The construction suggested by Lemma 1 enables transformation of a problem of maximization of E_2 to a problem of maximization of $\hat{E}_1$; hence randomness can be introduced. To summarize, given a quadratic function of the form E_1 or E_2, it is possible to construct a neural network which will perform a random local search for the maximum.

A rich class of optimization problems can be represented by quadratic functions [4]. A problem which not only is representable by a quadratic function but actually is equivalent to it is that of finding a minimum cut (MC) in a graph [4], [7]. In what follows, we present the equivalence between the MC problem and neural networks (Theorem 4 and 5) and also show how neural networks relate to the directed min cut (DMC) problem (Theorem 6). To make the foregoing statements clear, let us start by defining the term cut in a graph.

Definition: Let $G=(V,E)$ be a weighted and undirected graph, with W being an $n\times n$ symmetric matrix of weights of the edges of G. Let V_1 be a subset of V, and let $V_{-1}=V-V_1$. The *set of edges* each of which is incident at one node in V_1 and at one node in V_{-1} is called a *cut* of the graph G. A *minimum cut* in a graph is a cut for which the sum of the corresponding edge weights is minimal over all V_1.

Theorem 4 [4], [7]: Let $G=(V,E)$ be a weighted and undirected graph, with W the matrix of its edge weights. Then the MC problem in G is equivalent to $\max Q_G(X)$, where $X\in\{-1,1\}^n$, and

$$Q_G(X)\stackrel{\text{def}}{=}\sum_{i=1}^{n}\sum_{j=1}^{n}W_{i,j}X_iX_j.$$

The foregoing theorem can be generalized to neural networks.

Theorem 5: Let $N=(W,T)$ be a neural network with W being an $n\times n$ zero diagonal matrix. Let G be a weighted graph with $(n+1)$ nodes with its weight matrix W_G being

$$W_G=\begin{pmatrix} W & T\\ T^T & 0\end{pmatrix}.$$

The problem of finding a state V in N for which E_1 is a global maximum is equivalent to the MC problem in the corresponding graph G.

Proof: Note that the graph G is built out of N by adding one node to N and connecting it to the other n nodes with the edge connected to node i having a weight T_i (the corresponding threshold). Clearly, if the state of the added node is constrained to -1, then for all $X\in\{-1,1\}^n$,

$$Q_G(X,-1)=E_1(X).$$

Hence the equivalence follows from Theorem 4. Note that the state of node $(n+1)$ need not be constrained to -1. There is a symmetry in the cut; that is, $Q_G(X)=Q_G(-X)$ for all $X\in\{-1,1\}^{n+1}$. Thus if a minimum cut is achieved with the state of node $(n+1)$ being 1, then a minimum is also achieved by the cut obtained by interchanging V_1 and V_{-1} (resulting in $X_{n+1}=-1$).

What about directed graphs? Is it possible to design a neural network which will perform a local search for a minimum cut in a directed graph?

Definition: Let $G=(V,E)$ be a weighted and directed graph. Each edge has a direction and a weight. The weights of the directed edges (arcs) can be represented by an $n\times n$ matrix W in which $W_{i,j}$ is the weight of the arc from i to j. Let V_1 be a subset of V, and let $V_{-1}=V-V_1$. The *set of arcs* each of which has its tail at a node in V_1 and its head at a node in V_{-1} is called a *directed cut of G*.

Theorem 6 [1]: Let $G=(V,E)$ be a weighted directed graph with W the matrix of its edge weights (W is not necessarily symmetric). The network $N=(\tilde{W},T)$ performs a local search for a DMC of G where

$$\tilde{W}_{ij}=\frac{1}{2}(W_{ij}+W_{ji})$$

$$T_k=\frac{1}{2}\sum_{i=1}^{n}(W_{ki}-W_{ik}).$$

The MC problem as defined in the paper is NP-hard [2]. The importance of the relation between the MC problem and neural networks lies in the fact that the MC problem can be viewed as a generic graph problem which can be mapped to the model. Thus theoretically one can transform every NP-hard problem to the MC problem and use the corresponding neural network to perform a local search algorithm. The problem with this approach is that only the problem is mapped while the algorithm for solving the problem is imposed by the way the model is operating. Theorem 6 is an example of programming the network to perform a specific local search algorithm for solving the DMC problem. It was relatively easy to find such a mapping, probably because the algorithm we chose is the one performed by the network for the MC problem.

