Asymptotic Orders of Reachability in Perturbed Linear Systems

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Abstract—A framework for studying asymptotic orders of reachability in perturbed linear, time-invariant systems is developed. The systems of interest are defined by matrices that have asymptotic expansions in powers of a perturbation parameter ϵ about the point 0. The reachability structure is exposed via the Smith form of the reachability matrix. This approach is used to provide insight into the kinds of inputs needed to reach weakly reachable target states, into the structure of high-gain feedback for pole placement, and into the types of inputs that steer trajectories arbitrarily close to almost (A, B)-invariant subspaces and almost (A, B)-controllability subspaces.

I. INTRODUCTION

A. Motivation

IN this paper, we develop and apply a theory of asymptotic orders of reachability in linear time-invariant systems parametrized by some small variable ϵ. The approach draws in part on the algebraic formulation of [1], [2]. To provide a motivation for the key issues in our approach, consider the following discrete-time system as an example.

Example 1.1:

$$x[k+1]=\begin{bmatrix} 1 & 1 \\ 0.01 & 2 \end{bmatrix} x[k]+\begin{bmatrix} 1 \\ 0.01 \end{bmatrix} u[k].$$

This system is reachable but the reachability matrix

$$[b\,|\,Ab]=\begin{bmatrix} 1 & 1.01 \\ 0.01 & 0.03 \end{bmatrix}$$

is not very far from a singular matrix, in that its condition number is approximately 10^4. This leads to numerical difficulties in determining reachability, as shown in [3]. Also, consider the minimum energy control problem for this system. The minimum energy control to reach $x[2] = [1\ 0]'$ (where $'$ denotes the transpose) from $x[0] = 0$ is $u[1] = -0.5$ and $u[2] = 1.5$, while the minimum energy control for $x[2] = [1\ 1]'$ is $u[1] = 49.7$ and $u[2] = -49$. This order of magnitude difference between controls is another indication of near unreachability. Still further indications may be obtained, for example, by considering how small a perturbation of the system matrices suffices to destroy reachability (in this case, of the order of 0.01), or by examining the magnitude of feedback gain required to shift

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poles by various amounts (in this case, to move the eigenvalues by 2, feedback gains of magnitude approximately 10^2 are required, as illustrated in Example 3.1). □

Our treatment of problems of this type is qualitative rather than numerical in nature: we assume that small values in the system are modeled by functions of a small parameter ϵ, which implicitly indicates the presence of different orders of coupling among state variables and inputs. The formulation that we use permits the state space to be decomposed according to the "asymptotic orders of reachability" of different target states. Specifically, we consider continuous-time and discrete-time systems of the form

$$\dot{x}(t)=A(\epsilon)x(t)+B(\epsilon)u(t) \tag{1.1}$$

$$x[k+1]=A(\epsilon)x[k]+B(\epsilon)u[k]. \tag{1.2}$$

Here $A(\epsilon)$ and $B(\epsilon)$, in general, have entries from the field $L(\epsilon)$ of functions $l(\epsilon)$ that have asymptotic expansions (see [4]) of the form

$$l(\epsilon)=\sum_{-k}^{\infty} l_i\epsilon^i \qquad (\epsilon \to 0) \tag{1.3}$$

for some finite k, so,

$$A(\epsilon) : L^n(\epsilon) \to L^n(\epsilon), \quad B(\epsilon) : L^m(\epsilon) \to L^n(\epsilon). \tag{1.4}$$

(Strictly speaking, we should write $\approx$ instead of $=$ in (1.3) to emphasize that the series on the right is asymptotic and not necessarily convergent [4], but this abuse of notation is common.)

Defining these systems over $L(\epsilon)$ permits us to examine the effect or necessity of high gain feedback. However, many of our results will involve matrices over the ring $T(\epsilon)$ of functions $t(\epsilon)$ that have asymptotic expansions of the form

$$t(\epsilon)=\sum_{0}^{\infty} t_i\epsilon^i \qquad (\epsilon \to 0). \tag{1.5}$$

For verification that $T(\epsilon)$ is a ring, see [4, p. 15]. The ring $T(\epsilon)$ is easily shown to be a principal ideal domain and $L(\epsilon)$ is its field of fractions. This allows us to use various results on matrix canonical forms such as the Smith form [5, App. B.2] in our development. (The symbols $L(\epsilon)$ and $T(\epsilon)$ have been chosen to serve as mnemonics for "Laurent-series like" and "Taylor-series like.")

The above formulation strongly suggests connections with work on parametrized linear systems [6], [7], and more generally with studies of systems over rings [8], [9]. The approach in this paper owes something to our earlier work [1], [2] on an algebraic framework for multiple time scale decomposition in singularly perturbed systems, and therefore takes a relatively independent track. The important task of making and exploiting explicit connections with the literature on systems over rings is left to future work. (Sontag [10] has shown us that explicit connections are not only possible but may be quite fruitful.)

This work was particularly motivated by the numerical prob-

lems encountered in various pole placement methods and in evaluating system reachability. Pole placement and related numerical issues are addressed using various approaches in the current literature [11]-[14]. In multiinput systems, unlike single-input systems, the feedback matrix that produces a given set of poles is not unique, and the additional degrees of freedom may be used to attain other control objectives (see [14]). One may, for example, attempt to minimize the maximum feedback gain; [12] addresses this problem via numerical examples involving balancing the A and B matrices and redistribution of the feedback task among the inputs. These examples contain some intuitive ideas, but have not led to systematic procedures that work well for well-defined and substantial classes of systems. One of our objectives here is to suggest an analytical approach to understanding and structuring feedback gains for pole placement.

Another area of numerical work involves criteria to measure controllability. Boley and Lu [16] use the "distance to the nearest uncontrollable system" as a criterion. They define this by the minimum norm perturbation that would make a system uncontrollable. They also relate this concept to state feedback by measuring the amount that the eigenvalues move due to state feedback of bounded magnitude. Connections may also be made to the literature on balanced realizations [15] where the singular values of the controllability Grammian are used to indicate nearness to uncontrollability.

The issue of controllability in perturbed systems of the form (1.1) has been examined by Chow [17]. He defines a system to be strongly controllable if the system is controllable at $\epsilon = 0$. Otherwise, he calls it weakly controllable and concludes that pole placement of such systems will require controls with large gains. Chow looks at systems with two time scales (slow and fast), and he proves that a necessary and sufficient condition for such a singularly perturbed system to be strongly controllable is the controllability of its slow and fast subsystems.

Our analysis goes further than Chow's in that we examine the relative orders of reachability of different parts of the state space. As already mentioned, the methods we use have some similarity to those used by Lou et al. [1], [2], who relate the multiple time scale structure of the system (1.1) to the invariant factors of $A(\epsilon)$, when this matrix has entries from the ring of functions analytic at $\epsilon = 0$. The results in [1], [2] actually hold for $A(\epsilon)$ defined over the considerably more general ring $T(\epsilon)$ used in this paper, although this fact was not recognized there. The Smith decomposition of $A(\epsilon)$ plays a key role in the analysis of [1], [2], while the Smith decomposition of the reachability matrix is central to the development in this paper. While the primary focus of the work in [1], [2] is on time scale structure, some attention is paid there to control. In particular, [1] gives results on the use of feedback in (1.1) to change the time scale structure of the system. The work in [18] may be seen as a continuation of the work in [1], [2] in that it analyzes the effect of control and feedback on the system of (1.1). This paper is based on the work in [18].

