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REACHABILITY, OBSERVABILITY, AND MINIMALITY FOR SHIFT-INVARIANT TWO-POINT BOUNDARY-VALUE DESCRIPTOR SYSTEMS*

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Abstract. In this paper we study the system-theoretic properties of two related classes of shift-invariant two-point boundary-value descriptor systems (TPBVDSs), namely *displacement* systems for which Green's function is shift-invariant, and *stationary* systems for which the input-output map is stationary. For such systems it is possible to obtain detailed characterizations of the properties of weak reachability and observability introduced in [16] and of minimality as well. An important difference, that has also been noted before in a different context [9], is that there is a certain level of nonuniqueness in minimal realizations. Another property that is studied in this paper is that of extendibility, i.e., the concept of considering a TPBVDS as being defined on a sequence of intervals of increasing length. Necessary and sufficient conditions for extendibility are given.

1. Introduction

It has been long recognized [11], [12] that discrete-time descriptor systems, which may exhibit noncausal behavior, generally require the specification of boundary conditions in order to be well-posed. For this reason such

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models are a natural class for describing phenomena in which the independent variable is space, rather than time. Indeed, it was precisely this motivation that led Krener [9], [10] to investigate the system-theoretic properties of standard (i.e., nondescriptor) boundary-value models in continuous-time and Adams and coworkers [1]-[3], [16] to investigate estimation problems for rather general classes of boundary-value models.

In this paper we continue the development of a system theory for two-point boundary-value descriptor systems (TPBVDSs) begun in [13] and [17]. Much of our motivation stems from a desire to analyze the properties of optimal estimators for noncausal models as developed in [1]-[3], [14], and [15]. For example, a new class of generalized Riccati equations arises in [14] and [15], and it is of interest to determine conditions under which positive-definite solutions exist and the implications for filter stability. A second motivation for our work is to parallel and go beyond the theory of Krener [9], [10] and Gohberg and Kaashoek [5]-[7], for descriptor models in discrete time. Reference [17] and this paper carry out that parallel, with important differences caused by the possible singularity of the system matrices. In subsequent papers we will go beyond the system-theoretic issues investigated in [5]-[7], [9], and [10] and will consider questions such as stability, Lyapunov equations, and Riccati equations.

In the next section we introduce the class of TPBVDSs, and define and characterize two notions of shift-invariant systems, namely *displacement* and *stationary systems*. As with the systems considered in [5]-[7], [9], and [10], shift invariance requires the system matrices to have certain properties which facilitate our analysis. In Section 3 we review the notions of inward and outward processes introduced for TPBVDSs in [17] and show in particular that the inward process has a simple form for displacement systems. We also introduce the notion of *extendibility* and characterize this property for displacement and stationary systems. Section 4 develops the properties of reachability and observability for displacement systems, while in Section 5 we present minimality results for stationary systems. Some extensions are presented in Section 6, and we conclude with a brief discussion in Section 7.

2. Shift-invariant two-point boundary-value descriptor systems

A TPBVDS is described by the following dynamic equation, boundary condition, and output:

$$Ex(k+1) = Ax(k) + Bu(k), \qquad 0 \le k \le N-1, \tag{2.1}$$

$$V_i x(0) + V_f x(N) = v, \tag{2.2}$$

$$y(k) = Cx(k), \qquad k = 0, 1, \ldots, N. \tag{2.3}$$

Here x and v are n-dimensional, u is m-dimensional, y is p-dimensional, and E, A, B, V_i, V_f, and C are constant matrices. We also assume that $N \geq 2n$ so that all modes can be excited and observed. In [17] it is shown that if (2.1)-(2.2) is well-posed, we can assume, without loss of generality, that (2.1)-(2.2) is in *normalized form*, i.e., that there exist scalars α and β such that

$$\alpha E + \beta A = I \tag{2.4}$$

(this is referred to as the *standard form* for the pencil $\{E, A\}$) and in addition

$$V_i E^N + V_f A^N = I. \tag{2.5}$$

Note that (2.4) implies that E and A commute and also that $\{E^k, A^k\}$ is regular for all $k \geq 0$ (see [17]).

As derived in [17], the map from $\{u, v\}$ to x has the following form:

$$x(k) = A^k E^{N-k} v + \sum_{j=0}^{N-1} G(k, j) B u(j), \tag{2.6}$$

where Green's function $G(k, j)$ is given by

$$G(k, j) = \begin{cases} A^k (A - E^{N-k}(V_i A + \omega V_f E) E^k) E^{j-k} A^{N-j-1} \Gamma^{-1}, & j \geq k, \\ E^{N-k}(\omega E - A^k (V_i A + \omega V_f E) A^{N-k}) E^j A^{k-j-1} \Gamma^{-1}, & j < k, \end{cases} \tag{2.7}$$

and where ω is any number for which Γ is invertible, where

$$\Gamma \triangleq \omega E^{N+1} - A^{N+1}. \tag{2.8}$$

In marked contrast to the case for causal systems ($E = I$, $V_f = 0$), $G(k, j)$ does *not*, in general, depend on the difference in its arguments. Borrowing terminology from [5]-[7] we introduce:

Definition 2.1. The TPBVDS (2.1)-(2.2) is a *displacement system* if (with the usual abuse of notation)

$$G(k, j) = G(k - j), \qquad 0 \leq k \leq N, \quad 0 \leq j \leq N - 1. \tag{2.9}$$

With $v = 0$ in (2.2), we have that (2.1)-(2.3) define a linear map of the form

$$y(k) = \sum_{j=0}^{N-1} W(k, j) u(j), \tag{2.10}$$

$$W(k, j) = C G(k, j) B. \tag{2.11}$$

Definition 2.2. The TPBVDS (2.1)-(2.3) is *stationary* if (again with the usual abuse of notation)

$$W(k, j) = W(k - j), \qquad 0 \leq k \leq N, \quad 0 \leq j \leq N - 1. \tag{2.12}$$

Theorem 2.1. *The TPBVDS* (2.1)-(2.3) *is stationary if and only if*

$$O_s[V_i, E]R_s = O_s[V_i, A]R_s = 0, \tag{2.13a}$$

$$O_s[V_f, E]R_s = O_s[V_f, A]R_s = 0, \tag{2.13b}$$

where $[X, Y] = XY - YX$, *and*

$$R_s = [A^{n-1}B | EA^{n-2}B | \cdots | E^{n-1}B], \tag{2.14}$$

$$O_s = \begin{bmatrix} CA^{n-1} \\ CEA^{n-2} \\ \vdots \\ CE^{n-1} \end{bmatrix}. \tag{2.15}$$

Corollary. *The TPBVDS* (2.1)-(2.2) *is a displacement system if and only if*

$$[V_i, E] = [V_i, A] = 0, \tag{2.16a}$$

$$[V_f, E] = [V_f, A] = 0. \tag{2.16b}$$

The matrices R_s and O_s in (2.14), (2.15) are the *strong reachability* and *strong observability* matrices of the TPBVDS (see Section 4). Thus (2.13) states that V_i and V_f must commute with E and A except for parts that are either in the left nullspace of R_s or the right nullspace of O_s. If R_s and O_s are of full rank—i.e., if the TPBVDS is *strongly reachable* and *strongly observable*—V_i and V_f must commute with E and A. Turning to the corollary, we see that this is precisely the condition for a TPBVDS to be displacement, so that a displacement system is always stationary. Furthermore, the only way in which a TPBVDS can be stationary without being displacement is if the system is not strongly reachable or strongly observable. The results of causal system theory might then suggest that this distinction between displacement and stationary is a trivial artifact caused by the use of possible nonminimal realizations. However, as in the case of continuous-time boundary-value systems [10], the story is different for TPBVDSs. Specifically, as is shown in Section 5, a TPBVDS can be minimal without being strongly reachable or strongly observable.

Proof of the Corollary. We assume that Theorem 2.1 holds. From the theorem, a TPBVDS is displacement if and only if (2.13) holds with R_s and O_s defined with $C = B = I$. However, thanks to the generalized Cayley-Hamilton theorem for pencils in standard form [17], the matrices $\{A^k E^{n-k-1} | 0 \leq k \leq n-1\}$ span the same set as $\{E^k A^j | k, j \geq 0\}$. Thus R_s and O_s are of full rank, so that (2.13) is equivalent to (2.16). □

Proof of Theorem 2.1. What we must show is that (2.13) is equivalent to

$$W(k+1, j+1) = W(k, j) \tag{2.17}$$

for $0 \leq k \leq N-1$, $0 \leq j \leq N-2$. Then, using (2.7), the commutativity of E and A, and performing some algebra we find that (2.17) is equivalent to

$$CA^{k+1}E^{N-k-1}[V_iA+\omega V_fE]A^{N-j-2}E^{j+1}\Gamma^{-1}B$$
$$= CA^kE^{N-k}[V_iA+\omega V_fE]A^{N-j-1}E^j\Gamma^{-1}B. \tag{2.18}$$

From the Cayley-Hamilton theorem and the fact that $N \geq 2M$, we find that (2.18) is equivalent to

$$O_sA[V_iA+\omega V_fE]E\Gamma^{-1}R_s = O_sE[V_iA+\omega V_fE]A\Gamma^{-1}R_s. \tag{2.19}$$