An *open problem* is the following: there are many known local search algorithms for solving hard problems that have good performance; find a *known* local search algorithm which can be mapped to the neural network model.

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Sampling Theorems for Two-Dimensional Isotropic Random Fields

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***Abstract*—New sampling theorems are developed for isotropic random fields and their associated Fourier coefficient processes. A wavenumber-limited isotropic random field $z(\vec{r})$ is considered whose spectral density function is zero outside a disk of radius B centered at the origin of the**

Manuscript received September 3, 1986; revised February 19, 1988. This work was supported in part by the National Science Foundation under Grant ECS-83-12921 and in part by the Army Research Office under Grants DAAG-84-K-0005 and DAAL 03-86-K-0171.

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IEEE Log Number 8824062.

0018-9448/88/0900-1092$01.00 ©1988 IEEE

wavenumber plane. $z(\vec{r})$ can be reconstructed in the mean-square sense from its observation on the countable number of circles of radii $r_i = i\pi/B$, $i \in N$, or of radii $r_i = a_{i,n}/B$, $i \in N$, where $a_{i,n}$ denotes the ith zero of the nth-order Bessel function $J_n(x)$, and n is arbitrary.

I. Introduction

Spatially distributed random processes arise in various fields including image processing, meteorology, geophysical signal processing, oceanography, and optical processing. Since it is not possible to obtain observations at every point in space, one is led in practice to deal with sampled versions of these processes. Petersen and Middleton [1] extended the one-dimensional (1D) Shannon sampling theorem to m-dimensional Euclidean spaces. In particular, they developed efficient *point* sampling and reconstruction techniques for wavenumber-limited homogeneous random fields which minimize the number of sample points required per unit area to reconstruct a given field in the sense of a vanishing mean-square error.

Here, by contrast, we present a new reconstruction procedure for two-dimensional (2D) wavenumber-limited *isotropic* random fields sampled along circles in the Euclidean plane, where the reconstruction is to be understood in a mean-square sense. Isotropic fields are characterized by the fact that their mean value is a constant independent of position and their autocovariance function is invariant under all rigid body motions, i.e., translations and rotations. We show in Section II that the invariance of isotropic covariance functions under all rigid body motions implies that the power spectra of such fields are circularly symmetric in the wavenumber plane; some authors have used this latter fact to define the notion of isotropy (e.g., [1]). In some sense, isotropy is the natural extension of the notion of stationarity in one dimension. Furthermore, isotropic random fields arise in a number of practical problems such as the black body radiation problem [2], the study of underwater ambient noise in horizontal planes parallel to the surface of the ocean [3], and the investigation of temperature and pressure distributions at constant altitude in the atmosphere [4]. The importance of the sampling techniques that we develop stems from the fact that in some applications such as the mapping of the gravitational fields of planets using orbiting satellites [5], sampling along circles is more natural than sampling at discrete points on a fixed lattice. Furthermore, these new sampling schemes have been found useful in developing high-resolution spectral estimation techniques for isotropic random fields [6]. Finally, as will be explained later, our sampling techniques can be implemented in practice with a small mean-square error by sampling the isotropic fields at a discrete set of points along circles rather than continuously along each circle.

An important property of isotropic fields is that, when they are expanded in a Fourier series in terms of the polar coordinate angle θ, the Fourier coefficient processes of different orders are uncorrelated [7]. We consider here a wavenumber-limited isotropic random field $z(\vec{r})$,[1] whose spectral density function is zero outside a disk of radius B centered at the origin of the wavenumber plane. By developing sampling and reconstruction techniques for the Fourier coefficient processes $z_k(r)$ associated with $z(\vec{r})$, we prove that the process $z(\vec{r})$ can be reconstructed in a mean-square sense from its observation on the countable set of circles of radii $r_i = i\pi/B$, $i \in N$, or of radii $r_i = a_{i,n}/B$, $i \in N$, where $a_{i,n}$ is the ith zero of the nth-order Bessel function $J_n(x)$ and the index n of $J_n(x)$ can be selected arbitrarily.