B. Outline

Throughout the paper, $\mathcal{C}_k(\epsilon)$ will denote the k-step reachability matrix

$$\mathcal{C}_k(\epsilon) = [B(\epsilon)|A(\epsilon)B(\epsilon)| \cdots |A^{k-1}(\epsilon)B(\epsilon)] \qquad (1.6)$$

where $A(\epsilon)$, $B(\epsilon)$ are as in (1.1)-(1.5). We shall simply write $\mathcal{C}(\epsilon)$ for $\mathcal{C}_n(\epsilon)$, and call this *the* reachability matrix. We also assume throughout that $(A(\epsilon), B(\epsilon))$ is reachable for all $\epsilon \in (0, a)$, some $a \in \mathbb{R}^+$, or equivalently $\mathcal{C}(\epsilon)$ is full row rank for all $\epsilon \in (0, a)$, $a \in \mathbb{R}^+$.

In Section II, we develop a theory of orders of reachability. We start with discrete-time systems and illustrate that the orders of reachability can be recovered from the Smith decomposition of the reachability matrix. We define a standard form which displays these orders explicitly. Also, we show that equivalent results hold

for continuous-time systems. In Section III, this theory is extended to pole placement by full state feedback for systems with entries over $T(\epsilon)$. We also provide a computationally and numerically reasonable algorithm for pole placement. Section IV develops connections with Willems' work on "almost invariance" [19], [20]. We show how to find an input that steers the trajectories of a system arbitrarily close to an almost (A, B)-invariant subspace and show that the subspace that a sequence of (A, B)-controllability subspaces converges to is almost (A, B)-invariant. In Section V, we summarize our results and suggest problems for further research.

II. ORDERS OF REACHABILITY

A. ϵ^j-Reachability for Discrete-Time Systems

We start by developing our theory of asymptotic orders of reachability for systems of the form (1.2) in an analogous way to existing linear control theory. In order to provide a motivation for our approach, let us start with the following counterpart of Example 1.1.

Example 2.1:

$$x[k+1] = \begin{bmatrix} 1 & 1 \\ \epsilon & 2 \end{bmatrix} x[k] + \begin{bmatrix} 1 \\ \epsilon \end{bmatrix} u[k]$$

so

$$\mathcal{C}(\epsilon) = \begin{bmatrix} 1 & 1+\epsilon \\ \epsilon & 3\epsilon \end{bmatrix}.$$

This system is reachable for all $\epsilon \in (0, 2)$. The minimum energy control sequence needed to go from the origin to $x[2] = [1 \ 0]'$ is $u[1] = -1/(2 - \epsilon)$ and $u[2] = 3/(2 - \epsilon)$, which are O(1), [4]. The minimum energy control sequence for $x[2] = [1 \ 1]'$ is $u[1] = (-\epsilon + 1)/\epsilon(2 - \epsilon)$ and $u[2] = (2\epsilon - 1)/\epsilon(2 - \epsilon)$, which are O(1/$\epsilon$). ☐

This characterization of target states by the order of control needed to reach them is now generalized as follows for the discrete-time system (1.2).

Definition 2.2: $x(\epsilon) \in T^n(\epsilon)$ is ϵ^j-*reachable* if there exists an O(1/ϵ^j) input sequence $\mathcal{U}(\epsilon) \equiv [u'[n - 1] \cdots u'[0]]'$ such that $x(\epsilon)$ is reached from zero in n steps using $\mathcal{U}(\epsilon)$ [i.e., $x(\epsilon) = \mathcal{C}(\epsilon)\mathcal{U}(\epsilon)$].

Let $\mathcal{X}^j$ be the set of all ϵ^j-reachable states, then $\mathcal{X}^0 \subset \mathcal{X}^1 \subset \mathcal{X}^2 \subset \cdots$ and $\mathcal{X}^j$ is a $T(\epsilon)$-submodule of $T^n(\epsilon)$. We term $\mathcal{X}^j$ the ϵ^j-*reachable submodule*. ☐

Note that if $x(\epsilon)$ is ϵ^j-reachable, then $(1/\epsilon)x(\epsilon)$ is not necessarily ϵ^j-reachable. Thus, if we had considered target states in $L^n(\epsilon)$ in Definition 2.2, then the set of ϵ^j-reachable states would not be $L(\epsilon)$-subspaces.

In Example 2.1, $\mathcal{X}^0 = \text{Im} [1 \ 0]' + \epsilon T^2(\epsilon)$, $\mathcal{X}^1 = \mathcal{X}^2 = \cdots = T^2(\epsilon)$.

An interesting property of the set of ϵ^j-reachable submodules is that all the structure is embedded in the ϵ^0-reachable submodule. First of all, note that $\mathcal{X}^0$ is the restriction to $T^n(\epsilon)$ of the image of the reachability matrix under the set of all control sequence vectors $\mathcal{U}(\epsilon)$ in $T^{mn}(\epsilon)$. Also, the ϵ^j-reachable submodule is simply obtained by scaling the ϵ^{j-1}-reachable submodule by $1/\epsilon$. To state this formally.

Proposition 2.3: $\mathcal{X}^0 = \{\mathcal{C}(\epsilon)T^{mn}(\epsilon)\} \cap T^n(\epsilon)$ and $\mathcal{X}^j = (1/\epsilon)\{\mathcal{X}^{j-1} \cap \epsilon T^n(\epsilon)\} = (1/\epsilon^i)\{\mathcal{X}^{j-i} \cap \epsilon^i T^n(\epsilon)\}$, for nonnegative integers i, j and $j \geq i$.

Proof: By Definition 2.2, $\mathcal{X}^0 = \{\mathcal{C}(\epsilon)T^{mn}(\epsilon)\} \cap T^n(\epsilon)$, or in general $\mathcal{X}^j = \{\mathcal{C}(\epsilon)(1/\epsilon^j)T^{mn}(\epsilon)\} \cap T^n(\epsilon)$. Then,

$$\frac{1}{\epsilon^i}\{\mathcal{X}^{j-i} \cap \epsilon^i T^n(\epsilon)\} = \frac{1}{\epsilon^i}\left\{\left\{\frac{1}{\epsilon^{j-i}}\mathcal{C}(\epsilon)T^{mn}(\epsilon)\right\} \cap \epsilon^i T^n(\epsilon)\right\}$$

$$= \left\{\frac{1}{\epsilon^j}\mathcal{C}(\epsilon)T^{mn}(\epsilon)\right\} \cap T^n(\epsilon) = \mathcal{X}^j. \quad ☐$$

The structure of the ϵ^j-reachable submodules is not always as easily obtained by inspection of the pair $(A(\epsilon), B(\epsilon))$ as it was in Example 2.1. To illustrate this, consider an ϵ perturbation of Example 2.1.