Define the strong reachability subspace

$$\mathscr{R}_s = \mathrm{Im}(R_s). \tag{2.20}$$

Then the generalized Cayley-Hamilton theorem implies that $\mathscr{R}_s$ is A- and E- and therefore also Γ-invariant. Furthermore, for almost all ω, Γ is invertible so that the range of $\Gamma^{-1}\mathscr{R}_s$ is $\mathscr{R}_s$. Since this does not depend on ω, we can deduce that (2.19) is equivalent to the following pair of equalities:

$$O_s[AV_iE-EV_iA]A\Gamma^{-1}R_s=0, \tag{2.21}$$

$$O_s[AV_fE-EV_fA]E\Gamma^{-1}R_s=0. \tag{2.22}$$

Since $\{E^N, A^N\}$ is regular, $\mathscr{R}_s = \mathrm{Im}([A^NR_s \mid E^NR_s])$ so that (2.21) is equivalent to

$$O_s[AV_iE-EV_iA]A\Gamma^{-1}A^NR_s=0, \tag{2.23}$$

$$O_s[AV_iE-EV_iA]A\Gamma^{-1}E^NR_s=0. \tag{2.24}$$

In a similar fashion we have that (2.22) is equivalent to the pair of equalities

$$O_s[AV_fE-EV_fA]E\Gamma^{-1}A^NR_s=0, \tag{2.25}$$

$$O_s[AV_fE-EV_fA]E\Gamma^{-1}E^NR_s=0. \tag{2.26}$$

Using the commutativity of E and A together with (2.5), we see that (2.25) is equivalent to

$$O_s[-AV_iE+EV_iA]E^{N+1}\Gamma^{-1}R_s=0. \tag{2.27}$$

Using the definition of Γ, we see that (2.23) and (2.27) imply

$$O_s[AV_iE-EV_iA]R_s=0. \tag{2.28}$$

In a similar fashion (2.24) and (2.27) can be shown to imply

$$O_s[AV_fE-EV_fA]R_s=0. \tag{2.29}$$

The E- and A-invariance of $\mathscr{R}_s$ then imply that (2.28), (2.29) are, in fact, equivalent to (2.21), (2.22).

Finally, note that thanks to the commutativity of E and A, (2.13a) implies (2.28) and (2.13b) implies (2.29). To see that the reverse of these implications holds, assume that $\alpha \neq 0$ in (2.4) (if $\alpha = 0$, reverse the role of E and A). Then $E = \gamma I + \delta A$ with $\gamma \neq 0$. Substituting this into (2.28) yields (2.13a). Similarly (2.29) implies (2.13b). □

The characterization of the displacement property in (2.13) simplifies many computations. In particular, it is not difficult to check that Green's function of a displacement system is

$$G(k)=\begin{cases} V_iA^{k-1}E^{N-k}, & k>0, \\ -V_fE^{-k}A^{N+k-1}, & k\leq 0. \end{cases} \tag{2.30}$$

Similarly, the weighting pattern of a stationary TPBVDS is given by

$$W(k)=\begin{cases} CV_iA^{k-1}E^{N-k}B, & k>0, \\ -CV_fE^{-k}A^{N+k-1}B, & k\leq 0. \end{cases} \tag{2.31}$$

3. Inward processes, outward processes, and extendibility

As discussed in [10], inward and outward processes play an important role in the analysis of TPBVDSs. The outward process, which expands outward toward the boundaries, summarizes what we need to know about the input inside any interval in order to determine x outside the interval. The inward process uses input values near the boundary to propagate the boundary condition inward.

The outward process has a simple definition and characterization [17]:

$$z_o(k,j)=E^{j-k}x(j)-A^{j-k}x(k), \qquad k<j. \tag{3.1}$$

It is possible to express $z_o(k,j)$ in terms of the intervening inputs:

$$z_o(k,j)=\sum_{r=k}^{j-1} E^{r-k}A^{j-r-1}Bu(r) \tag{3.2}$$

and to write outward recursions (k decreasing and j increasing). In general $z_o(k,j)$ can only be propagated in an outward direction. Note also that $z_o(k,j)$ does not involve the boundary matrices V_i and V_f.

The inward process $z_i(K,L)$, $K\leq L$, is a function of the boundary value v and the inputs $\{u(0),\ldots,u(K-1)\}$ and $\{u(L),\ldots,u(N-1)\}$ so that the TPBVDS (2.1) with boundary condition

$$V_i(K,L)x(K)+V_f(K,L)x(L)=z_i(K,L) \tag{3.3}$$

yields the same solution as (2.1), (2.2) for $K\leq k\leq L$ see [17]. Here $V_i(K,L)$ and $V_f(K,L)$ are assumed to be such that (2.1), (3.3) is in normalized form, i.e.,

$$V_i(K,L)E^{L-K}+V_f(K,L)A^{L-K}=I. \tag{3.4}$$

Note in particular the starting values and the "final values"

$$z_i(0,N)=v, \qquad V_i(0,N)=V_i, \qquad V_f(0,N)=V_f, \tag{3.5}$$

$$z_i(k,k)=x(k) \qquad \text{for all } k. \tag{3.6}$$

For the general TPBVDS (2.1), (2.2) there are no simple formulas or recursions for z_i, V_i, and V_f (see [17]). However, we do have the following for displacement systems:

Proposition 3.1. *Assume that* (2.1)-(2.2) *is a displacement system. Then for* $k \leq j$

$$V_i(k, j) = V_i E^{N-j+k}, \tag{3.7}$$

$$V_f(k, j) = V_f A^{N-j+k}, \tag{3.8}$$

$$z_i(k, j) = E^{N-j} A^k v + V_i E^{N-j} z_o(0, k) - V_f A^k z_o(j, N) \tag{3.9}$$

$$= E^{N-j} A^k v + V_i E^{N-j} \sum_{r=0}^{k-1} E^r A^{k-r-1} B u(r)$$

$$- V_f A^k \sum_{s=1}^{N-1} E^{s-1} A^{N-s-1} B u(s). \tag{3.10}$$

Proof. First, (2.5) guarantees that the definitions in (3.7), (3.8) yield TPBVDSs in normalized form for all $k \leq j$. Equations (3.9) and (3.10) are then obtained by replacing $x(k)$ and $x(j)$ in

$$z_i(k, j) = V_i E^{N-j+k} x(k) + V_f A^{N-j+k} x(j) \tag{3.11}$$

by their expressions in terms of v and u in (2.6) using (2.30). Thus we have from (3.10) that $z_i(k, j)$ depends only on v and the values of u *off* the interval $[k, j]$. Finally, to show that (2.1), (3.3) yields the same solution, we note that the pair of relations (3.1) and (3.11) have a simple inverse:

$$\begin{bmatrix} x(k) \\ x(j) \end{bmatrix} = \begin{bmatrix} -V_f A^{N-j+k} & E^{j-k} \\ V_i E^{N-j+k} & A^{j-k} \end{bmatrix} \begin{bmatrix} z_o(k, j) \\ z_i(k, j) \end{bmatrix}. \tag{3.12}$$

Thus $x(j)$ and $x(k)$ can be obtained completely from $z_i(k, j)$ as we have defined it, and the outward process, which is not changed by restricting the size of the interval. Thus the values of x at these two points are correct, and therefore by moving in one step at a time we conclude that (2.1), (3.3), with the choice of V_i and V_f given by (3.7), (3.8) yields the correct solution. □

An important interpretation of the inward process is that the Green's function for the system (2.1), (3.3) on the smaller interval $[K, L]$ is the *restriction* of the Green's function of the original system (2.1)-(2.2) defined on $[0, N]$. A logical question then is whether we can also move the boundary conditions outward to obtain an *extension* of the Green's function.

Definition 3.1. A stationary TPBVDS (2.1)-(2.2) is *input-output extendible* if, given any $K \le 0 < N \le L$, there exists a TPBVDS over $[K, L]$ with the same dynamics as in (2.1) but with new boundary matrices such that:

(i) The new system is stationary.
(ii) The weighting pattern $W(k-j)$ of the original system is the restriction of the weighting pattern $W_e(k-j)$ of the new extended system, i.e., $W(k-j) = W_e(k-j)$, $1-N \le k-j \le N$.

Definition 3.2. A displacement TPBVDS (2.1)-(2.2) is *extendible* if, given any $K \le 0 < N \le L$, there exists a TPBVDS over $[K, L]$ with the same dynamics as in (2.1) but with new boundary matrices such that:

(i) This new system has the displacement property.
(ii) The Green's function $G(k-j)$ of the original system is the restriction of the weighting pattern $G_e(k-j)$ of the new extended system, i.e., $G(k-j) = G_e(k-j)$, $1-N \le k-j \le N$.

For any matrix F, define its *Drazin Inverse* [4], F^{D}, and its *invertible modification* $\tilde{F}$ as follows. Let T be an invertible matrix such that

$$F = T\begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix} T^{-1}, \tag{3.13}$$

where M is invertible and N is nilpotent (e.g., real Jordan form will do). Then

$$F^{\mathrm{D}} = T\begin{bmatrix} M^{-1} & 0 \\ 0 & 0 \end{bmatrix} T^{-1}, \tag{3.14}$$

$$\tilde{F} = T\begin{bmatrix} M & 0 \\ 0 & N+I \end{bmatrix} T^{-1}. \tag{3.15}$$

These matrices have a number of important properties:

(i) F^{D} and $\tilde{F}$ commute with each other and with F.
(ii) If F is invertible, $F^{\mathrm{D}} = F^{-1}$, and $\tilde{F} = F$.
(iii) If μ is the degree of nilpotency of N, then for $k \ge \mu$

$$F^{k+1}F^{\mathrm{D}} = F^k, \qquad F^k\tilde{F} = F^{k+1}. \tag{3.16}$$

(iv)
$$F^{\mathrm{D}}F = F^{\mathrm{D}}\tilde{F}. \tag{3.17}$$

(v) The condition

$$\operatorname{Ker}(F^{\mu}) \subset \operatorname{Ker}(G) \tag{3.18}$$

is equivalent to

$$GF^{\mathrm{D}}F = G. \tag{3.19}$$

(vi) If $\mathscr{H}$ is an F-invariant subspace, then $F^{\mathrm{D}}F\mathscr{H} \subset \mathscr{H}$ and is also F-invariant.