Section II contains some standard properties of the Fourier expansions of isotropic random fields. The main results of this correspondence are described in Section III where two procedures for sampling wavenumber-limited isotropic random fields are developed. Finally, Section IV contains conclusions.

[1]Throughout this paper we use $\vec{r}$ to denote a point in 2D Cartesian space. The polar coordinates of this point are denoted by r and θ.

II. Fourier Series for Isotropic Fields

The covariance function

$$K(\vec{r}) = E[z(\vec{v})z(\vec{v}+\vec{r})] \tag{2.1}$$

of any zero-mean isotropic random field $z(\vec{r})$ is a function of r only, so that, by abuse of notation, we can write

$$K(\vec{r}) = K(r). \tag{2.2}$$

Such a field can be expanded into a Fourier series of the form [7], [8]

$$z(\vec{r}) = \sum_{n=-\infty}^{\infty} z_n(r)e^{jn\theta} \tag{2.3}$$

$$z_n(r) = \frac{1}{2\pi}\int_0^{2\pi} z(\vec{r})e^{-jn\theta}\,d\theta \tag{2.4}$$

where the Fourier coefficient processes of different orders are uncorrelated, i.e.,

$$E[z_n(r)z_m(s)] = 0 \tag{2.5}$$

for $n \neq m$. If we assume that $K(r)$ has a Hankel transform [9], i.e., if $K(r) \in L_1(rdr)$, and that $K(|\vec{r}-\vec{s}|)$ is the kernel of a self-adjoint nonnegative definite operator defined over $L_2(rdrd\theta)$, then it can be shown that the covariance function $k_n(r,s)$ of the nth-order Fourier coefficient $z_n(r)$ is given by [7], [8]

$$\begin{aligned} k_n(r,s) &= E[z_n(r)z_n(s)] \\ &= \int_0^{\infty} J_n(\lambda r)J_n(\lambda s)S(\lambda)\lambda\,d\lambda. \end{aligned} \tag{2.6}$$

In (2.6) $J_n(\cdot)$ is the Bessel function of order n and $S(\lambda)$ is the power spectrum associated with $z(\vec{r})$, i.e.,

$$\begin{aligned} S(\vec{\lambda}) &= \int_{R^2} K(\vec{r})e^{-j\vec{\lambda}\cdot\vec{r}}\,d\vec{r} \\ &= 2\pi\int_0^{\infty} K(r)J_0(\lambda r)r\,dr \\ &= S(\lambda) \end{aligned} \tag{2.7}$$

where $\lambda = |\vec{\lambda}|$ is the magnitude of the wave vector $\vec{\lambda}$, and we have taken advantage of the circular symmetry of $K(\vec{r})$. Note that (2.6) implies that $S(\lambda)$ can be recovered from $k_n(r,s)$ for an arbitrary value of n by taking the nth-order Hankel transform [9] of $k_n(r,s)$ with respect to the variable r and dividing by $J_n(\lambda s)/2\pi$. This fact will be useful in explaining the results presented in the next section.

III. Sampling Theorems

We shall develop two different procedures for sampling and reconstructing the Fourier coefficient processes associated with a given isotropic random field. Using the sampling theorems for the Fourier processes, we show that a wavenumber-limited isotropic random field can be reconstructed from its observations on a countably infinite number of concentric circles with a vanishing mean-square error.

A. Sampling the Covariance Functions of the Fourier Processes

Let us begin by presenting two different sampling procedures for the covariance function of the nth-order Fourier coefficient process.

Theorem 1: The nth-order Fourier coefficient process covariance function $k_n(r,s)$ of an isotropic random field $z(\vec{r})$, whose spectral density function $S(\lambda)$ is wavenumber-limited to the region $\lambda < B$, can be reconstructed exactly from the sample values of the mth-order Fourier coefficient process covariance

function $k_m(r,s)$ taken over a lattice of points $\{(a_{i,m}/B, a_{j,m}/B);\ i,j \in \boldsymbol{N}\}$, where $a_{i,m}$ is the ith zero of the mth-order Bessel function $J_m(x)$. Here m may equal n.