Example 2.4:

$$x[k+1] = \begin{bmatrix} 1 & 1 \\ -\epsilon & 2 \end{bmatrix} x[k] + \begin{bmatrix} 1 \\ \epsilon \end{bmatrix} u[k]$$

for which

$$\mathcal{C}(\epsilon) = \begin{bmatrix} 1 & 1+\epsilon \\ \epsilon & \epsilon \end{bmatrix}.$$

This system is reachable for all $\epsilon \in (0, \infty)$. In this case, we find that $x[2] = [1\ 0]'$ is ϵ-reachable, and $x[2] = [1\ 1]'$ is ϵ^2-reachable. Therefore, even an ϵ perturbation may cause drastic changes in our submodules.

B. Smith Decomposition of $\mathcal{C}(\epsilon)$

The key element in our results is the Smith decomposition of $\mathcal{C}(\epsilon)$, since in effect this tells us how $\mathcal{C}(\epsilon)$ becomes singular as $\epsilon \downarrow 0$. The $n \times mn$ matrix $\mathcal{C}(\epsilon)$, which has been assumed to have full row rank for $\epsilon \in (0, a)$, has a Smith decomposition [1, 2, 5, 21, 22]

$$\mathcal{C}(\epsilon) = P(\epsilon)D(\epsilon)Q(\epsilon) \qquad (2.1)$$

where $P(\epsilon)$, $n \times n$, is unimodular over $T(\epsilon)$, $Q(\epsilon)$, $n \times mn$ is finite and full row rank at $\epsilon = 0$, right-invertible over $T(\epsilon)$;

$$D(\epsilon) = \text{diag }\{\epsilon^{-h}I_{p_{-h}}, \epsilon^{-h+1}I_{p_{-h+1}}, \cdots, I_{p_0}, \cdots, \epsilon^k I_{p_k}\} \qquad (2.2)$$

is $n \times n$, where I_{p_i} denotes a $p_i \times p_i$ identity matrix with $p_i = 0$ corresponding to absence of the ith block, and with $p_k \neq 0$. We shall term k the *order of reachability* of the system, for reasons that will become clear. The numbers p_i, and hence $D(\epsilon)$, are unique, although $P(\epsilon)$ and $Q(\epsilon)$ are not.

For the remainder of this section, we will assume without loss of generality that $h = 0$, as this can simply be achieved by scaling the input by ϵ^h. Now, from Proposition 2.3 and (2.1), $\mathcal{X}^j = P(\epsilon)\mathcal{Y}^j$ where

$$\mathcal{Y}^j = \mathcal{E}_j + \epsilon \mathcal{E}_{j+1} + \cdots + \epsilon^{k-1-j}\mathcal{E}_{k-1} + \epsilon^{k-j}T^n(\epsilon) \qquad (2.3)$$

and $\mathcal{E}_i = \text{Im } [I_{n_i}\ 0]'$, $n_i = p_0 + \cdots + p_i$. In fact, $\mathcal{Y}^j$ is just the ϵ^j-reachable submodule of the new description obtained through similarity transformation by $P(\epsilon)$, and its structure immediately follows from the fact that the reachability matrix of the transformed system is $D(\epsilon)Q(\epsilon)$ with $Q(\epsilon)$ right invertible over $T(\epsilon)$. This transformed system is examined further in the next subsection.

C. Standard Form

Consider a pair $(A(\epsilon), B(\epsilon))$ with a Smith decomposition of its reachability matrix defined as above. We will term such a system an ϵ^k-*reachable system with (reachability order) indexes* n_0, $\cdots$, n_k. Let $\bar{A}(\epsilon) = P^{-1}(\epsilon)A(\epsilon)P(\epsilon)$ and $\bar{B}(\epsilon) = P^{-1}(\epsilon)B(\epsilon)$. The pair $(\bar{A}(\epsilon), \bar{B}(\epsilon))$ will be called a *standard form* for $(A(\epsilon), B(\epsilon))$.

The system in Example 2.1 is already in standard form because it has a Smith decomposition with $P(\epsilon) = I$. For the system in Example 2.4, a Smith decomposition of the reachability matrix is

$$\mathcal{C}(\epsilon) = \begin{bmatrix} 1 & 0 \\ \epsilon & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \epsilon^2 \end{bmatrix} \begin{bmatrix} 1 & 1+\epsilon \\ 0 & -1 \end{bmatrix} = P(\epsilon)D(\epsilon)Q(\epsilon).$$

The structure of $D(\epsilon)$ uncovers the previously hidden ϵ^2 structure.

To see this more explicitly, transform the system by $P(\epsilon)$

$$y[k+1] = \begin{bmatrix} 1+\epsilon & 1 \\ -\epsilon^2 & 2-\epsilon \end{bmatrix} y[k] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u[k].$$

A standard form for a system is termed a *proper standard form* if $\bar{A}(\epsilon)$ has the following structure:

$$\bar{A}(\epsilon) = \begin{bmatrix} A_{0,0}(\epsilon) & 1/\epsilon A_{0,1}(\epsilon) & \cdots & 1/\epsilon^k A_{0,k}(\epsilon) \\ \epsilon A_{1,0}(\epsilon) & A_{1,1}(\epsilon) & \cdots & 1/\epsilon^{k-1}A_{1,k}(\epsilon) \\ & & \ddots & \vdots \\ \epsilon^k A_{k,0}(\epsilon) & \epsilon^{k-1}A_{k,1}(\epsilon) & \cdots & A_{k,k}(\epsilon) \end{bmatrix} \begin{matrix} \}p_0 \\ \}p_1 \\ \vdots \\ \}p_k \end{matrix}$$

$$(2.4a)$$

where the $A_{i,j}(\epsilon)$ are over $T(\epsilon)$, and $n_i = \Sigma_{j=0}^i p_j$. Note that in this case, due to the structure of $\bar{A}(\epsilon)$ and $\mathcal{C}(\epsilon)$, satisfying our assumption that $h = 0$ in (2.2) only requires scaling $\bar{B}(\epsilon)$ or equivalently $B(\epsilon)$ such that its leading order term is ϵ^0. Then, $\bar{B}(\epsilon)$ has the structure

$$\bar{B}(\epsilon) = \begin{bmatrix} B_0(\epsilon) \\ \epsilon B_1(\epsilon) \\ \vdots \\ \epsilon^k B_k(\epsilon) \end{bmatrix} \begin{matrix} \}p_0 \\ \}p_1 \\ \vdots \\ \}p_k \end{matrix} \qquad (2.4b)$$

Example 2.1 and the transformed version of Example 2.4 are both in proper standard form. In fact, the next result shows that finding one proper standard form is enough to conclude that all standard forms of a pair are proper.

Proposition 2.5: If a pair $(A(\epsilon), B(\epsilon))$ has a proper standard form, then all standard forms of $(A(\epsilon), B(\epsilon))$ are proper.