Theorem 3.1. *A stationary weighting pattern is extendible if and only if*

$$O_s[V_i - V_i E^D E]R_s = 0, \tag{3.20a}$$

$$O_s[V_f - V_f A^D A]R_s = 0. \tag{3.20b}$$

Corollary. *A displacement TPBVDS is extendible if and only if*

$$V_i - V_i E^D E = 0, \tag{3.21a}$$

$$V_f - V_f A^D A = 0. \tag{3.21b}$$

The corollary follows from the theorem exactly as in the proof of the corollary to Theorem 2.1. Note also that, thanks to (3.18) and (3.19), (3.21) is equivalent to

$$\operatorname{Ker}(E^n) \subset \operatorname{Ker}(V_i), \tag{3.22a}$$

$$\operatorname{Ker}(A^n) \subset \operatorname{Ker}(V_f), \tag{3.22b}$$

while (3.20) states that (3.22) holds modulo the strongly unreachable and unobservable subspaces. The interpretation of (3.22) is straightforward. From (3.7), (3.8) we see that moving in the boundaries involves left multiplication of V_i and V_f by powers of E and A, respectively. Thus if we first extend outward sufficiently far and then move back inward, we will have annihilated the parts of V_i and V_f acting on the nilpotent portions of E and A, respectively. Recovering the original system then requires that there is no such portion of V_i or V_f to annihilate, which is what (3.22) states.

Proof of Theorem 3.1. Suppose that we begin with a stationary TPBVDS (2.1)-(2.3). Then extendibility is equivalent to the existence, for each $K \geq N$, of matrices $V_{i,K}$ and $V_{f,K}$ so that the TPBVDS (2.1), (2.3) with

$$V_{i,K}\, x(0) + V_{f,K}\, x(K) = v \tag{3.23}$$

is stationary and in normalized form, and the weighting pattern of (2.1), (3.23), (2.3) agrees with that of (2.1)-(2.3) over the range of lags arising in (2.1)-(2.3) (obviously $V_{i,N} = V_i$ and $V_{f,N} = V_f$).

Let us first show the necessity of (3.20). In particular, since (2.1), (3.23), (2.3) is stationary for each K, we can use (2.31) to write the weighting pattern matching condition:

$$CV_{i,K}A^{k-1}E^{K-k}B = CV_iA^{k-1}E^{N-k}B, \qquad \forall K \geq N, \quad k = 1, \ldots, N, \tag{3.24a}$$

$$CV_{f,K}E^{-k}A^{K+k-1}B = CV_fE^{-k}A^{N+k-1}B, \qquad \forall K \geq N, \quad k = -N+1, \ldots, 0. \tag{3.24b}$$

Writing $K = N + j$ and using Cayley-Hamilton and (2.13a) we find that (3.24a) is equivalent to

$$O_s[V_{i,N+j}E^j - V_i]R_s = 0, \qquad \forall j \geq 0. \tag{3.25}$$

Thus we see that for each j there exists a matrix M_j so that

$$O_s M_j R_s = 0, \tag{3.26}$$

$$\mathrm{Ker}(E^j) \subset \mathrm{Ker}(V_i + M_j). \tag{3.27}$$

Applying (3.18), (3.22) to (3.27) we see that (3.26) yields

$$O_s[V_i - V_i E^{\mathrm{D}} E] R_s = O_s M_j E^{\mathrm{D}} E R_s. \tag{3.28}$$

Since $\mathscr{R}_s$ is E-invariant, property (vi) implies that there exists a matrix Q so that

$$E^{\mathrm{D}} E R_s = R_s Q. \tag{3.29}$$

Equation (3.20a) then follows from (3.26) and (3.27). Similarly (3.26b) implies (3.20b).

To show the sufficiency of (3.20) we need to construct matrices $V_{i,K}$ and $V_{f,K}$. It is straightforward to check that several alternate forms for the solutions can be given. For example,

$$V_{i,K} = V_i \tilde{E}^{N-K}, \tag{3.30a}$$

$$V_{f,K} = V_f \tilde{A}^{N-K} \tag{3.30b}$$

have the required properties. Alternatively,

$$V_{i,K} = P(E^{\mathrm{D}})^K, \tag{3.31a}$$

$$V_{f,K} = (I - P)(A^{\mathrm{D}})^K \tag{3.31b}$$

with

$$P = V_i E^N \tag{3.32}$$

and indeed both of these forms yield the correct boundary matrices for $K < N$ as well. □

The matrix P introduced in (3.32) is an extremely useful one for simplifying the description of extendible systems. In particular, the specification in (2.1)-(2.3) is equivalent to providing a 7-tuple

$$(C, V_i, V_f, E, A, B, N) \tag{3.33}$$

so that the TPBVDS is in normalized form over $[0, N]$. For an extendible system, we need only

$$(C, P, E, A, B) \tag{3.34}$$

with the requirement that (E, A) be in standard form. The corresponding 7-tuple over $[0, N]$ is then

$$(C, P(E^{\mathrm{D}})^N, (I-P)(A^{\mathrm{D}})^N, E, A, B, N). \tag{3.35}$$

Also, the weighting pattern $W(k)$, $k \in (-\infty, \infty)$ can be conveniently expressed in terms of P:

$$W(k) = \begin{cases} CPE^{\mathrm{D}}(AE^{\mathrm{D}})^{k-1}B, & k > 0, \\ -C(I-P)A^{\mathrm{D}}(EA^{\mathrm{D}})^{-k}B, & k \leq 0. \end{cases} \tag{3.36}$$

Finally, it is worth noting that while not all stationary or displacement systems are extendible, for any such system we can find one that is extendible and is "almost the same" in that the responses—$y(k)$ for stationary systems, $x(k)$ for displacement systems—are identical for the original and modified systems for $k \in [n, N-n]$. In fact, by examining (2.32) and (2.33) we see that such a TPBVDS can be obtained by replacing V_i and V_f by the lowest rank matrices $\tilde{V}_i$ and $\tilde{V}_f$ satisfying

$$\tilde{V}_i E^n = V_i E^n, \tag{3.37a}$$

$$\tilde{V}_f A^n = V_f A^n. \tag{3.37b}$$

These choices guarantee that the $\tilde{V}_i$ and $\tilde{V}_f$ annihilate the nilpotent portions of E and A which affect behavior only near the boundary.

4. Reachability and observability

As discussed in [17], there are two notions for both reachability and observability. In this section we review these definitions and present additional results for displacement systems.

Definition 4.1. The system (2.1)-(2.2) is *strongly reachable* on $[K, L]$ if the map

$$\{u(k) \colon k \in [K, L]\} \to z_{\mathrm{o}}(K, L) \tag{4.1}$$

is onto. The system is *strongly reachable* if it is strongly reachable on some interval.

From (3.2) we can write

$$z_{\mathrm{o}}(K, L) = R_{\mathrm{s}}(L-K) \begin{bmatrix} u(K) \\ \vdots \\ u(L-1) \end{bmatrix}, \tag{4.2}$$

$$R_{\mathrm{s}}(j) = [A^{j-1}B \,|\, EA^{j-2}B \,|\, \cdots \,|\, E^{j-1}B]. \tag{4.3}$$

Note that $R_{\mathrm{s}} = R_{\mathrm{s}}(n)$. Furthermore, a TPBVDS is strongly reachable if and only if R_{s} has full rank, and the strongly reachable spaces have the usual nesting property, i.e.,

$$\mathcal{R}_{\mathrm{s}}(k) = \mathrm{Im}[R_{\mathrm{s}}(k)] \subset \mathrm{Im}[R_{\mathrm{s}}(k+1)] = \mathcal{R}_{\mathrm{s}}(k+1). \tag{4.4}$$

Definition 4.2. The system (2.1)-(2.3) is *strongly observable* on $[K, L]$ if the map

$$z_i(K, L) \rightarrow \{y(k) : k \in [K, L]\} \tag{4.5}$$

defined by (2.1), (3.3), and (2.3) with $u \equiv 0$ on $[K, L]$ is one to one. The system is *strongly observable* if it is strongly observable on some $[K, L]$.