Proof: Over the interval $0<\lambda<B$, $J_n(\lambda r)$ can be expanded into a Fourier–Bessel series of the form [10]

$$J_n(\lambda r) = \sum_{i=1}^{\infty} b_{i,m}^n(r) J_m\left(\frac{a_{i,m}\lambda}{B}\right), \qquad 0<\lambda<B \tag{3.1}$$

where

$$b_{i,m}^n(r) = \frac{2\int_0^B J_n(\lambda r) J_n\left(\frac{a_{i,m}\lambda}{B}\right)\lambda\, d\lambda}{B^2 J_{m+1}(a_{i,m}) J_{m-1}(a_{i,m})}. \tag{3.2}$$

Substituting (3.1) for $J_n(\lambda r)$ and $J_n(\lambda s)$ into (2.6) yields the desired result

$$k_n(r,s) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} b_{i,m}^n(r) b_{j,m}^n(s) k_m\left(\frac{a_{i,m}}{B}, \frac{a_{j,m}}{B}\right). \tag{3.3}$$

Note that, according to Theorem 1, it is possible to reconstruct the covariance functions of the Fourier coefficient processes of *all* orders given sampled values of the covariance function of a *single* Fourier coefficient process of *any* order. This should not come as a surprise: if one can reconstruct $k_m(r,s)$ exactly from its sample values on the grid $\{(a_{i,m}/B, a_{j,m}/B);\ i,j \in \boldsymbol{N}\}$, then, as mentioned earlier, one can easily compute $S(\lambda)$. Given $S(\lambda)$, one can then evaluate $k_n(r,s)$ for all n via (2.6). We now state and prove a second sampling theorem for the covariance function $k_n(r,s)$ of the nth-order Fourier coefficient process. In this case we use samples of $k_n(r,s)$, rather than of $k_m(r,s)$.

Theorem 2: The nth-order Fourier coefficient process covariance function $k_n(r,s)$ of an isotropic random field $z(\vec{r})$, whose spectral density function $S(\lambda)$ is wavenumber-limited to the region $\lambda<B$, can be reconstructed exactly from its own sample values taken over a lattice of points $\{i\pi/B, j\pi/B);\ i,j \in \boldsymbol{N}\}$, or over a lattice of points $\{(a_{i,m}/B, a_{j,m}/B);\ i,j \in \boldsymbol{N}\}$, where $a_{i,m}$ is the ith zero of the mth-order Bessel function $J_m(x)$.

Proof: Consider the identity (see the Appendix)

$$J_n(\lambda r) = \sum_{i=0}^{\infty} d_i^n(r) J_n\left(\frac{i\pi}{B}\lambda\right), \qquad 0<\lambda<B \tag{3.4}$$

where

$$d_i^n(r) = \frac{1}{(1+\delta_{0,n})}\left((-1)^n \frac{\sin\left(B\left(r+\frac{i\pi}{B}\right)\right)}{B\left(r+\frac{i\pi}{B}\right)} + \frac{\sin\left(B\left(r-\frac{i\pi}{B}\right)\right)}{B\left(r-\frac{i\pi}{B}\right)}\right) \tag{3.5}$$

and where $\delta_{0,n}$ denotes the Kronecker delta function, i.e., $\delta_{0,n}=1$ if $n=0$ and $\delta_{0,n}=0$ otherwise. Substituting (3.4) into (2.6) yields

$$k_n(r,s) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} d_i^n(r) d_j^n(s) k_n\left(\frac{i\pi}{B}, \frac{j\pi}{B}\right). \tag{3.6}$$

Similarly, by substituting the identity (see the Appendix)

$$J_n(\lambda r) = \sum_{i=1}^{\infty} c_{i,m}^n(r) J_n\left(\frac{a_{i,m}}{B}\lambda\right), \qquad 0<\lambda<B \tag{3.7}$$

where

$$c_{i,m}^n(r) = \frac{2a_{i,m}J_m(Br)}{\left(a_{i,m}^2 - B^2r^2\right)J_{m+1}(a_{i,m})}\left(\frac{a_{i,m}}{r}\right)^{|m-n|} \tag{3.8}$$

into (2.6) we obtain

$$k_n(r,s) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} c_{i,m}^n(r) c_{j,m}^n(s) k_n\left(\frac{a_{i,m}}{B}, \frac{a_{j,m}}{B}\right). \tag{3.9}$$

Observe that Theorem 2 asserts that the same sampling grid can be used for all of the Fourier coefficient process covariance functions. The sampling grid can be selected to be $\{(i\pi/B, j\pi/B)\}$ or $\{(a_{i,m}/B, a_{j,m}/B)\}$ where m is fixed but arbitrary. This fact will prove useful in deriving sampling theorems for isotropic random fields.