Proof: Let $\mathcal{C}(\epsilon) = P_1(\epsilon)D(\epsilon)Q_1(\epsilon) = P_2(\epsilon)D(\epsilon)Q_2(\epsilon)$, then $\bar{A}_i(\epsilon) = P_i^{-1}(\epsilon)A(\epsilon)P_i(\epsilon)$, $\bar{B}_i(\epsilon) = P_i^{-1}(\epsilon)B(\epsilon)$ for $i = 1, 2$ are two standard forms. Suppose that the pair $(\bar{A}_1(\epsilon), \bar{B}_1(\epsilon))$ is a proper standard form. Let $\tilde{A}_i(\epsilon) = D^{-1}(\epsilon)\bar{A}_i(\epsilon)D(\epsilon)$, $\tilde{B}_i(\epsilon) = D^{-1}(\epsilon)\bar{B}_i(\epsilon)$ for $i = 1, 2$. Note $\tilde{A}_1(\epsilon)$ and $\tilde{B}_1(\epsilon)$ are both over $T(\epsilon)$. We wish to show that the same is true for $\tilde{A}_2(\epsilon)$ and $\tilde{B}_2(\epsilon)$. Let $R(\epsilon) = D^{-1}(\epsilon)P_2^{-1}(\epsilon)P_1(\epsilon)D(\epsilon)$, then $R(\epsilon)$ is invertible over $L(\epsilon)$, and $Q_2(\epsilon) = R(\epsilon)Q_1(\epsilon)$. But then $R(\epsilon) = Q_2(\epsilon)Q_1^+(\epsilon)$ and $R^{-1}(\epsilon) = Q_1(\epsilon)Q_2^+(\epsilon)$, where $Q_i^+(\epsilon)$ denotes the right inverse of $Q_i(\epsilon)$, which exists over $T(\epsilon)$. Thus, $R(\epsilon)$ is unimodular. Since $(\tilde{A}_1(\epsilon), \tilde{B}_1(\epsilon))$ is over $T(\epsilon)$ and $\tilde{A}_2(\epsilon) = R(\epsilon)\tilde{A}_1(\epsilon)R^{-1}(\epsilon)$, $\tilde{B}_2(\epsilon) = R(\epsilon)\tilde{B}_1(\epsilon)$, the pair $(\tilde{A}_2(\epsilon), \tilde{B}_2(\epsilon))$ is also over $T(\epsilon)$. Therefore, $(\bar{A}_2(\epsilon), \bar{B}_2(\epsilon))$ is a proper standard form. □

A pair $(A(\epsilon), B(\epsilon))$ is termed *proper* if it has a proper standard form. Thus, the systems in both Examples 2.1 and 2.4 are proper. It turns out that the condition that the coefficients of the characteristic polynomial of $A(\epsilon)$ are over $T(\epsilon)$ is necessary and sufficient for a system to be proper. In general, we have the following.

Proposition 2.6: The following statements are equivalent.

1) $(A(\epsilon), B(\epsilon))$ is proper.

2) $\mathcal{C}_i(\epsilon) = [B(\epsilon) | \cdots | A^{i-1}(\epsilon)B(\epsilon)]$ for all positive integers i is over $T(\epsilon)$. To say this more simply, $\mathcal{C}_\infty(\epsilon)$ is over $T(\epsilon)$.

3) The coefficients of the characteristic polynomial $\sigma(A(\epsilon))$ of $A(\epsilon)$ are over $T(\epsilon)$.

Proof: $(1 \rightarrow 2)$ Follows from the definition of a proper form and the structure in (2.4).

$(2 \rightarrow 3)$ It is not hard to show, using [8, Theorem 1.15], that $T(\epsilon)$ is *completely integrally closed*. Since $\mathcal{C}_\infty(\epsilon)$ is over $T(\epsilon)$, the map $\{A^{i-1}(\epsilon)B(\epsilon)\}_{i=1}^\infty$ is over $T(\epsilon)$. Finally, by invoking [8, Theorem 4.17], we achieve the desired result. An alternative proof may be obtained by working with the Jordan form of $A(\epsilon)$, and using results in [23].

$(3 \rightarrow 1)$ Let $\bar{A}(\epsilon) = D^{-1}(\epsilon)\bar{A}(\epsilon)D(\epsilon)$, $\bar{B}(\epsilon) = D^{-1}(\epsilon)\bar{B}(\epsilon)$. Since $\bar{\mathcal{C}}_n(\epsilon) = Q(\epsilon)$ is over $T(\epsilon)$, $\bar{B}(\epsilon)$ is also over $T(\epsilon)$. Since the coefficients of $\sigma(\bar{A}(\epsilon))$ are over $T(\epsilon)$, it follows from the Cayley–

Hamilton theorem that $\tilde{\mathcal{C}}_\infty(\epsilon)$ is over $T(\epsilon)$. In particular, $\tilde{A}(\epsilon)\tilde{\mathcal{C}}_n(\epsilon)$ is over $T(\epsilon)$ and since $Q(\epsilon)$ is right invertible over $T(\epsilon)$, $\tilde{A}(\epsilon)$ is over $T(\epsilon)$. Therefore, $(\tilde{A}(\epsilon), \tilde{B}(\epsilon))$ is a proper standard form. □

As an immediate consequence of Statement 2 of Proposition 2.6 we have the following important property of proper systems.

Corollary 2.7: Given a proper pair $(A(\epsilon), B(\epsilon))$, $x \in \mathfrak{X}^j$ iff x is reachable with $O(1/\epsilon^j)$ control in p steps, for all $p \geq n$. □

For proper systems, therefore, it suffices to work with the Smith structure of $\mathcal{C}_n(\epsilon) = \mathcal{C}(\epsilon)$.

Let us also supplement Proposition 2.6 with the following.

Corollary 2.8: $\tilde{\mathcal{C}}_\infty(\epsilon)$ is over $T(\epsilon)$ iff $\tilde{\mathcal{C}}_{n+1}(\epsilon)$ is over $T(\epsilon)$.

Proof: ($\rightarrow$) Since $\tilde{\mathcal{C}}_{n+1}(\epsilon) = [\tilde{B}(\epsilon)|\tilde{A}(\epsilon)\tilde{\mathcal{C}}_n(\epsilon)]$, and $\tilde{\mathcal{C}}_n(\epsilon)$ is right invertible over $T(\epsilon)$, $\tilde{A}(\epsilon)$ are $\tilde{B}(\epsilon)$ are over $T(\epsilon)$. Thus, $\tilde{\mathcal{C}}_\infty(\epsilon)$ is over $T(\epsilon)$.

($\leftarrow$) Trivial. □

The standard form will prove to be very useful to us, especially for finding feedback to place eigenvalues (Section III). In the Appendix we develop an algorithm to get to a standard form without first constructing the reachability matrix and then explicitly determining its Smith decomposition in order to obtain the transformation matrix $P(\epsilon)$. The algorithm works directly on the pair $(A(\epsilon), B(\epsilon))$, and is a natural extension of the recommended procedure [3] for testing reachability of a constant pair (A, B).

D. Continuous Time

A natural counterpart to Definition 2.2 for continuous time is as follows.

Definition 2.9: $x \in T^n(\epsilon)$ is ϵ^j-reachable if $\exists \tau \in \mathbb{R}^+$ and $u(t) \in (1/\epsilon^j)T^m(\epsilon)$, $\forall t \in [0, \tau]$, such that $x(\tau) = x$, with $x(0) = 0$. Let $\mathfrak{X}^j$ be the set of all ϵ^j-reachable states, then $\mathfrak{X}^0 \subset \mathfrak{X}^1 \subset \mathfrak{X}^2 \subset \cdots$ and $\mathfrak{X}^j$ is an $T(\epsilon)$-submodule of $T^n(\epsilon)$. We term $\mathfrak{X}^j$ the ϵ^j-reachable submodule. □

These submodules have properties analogous to those of discrete time as the following proposition and corollary show (the proofs are given in detail in [18]).