With $u \equiv 0$, we have

$$\begin{bmatrix} y(K) \\ \vdots \\ y(L) \end{bmatrix} = O_s(L-K) z_i(K, L), \tag{4.6}$$

$$O_s(j) = \begin{bmatrix} CE^j \\ CAE^{j-1} \\ \vdots \\ CA^j \end{bmatrix}. \tag{4.7}$$

Note that $O_s = O_s(n-1)$. Furthermore, a TPBVDS is strongly observable if and only if O_s has full rank. In addition, the strong unobservability subspaces have the usual nesting property

$$\mathcal{O}_s(k+1) = \operatorname{Ker}(O_s(k+1)) \subset \mathcal{O}_s(k) = \operatorname{Ker}(O_s(k)). \tag{4.8}$$

For future reference we define the strongly unobservable subspace

$$\mathcal{O}_s = \operatorname{Ker}(O_s). \tag{4.9}$$

Definition 4.3. The system (2.1)-(2.2) is *weakly reachable* off $[K, L]$ if the map

$$\{u(k) \colon k \in [0, K-1] \cup [L, N-1]\} \rightarrow z_i(K, L) \tag{4.10}$$

with $v = 0$ is onto. The *weakly reachable* subspace $\mathcal{R}_w(K, L)$ is the range of this map.

In contrast to strong reachability, weak reachability involves the boundary matrices as well as E, A, and B. For the general TPBVDS (2.1)-(2.2) the investigation of weak reachability, in [17], is somewhat complicated since no simple form exists for the inward process. The situation is far simpler for displacement and extendible displacement systems.

Proposition 4.1. *For $K, L \in [n, N-n]$ we have that for a displacement system*

$$\mathcal{R}_w(K, L) \equiv \operatorname{Im}[V_i E^N R_s \mid V_f A^N R_s] = V_i E^N \mathcal{R}_s + V_f A^N \mathcal{R}_s. \tag{4.11}$$

Proof. From (3.9) and (4.2) we see that

$$\mathcal{R}_w(K, L) = V_i E^{N-L} \mathcal{R}_s(K) + V_f A^K \mathcal{R}_s(N-L). \tag{4.12}$$

Thus, for K, $N-L \geq n$, what we would like to show is that

$$V_i E^{N-L} \mathcal{R}_s + V_f A^K \mathcal{R}_s = V_i E^N \mathcal{R}_s + V_f A^N \mathcal{R}_s. \tag{4.13}$$

This follows easily from the fact that

$$E^s \mathcal{R}_s = E^r \mathcal{R}_s, \qquad A^s \mathcal{R}_s = A^r \mathcal{R}_s \qquad \text{for} \quad s, r \geq n, \tag{4.14}$$

which in turn follows from the E- and A-invariance of $\mathcal{R}_s$ and dimension counting. □

Thus we see that far enough from the boundaries the weakly reachable space for a displacement system is constant. However, the displacement property by itself is not enough to guarantee that the weakly reachable spaces near the boundaries are contained in the space defined in (4.11). This property does hold if the TPBVDS is extendible.

Proposition 4.2. *Define the following matrix and subspace*

$$R_w = [\, V_i R_s \,|\, V_f R_s], \qquad \mathcal{R}_w = \mathrm{Im}[R_w]. \tag{4.15}$$

Then for an extendible displacement system the following two properties hold:

$$\mathcal{R}_w(K, L) = \mathcal{R}_w \qquad \textit{for} \quad K, L \in [n, N-n], \tag{4.16}$$

$$\mathcal{R}_w(K, L) \subset \mathcal{R}_w \qquad \textit{for all } K, L. \tag{4.17}$$

Proof. Thanks to the invariance of $\mathcal{R}_s$ and the fact that $\mathcal{R}_s(j) \subset \mathcal{R}_s$, (4.17) follows if we can show (4.16). However, for an extendible displacement system, (3.22) holds, which, since $N \geq 2n$, implies

$$V_i E^N \mathcal{R}_s = V_i \mathcal{R}_s, \qquad V_f A^N \mathcal{R}_s = V_f \mathcal{R}_s \tag{4.18}$$

proving the proposition. □

For a nonextendible system, $\mathcal{R}_w$ may be larger than $\mathcal{R}_w(K, L)$ for *any* K and L. However, recall that in this case the weakly reachable space need not be contained in the weakly reachable space (4.11) far from the boundary. The net effect is that the *union* of the weakly reachable subspaces over all (K, L) pairs can be larger that any one of them. In fact, by examining the range of the mapping in (3.10) for all values of k and j and using Cayley-Hamilton, we have the following:

Proposition 4.3. *For any displacement system*

$$\bigcup_{K,L} \mathcal{R}_w(K, L) = \mathcal{R}_w. \tag{4.19}$$

We call a system *weakly reachable* if

$$\bigcup_{K,L} \mathcal{R}_w(K, L) = \mathbb{R}^n \tag{4.20}$$

(note that in [17], we called a system weakly reachable if $\mathcal{R}_w(K, L) = \mathbb{R}^n$ for all $K, L \in [n, N-n]$; we see later why this new definition of weak reachability is more appropriate). Clearly, a displacement system is weakly reachable if $\mathcal{R}_w = \mathbb{R}^n$.

In analogy with the strong reachability result in [17], we state without proof the following.

Proposition 4.4. *A displacement system is weakly reachable if and only if the matrix* $[sE - tA | V_i B | V_f B]$ *has full rank for all* $(s, t) \neq (0, 0)$.

Another important fact follows, which justifies our use of the terms "strong" and "weak."

Proposition 4.5. *The following inclusion always holds*:

$$\mathcal{R}_s \subset \mathcal{R}_w. \tag{4.21}$$

Proof. The E- and A-invariance of $\mathcal{R}_s$ guarantees the existence of M_1 and M_2 so that

$$R_s M_1 = E^N R_s, \qquad R_s M_2 = A^N R_s. \tag{4.22}$$

Therefore, since $V_i E^N + V_f A^N = I$,

$$\begin{aligned}\mathcal{R}_s = \operatorname{Im}[R_s] &= \operatorname{Im}[(V_i E^N + V_f A^N) R_s] = \operatorname{Im}[V_i R_s M_1 + V_f R_s M_2] \\ &\subset \operatorname{Im}[V_i R_s | V_f R_s] = \mathcal{R}_w. \qquad \square\end{aligned} \tag{4.23}$$

As we would expect, there is a dual set of concepts and results for weak observability:

Definition 4.4. The system (2.1)-(2.3) is *weakly observable* off $[K, L]$ if the map

$$z_o(K, L) \to \{y(k) \colon k \in [0, K] \cup [L, N]\} \tag{4.24}$$

with $v = 0$ and $u(j) = 0, j \in [0, K-1] \cup [L, N-1]$ is one to one. The *weakly unobservable subspace* $\mathcal{O}_w(K, L)$ is the kernel of this map.

Proposition 4.6. *For* $K, L \in [n, N-n]$ *we have that for a displacement system*

$$\mathcal{O}_w(K, L) = \operatorname{Ker}\begin{bmatrix} O_s E^N V_i \\ O_s A^N V_f \end{bmatrix}. \tag{4.25}$$

Proposition 4.7. *Define the following matrix and subspace:*

$$O_w = \begin{bmatrix} O_s V_i \\ O_s V_f \end{bmatrix}, \qquad \mathcal{O}_w = \mathrm{Ker}[O_w], \tag{4.26}$$

then for an extendible displacement system we have

$$\mathcal{O}_w(K, L) = \mathcal{O}_w \qquad for \quad K, L \in [n, N-n], \tag{4.27}$$

$$\mathcal{O}_w \subset \mathcal{O}_w(K, L) \qquad for\ all\ K, L. \tag{4.28}$$

Thus, not only is the dimension of the weakly unobservable space constant for K, L far from the boundaries (which is always the case [17]), the space itself is constant. Furthermore, in the extendible case we have a simpler form for the unobservable space, as well as the nesting property.

Proposition 4.8. *For any displacement system*

$$\bigcap_{K,L} \mathcal{O}_w(K, L) = \mathcal{O}_w. \tag{4.29}$$

We call a system *weakly observable* if

$$\bigcap_{K,L} \mathcal{O}_w(K, L) = \{0\} \tag{4.30}$$

(again this is in contrast with the weak observability definition in [17] where we required that $\mathcal{O}_w(K, L) = \{0\}$ for all $K, L \in [n, N-n]$). Clearly, a displacement system is weakly observable if $\mathcal{O}_w = \{0\}$.

Proposition 4.9. *A displacement system is weakly observable if and only if the matrix*

$$\begin{bmatrix} sE - tA \\ CV_i \\ CV_f \end{bmatrix}$$

has full rank for all $(s, t) \neq (0, 0)$.

Proposition 4.10. *The following inclusion always holds:*

$$\mathcal{O}_w \subset \mathcal{O}_s. \tag{4.31}$$

5. Minimality

In this section we present minimality results for stationary and extendible stationary systems. These results are analogous to those in [10], with differences due to possible singularity of E and A.

Definition 5.1. A stationary TPBVDS is *minimal* if x has the lowest dimension among all TPBVDSs having the same weighting pattern.

Theorem 5.1. *A stationary TPBVDS is minimal if and only if*

$$\text{(a)}\quad \mathcal{R}_w = \mathbb{R}^n, \tag{5.1}$$

$$\text{(b)}\quad \mathcal{O}_w = \{0\}, \tag{5.2}$$

$$\text{(c)}\quad \mathcal{O}_s \subset \mathcal{R}_s. \tag{5.3}$$

Note that this theorem is concerned with *stationary* rather than displacement TPBVDSs. Thus, $\mathcal{R}_w$ may differ from the weak reachability subspace and $\mathcal{O}_w$ from the weak observability subspace. Nevertheless, these spaces are the keys to minimality. Also, as we will see, we need to introduce three different Hankel matrices and, as in [10], we may have a certain level of nonuniqueness in minimal realizations that is not present in the causal case.

