Eigenstructure Approach for Array Processing with Unknown Intensity Coefficients

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Abstract—**The eigenstructure approach for array processing is examined for the general case in which it is required to estimate several parameters related to the directional patterns of the sources and the sensors as well as parameters related to the location of the sources. The assumption of the MUSIC algorithm that any given source is observed by all the sensors with the same intensity is removed, and hence, the proposed technique is useful for localizing emitters in the near field of the array and for using sensors with known phase but unknown gain pattern. The resulting method is illustrated by simple examples, which are also used to show that the standard MUSIC algorithm does not work when the assumption of equal intensities is violated.**

I. Introduction

In this paper we are concerned with the problem of localizing several radiating sources by observation of their signals at spatially separated sensors. This is a problem of considerable importance, occurring in a variety of fields ranging from radar, sonar, and oceanography to seismology and radio-astronomy. In recent years, there has been a growing interest in eigenstructure based methods, perhaps due to the introduction of the MUSIC method by Schmidt [1], which is a technique that can be applied to general array configurations and which is relatively simple and efficient. A comprehensive discussion of the MUSIC method may be found in [1], while [3] contains a literature survey of most of the recently published results.

An assumption common to all previously published contributions in this area is that any given source is observed by all the sensors with the same intensity. This assumption is reasonable only if the sources are in the far field of the array and the sensors have identical radiation patterns. In this paper we illustrate a potential problem with the basic MUSIC method when this assumption is violated. To remedy this problem, we remove the assumption of equal intensity and thus extend the applicability of the MUSIC technique, or any other eigenstructure approach, to the case of near field sources and/or sensors with unknown gain patterns. The proposed approach does not extend to the case of sensor phase uncertainty. See [5] for a different approach that is capable of direction finding with sensor gain and phase uncertainty. However, the algorithm in [5] is limited to sources in the far field or sources in the near field with omnidirectional gain pattern.

The paper is organized as follows. The problem formulation and proposed solution are described in Section II, which also contains a condition for unique solution. In Section III we illustrate through examples that the MUSIC method breaks down when the signals are observed with unequal intensities, while the proposed technique performs well. However, since in our approach there are more degrees of freedom, spurious estimates may be generated. We indicate how postprocessing can eliminate the phantom results. Section IV contains some conclusions.

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II. Problem Formulation and Solution

Consider N radiating sources with an arbitrary radiation pattern observed by an array of M sensors. The signal at the output of the mth sensor can be described by

$$x_m(t) = \sum_{n=1}^{N} \alpha_{mn} s_n(t - \tau_{mn}) + v_m(t); \quad m = 1, 2, \cdots, M; \quad -T/2 \le t \le T/2 \tag{1}$$

where $\{s_n(t)\}_{n=1}^{N}$ are the radiated signals, $\{v_m(t)\}_{m=1}^{M}$ are additive noise processes, and T is the observation interval. The intensities α_{mn} (which are assumed to be real) and the delays τ_{mn} are parameters related to the directional patterns and relative location of the nth source and the mth sensor.

A convenient separation of the parameters to be estimated is obtained by using Fourier coefficients defined by

$$X_m(\omega_l) = \frac{1}{\sqrt{T}} \int_{-T/2}^{T/2} x_m(t) e^{-j\omega_l t}\, dt,$$

where $\omega_l = (2\pi/T)(l_1 + l)$, $l = 1, 2, \cdots, L$; and l_1 is a constant. In principle, the number of coefficients required to capture all the signal information is infinite. However, if we consider signals with energy concentrated in a finite spectral band, we can use only $L < \infty$ coefficients.

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Taking the Fourier coefficients of (1), we obtain

$$X_m(\omega_l) = \sum_{n=1}^{N} \alpha_{mn} e^{-j\omega_l \tau_{mn}} S_n(\omega_l) + V_m(\omega_l); \quad l = 1, 2, \cdots, L; \tag{2}$$

where $S_n(\omega_l)$ and $V_m(\omega_l)$ are the Fourier coefficients of $s_n(t)$ and $v_m(t)$, respectively. Equation (2) may be expressed using vector notation as follows:

$$X(\omega_l) = A(\omega_l) S(\omega_l) + V(\omega_l); \quad l = 1, 2, \cdots, L; \tag{3}$$