Proposition 2.10: Given a continuous-time system described by the pair $(A(\epsilon), B(\epsilon))$, then $\mathfrak{X}^0 = \langle A(\epsilon)|\mathfrak{B}_\epsilon\rangle \cap T^n(\epsilon)$ where $\langle A(\epsilon)|\mathfrak{B}_\epsilon\rangle \equiv \Sigma_1^n A^{i-1}(\epsilon)\mathfrak{B}_\epsilon$ and $\mathfrak{B}_\epsilon$ is the image of $B(\epsilon)$ over $T(\epsilon)$. □

Corollary 2.11: $\mathfrak{X}^0 = P(\epsilon)D(\epsilon)T^n(\epsilon)$ where $\mathcal{C}(\epsilon) = P(\epsilon)D(\epsilon)Q(\epsilon)$ is a Smith decomposition for the reachability matrix. □

Using the iterative relation $\mathfrak{X}^{j+1} = (1/\epsilon)\{\mathfrak{X}^j \cap \epsilon T^n(\epsilon)\}$ (Proposition 2.3), we can recover all the other reachability submodules from the Smith decomposition of the reachability matrix and Corollary 2.11. Therefore, all our results for discrete time also hold for continuous time.

One important difference exists, however. By an ϵ-dependent change of time scale in continuous time, we can satisfy Statement 3 of Proposition 2.6, so there is no loss of generality in assuming that a continuous-time system (1.1) is proper. In discrete time, by contrast, an assumption that (1.2) is proper *is* restrictive.

III. SHIFTING EIGENVALUES BY $O(1)$ USING FULL STATE FEEDBACK

In this section, we restrict our attention to reachable systems over $T(\epsilon)$. These systems are proper and all eigenvalues of $A(\epsilon)$ are continuous at $\epsilon = 0$. We address the problem of arbitrarily shifting the limiting values of these eigenvalues as $\epsilon \rightarrow 0$, using full state feedback. In other words, we wish to find $F(\epsilon)$ over $L(\epsilon)$ such that $A_F(\epsilon) = A(\epsilon) + B(\epsilon)F(\epsilon)$ has the desired eigenvalues as $\epsilon \rightarrow 0$.

Example 3.1: The eigenvalues of $A(\epsilon)$ in Example 2.1 are at $\lambda_1 = 1 + O(\epsilon)$ and $\lambda_2 = 2 + O(\epsilon)$. A state feedback of $[2\ 4]$ shifts these eigenvalues to $3 + O(\epsilon)$ and $2 + O(\epsilon)$. It is not hard to see that there is no $O(1)$ state feedback that can arbitrarily place λ_2 as $\epsilon \rightarrow 0$. However, a state feedback gain of $[5\ -1/\epsilon]$ shifts the

eigenvalues to $3 + O(\epsilon)$ and $4 + O(\epsilon)$. Here both eigenvalues are moved as $\epsilon \rightarrow 0$, but an $O(1/\epsilon)$ feedback gain has to be used. Note that the closed-loop system

$$A_F(\epsilon) = \begin{bmatrix} 6 & 1-1/\epsilon \\ 6\epsilon & 1 \end{bmatrix}, \qquad B(\epsilon) = \begin{bmatrix} 1 \\ \epsilon \end{bmatrix}$$

is *not* over $T(\epsilon)$ but it is ϵ-reachable with the same indexes, $n_0 = 1$ and $n_1 = 2$, as the original system, and is in proper standard form. □

We shall now show that, for systems over $T(\epsilon)$, the order of feedback gain necessary and sufficient to place the limiting values of all eigenvalues as $\epsilon \rightarrow 0$ is directly given by the order of reachability of the system. Let us start by looking at ϵ^0-reachable systems. In all that follows, Λ denotes a self-conjugate set of n numbers, $\lambda(A)$ denotes the spectrum of A, and Z denotes the set of all integers. Define

$$\alpha = \min_{r \in Z} \{r \,|\, \forall \Lambda, \exists F(\epsilon) = O(1/\epsilon^r),$$

$$\text{s.t. } \lambda(A(\epsilon) + B(\epsilon)F(\epsilon))|_{\epsilon=0} = \Lambda\}. \quad (3.1)$$

Hence, α is the smallest order of feedback gain that will produce arbitrary placement of the limiting eigenvalues as $\epsilon \rightarrow 0$.

Lemma 3.2: The pair $(A(\epsilon), B(\epsilon))$, over $T(\epsilon)$, is ϵ^0-reachable iff $\alpha = 0$.

Proof: ($\rightarrow$) If the pair $(A(\epsilon), B(\epsilon))$ is ϵ^0-reachable, then, $\mathcal{C}(\epsilon)|_{\epsilon=0}$ has full row rank. Thus, the pair $(A(0), B(0))$ is reachable, and $\forall \Lambda$, $\exists F: \mathbb{R}^n \rightarrow \mathbb{R}^n$ s.t. $\lambda(A(\epsilon) + B(\epsilon)F)|_{\epsilon=0} = \lambda(A(0) + B(0)F) = \Lambda$. Hence $\alpha \leq 0$. Now assume $\alpha < 0$. Then, $\lim_{\epsilon \downarrow 0} F(\epsilon) = 0$ for those $F(\epsilon)$ of $O(1/\epsilon^\alpha)$ that produce arbitrary placement of the limiting eigenvalues as $\epsilon \rightarrow 0$ according to (3.1). But then $\lim_{\epsilon \downarrow 0} (A(\epsilon) + B(\epsilon)F(\epsilon)) = A(0)$, so no limiting eigenvalue as $\epsilon \rightarrow 0$ is moved, which is a contradiction. We conclude that $\alpha = 0$.

($\leftarrow$) Conversely, assume that $\alpha = 0$, then $\forall \Lambda$, $\exists F = F(\epsilon)|_{\epsilon=0}$ s.t. $\lambda(A(0) + B(0)F) = \Lambda$. Thus, the pair $(A(0), B(0))$ is reachable, and $\mathcal{C}(\epsilon)|_{\epsilon=0}$ has full row rank, so the pair $(A(\epsilon), B(\epsilon))$ is ϵ^0-reachable. □

Proposition 3.3: The pair $(A(\epsilon), B(\epsilon))$, over $T(\epsilon)$, is ϵ^k-reachable iff $\alpha = k$.