Proof. We begin with the description of reduction procedures if any of the conditions (5.1)-(5.3) are not satisfied. Consider first the case in which $\mathcal{R}_w \neq \mathbb{R}^n$. Suppose further that $\mathcal{R}_w$ is A- and E-invariant. In this case let $\mathcal{R}_2$ be any subspace such that

$$\mathcal{R}_w \oplus \mathcal{R}_2 = \mathbb{R}^n. \tag{5.4}$$

Performing a similarity transformation compatible with (5.4) we can assume that

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \qquad E = \begin{bmatrix} E_{11} & E_{12} \\ 0 & E_{22} \end{bmatrix}, \qquad V_i = \begin{bmatrix} V_{11}^i & V_{12}^i \\ V_{21}^i & V_{22}^i \end{bmatrix}, \tag{5.5a}$$

$$V_f = \begin{bmatrix} V_{11}^f & V_{12}^f \\ V_{21}^f & V_{22}^f \end{bmatrix}, \qquad C = [C_1 \,|\, C_2], \qquad B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}. \tag{5.5b}$$

The 0-blocks in A and E follow from the assumed invariance, the 0-block in B from the fact that $\mathrm{Im}[B] \subset \mathcal{R}_s$ (Cayley-Hamilton) and $\mathcal{R}_s \subset \mathcal{R}_w$ (Proposition 4.5 which holds for *any* TPBVDS). Furthermore, using (4.17) we can conclude that

$$V_{21}^i A_{11}^k E_{11}^j B_1 = V_{21}^f A_{11}^k E_{11}^j B_1 = 0. \tag{5.6}$$

From the form (2.31) for the weighting pattern of a stationary system we can then conclude that the weighting pattern of our system is given by

$$W(k) = \begin{cases} C_1 V_{11}^i A_{11}^{k-1} E_{11}^{N-k} B_1, & k > 0, \\ -C_1 V_{11}^f E_{11}^{-k} A_{11}^{N+k-1} B_1, & k \le 0, \end{cases} \tag{5.7}$$

so that we have apparently reduced our system to $(C_1, V_{11}^i, V_{11}^f, E_{11}, A_{11}, B_1)$. Note that assuming that E and A are in standard form, so are E_{11} and A_{11}. However, the boundary matrices V_{11}^i and V_{11}^f need not be normalized and indeed there is no guarantee that this TPBVDS is well-posed. However, thanks to the following result, we can modify the boundary matrices in order to make it well-posed while leaving the weighting pattern unchanged.

Lemma 5.1. *Consider a (possibly not well-posed) TPBVDS* (2.1)-(2.2), *with E and A in standard form and for which* (2.13) *and the following hold*:

$$O_s(V_iE^N + V_fA^N)R_s = O_sR_s. \tag{5.8}$$

Then we can find $\tilde{V}_i$, $\tilde{V}_f$ so that (2.5) *and* (2.13) *hold for $\tilde{V}_i$ and $\tilde{V}_f$, and furthermore*

$$O_s V_iR_s = O_s \tilde{V}_iR_s, \qquad O_s V_fR_s = O_s \tilde{V}_fR_s. \tag{5.9}$$

Proof of Lemma 5.1. Let

$$X = I - [V_iE^N + V_fA^N] \tag{5.10}$$

so that

$$O_sXR_s = 0. \tag{5.11}$$

Let a and b be any scalars such that $(aE^N + bA^N)$ is invertible, and then take

$$\tilde{V}_i = V_i + aX(aE^N + bA^N)^{-1}, \tag{5.12a}$$

$$\tilde{V}_f = V_f + bX(aE^N + bA^N)^{-1}. \tag{5.12b}$$

From (5.11) and the A- and E-invariance of $\mathcal{R}_s$ and $\mathcal{O}_s$ we have

$$O_sA^kE^jXA^rE^sR_s = 0, \qquad k, j, r, s \geq 0, \tag{5.13}$$

from which we can easily check that (2.13) holds for $\tilde{V}_i$ and $\tilde{V}_f$ and that (5.9) holds as well. Finally, (2.5) can also be checked by direct calculation. □

The significance of this lemma can be seen as follows. Note first that the block elements of the matrix $O_s V_iR_s$ are $\{CA^kE^{n-k-1}V_iA^jE^{n-j-1}B, 0 \leq k, j \leq n-1\}$. Furthermore, thanks to (2.13), for a stationary system these are the same as $\{CV_iA^mE^{2n-m-2}B, 0 \leq m \leq 2n-2\}$. From Cayley-Hamilton and (2.31), we can conclude that $O_s V_iR_s$ provides us with a complete specification of $W(k)$, $k > 0$—i.e., given $O_s V_iR_s$ we can completely determine $W(k)$, $k > 0$. Similarly, $O_s V_fR_s$ specifies $W(k)$, $k \leq 0$. Thus what Lemma 5.1 says is that as long as (2.13) and (5.18) hold, we can modify the boundary matrices to obtain a stationary TPBVDS in normalized form with the same weighting pattern.

To apply this lemma to our reduced system, we must show that (2.13) and (5.8) hold for this system. Note first that (2.13) holds for our original system (5.5). This together with (5.6) allows us to conclude that (2.13) holds for the reduced system as well. Furthermore, (5.5), (5.6) also imply much more:

$$CA^kE^jB = C_1A_{11}^kE_{11}^jB_1, \tag{5.14a}$$

$$CV_iA^kE^jB = C_1V_{11}^iA_{11}^kE_{11}^jB_1, \tag{5.14b}$$

$$CV_fA^kE^jB = C_1V_{11}^fA_{11}^kE_{11}^jB_1. \tag{5.14c}$$

Therefore, since our original system was assumed to be in normalized form

$$C_1[V_{11}^i E_{11}^N + V_{11}^f A_{11}^N] A_{11}^k E_{11}^j B_1 = C[V_i E^N + V_f A^N] A^k E^j B$$
$$= CA^k E^j B = C_1 A_{11}^k E_{11}^j B_1 \qquad (5.15)$$

from which we conclude that

$$O_s^1[V_{11}^i E_{11}^N + V_{11}^f A_{11}^N] R_s^1 = O_s^1 R_s^1, \qquad (5.16)$$

where O_s^1 and R_s^1 are the strong observability and reachability matrices for (5.8).

Thus we have provided a procedure for reducing the dimension of a stationary TPBVDS if $\mathcal{R}_w \neq \mathbb{R}^n$, provided that $\mathcal{R}_w$ is E- and A-invariant. If this is not the case, we use the following.

Lemma 5.2. *Consider any stationary TPBVDS. Then we can construct another TPBVDS with the same weighting pattern, dimension, and $\mathcal{R}_w$ so that $\mathcal{R}_w$ is E- and A-invariant.*

Proof. To begin, assume $\alpha E + \beta A = I$ with $\alpha \neq 0$. If $\alpha = 0$ reverse the roles of E and A in what follows. The key to this result is (2.13) from which we can deduce that

$$A\mathcal{R}_w \subset \mathcal{R}_w + \mathcal{O}_s. \qquad (5.17)$$

Also recall that $\mathcal{R}_s$ is A-invariant. A basic result [18] is the following: Let D be a matrix such that

$$\mathrm{Im}(D) = \mathcal{O}_s. \qquad (5.18)$$

Then, thanks to Proposition 4.5, we can find a matrix F so that

$$F\mathcal{R}_s = \{0\}, \qquad (5.19)$$

$$(A + DF)\mathcal{R}_w \subset \mathcal{R}_w. \qquad (5.20)$$

Suppose we then let

$$\tilde{A} = A + DF, \qquad \tilde{E} = E - (\beta/\alpha)DF \qquad (5.21)$$

(so that $\alpha\tilde{E} + \beta\tilde{A} = I$). From (5.18), (5.20), and (5.21) we see that

$$C\tilde{A}^k\tilde{E}^j = CA^kE^j, \qquad \tilde{A}^k\tilde{E}^jB = A^kE^jB \qquad (5.22)$$

for all k and j, so that $O_s = \tilde{O}_s$, $R_s = \tilde{R}_s$, where $\tilde{O}_s$, $\tilde{R}_s$ are the strong observability and reachability matrices for $(C, \tilde{E}, \tilde{A}, B)$. Also, since the original system is stationary and D and F satisfy (5.18), (5.20),

$$\tilde{O}_s[\tilde{A}, V_i]\tilde{R}_s = \tilde{O}_s[\tilde{A}, V_f]\tilde{R}_s = 0, \qquad (5.23a)$$

$$\tilde{O}_s[\tilde{E}, V_i]\tilde{R}_s = \tilde{O}_s[\tilde{E}, V_f]\tilde{R}_s = 0. \qquad (5.23b)$$

Furthermore,

$$O_s(V_i\tilde{E}^N + V_f\tilde{A}^N)R_s = O_s(V_iE^N + V_fA^N)R_s = O_sR_s. \qquad (5.24)$$

Thus applying Lemma 5.1, we can construct $\tilde{V}_i$, $\tilde{V}_f$ so that the system $(C, \tilde{E}, \tilde{A}, \tilde{V}_i, \tilde{V}_f, B)$ is in normalized form and has the same weighting pattern as the original system. The only remaining thing to be checked is that $\tilde{\mathscr{R}}_w$ for this system is equal to the original $\mathscr{R}_w$.