where

$$X(\omega_l) = [X_1(\omega_l), X_2(\omega_l), \cdots, X_M(\omega_l)]^T,$$
$$S(\omega_l) = [S_1(\omega_l), S_2(\omega_l), \cdots, S_N(\omega_l)]^T,$$
$$V(\omega_l) = [V_1(\omega_l), V_2(\omega_l), \cdots, V_M(\omega_l)]^T,$$
$$A(\omega_l) = [\boldsymbol{a}_l(\theta_2), \boldsymbol{a}_l(\theta_2), \cdots, \boldsymbol{a}_l(\theta_N)],$$
$$\boldsymbol{a}_l(\theta_n) = [\alpha_{1n} e^{-j\omega_l \tau_{1n}}, \alpha_{2n} e^{-j\omega_l \tau_{2n}}, \cdots, \alpha_{Mn} e^{-j\omega_l \tau_{Mn}}]^T.$$

We use θ_n to represent all the parameters of interest associated with the nth signal, namely, $\{\alpha_{nm}\}_{m=1}^{M}$ and $\{\tau_{mn}\}_{m=1}^{M}$. Our main goal is to estimate the set $\{\theta_n\}_{n=1}^{N}$. Note that if the spectrum of the signals is concentrated around ω_1, with a bandwidth that is small compared to $2\pi/T$, then (3) reduces to a single relation between the observation vector $X(\omega_1)$ and the parameters, i.e., $L = 1$. In this case, it is customary to use many short observation intervals or simply time samples, and the model becomes

$$X(j) = AS(j) + V(j); \quad j = 1, 2, \cdots, J; \tag{4}$$

where the dependence on the single frequency ω_1 is suppressed, and j denotes the index of the different samples. Note that the main difference between the narrow-band case and the wide-band case is that A is the same in all the J equations specified by (4), while $A(\omega_l)$ is different in each of the L equations given by (3). However, the estimation procedure discussed here is equally applicable to both cases. In this paper we concentrate on the narrow-band case. The modification for the wide-band case, using, for example, [2] or [4], is straightforward.

Note that since neither A nor $S(j)$ are known in advance, they cannot be uniquely determined. One can left multiply $S(j)$ by an $N \times N$ complex (diagonal or any nonsingular) matrix, and right multiply A by the inverse of this nonsingular matrix, without changing the received signals $X(j)$. Hence, without loss of generality, we select the norm of every column of A to be one.

The following assumptions are made:

a) the signals and noises are stationary over the observation interval;
b) the number of sources is known;
c) the columns of A are linearly independent;
d) the signals are not perfectly correlated; and
e) the noise covariance matrix is known except for a multiplicative constant σ^2.

The correlation matrices of the signal, noise, and observation vectors are given, respectively, by

$$R_s = E\{SS^H\},$$
$$\sigma^2 \Sigma_0 = E\{V^H\},$$
$$R_x = E\{XX^H\} = AR_sA^H + \sigma^2\Sigma_0, \tag{5}$$

where $(\cdot)^H$ represents the Hermitian transpose operation. The following theorem forms the basis for the eigenstructure approach.

Theorem: Let λ_i and $\boldsymbol{u}_i$, $i = 1, 2, \cdots, M$ be the eigenvalues and corresponding eigenvectors of the matrix pencil (R_x, Σ_0) (i.e., the solutions of $R_x \boldsymbol{u} = \lambda \Sigma_0 \boldsymbol{u}$), where the λ_i's are listed in descending order. Then,

1) $\lambda_{N+1} = \lambda_{N+2} = \cdots = \lambda_M = \sigma^2$; and
2) each of the columns of A is orthogonal to the matrix $U = [\boldsymbol{u}_{N+1}, \boldsymbol{u}_{N+2}, \cdots, \boldsymbol{u}_M]$.

Proof: See [2].

This theorem suggests that reasonable estimates of the parameters $[\theta_n\}_{n=1}^{N}$ may be obtained by first generating an estimate $\hat{U}$ of U and then searching over all possible values of θ_n for vectors $\boldsymbol{a}(\theta_n)$ that are nearly orthogonal to $\hat{U}$. This may be written as

$$\hat{\theta}_n = \arg\min_{\theta_n} \left\| \hat{U}^H \boldsymbol{a}(\theta_n) \right\|^2, \tag{6}$$

