-l})^T \end{bmatrix} \tag{6.27}$$

where for convenience it has been assumed that $k \geq l$. Since the limit obtained in (6.27) depends only on $k - l$, we can therefore conclude that independently of the choice of boundary variance Q, an internally stable TPBVDS converges to a stochastically stationary system as $N \to \infty$. This stochastically stationary system is separable into forward and backward causal components, which

are, however, correlated through the input noise $u(k)$. This last fact can be seen from (6.27), where if we denote by $x^*(k)$ the limiting process obtained by letting $N \to \infty$, and by shifting the left boundary of the interval of definition to $-\infty$, the cross-correlation $R_{fb}^*(k - l)$ between the forward component $x_f^*(k)$ and the backward component $x_b^*(l)$ is nonzero for $k \geq l$, since both of these processes depend on the noise over interval $[l, k]$, whereas the cross-correlation between $x_b^*(k)$ and $x_f^*(l)$ is zero, since they depend on the noise over disjoint intervals.

VII. Conclusions

In this paper, in spite of the fact that two-point boundary-value descriptor systems are defined only over a finite interval, we have been able to introduce a concept of internal stability for these systems and to develop a corresponding generalized Lyapunov stability theory. As was mentioned in the Introduction, this paper is part of a larger effort devoted to the study of the system properties, and the developement of estimation algorithms for TPBVDS's. In particular, the smoothing problem for TPBVDS's was examined in [25], [4], where it was shown that the smoother itself is a TPBVDS which can then be decoupled into forward and backward stable components through the introduction of generalized Riccati equations that were studied in [25] and [4]. An interesting question which arises in this context is whether for a strongly reachable and observable TPBVDS, the smoother is internally stable in the sense discussed in this paper. It turns out that this is the case, as will be proved in a subsequent report. In other words, the concept of internal stability developed here for TPBVDS's appears to be the natural generalization of the corresponding notion for standard causal state-space models and leads to just as rich a set of system-theoretic results.

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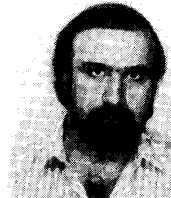
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