From the definition of $\tilde{V}_i$, $\tilde{V}_f$ we have for $r_1, r_2 \in \mathscr{R}_s = \tilde{\mathscr{R}}_s$

$$\tilde{V}_i r_1 + \tilde{V}_f r_2 = V_i r_1 + V_f r_2 + q, \tag{5.25}$$

$$\begin{aligned} q = &-(a\tilde{E}^N + b\tilde{A}^N)^{-1}(ar_1 + br_2) \\ &+ [V_i\tilde{E}^N + V_f\tilde{A}^N](a\tilde{E}^N + b\tilde{A}^N)^{-1}(ar_1 + br_2). \end{aligned} \tag{5.26}$$

Since $\mathscr{R}_s$ is $\tilde{E}$- and $\tilde{A}$-invariant, the first term in (5.22) is in $\mathscr{R}_s$. Also, thanks to (5.22)

$$\tilde{E}^N r = E^N r, \qquad \tilde{A}^N r = A^N r, \qquad r \in \mathscr{R}_s. \tag{5.27}$$

Thus, since the original system is in normalized form, we can conclude that $q = 0$ and, thanks to the definition (4.17) of $\mathscr{R}_w$, Lemma 5.2 is proved. □

Reducing the dimension of the realization if (5.2) is violated is the dual of what we have just considered, and we omit the details. When (5.3) is not satisfied there is a subspace $\mathscr{X} \neq \{0\}$ such that

$$\mathscr{R}_s \oplus \mathscr{X} = \mathscr{R}_s + \mathscr{O}_s. \tag{5.28}$$

Let $\mathscr{W}$ be any subspace such that $\mathscr{W} \oplus \mathscr{X} = \mathbb{R}^n$ and perform a similarity transformation of the TPBVDS to represent it in a basis compatible with (5.4). This yields a model as in (5.5b), (5.5c) with the additional fact that $C_2 = 0$. To put the reduced system in normalized form we apply Lemma 5.1.

What remains to show is that two stationary TPBVDSs with the same weighting pattern and both satisfying (5.1)-(5.3) must have the same dimension and consequently are minimal. Consider two such systems $(C_j, E_j, A_j, V_j^i, V_j^f, B_j)$, $j = 1, 2$, and without loss of generality assume that both are in normalized form with the same α and β. What we know is that

$$C_1 V_1^i A_1^{k-1} E_1^{N-k} B_1 = C_2 V_2^i A_2^{k-1} E_2^{N-k} B_2, \qquad 0 < k \leq N, \tag{5.29a}$$

$$C_1 V_1^f E_1^{-k} A_1^{N+k-1} B_1 = C_2 V_2^f E_2^{-k} A_2^{N+k-1} B_2, \qquad 1 - N \leq k \leq 0. \tag{5.29b}$$

The following lemma allows us to conclude considerably more:

Lemma 5.3. *Let $\{E_i, A_i\}$, $i = 1, 2$, be two regular pencils so that $\alpha E_i + \beta A_i = I$, $i = 1, 2$, and $\dim(E_i) = n_i$. Assume that $N \geq 2\max(n_1, n_2)$. Also assume that for some matrices $\{M_i, N_i\}$, $i = 1, 2$,*

$$M_1 A_1^k E_1^{N-1-k} N_1 = M_2 A_2^k E_2^{N-1-k} N_2, \qquad 0 \leq k \leq N - 1. \tag{5.30}$$

Then for all K, L,

$$M_1 A_1^K E_1^L N_1 = M_2 A_2^K E_2^L N_2. \tag{5.31}$$

Proof. Note first that for $K+L \leq N-1$ we can write

$$E_i^K A_i^L = E_i^K A_i^L (\alpha E_i + \beta A_i)^{N-1-K-L} \tag{5.32}$$

and in this case (5.31) follows directly from (5.30). For $K+L \geq N$, let us assume for simplicity that $\alpha \neq 0$. From what we have first shown for $K+L \leq N-1$, we know that for $0 \leq k \leq N-1$

$$M_1 A_1^k N_1 = M_2 A_2^k N_2. \tag{5.33}$$

From results on the causal partial realization problem [8] and the fact that $N \geq 2n_i$, we can conclude that (5.33) then holds for all $k \geq 0$. Equation (5.31) then follows since we can write E_i as $(I-\beta A_i)/\alpha$. $\square$

We now have

$$C_1 V_1^i A_1^K E_1^L B_1 = C_2 V_2^i A_2^K E_2^L B_2, \tag{5.34a}$$

$$C_1 V_1^f A_1^K E_1^L B_1 = C_2 V_2^f A_2^K E_2^L B_2 \tag{5.34b}$$

for all K and L, and, since (2.5) holds for both systems, we can conclude that

$$C_1 A_1^K E_1^L B_1 = C_2 A_2^K E_2^L B_2 \tag{5.35}$$

for all K and L. We now introduce three different Hankel matrices. For simplicity, let us assume that N is odd. Then (5.34), (5.35) imply that

$$\begin{aligned} H_{\text{in}} &= O_{\text{s}}^1[(N-1)/2] R_{\text{w}}^1[(N+1)/2] \\ &= O_{\text{s}}^2[(N-1)/2] R_{\text{w}}^2[(N+1)/2], \end{aligned} \tag{5.36}$$

$$\begin{aligned} H_{\text{out}} &= O_{\text{w}}^1[(N-1)/2] R_{\text{s}}^1[(N+1)/2] \\ &= O_{\text{w}}^2[(N-1)/2] R_{\text{s}}^2[(N+1)/2], \end{aligned} \tag{5.37}$$

$$\begin{aligned} H_{\text{s}} &= O_{\text{s}}^1[(N-1)/2] R_{\text{s}}^1[(N+1)/2] \\ &= O_{\text{s}}^2[(N-1)/2] R_{\text{s}}^2[(N+1)/2], \end{aligned} \tag{5.38}$$

where for $j=1, 2$

$$R_{\text{w}}^j = [\, V_j^i R_{\text{s}}^j[(N+1)/2] \mid V_j^f R_{\text{s}}^j[(N+1)/2]], \tag{5.39a}$$

$$O_{\text{w}}^j = \begin{bmatrix} O_{\text{s}}^j[(N-1)/2] V_j^i \\ O_{\text{s}}^j[(N-1)/2] V_j^f \end{bmatrix}, \tag{5.39b}$$

and R_{s}^j and O_{s}^j denote the strong reachability and observability matrices for system j. The first two of the three Hankel matrices (5.36)-(5.38) have simple interpretations. Specifically, assume that $(N-1)$ is divisible by 4 and let $k=(N-1)/4$ and $l=3(N-1)/4$. Then H_{in} corresponds to the input-output map resulting from applying inputs off the interval $[k, l]$ and observing it on the interval. H_{out} corresponds to driving the system on $[k, l]$ and then observing it outside this interval.

Since both systems satisfy (5.1)-(5.3) and $N \geq 2n_i$, we have that $O^i_w[(N-1)/2]$ is full rank. Consequently from (5.39) we can find a matrix U so that

$$R^2_s[(N+1)/2] = UR^1_s[(N+1)/2]. \tag{5.40}$$

Similarly, we can obtain an analogous expression for R^1_s in terms of R^2_s. These allow us to conclude that

$$\text{rank}(R^1_s[(N+1)/2]) = \text{rank}(R^2_s[(N+1)/2]) = \rho \tag{5.41}$$

and in an analogous way we can show that

$$\text{rank}(O^1_s[(N-1)/2]) = \text{rank}(O^2_s[(N-1)/2]) = \omega. \tag{5.42}$$

Finally, condition (5.3) together with (5.38) imply that

$$\rho - (n_1 - \omega) = \text{rank } H_s = \rho - (n_2 - \omega) \tag{5.43}$$

from which we see that

$$n_1 = n_2. \qquad \square \tag{5.44}$$

Corollary 5.1. *Let* $(C_j, E_j, A_j, V^i_j, V^f_j, B_j)$, $j = 1, 2$, *be two minimal realizations, where* $\{E_j, A_j\}$, $j = 1, 2$, *are in standard form for the same* α *and* β. *Then there exists an invertible matrix* T *so that*

$$B_2 = TB_1, \tag{5.45a}$$

$$C_2 = C_1 T^{-1}, \tag{5.45b}$$

$$O^1_s(V^i_1 - T^{-1} V^i_2 T)R^1_s = 0, \tag{5.46a}$$

$$O^1_s(V^f_1 - T^{-1} V^f_2 T)R^1_s = 0, \tag{5.46b}$$

$$(A_1 - T^{-1} A_2 T)R^1_s = 0, \tag{5.47a}$$

$$(E_1 - T^{-1} E_2 T)R^1_s = 0, \tag{5.47b}$$

$$O^1_s(A_1 - T^{-1} A_2 T) = 0, \tag{5.47c}$$

$$O^1_s(E_1 - T^{-1} E_2 T) = 0, \tag{5.47d}$$

where R^1_s *and* O^1_s *are the strong reachability and observability matrices for system* 1.