B. Sampling Isotropic Fields

Theorem 2 can now be used to prove the following important result.

Theorem 3: The nth-order Fourier coefficient process $z_n(r)$ corresponding to an isotropic random field $z(\vec{r})$ with a wavenumber-limited spectrum $S(\lambda)$ that vanishes identically for all $\lambda>B$ can be reconstructed with zero mean-square error from its samples $\{z_n(a_{i,m}/B);\ i \in \boldsymbol{N}\}$, where $a_{i,m}$ is the ith zero of the mth-order Bessel function $J_m(x)$ as

$$z_n(r) = \sum_{i=1}^{\infty} c_{i,m}^n(r) z_n\left(\frac{a_{i,m}}{B}\right), \tag{3.10}$$

and from its samples $\{z_n(i\pi/B);\ i \in \boldsymbol{N}\}$ as

$$z_n(r) = \sum_{i=0}^{\infty} d_i^n(r) z_n\left(\frac{i\pi}{B}\right) \tag{3.11}$$

where the series in (3.10) and (3.11) converge in the mean-square sense, and $d_i^n(r)$ and $c_{i,m}^n(r)$ are defined by (3.5) and (3.8), respectively.

Proof: To show that the series on the right side of (3.10) converges in mean to $z_n(r)$, let us denote by $\hat{z}_{n,I}^B(r)$ its partial sum

$$\hat{z}_{n,I}^B(r) = \sum_{i=1}^{I} c_{i,m}^n(r) z_n\left(\frac{a_{i,m}}{B}\right). \tag{3.12}$$

From (3.1) and (2.6), it can be shown that

$$k_n(r,s) = \sum_{i=1}^{\infty} c_{i,m}^n(s) k_n\left(r, \frac{a_{i,m}}{B}\right) \tag{3.13}$$

$$k_n\left(\frac{a_{i,m}}{B}, r\right) = \sum_{j=1}^{\infty} c_{j,m}^n(r) k_n\left(\frac{a_{i,m}}{B}, \frac{a_{j,m}}{B}\right). \tag{3.14}$$

Using the above two equations, we obtain

$$\begin{aligned}\lim_{I\to\infty} E\left[z_n(r)\left(z_n(r) - \hat{z}_{n,I}^B(r)\right)\right] &= \lim_{I\to\infty}\left(k_n(r,r) - \sum_{i=1}^{I} c_{i,m}^n(r) k_n\left(r, \frac{a_{i,m}}{B}\right)\right)\\ &= 0\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}\lim_{I\to\infty} E\left[\hat{z}_{n,I}^B(r)\left(z_n(r) - \hat{z}_{n,I}^B(r)\right)\right] &= \lim_{I\to\infty}\left(\sum_{i=1}^{I}\sum_{j=I+1}^{\infty} c_{i,m}^n(r) c_{j,m}^n(r) k_n\left(\frac{a_{i,m}}{B}, \frac{a_{j,m}}{B}\right)\right)\\ &= 0.\end{aligned} \tag{3.16}$$

Combining (3.15) with (3.16), it follows that

$$\lim_{I\to\infty} E\left[\left|z_n(r) - \hat{z}_{n,I}^B(r)\right|^2\right] = 0. \tag{3.17}$$

By using a similar approach it can be shown that the right side of (3.11) converges in mean to $z_n(r)$.

Note that Theorem 3 shows that the *nonstationary* 1D process $z_n(r)$ can be reconstructed from its sample values at the points $\{a_{i,m}/B\colon i \in N\}$ or $\{i\pi/B\colon i \in N\}$. Observe also that the weighting functions $c_{i,m}^n(r)$ and $d_i^n(r)$ used in the reconstruction of $z_n(r)$ from its sample values (see (3.10), (3.11)) are orthogonal in the sense of [1], i.e.,

$$c_{i,m}^n\left(\frac{a_{l,m}}{B}\right) = 0, \qquad \text{for } l \neq i$$

$$d_i^n\left(\frac{l\pi}{B}\right) = 0, \qquad \text{for } l \neq i$$

which guarantees the linear independence of the sample values of $z_n(r)$. As pointed out in [1], the weighting functions need not be orthogonal to achieve zero mean-square error. However, it seems that the convergence of the series is more rapid near the sampling points when orthogonal weights are used [1].