Proof: ($\rightarrow$) If the pair $(A(\epsilon), B(\epsilon))$ is ϵ^k-reachable, then the pair $\tilde{A}(\epsilon) = D^{-1}(\epsilon)P^{-1}(\epsilon)A(\epsilon)P(\epsilon)D(\epsilon)$, $\tilde{B}(\epsilon) = D^{-1}(\epsilon)P^{-1}(\epsilon)B(\epsilon)$ is ϵ^0-reachable and is over $T(\epsilon)$ (Proposition 2.6). Thus, by Lemma 3.2, $\forall \Lambda$, $\exists$ an $O(1)\tilde{F}(\epsilon)$ s.t. $\lambda(\tilde{A}(\epsilon) + \tilde{B}(\epsilon)\tilde{F}(\epsilon))|_{\epsilon=0} = \Lambda$. Let $F(\epsilon) = \tilde{F}(\epsilon)D^{-1}(\epsilon)P^{-1}(\epsilon)$, then $F(\epsilon)$ is $O(1/\epsilon^k)$. Since $(\tilde{A}_{\tilde{F}}(\epsilon), \tilde{B}(\epsilon))$ is proper, the coefficients of $\sigma(\tilde{A}_{\tilde{F}}(\epsilon))$ are over $T(\epsilon)$. Thus,

$$\lim_{\epsilon \downarrow 0} \lambda(\tilde{A}(\epsilon) + \tilde{B}(\epsilon)\tilde{F}(\epsilon)) = \lim_{\epsilon \downarrow 0} \lambda(A(\epsilon) + B(\epsilon)F(\epsilon)) \quad (3.2)$$

and $\alpha \leq k$. To see that the equality must hold, note first that

$$\tilde{A}(\epsilon) = \begin{bmatrix} A_{0,0}(\epsilon) & \epsilon A_{0,1}(\epsilon) & \cdots & \epsilon^k A_{0,k}(\epsilon) \\ A_{1,0}(\epsilon) & A_{1,1}(\epsilon) & \cdots & \epsilon^{k-1} A_{1,k}(\epsilon) \\ \vdots & & \ddots & \vdots \\ A_{k,0}(\epsilon) & A_{k,1}(\epsilon) & \cdots & A_{k,k}(\epsilon) \end{bmatrix} \quad (3.3)$$

$$\underbrace{\qquad\qquad\qquad\qquad}_{n - n_{k-1} \text{ columns}}$$

where the $A_{i,j}(\epsilon)$ are over $T(\epsilon)$. Now, if $\alpha < k$, then the last $n - n_{k-1}$ columns of $\tilde{F}(0) = \lim_{\epsilon \downarrow 0} F(\epsilon)P(\epsilon)D(\epsilon) = 0$ for those $F(\epsilon)$ of $O(1/\epsilon^\alpha)$ that produce arbitrary eigenvalue placement according to (3.1). But then

$$\lim_{\epsilon \downarrow 0} (\tilde{A}(\epsilon) + \tilde{B}(\epsilon)\tilde{F}(\epsilon)) = \begin{bmatrix} * & 0 \\ * & A_{k,k}(0) \end{bmatrix} \quad (3.4)$$

were $*$ denotes some constant entries, and the limiting eigenvalues corresponding to $A_{k,k}(\epsilon)$ are *not* moved, which is a contradiction. We conclude that $\alpha = k$.

($\leftarrow$) Clearly, the pair $(A(\epsilon), B(\epsilon))$ is ϵ^j-reachable for some j. By the first part of this proof, $\alpha = j$. Hence, $j = k$ and the pair is ϵ^k-reachable. $\square$

Note that if some pair $(A(\epsilon), B(\epsilon))$ over $T(\epsilon)$ is ϵ^0-reachable, then the closed-loop pair $(A_F(\epsilon), B(\epsilon))$, where $A_F(\epsilon) = A(\epsilon) + B(\epsilon)F(\epsilon)$, is ϵ^0-reachable for all $F(\epsilon)$ of $O(1)$. Thus, we have the following result.

Corollary 3.4: Given a pair $(A(\epsilon), B(\epsilon))$ over $T(\epsilon)$, the reachability indexes n_i, as defined in Section II-C, are invariant under any feedback of the form $F(\epsilon) = \tilde{F}(\epsilon)D^{-1}(\epsilon)P^{-1}(\epsilon)$ where $\tilde{F}(\epsilon)$ is $O(1)$. Also, the closed-loop pair is proper. $\square$

The ϵ^j-reachable submodules of the standard form are uniquely determined by the indexes, and the ϵ^j-reachable submodules of the original system are uniquely determined by the ϵ^j-reachable submodules of the standard form, via $P(\epsilon)$. Thus:

Corollary 3.5: Given a pair $(A(\epsilon), B(\epsilon))$ over $T(\epsilon)$, the ϵ^j-reachability submodules are invariant under any feedback of the form $F(\epsilon) = \tilde{F}(\epsilon)D^{-1}(\epsilon)P^{-1}(\epsilon)$, where $\tilde{F}(\epsilon)$ is $O(1)$. $\square$

For the more general class of proper systems over $L(\epsilon)$, the orders of feedback gains do *not* necessarily match the orders of reachability. Let us consider the following example.

Example 3.6: The pair

$$A(\epsilon) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1/\epsilon \\ 0 & 2\epsilon & 0 \end{bmatrix}, \quad B(\epsilon) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

corresponds to an ϵ-reachable system in proper standard form. Let

$$F(\epsilon) = \begin{bmatrix} f_1 & f_2 & 0 \\ f_3 & f_4 & 0 \end{bmatrix},$$

where the f_i are all scalar constants, then $\det(\lambda I - A_F(\epsilon)) = \lambda^3 - (f_1 + f_4)\lambda^2 + (f_1 f_4 - f_2 f_3 - 2)\lambda + 2f_1$. Clearly, $f_i \in \mathbb{R}$ can be chosen appropriately to match any third degree polynomial with real coefficients. Therefore, all eigenvalues of $A(\epsilon)$ can be arbitrarily moved as $\epsilon \to 0$ using only $O(1)$ feedback gains. What happens in this example is that an $O(1)$ gain for the third state component produces an $O(1/\epsilon)$ input for the second component. Therefore, even with $O(1)$ gains, the input values themselves will be $O(1/\epsilon)$, as would be expected when producing shifts in the limiting eigenvalues for this ϵ-reachable system. $\square$

The overall effect of $O(1)$ feedback on the eigenvalues, even for systems over $T(\epsilon)$, is a more subtle issue than the order of feedback necessary to shift the limiting eigenvalues. Consider the following example.

Example 3.7: Let

$$A(\epsilon) = \begin{bmatrix} 0 & 1 \\ \epsilon & 0 \end{bmatrix}, \quad B(\epsilon) = \begin{bmatrix} 1 \\ \epsilon \end{bmatrix}.$$

The reachability order indexes are $n_0 = 1$ and $n_1 = 2$. The eigenvalues of $A(\epsilon)$ are at $\pm\sqrt{\epsilon}$. Feedback of $[-1 \; -1]$ moves the eigenvalues to -1 and $-\epsilon$. Thus, the effect of feedback is larger than $O(\epsilon)$, namely $O(\sqrt{\epsilon})$. (It is worth noting that the original system did not have well-behaved time scale structure in the sense of [1, 2], and that the feedback produces well-behaved time scale structure.) $\square$

We leave these problems for further research. Section V suggests some potential extensions.