Proof. Since $\mathcal{R}^i_s(\mathcal{O}^i_s)$ have the same dimension, we can find invertible $T(W)$ so that

$$R^2_s = TR^1_s, \tag{5.48}$$

$$O^2_s W = O^1_s. \tag{5.49}$$

From (5.38) we can then conclude that

$$O_s^2[W-T]R_s^1=0. \tag{5.50}$$

Assume that we have chosen a basis for each of the two systems compatible with the decomposition $\mathcal{O}_s \oplus [\mathcal{O}_s^{\perp} \cap \mathcal{R}_s] \oplus \mathcal{R}_s^{\perp}$. The requirement (5.48) implies that T must have the form

$$T=\begin{bmatrix} T_1 & T_2 & * \\ T_3 & T_4 & * \\ 0 & 0 & * \end{bmatrix}, \tag{5.51}$$

where T_1, T_2, T_3, and T_4 are fixed and $*$ are arbitrary. Similarly, (5.49) implies that W is given by

$$W=\begin{bmatrix} * & * & * \\ 0 & W_1 & W_2 \\ 0 & W_3 & W_4 \end{bmatrix}. \tag{5.52}$$

Finally, by direct computation we can check that (5.50) implies

$$W_1=T_4, \qquad T_3=W_3=0 \tag{5.53}$$

so that with the indicated degrees of freedom we can take

$$W=T=\begin{bmatrix} T_1 & T_2 & * \\ 0 & T_4 & W_2 \\ 0 & 0 & W_4 \end{bmatrix}. \tag{5.54}$$

Proceeding with the proof, note that (5.45a), (5.45b) follow from (5.49), (5.50), and (2.4). Also, the equality of the weighting patterns of the two systems is equivalent to

$$O_s^1 V_1^i R_s^1 = O_s^2 V_2^i R_s^2, \tag{5.55a}$$

$$O_s^1 V_1^f R_s^1 = O_s^2 V_2^f R_s^2 \tag{5.55b}$$

from which (5.46) follows. From the invariance of $\mathcal{R}_s$ and Cayley-Hamilton we conclude that

$$A_2 R_s^2 = TA_1 R_s^1, \qquad E_2 R_s^2 = TE_2 R_s^2 \tag{5.56}$$

from which (5.47a) and (5.47b) follow. Equations (5.47c) and (5.47d) are verified in a similar fashion. □

Corollary 5.2. *Every extendible TPBVDS has a minimal realization that is also extendible.*

Proof. This result follows once we show the following. Assume that we have two realizations $(C_j, E_j, A_j, V_j^i, V_j^f, B_j)$, $j = 1, 2$. Then if one of these is extendible, so is the other. First, it is not difficult to obtain the following generalization of Lemma 5.3: if (5.30) holds, then for all P, Q, K, $L \geq 0$

$$M_1(A_1^{\mathrm{D}})^P(E_1^{\mathrm{D}})^Q A_1^K E_1^L N_1 = M_2(A_2^{\mathrm{D}})^P(E_2^{\mathrm{D}})^Q A_2^K E_2^L N_2. \tag{5.57}$$

Thus not only do we have that (5.55) holds but also

$$O_{\mathrm{s}}^1 V_1^i E_1^{\mathrm{D}} E R_{\mathrm{s}}^1 = O_{\mathrm{s}}^2 V_2^i E_2^{\mathrm{D}} E_2 R_{\mathrm{s}}^2, \tag{5.58a}$$

$$O_{\mathrm{s}}^1 V_1^f A_1^{\mathrm{D}} A R_{\mathrm{s}}^1 = O_{\mathrm{s}}^2 V_2^f A_2^{\mathrm{D}} A_2 R_{\mathrm{s}}^2. \tag{5.58b}$$

The result then follows from the characterization of extendibility in (3.20). □

Let us make several comments about these results. The proof we have given here is much in the spirit of Krener [10], although we have taken some care in the necessity portion to put our reduced or modified systems back into normalized form and to verify that the resulting systems had the desired properties (Lemmas 5.1 and 5.2). Also, as noted previously, it is the spaces $\mathcal{R}_{\mathrm{w}}$ and $\mathcal{O}_{\mathrm{w}}$ that play a critical role in minimality, although these need not be the weak reachability or unobservability spaces unless the system is displacement. The key point here appears to be that the difference between the stationarity property and displacement property seems unimportant when we look only at input-output behavior since it is the *projection* of the weak reachability and unobservability spaces, as seen through O_{s} and R_{s}, that are important, and these are the same as the projections of $\mathcal{R}_{\mathrm{w}}$ and $\mathcal{O}_{\mathrm{w}}$. Finally, note that if we start with a displacement system and follow the reduction procedure described in the proof, we do not necessarily end up with a displacement system. However, thanks to Corollary 5.1, there may be a certain level of nonuniqueness in minimal realizations—both in the state space isomorphism T and, more importantly, in the boundary and system matrices. A conjecture that remains open is that we can use this freedom to choose a minimal realization that is also a displacement system.

6. Block standard and normalized forms

A well-known result for causal systems is the following. Suppose that A is block diagonalized with no common eigenvalues among the blocks. Then reachability and observability of the entire system is equivalent to the reachability and observability of all of the individual subsystems defined by the block structure of A. The same type of result is easily shown to hold as well for TPBVDSs.

Definition 6.1. The regular pencil $\{E, A\}$ is in block standard form (BSF) if

$$E = \mathrm{diag}(E_1, E_2, \ldots, E_M), \tag{6.1}$$

$$A = \mathrm{diag}(A_1, A_2, \ldots, A_M), \tag{6.2}$$

where each $\{E_i, A_i\}$ pair is in standard form, i.e., there exist α_i, β_i such that

$$\alpha_i E_i + \beta_i A_i = I, \qquad i = 1, \ldots, M, \tag{6.3}$$

and furthermore $\{E_i, A_i\}$ and $\{E_j, A_j\}$, $i \neq j$, have no eigenmode in common. That is for any pair $(s, t) \neq (0, 0)$, $|sE_i + tA_i| = 0$ for at most one value of i.

In (6.1), (6.2) E and A commute, and, as in [17], we can readily check that well-posedness of (2.1)–(2.3) when E and A commute is equivalent to the invertibility of $V_i E^N + V_f A^N$. Consequently, we can premultiply (2.2) by the inverse of this matrix to obtain a generalization of normalized form:

Definition 6.2. The TPBVDS (2.1)–(2.2) is in block normalized form (BNF) if $\{E, A\}$ is in BSF and (2.5) holds.

In general, there is no reason for V_i and V_f also to be block-diagonal for a system in BNF. However, in the stationary case we have the following result:

Theorem 6.1. *A TPBVDS in BNF is stationary if and only if it has a representation where V_i and V_f are in the same block diagonal form as E and A, i.e.,*

$$V_i = \operatorname{diag}(V_1^i, \ldots, V_M^i), \tag{6.4}$$

$$V_f = \operatorname{diag}(V_1^f, \ldots, V_M^f), \tag{6.5}$$

and, moreover, each of the subsystems $(C_k, E_k, A_k, V_k^i, V_k^f, B_k)$ is stationary.

Corollary 6.1. *A TPBVDS in BNF is displacement if and only if V_i and V_f are in the same block-diagonal form* (6.4), (6.5) *as E and A, and, moreover, each of the subsystems is displacement.*

Proof of Theorem 6.1. Consider a TPBVDS in BNF. We first prove the following:

Lemma 6.1. *The strong reachability and observability matrices of the overall system have the following form:*

$$R_s = \operatorname{diag}(R_s^1, \ldots, R_s^M) \cdot W, \tag{6.6}$$

$$O_s = V \cdot \operatorname{diag}(O_s^1, \ldots, O_s^M), \tag{6.7}$$

where W and V are invertible matrices and R_s^k and O_s^k are strong reachability and strong observability matrices of the kth block of the system.

Proof. We begin by putting the pencil in standard form by premultiplying E and A by $(\alpha E+\beta A)^{-1}$ for some α and β. Note that $(\alpha E+\beta A)^{-1}$ is block-diagonal, as are the new E and A matrices. Indeed, all we have done is to modify the system so that (6.3) is satisfied with all α_i equal to α and all β_i equal to β. Assume $\alpha \neq 0$ (otherwise reverse the roles of E and A). In this case the condition that no two blocks of E and A have the same eigenmode now implies that no two blocks of A have the same eigenvalue. Also

$$\mathscr{R}_s = \operatorname{Im}[B|AB|\cdots|A^{n-1}B]. \tag{6.8}$$

Equation (6.6) then follows from the usual causal system result, and (6.7) can be verified similarly. □

Note that Lemma 6.1 demonstrates the equivalence of strong reachability/observability of the overall system and of all of the subsystems. Also, since every block is in standard form we can see that the strong reachability and observability spaces are E- and A-invariant.

An examination of the proof of Theorem 2.1 shows that if we assume that E and A commute, the necessary and sufficient conditions for stationarity are (2.28) and (2.29). Let V^i_{kl} and V^f_{kl} denote the kl-blocks of V_i and V_f, respectively, then let $\tilde{V}_i$ and $\tilde{V}_f$ be obtained from V_i and V_f by nulling the off-diagonal blocks. What we wish to show is that $(C, E, A, \tilde{V}_i, \tilde{V}_f, B)$ is in BNF and has the same weighting pattern.