where $\|\cdot\|$ denotes the Euclidean norm. As already mentioned, since there is an extra degree of freedom, there is no loss of generality in assuming that $\|\boldsymbol{a}(\theta_n)\| = 1$. This also eliminates the trivial solution of (6). Note that (6) requires a multidimensional search over the parameters $\{\alpha_{mn}\}$ and $\{\tau_{mn}\}$, in contrast with the basic MUSIC method which assumes that all the parameters $\{\alpha_{mn}\}$ are equal to one, or alternatively that they are known and stored in large calibration tables. The multidimensional search can be considerably simplified by decomposing $\boldsymbol{a}(\theta_n)$ as follows:

$$\boldsymbol{a}(\theta_n) = \Gamma(\boldsymbol{\tau}_n) \cdot \boldsymbol{\alpha}_n,$$

where

$$\boldsymbol{\alpha}_n = (\alpha_{1n}, \alpha_{2n}, \cdots, \alpha_{Mn})^T,$$
$$\Gamma(\boldsymbol{\tau}_n) = \text{diag}\,(e^{-j\omega_1 \tau_{1n}}, e^{-j\omega_1 \tau_{2n}}, \cdots, e^{-j\omega_1 \tau_{Mn}}),$$

and

$$\boldsymbol{\tau}_n = (\tau_{1n}, \tau_{2n}, \cdots, \tau_{Mn})^T.$$

Using this notation, (6) becomes

$$\hat{\theta}_n = \arg\min_{\alpha_n, \tau_n} \boldsymbol{\alpha}_n^T \Gamma^H(\boldsymbol{\tau}_n) \hat{U}\hat{U}^H \Gamma(\boldsymbol{\tau}_n) \boldsymbol{\alpha}_n. \tag{7}$$

The minimum must be found under the following constraint: 1) $\boldsymbol{\alpha}_n$ is a *real* vector; 2) $\|\boldsymbol{\alpha}_n\| = 1$; and 3) $\boldsymbol{\tau}_n$ is in the space induced by all possible source locations. Hence,

$$\hat{\tau}_n = \arg\min_{\tau_n} \delta^{\min}\left\{C(\tau_n)\right\}, \tag{8.a}$$

$$\hat{\alpha}_n = W^{\min}, \tag{8.b}$$

where $\delta^{\min}\{C(\tau_n)\}$ is the smallest eigenvalue of the matrix $C(\tau_n)$ given by

$$C(\tau_n) = Re\left\{\Gamma^H(\tau_n)\hat{U}\hat{U}^H\Gamma(\tau_n)\right\}, \tag{9}$$

and $W^{\min}$ is the associated normalized eigenvector. Equation (8) requires a search over the space of vectors τ_n, induced by all possible individual source locations. In the basic MUSIC method, (8) is simply

$$\hat{\tau}_n = \arg\min_{\tau_n} \left\|\hat{U}^H a(\tau_n)\right\|^2,$$

where the vector $a(\tau_n)$ is only a function of the delays τ_n and not of the intensities. That means that for every possible τ_n, the intensities are assumed to be equal to one, or a corresponding vector α_n is known (stored in memory) and is used to construct $a(\tau_n)$.

The proposed algorithm may be summarized as follows.

a) Estimate the observation covariance matrix

$$\hat{R}_x = \frac{1}{J}\sum_{j=1}^{J} X(j)X(j)^H.$$

b) Find the $M - N$ eigenvectors, $\{\hat{u}_i\}$, corresponding to the smallest $M - N$ eigenvalues of the pencil $(\hat{R}_x, \Sigma_0)$, and construct the matrix

$$\hat{U} = [\hat{u}_{N+1}, \hat{u}_{N+2}, \cdots, \hat{u}_M].$$

c) Evaluate, for all possible source locations, the "spatial spectrum" given by

$$P(\tau) = \frac{1}{\delta^{\min}\left\{C(\tau)\right\}},$$

where $C(\tau)$ is defined by (9).

d) Select the N highest peaks of $P(\tau)$. The corresponding values of τ describe the source locations, and the corresponding eigenvectors describe the intensity vectors $\{\alpha_n\}$.

This conceptually simple algorithm requires, in step c), more computational effort than the basic MUSIC method. However, the results, illustrated in the next section, justify this effort.

Also note that the computational requirements, associated with the proposed approach, are typically less than the requirements associated with the iterative maximum likelihood solution described in [6].