An extension of Algorithm A.3 can be used to compute the feedback matrix necessary to shift eigenvalues by some desired amount. Application of Algorithm A.3 produces a pair $(A_k(\epsilon), B_k(\epsilon))$, where $A_k(\epsilon) = S^{-1}(\epsilon)A(\epsilon)S(\epsilon)$, $B_k(\epsilon) = S^{-1}(\epsilon)B(\epsilon)$, where $(A_k(0), B_k(0))$ is reachable and $S(\epsilon)$ is the product of all the similarity transformations used to achieve the final pair. From the pair $(A_k(0), B_k(0))$, we can compute a feedback matrix F such that the eigenvalues of $A_{kF}(0) = A_k(0) + B_k(0)F$ are as desired. We have that $\lambda(A_{kF}(\epsilon))|_{\epsilon=0} = \lambda(A_{kF}(0))$ and that $(A_{kF}(\epsilon), B(\epsilon))$ is proper. Let $F(\epsilon) = FS^{-1}(\epsilon)$ and $A_F(\epsilon) = A(\epsilon) + B(\epsilon)F(\epsilon)$. Since

$S(\epsilon)$ is invertible for $\epsilon \in (0, a)$ for some $a \in \mathbb{R}^+$, $(A_F(\epsilon), B(\epsilon))$ is also proper. Therefore, as in the proof of Proposition 3.3, the eigenvalues of $A_F(\epsilon)$ are as desired.

This algorithm was applied in [18] to a fifth order, weakly reachable system over $\mathbb{R}$ with one input. The system was first parametrized by replacing certain small entries by (constant multiples of) powers of ϵ. The feedback gain to place the limiting eigenvalues calculated for the parametrized system by the above approach was evaluated at the specific value of ϵ corresponding to the original system. This approach produced far better numerical results than calculating the feedback directly for the given system. Similar concerns have been expressed by authors interested in numerical issues of multivariable pole placement for linear time-invariant systems (as explained in Section I-A). Our approach would attempt to address those issues by scaling the pair (A, B) appropriately. Unfortunately, (A, B) has to be parametrized by ϵ first if ϵ does not represent some (small) physical parameter. Further study of this problem is left for future research, although some heuristic suggestions for parametrizations are made in Section V.

IV. ALMOST INVARIANT SUBSPACES

A. $(A(\epsilon)), B(\epsilon))$-Invariance and Almost (A, B)-Invariance

In this section, we use our framework to provide some new insights on the notions of almost (A, B)-invariance and almost (A, B)-controllability, introduced into the geometric approach to linear systems [24] by Willems [19]. These concepts have applications to disturbance decoupling, robustness, noisy gain stabilization, and cheap control.

To provide orientation and give the flavor of our approach, we consider the following example.

Example 4.1: Let

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

It is easy to see from the results in [19] that $\mathcal{V}_a = \text{Im} \, [1 \; 0]'$ is an almost (A, B)-invariant subspace. Consider the $L(\epsilon)$-subspace $\mathcal{V}_\epsilon$ generated by $[1 \; \epsilon]'$. Since

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ \epsilon \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \epsilon \end{bmatrix}(1/\epsilon) + \begin{bmatrix} 1 \\ 0 \end{bmatrix}(-1/\epsilon)$$

this subspace is an (A, B)-invariant $L(\epsilon)$-subspace [24]. As $\epsilon \to 0$, $\mathcal{V}_\epsilon \to \text{Im} \, [1 \; 0]'$ (over $\mathbb{R}$), which is the almost (A, B)-invariant subspace identified above. So we have found an (A, B)-invariant $L(\epsilon)$-subspace $\mathcal{V}_\epsilon$ that converges asymptotically to an almost (A, B)-invariant subspace. Using the relation $(-1/\epsilon) = -F(\epsilon)[1 \; \epsilon]'$ with $F(\epsilon) = [1/\epsilon \; 0]$, $\mathcal{V}_\epsilon$ is $A_F(\epsilon)$ invariant, where

$$A_F(\epsilon) = A + BF(\epsilon) = \begin{bmatrix} 1/\epsilon & 0 \\ 1 & 0 \end{bmatrix}.$$

Furthermore, $\mathcal{V}_\epsilon$ is a coasting subspace [19], i.e., it is (A, B)-invariant but has no (A, B)-controllable part, whereas $\mathcal{V}_a$ is a sliding subspace [19], i.e., it is almost (A, B)-invariant but it has no (A, B)-invariant part.

Note that an eigenvalue of $A_F(\epsilon) \to +\infty$ as $\epsilon \to 0$. On the other hand, consider the (A, B)-invariant $L(\epsilon)$-subspace $\hat{\mathcal{V}}_\epsilon$ generated by $[1 \; -\epsilon]'$. As $\epsilon \to 0$, $\hat{\mathcal{V}}_\epsilon \to \mathcal{V}_a$ also. By going through the above procedure, we get $\hat{F}(\epsilon) = [-1/\epsilon \; 0]$ and

$$A_{\hat{F}}(\epsilon) = \begin{bmatrix} -1/\epsilon & 0 \\ 1 & 0 \end{bmatrix}.$$

Now the eigenvalue of $A_{\hat{F}}(\epsilon)$ that blows up approaches $-\infty$ as $\epsilon \to 0$. $\square$

We proceed with proving some results related to the above observations.

Definition 4.2: A subspace $\mathcal{V}_\epsilon \subset L^n(\epsilon)$ is $(A(\epsilon), B(\epsilon))$-*invariant* if $\exists F(\epsilon): L^n(\epsilon) \to L^m(\epsilon)$ s.t. $A_F(\epsilon)\mathcal{V}_\epsilon \subset \mathcal{V}_\epsilon$, where $A_F(\epsilon) = A(\epsilon) + B(\epsilon)F(\epsilon)$. We denote the family of $(A(\epsilon), B(\epsilon))$-invariant $L(\epsilon)$-subspaces by V_ϵ. $\square$

A straightforward consequence of this definition is the following well-known result [24].

Proposition 4.3: $\mathcal{V}_\epsilon \in V_\epsilon$ iff $A(\epsilon)\mathcal{V}_\epsilon \subset \mathcal{V}_\epsilon + \mathcal{B}$, where $\mathcal{B} = B(\epsilon)L^m(\epsilon)$. $\square$

Let $\mathcal{V}_\epsilon \subset L^n(\epsilon)$ and $V(\epsilon) = [v_1(\epsilon)| \cdots |v_\mu(\epsilon)]$ be a matrix such that its columns form a basis over $L(\epsilon)$ for $\mathcal{V}_\epsilon$. Let $V(\epsilon) = P_V(\epsilon)D_V(\epsilon)Q_V(\epsilon)$ be a Smith decomposition of $V(\epsilon)$ such that $P_V(\epsilon)$ is $n \times \mu$, $D_V(\epsilon)$ and $Q_V(\epsilon)$ are $\mu \times \mu$. Then the columns $p_i(\epsilon)$ of $P_V(\epsilon)$ form a basis over $L(\epsilon)$ for $\mathcal{V}_\epsilon$ such that $p_i(\epsilon) \in T^n(\epsilon)$ and the columns of $P_V(0)$ are a basis over $\mathbb{R}$. We use this for the existence of the desired basis in the following definition.