That it is in BNF follows immediately since we have not changed E and A and

$$\tilde{V}_i E^N + \tilde{V}_f A^N = V_i E^N + V_f A^N = I. \tag{6.9}$$

Thus what we need to show is that

$$O_s \tilde{V}_i R_s = O_s V_i R_s, \tag{6.10}$$

$$O_s \tilde{V}_f R_s = O_s V_f R_s. \tag{6.11}$$

We focus on (6.10) as (6.11) follows similarly. Thanks to (6.6) we need to show that

$$O^k_s V^i_{kj} R^j_s = 0, \qquad j \neq k. \tag{6.12}$$

From (2.28) we immediately find that for $j \neq k$

$$O^k_s[E_k V^i_{kj} A_j] R^j_s = O^k_s[A_k V^i_{kj} E_j] R^j_s. \tag{6.13}$$

Recall that $\{E_j, A_j\}$ and $\{E_k, A_k\}$ are in standard form, and indeed by a block-diagonal transformation we can assume that $\alpha E_j+\beta A_j=\alpha E_k+\beta A_k=I$ for a single given pair α and β. Furthermore, assume that $\alpha \neq 0$ (otherwise reverse the roles of E and A), so that (6.13) implies that

$$O^k_s[V^i_{kj} A_j] R^j_s = O^k_s[A_k V^i_{kj}] R^j_s. \tag{6.14}$$

Since $\mathcal{R}_s^j$ is A_j-invariant and $\mathcal{O}_s^k$ is A_k-invariant, we have that (6.14) implies that

$$O_s^k[V_{kj}^i p(A_j)]R_s^j = O_s^k[p(A_k)V_{kj}^i]R_s^j \tag{6.15}$$

for any polynomial p. Take any generalized eigenvector v of A_j in $\mathcal{R}_s^j$ corresponding to the eigenvalue λ_j of A_j, then there is an integer m so that

$$(\lambda_j I - A_j)^m v = 0. \tag{6.16}$$

Let $p(x) = (\lambda_j - x)^m$. Also, let w be any generalized left-eigenvector of A_k in $(\mathcal{O}_s^k)^\perp$ corresponding to the eigenvalue μ_k of A_k. Then, from (6.15) we have

$$0 = w'V_{kj}^i p(A_j)v = w'p(A_k)V_{kj}^i v = (\lambda_j - \mu_k)^m w'V_{kj}^i v. \tag{6.17}$$

Since $(\lambda_j - \mu_k)^m \neq 0$, we can conclude that

$$w'V_{kj}^i v = 0. \tag{6.18}$$

But, since $\mathcal{R}_s^j$ is A_j-invariant and $\mathcal{O}_s^k$ is A_k-invariant, the columns of R_s^j and rows of O_s^k are spanned by such v's and w's, respectively, yielding (6.12). □

Note that if the overall system is not both strongly reachable and observable, there is some freedom in the choice of V_i and V_f. What the theorem says is that we can always choose these to be block-diagonal. If, however, all of the subsystems *are* strongly reachable and observable, then the *only* possibility is for V_i and V_f to be block-diagonal. This is what happens in Corollary 6.1. Note also that since we can always take the boundary matrices to be block-diagonal, we see that minimality of the overall system is equivalent to minimality of all of the subsystems.

Note that for a stationary TPBVDS in BNF with V_i and V_f as in (6.4), (6.5), Theorem 6.1 and Theorem 2.1, applied to each subsystem, allow us to deduce that (2.13) holds, which in turn allows us to obtain the simple form for the weighting pattern given in (2.31). More importantly, Lemma 6.1 allows us to study reachability and observability of individual eigenmodes.

Theorem 6.2. *The zero and infinite modes of a minimal, extendible, stationary TPBVDS are strongly reachable and observable.*

Proof. We focus on reachability, as observability is proved similarly. Zero and infinite modes correspond to pairs of the form (s, t) with either $s = 0$ or $t = 0$ so that $|sE + tA| = 0$. As the two cases are identical, we focus on the case when $s = 0$ (zero eigenmode). From Lemma 6.1 and Theorem 6.1 we know that we need only look at the individual part of the system having eigenmode zero. That is, we need only consider the case of an extendible stationary TPBVDS with A being nilpotent. In this case since E and A are in standard form, E is invertible, so without loss of generality we can take $E = I$. Then since $A^N = 0$, the normalized form condition (2.5) implies that $V_i = I$. Also, (3.20) implies

$$O_s V_f R_s = 0 \tag{6.19}$$

so that by Corollary 5.1 we can set $V_f = 0$ without affecting the weighting pattern, minimality, or $\mathscr{R}_s$. In this case, however, (4.17) implies that $\mathscr{R}_w = \mathscr{R}_s$, so that (since $\mathscr{R}_w = \mathbb{R}^n$) the system is strongly reachable. □

Corollary 6.2. *The zero and infinite mode portions of the weakly reachable (observable) and strongly reachable (observable) subspaces of an extendible displacement system are identical.*

Proof. The argument exactly follows the proof of Theorem 6.2 up to (6.19). However, since we are now dealing with an extendible displacement system, we can immediately conclude (from (3.22)) that $V_f = 0$ (rather than resorting to minimality as in the theorem). From (4.17) we then have $\mathscr{R}_w = \mathscr{R}_s$. □

7. Conclusions

In this paper we have developed some of the system-theoretic properties of TPBVDSs. As we have seen, within the classes of stationary and displacement systems we can perform relatively simple and explicit computations that allow us to derive detailed characterizations of reachability, observability, and minimality. As had already been noted for continuous-time, nondescriptor boundary-value systems, minimality for TPBVDSs is a bit more complicated than for causal systems. Indeed, there is a certain degree of nonuniqueness in minimal realizations. One open problem that we have noted concerns whether we can use this freedom to guarantee that a displacement system always has a minimal realization that is also displacement.

Another concept that we have introduced and studied in this paper is extendibility, i.e., the idea of thinking of a TPBVDS as being defined on a sequence of intervals of increasing length. Once such a notion is introduced, it becomes possible to talk about asymptotic properties such as stability. This is one of the subjects of [15] in which we also examine the concept of stochastic stationarity for TPBVDSs driven by white noise. As might be expected, there is some relationship between stability and stochastic stationarity (although it is more complex than in the causal case), and a new type of generalized Lyapunov equation plays a central role in this relationship.

References

[1] M. B. Adams, Linear estimation of boundary value stochastic processes, Sc.D. Thesis, Dept. of Aeronautics and Astronautics, MIT, Cambridge, MA, April 1983.

[2] M. B. Adams, A. S. Willsky, and B. C. Levy, Linear estimation of boundary value stochastic processes, Part 1: The role and construction of complementary models, *IEEE Trans. Automat. Control*, vol. 29, no. 9, pp. 803-811, Sept. 1984.

[3] M. B. Adams, A. S. Willsky, and B. C. Levy, Linear estimation of boundary value stochastic processes, Part 2: 1-D smoothing problems, *IEEE Trans. Automat. Control*, vol. 29, no. 9, pp. 811-821, Sept. 1984.

[4] A. Ben-Israel and T. N. E. Greville, *Generalized Inverses—Theory and Applications*, Wiley-Interscience, New York, 1975.

[5] I. Gohberg and M. A. Kaashoek, On minimality and stable minimality of time varying linear systems with well posed boundary conditions, *Internat. J. Control*, vol. 43, no. 5, pp. 1401-1411, 1986.

[6] I. Gohberg and M. A. Kaashoek, Similarity and reduction for time-varying linear systems with well-posed boundary conditions, *SIAM J. Control Optim.*, vol. 24, no. 5, pp. 961-978, Sept. 1986.

[7] I. Gohberg, M. A. Kaashoek, and L. Lerer, Minimality and irreducibility of time-invariant linear boundary-value systems, *Internat. J. Control*, vol. 44, no. 2, pp. 363-379, 1986.

[8] I. Gohberg, M. A. Kaashoek, and L. Lerer, On minimality in the partial realization problem, *Systems Control Lett.*, vol. 9, pp. 97-104, 1987.

[9] A. J. Krener, Boundary value linear systems, *Asterisque*, vol. 75-76, pp. 149-165, Oct. 1980.

[10] A. J. Krener, Acausal realization theory, Part I: Linear deterministic systems, *SIAM J. Control Optim.*, vol. 25, no. 3, pp. 499-525, May 1987.

[11] D. G. Luenberger, Dynamic systems in descriptor form, *IEEE Trans. Automat. Control*, vol. 22, no. 3, pp. 312-321, June 1977.

[12] D. G. Luenberger, Time-invariant descriptor systems, *Automatica*, vol. 14, no. 5, pp. 473-480, Sept. 1978.

[13] R. Nikoukhah, System theory for two-point boundary value descriptor systems, M.Sc. Thesis, Dept. of Electrical Engineering and Computer Science, and Report LIDS-TH-1559, Lab. for Information and Decision Systems, MIT, Cambridge, MA, June 1986.

[14] R. Nikoukhah, A deterministic and stochastic theory for two-point boundary-value descriptor systems, Ph.D. Thesis, Dept. of Electrical Engineering and Computer Science, MIT, Cambridge, MA, Sept. 1988.

[15] R. Nikoukhah, M. B. Adams, A. S. Willsky, and B. C. Levy, Estimation for boundary-value descriptor systems, *Circuits Systems Signal Process.*, vol. 8, no. 1, pp. 25-48, 1989.

[16] R. Nikoukhah, B. C. Levy, and A. S. Willsky, Stability, stochastic stationarity, and generalized Lyapunov equations for two-point boundary-value descriptor systems, to appear in *IEEE Trans. Automat. Control*, vol. 34, no. 11, Nov. 1989.

[17] R. Nikoukhah, A. S. Willsky, and B. C. Levy, Boundary-value descriptor systems: well-posedness, reachability and observability, *Internat. J. Control*, vol. 46, no. 5, pp. 1715-1737, Nov. 1987.

[18] W. M. Wonham, *Linear Multivariable Control: A Geometric Approach*, Springer-Verlag, New York, 1979.