Before turning to some examples, it is appropriate to discuss the question of uniqueness. While sufficient conditions for uniqueness are still an open research problem, it is easy to derive a necessary condition following [1, p. 84]. Referring to the basic relation (5), we observe that R_x can be perfectly described by $2MN - N^2 + 1$ parameters. These parameters are the $N + 1$ different (real) eigenvalues and $2NM - N^2 - N$ parameters that define the N complex eigenvectors, associated with the signal subspace, that satisfy $N(N + 1)/2$ *complex* orthogonality constraints. On the other hand, we have MN unknown $\{\alpha_{mn}\}$, N constraints on $\{\alpha_{mn}\}$, $r \cdot N$ unknown location parameters ($r = 1$ for azimuth only system, $r = 2$ for azimuth and elevation system, etc.), N^2 unknown parameters that define the Hermitian matrix R_s, and a single unknown parameter σ^2. Thus, the problem is not strictly well posed unless

$$2MN - N^2 + 1 \geq MN - N + r \cdot N + N^2 + 1,$$

or

$$N \leq (M + 1 - r)/2. \tag{10}$$

However, even if this inequality is satisfied, one may obtain spurious results in *addition* to the desired solutions, as shown in the following section. It is then necessary to use postprocessing criteria to eliminate the undesired solutions.

III. Examples

To illustrate the behavior of the algorithms, let us consider two examples.

Example 1: Consider a uniform linear array of five sensors separated by one-half a wavelength of the actual narrow-band source signals. The sources are two narrow-band emitters located in the far-field of the array. In this case, if γ_n denotes the bearing of the nth source, ($n = 1, 2$) relative to the perpendicular to the array baseline, the differential delay is given by $\omega_1 \tau_{mn} = (m - 1)\pi \sin(\gamma_n)$. The intensity coefficients $\{\alpha_{mn}\}$ were chosen according to

$$\alpha_{mn} = 1 + \sqrt{12}\,\sigma_2 \cdot \beta_{mn}, \tag{11}$$

where the β_{mn}'s are independent, identically distributed random numbers whose distribution is uniform over the interval $[-0.5, 0.5]$, and where σ_2 is the standard deviation of α_{mn}. The signals, $S(j)$, and the noise, $V(j)$, are random complex Gaussian vectors with covariance matrices $\sigma_s^2 \cdot I$ and $\sigma_n^2 \cdot I$, respectively.

In the first experiment, we placed one source at $-5°$ and the other at $10°$, to demonstrate the "superresolution" performance of the algorithm (the Rayleigh resolution criterion for this array is $28.6°$). We then collected $J = 40$ snapshots with SNR $\triangleq 20 \log(\sigma_s/\sigma_n) = 30$ (dB). The standard deviation of the intensity coefficients was set at 0.5. (Note that the intensity coefficients were selected only once and they are the same for all the snapshots.) The "spatial spectrum"

$$P_I \triangleq -10 \log\left\{\delta^{\min}(\tau)\right\} \text{ (dB)},$$

of the proposed algorithm and the spatial spectrum of the MUSIC algorithm defined by

$$P_M \triangleq -20 \log\left\{\left\|\hat{U}^H a(\tau)\right\|\right\} \text{ (dB)};$$

is plotted in Fig. 1. We observe that the standard MUSIC

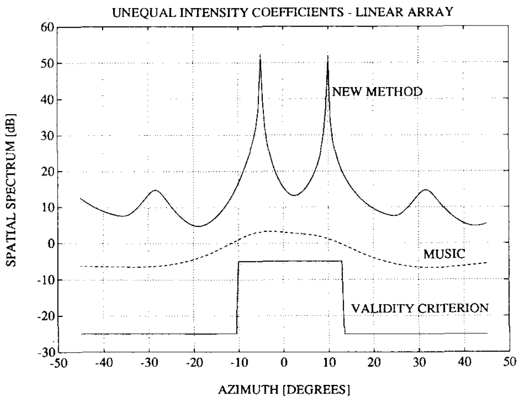

Fig. 1. Spatial spectrum of two far-field sources, with random intensity coefficients (linear array).

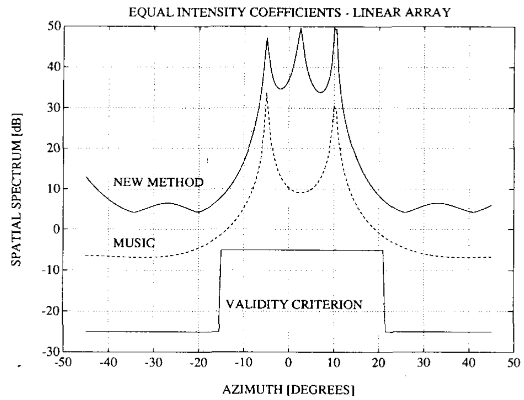

Fig. 2. Spatial spectrum of two far-field sources with equal intensity coefficients (linear array).