Comment: In practice, one is likely to use only a finite number of circles and approximate (3.10) and (3.11) by the partial sums

$$\hat{z}_{n,I}^B(r) = \sum_{i=1}^{I} c_{i,m}^n(r) z_n\left(\frac{a_{i,m}}{B}\right) \tag{3.18}$$

$$\hat{z}_{n,I}^F(r) = \sum_{i=1}^{I} d_i^n(r) z_n\left(\frac{i\pi}{B}\right). \tag{3.19}$$

The mean-square error in such approximations can be bounded by noting that

$$|J_n(x)| \leq \frac{1}{\sqrt{2}}, \qquad n \geq 1. \tag{3.20}$$

Hence using the fact that

$$E\left[\left|z_n(r) - \hat{z}_{n,I}^B(r)\right|^2\right] = \sum_{i=I+1}^{\infty} \sum_{j=I+1}^{\infty} k_n\left(\frac{a_{i,m}}{B}, \frac{a_{j,m}}{B}\right) c_{i,m}^n(r) c_{j,m}^n(r), \tag{3.21}$$

we obtain

$$E\left[\left|z_n(r) - \hat{z}_{n,I}^B(r)\right|^2\right] \leq \frac{S_{\max} B^2}{4}\left(\sum_{i=I+1}^{\infty} \left|c_{i,m}^n(r)\right|\right)^2, \tag{3.22}$$

where

$$S_{\max} = \max_{0 \leq \lambda \leq B} S(\lambda). \tag{3.23}$$

A similar bound can be obtained for the mean-square error that results from approximating $z_n(r)$ by $z_{n,I}^F(r)$.

Now recall that knowledge of $z(\vec{r})$ on a circle of radius r' is sufficient to compute *all* of the Fourier coefficient processes $z_n(r)$ at the location $r = r'$. Hence we have the following important result.

Theorem 4: Any isotropic random field $z(\vec{r})$ with a wavenumber-limited spectrum $S(\lambda)$ that vanishes identically for all $\lambda > B$ can be reconstructed with zero mean-square error from its samples on the countable set of circles of radii $r_i = a_{i,m}/B$, $i \in N$, where $a_{i,m}$ is the ith zero of the mth-order Bessel function $J_m(x)$, as

$$z(\vec{r}) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \sum_{i=1}^{\infty} c_{i,m}^n(r)\left\{\int_0^{2\pi} z\left(\frac{a_{i,m}}{B}, \phi\right) e^{-jn\phi}\, d\phi\right\} e^{jn\theta} \tag{3.24}$$

and from its samples on the countable number of circles of radii $r_i = i\pi/B$, $i \in N$, as

$$z(\vec{r}) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \sum_{i=0}^{\infty} d_i^n(r)\left\{\int_0^{2\pi} z\left(\frac{i\pi}{B}, \phi\right) e^{-jn\phi}\, d\phi\right\} e^{jn\theta}, \tag{3.25}$$

where the series in (3.24) and (3.25) converge in the mean-square sense.

Theorem 4 follows directly from Theorem 3 and (2.1) and (2.2) and is a generalization of a result in [6]. A natural question to be asked here, is which of the above two sampling schemes, i.e. the *Bessel sampling scheme* involving sampling on circles of radii $r_i = a_{i,m}/B$ or the *uniform sampling scheme* using circles of radii $r_i = i\pi/B$, is more efficient in terms of minimizing the number of sampling circles per unit radial length. This leads us to examine the distribution of the zeros $a_{i,m}$ of the mth-order Bessel function $J_m(x)$, along the positive real axis. For large i and a fixed value m, the approximate zeros of the mth-order Bessel function are given by the McMahon expansion [11]

$$a_{i,m} \approx \left(i + \frac{1}{2}m - \frac{1}{4}\right)\pi - \frac{(4m^2-1)}{8\pi\left(i + \frac{1}{2}m - \frac{1}{4}\right)} \cdots, \tag{3.26}$$

which shows that the separation $\Delta_{i,m}$ between two successive large zeros $a_{i+1,m}$ and $a_{i,m}$ of $J_m(x)$, with $i \gg m$, is approximately equal to

$$\Delta_{i,m} = a_{i+1,m} - a_{i,m} \approx \pi + \frac{(4m^2-1)}{8i^2} + O(i^{-3}). \tag{3.27}$$

In particular, two successive large zeros of $J_0(x)$ are separated by a distance slightly less than π, whereas two successive large zeros of $J_m(x)$ for $m \neq 0$, are separated by a distance slightly larger than π. As the order i of the zeros $a_{i,m}$ of $J_m(x)$ tends to infinity the separation $\Delta_{i,m}$ between successive zeros is asymptotically equal to π, for all m. Furthermore, examination of the small zeros of the mth-order Bessel function reveals that even for $i = 2$, $\Delta_{i,m}$ is approximately equal to π. Hence the Bessel sampling scheme is slightly more efficient than the uniform sampling scheme if the zeros of a large order Bessel function are used to generate the nonuniform circular sampling grid. However, the Bessel sampling scheme is primarily of theoretical interest, while the uniform sampling scheme is of more practical value since it does not require the knowledge of a large number of zeros of one of the Bessel functions.

Finally, observe that in practice one does not need to sample the field $z(\vec{r})$ continuously as a function of θ along any of the circles r_i. Note that along any of these circles $z(r_i, \theta)$ is a "stationary" process with covariance function

$$K(r_i; \theta, \phi) = E[z(r_i, \theta) z(r_i, \phi)] = \sum_{n=-\infty}^{\infty} k_n(r_i, r_i) e^{jn(\theta-\phi)}. \tag{3.28}$$

Examination of a plot of $J_n(x)$ [11] reveals that

$$J_n(x) \approx 0, \qquad \text{for } x \gg 1,\ n > x. \tag{3.29}$$

Hence, by using (2.6) and the Lebesgue dominated convergence theorem to interchange the operations of limit and integration, we obtain

$$k_n(r_i, r_i) \approx 0, \qquad \text{for } Br_i \gg 1,\ n > Br_i. \tag{3.30}$$

Equation (3.30) implies that along any circle of radius r_i, $z(r_i, \theta)$ can be approximated with a small mean-square error by the finite sum

$$z(r_i, \theta) \approx \sum_{n=-N}^{N} z_n(r_i) e^{jn\theta} \tag{3.31}$$

where $N \geq Br_i$. In particular, (3.24) and (3.25) can be approximated in the mean-square sense as

$$z(\vec{r}) \approx \sum_{i=1}^{\infty} \sum_{n=-N_i}^{N_i} f_{i,n}(r) z_n(r_i) e^{jn\theta} \tag{3.32}$$

where

$$r_i = \frac{i\pi}{B} \qquad f_{i,n}(r) = d_i^n(r) \qquad N_i \gg i\pi \tag{3.33}$$

or

$$r_i = \frac{a_{i,m}}{B} \qquad f_{i,n}(r) = c_{i,m}^n(r) \qquad N_i \gg a_{i,m}. \tag{3.34}$$

The coefficients $z_n(r_i)$, $-N_i \leq n \leq N_i$, can be determined by sampling $z(r_i, \theta)$ at $2N_i + 1$ points.

Comment: If a finite number I of circles is used in (3.32), then the mean-square error in approximating $z(\vec{r})$ by the partial sum

$$z(\vec{r}) \approx \sum_{i=1}^{I} \sum_{n=-N_i}^{N_i} f_{i,n}(r) z_n(r_i) e^{jn\theta} \tag{3.35}$$

can be shown to be bounded by

$$\begin{aligned} E\left[\left| z(\vec{r}) - \sum_{i=1}^{I} \sum_{n=-N_i}^{N_i} f_{i,n}(r) z_n(r_i) e^{jn\theta} \right|^2\right] &\leq \frac{SB^2}{4} \left(\sum_{n=-\infty}^{\infty} \left(\sum_{i=I+1}^{\infty} |f_{i,n}(r)| \right)^2 \right. \\ &\quad + \sum_{i=1}^{I} \sum_{j=1}^{I} \sum_{|n| > \max(N_i, N_j)} |f_{i,n}(r)| |f_{j,n}(r)| \\ &\quad \left. + \sum_{i=I+1}^{\infty} \sum_{j=1}^{I} \sum_{|n| > N_j} |f_{i,n}(r)| |f_{j,n}(r)| \right) \end{aligned} \tag{3.36}$$

where $S_{\max}$ is defined in (3.23).