approach fails to resolve the sources. On the other hand, the proposed method clearly resolves the sources, estimates the direction accurately, and above all obtains an accurate estimate of $\{\alpha_{mn}\}$ with maximum error of 2 percent (i.e., $\|\hat{\boldsymbol{\alpha}}_n - \boldsymbol{\alpha}_n\| / \|\boldsymbol{\alpha}_n\| \leq 0.02$). Notice that the spatial spectrum associated with the new method has additional low peaks at $-28°$ and $32°$. These phantom solutions can be easily removed using a "validity criterion" that requires all intensity coefficients to have the same sign. Hence, all solutions with associated $\hat{\boldsymbol{\alpha}}$ containing nonphysical coefficients are immediately eliminated. In Fig. 1, the validity criterion is high for acceptable solutions and low for nonphysical intensity coefficients.

Example 2: Consider exactly the same setting as in Example 1, except that now all the intensity coefficients are selected to be one, so that the intensities match the standard MUSIC assumptions. The results of applying both algorithms are shown in Fig. 2. As expected, the MUSIC algorithm performs well, but there is a potential problem with the proposed approach. In addition to the desired solutions at $-5°$ and $10°$, we obtain a phantom solution at exactly $2.5°$, which is not eliminated by the validity criterion. A simple analysis shows that the phantom solutions are expected to appear at $(\gamma_1 + \gamma_2)/2$ when the array is linear, $\boldsymbol{\alpha}_1 = \boldsymbol{\alpha}_2$, and two sources are close together. In general, ambiguous solutions occur whenever the surface spanned by $\boldsymbol{a}(\boldsymbol{\theta})$ ("array manifold") intersects, or is very close to, the signal subspace (the space spanned by the columns of A) in the more than N points [1].

Experiment 2 was repeated using a circular array of 6 sensors with equal spacing of one-half a wavelength. Again all intensity coefficients were equal to one. Fig. 3 shows the results. It is clear that now the ambiguous solution is detected by the validity criterion.

IV. Conclusions

In this paper, the eigenstructure approach has been used to obtain estimates of source locations as well as estimates of the intensity vectors $\{\boldsymbol{\alpha}_n\}$ simultaneously. We have

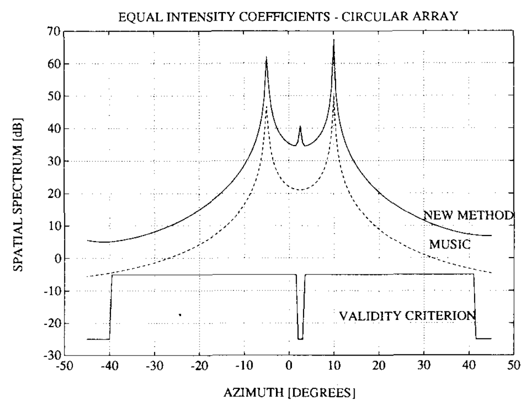

Fig. 3. Spatial spectrum of two far-field sources with equal intensity coefficients (circular array).

shown that the basic MUSIC method does not perform well when the vectors $\{\boldsymbol{\alpha}_n\}$ are not known *a priori*. The estimates of $\{\boldsymbol{\alpha}_n\}$ may be useful in their own right, but their estimation is essential, even if one is only interested in the source locations, in cases where it is not appropriate to assume omnidirectionality. For example, whenever a source is in the near field of the array, its radiation pattern can rarely be assumed omnidirectional. This is also important in applications in which it is unrealistic to assume that the radiation pattern of each sensor is accurately known (this usually requires frequent recalibration and a large memory).

We observed that in some cases, postprocessing is required to eliminate spurious solutions and also ambiguous solutions. The elimination of spurious solutions which are associated with nonphysical intensity vectors is relatively easy. On the other hand, elimination of ambiguous solutions that have acceptable intensity vectors is much more complicated, and requires a close examination of all results, as well as taking into account any available *a priori* knowledge. The appearance of ambiguous peaks in certain cases and their elimination is still an open subject of research.

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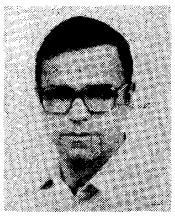

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Alan S. Willsky (S'70-M'73-SM'82-F'86), for a photograph and biography, see p. 812 of the May 1988 issue of this TRANSACTIONS.

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