IV. Conclusion

We have shown that a wavenumber-limited isotropic random field, with a power spectrum that is zero outside a disk of radius B centered at the origin of the wavenumber plane, can be reconstructed in a mean-square sense from its observation on the countable set of circles of radii $r_i = a_{i,m}/B$, $i \in N$, where $a_{i,m}$ is the ith zero of the mth-order Bessel function $J_m(x)$, or of radii $r_i = i\pi/B$, $i \in N$. This result is a direct consequence of the sampling theorems derived for the Fourier coefficient processes associated with the given isotropic random field.

The sampling schemes developed in this correspondence can be extended easily to isotropic random fields in higher dimensions, provided they are expanded in spherical harmonics instead of Fourier series. Finally, observe the parallel between our sampling procedures and the corresponding one-dimensional results. In one dimension, a stationary process can be reconstructed in the mean-square sense from its observation on a countable number of spheres in a space of dimension one, i.e., at a countably infinite number of points. In the general m-dimensional case, an isotropic random field can be reconstructed in the mean-square sense from its observation on a countably infinite number of spheres in the m-dimensional space.

Appendix

Proof of (3.4)

Let $p_B(\lambda)$ be the function

$$p_B(\lambda) = \begin{cases} 1, & \lambda < B \\ 0, & \text{otherwise} \end{cases}. \tag{A.1}$$

By using the identity [9, pp. 43, 99]

$$\int_{-\infty}^{\infty} J_n(\lambda r) e^{-jur} dr = \begin{cases} 2(-j)^n \dfrac{T_n\left(\dfrac{u}{\lambda}\right)}{(\lambda^2 - u^2)^{1/2}}, & 0 < |u| < \lambda \\ 0, & \text{otherwise} \end{cases} \tag{A.2}$$

where $T_n(x)$ is a Chebyshev polynomial of order n, it can be shown that the Fourier transform of $J_n(\lambda r) p_B(\lambda)$ with respect to r is band-limited to B radians per unit distance. Hence $J_n(\lambda r) p_B(\lambda)$ can be written as [10]

$$J_n(\lambda r) p_B(\lambda) = \sum_{i=0}^{\infty} d_i^n(r) J_n\left(\frac{i\pi}{B}\lambda\right), \qquad 0 < \lambda < B \tag{A.3}$$

where $d_i^n(r)$ is given in (3.5).

Proof of (3.7)

Consider the identity [11, p. 72],

$$J_l(\lambda r) = \sum_{i=1}^{\infty} c_{i,l}(r) J_l\left(\frac{a_{i,l}}{B}\lambda\right), \qquad 0 < \lambda < B \tag{A.4}$$

where

$$c_{i,l}(r) = \frac{2a_{i,l} J_l(Br)}{(a_{i,l}^2 - B^2 r^2) J_{l+1}(a_{i,l})}. \tag{A.5}$$

By repeatedly differentiating both sides of (A.5) with respect to λ and using the identity

$$\frac{d}{d\lambda} J_m(\lambda r) = r J_{m-1}(\lambda r) - \frac{m}{\lambda} J_m(\lambda r) \tag{A.6}$$

for the case where $n < l$ and the identity

$$\frac{d}{d\lambda} J_m(\lambda r) = -r J_{m+1}(\lambda r) + \frac{m}{\lambda} J_m(\lambda r) \tag{A.7}$$

for the case where $n > l$, we obtain

$$J_n(\lambda r) = \sum_{i=1}^{\infty} c_{i,l}^n(r) J_n\left(\frac{a_{i,l}}{B}\lambda\right), \qquad 0 < \lambda < B \tag{A.8}$$

where

$$c_{i,l}^n(r) = c_{i,l}(r) \left(\frac{a_{i,1}}{r}\right)^{|l-n|}. \tag{A.9}$$

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