Definition 4.4: Let $\mathcal{V}_\epsilon \subset L^n(\epsilon)$ and $\{v_1(\epsilon), \cdots, v_\mu(\epsilon)\}$ be a basis over $L(\epsilon)$ for $\mathcal{V}_\epsilon$ such that $v_i(\epsilon) \in T^n(\epsilon)$ and the set of vectors $\{v_1(0), \cdots, v_\mu(0)\}$ forms a basis over $\mathbb{R}$ for some $\mathcal{V}_a \subset \mathbb{R}^n$. Then we say that $\mathcal{V}_\epsilon$ converges asymptotically to $\mathcal{V}_a$, or $\mathcal{V}_\epsilon \xrightarrow[\epsilon \to 0]{} \mathcal{V}_a$ (this is convergence in the Grassmanian sense). $\square$

One can always construct a matrix $W(\epsilon)$ over $T(\epsilon)$, such that $W(0) = I$ and $v_i(\epsilon) = W(\epsilon)v_i(0)$. Thus, an alternate representation of $\mathcal{V}_\epsilon$ would be $W(\epsilon)\mathcal{V}_a$. We use these notions to connect our results to their counterparts in [19] and [24].

The following result enables us to establish a connection between our framework and the notion of almost (A, B)-invariance. It provides a method to compute approximations for the distributional inputs required to steer the trajectories of an almost (A, B)-invariant subspace exactly through that subspace. Using these high gain feedback approximations one can steer trajectories arbitrarily close to an almost (A, B)-invariant subspace.

Denote the family of almost (A, B)-invariant subspaces by V_a. We then have the following result.

Proposition 4.5: For a pair (A, B), if $\mathcal{V}_a \in V_a$ then $\exists \mathcal{V}_\epsilon \in V_\epsilon$ such that $\mathcal{V}_\epsilon \xrightarrow[\epsilon \to 0]{} \mathcal{V}_a$. $\square$

The proof is very similar in principle to that of Willems [19] and it is given in detail in [18]. However, note that the converse of the above proposition does not hold, although [19] claims that it does. To illustrate this, consider the following example.

Example 4.6: Let

$$A = \begin{bmatrix} 0_3 & 0_3 \\ I_3 & 0_3 \end{bmatrix} \text{ and } B = \begin{bmatrix} I_3 \\ 0_3 \end{bmatrix}.$$

Consider $\mathcal{V} = \{v_1(\epsilon), v_2(\epsilon), v_3(\epsilon)\}$ where $v_1(\epsilon) = [1\ 0\ 0\ 0\ \epsilon\ 0]'$, $v_2(\epsilon) = [0\ 0\ 0\ 1\ 0\ 0]'$, $v_3(\epsilon) = [0\ 1\ 0\ 0\ 0\ 1]'$ and $\{\cdot\}$ denotes span over $L(\epsilon)$. $\mathcal{V} \in V_\epsilon$ and $\mathcal{V} \xrightarrow[\epsilon \to 0]{} \mathcal{L}$ where $\mathcal{L} = \{v_1(0), v_2(0), v_3(0)\}$ and $\{\cdot\}$ now denotes span over $\mathbb{R}$. But $\mathcal{L}$ is *not* an almost (A, B)-invariant subspace (this can easily be tested using algorithms ISA and ACSA [19]). $\square$

Willems [19] poses the problem of finding an input that steers the trajectories of a system arbitrarily close to an almost (A, B)-invariant subspace. Our approach shows how this can be done. We show below how to construct an (A, B)-invariant $L(\epsilon)$-subspace that converges asymptotically to the almost (A, B)-invariant subspace. The desired input then follows on calculating the feedback that makes the (A, B)-invariant $L(\epsilon)$-subspace $A_F(\epsilon)$-invariant.

Recall from [19] that any almost (A, B)-invariant subspace $\mathcal{V}_a$ can be represented as $\mathcal{V}_a = \mathcal{V} + \mathcal{R}_a$ where $\mathcal{V}$ is (A, B)-invariant and $\mathcal{R}_a$ is almost (A, B)-controllable. Furthermore, any almost (A, B)-controllability subspace $\mathcal{R}_a$ can be represented as $\mathcal{R}_a = \mathcal{R}_o \oplus \mathcal{R}_s$ where $\mathcal{R}_o$ is the supremal (A, B)-controllability subspace in $\mathcal{R}_a$ and $\mathcal{R}_s$ is a sliding subspace. By a construction in the proof of Proposition 4.5 in [18], illustrated in the example below, we can find $\mathcal{V}_c \in V_\epsilon$ where $\mathcal{V}_c = Q(\epsilon)\mathcal{R}_s$, $Q(\epsilon)$ over $T(\epsilon)$ and $Q(0) = I$, where $\mathcal{V}_c$ is a coasting $L(\epsilon)$-subspace whose associated eigenvalues approach $-\infty$ as $\epsilon \to 0$. The feedback $F(\epsilon)$

that makes $\mathcal{V}_c$ an $A_F(\epsilon)$-invariant $L(\epsilon)$-subspace can be calculated and provides the desired input. Those eigenvalues of $A_F(\epsilon)$ that correspond to $\mathcal{R}_s$ approach $-\infty$ as $\epsilon \to 0$. This increases the magnitude of the feedback gains, and the generated inputs and their derivatives approach impulses in the limit. The eigenvalues corresponding to $\mathcal{R}_o$ can be assigned by the usual pole placement methods.

As an illustration of the procedure, consider the following example, which contains the essential features of the general case.

Example 4.7: Let

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathcal{V}_a = \{\bar{v}_1, \bar{v}_2\},$$

where

$$\bar{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } \bar{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$\mathcal{V}_a$ is an almost (A, B)-invariant subspace, and in fact it is a sliding subspace. Consider $\mathcal{V}_\epsilon = \{v_1(\epsilon), v_2(\epsilon)\}$, where $v_1(\epsilon) = [1\ -\epsilon\ \epsilon^2]'$ and $v_2(\epsilon) = [0\ 1\ -2\epsilon]'$. Note that $\mathcal{V}_\epsilon$ is a coasting $L(\epsilon)$-subspace, i.e., it is (A, B)-invariant but not (A, B)-controllable. Furthermore, $v_1(0) = \bar{v}_1$, $v_2(0) = \bar{v}_2$ and $\mathcal{V}_\epsilon \xrightarrow[\epsilon \to 0]{} \mathcal{V}_a$. Also, $v_i(\epsilon) = P(\epsilon)\bar{v}_i$, for $i = 1, 2$, where

$$P(\epsilon) = \begin{bmatrix} 1 & 0 & 0 \\ -\epsilon & 1 & 0 \\ 2\epsilon & -\epsilon^2 & 1 \end{bmatrix}$$

gets its lower triangular entries from a Pascal triangle construction with alternating signs (see [18]). Solving the equations

$$A(\epsilon)[v_1(\epsilon)|v_2(\epsilon)] = [v_1(\epsilon)|v_2(\epsilon)]g_v(\epsilon) + Bg_u(\epsilon),$$

$$g_u(\epsilon) = -F(\epsilon)[v_1(\epsilon)|v_2(\epsilon)]$$

yields $-F(\epsilon) = [2/\epsilon\ 1/\epsilon^2\ 0]'$ and

$$A_F(\epsilon) = A + BF(\epsilon) = \begin{bmatrix} -2/\epsilon & -1/\epsilon^2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

with $\mathcal{V}_\epsilon$ being $A_F(\epsilon)$-invariant. Note that the desired input $u(t) = F(\epsilon)x(t)$. On the other hand, the eigenvalues of $A_F(\epsilon)$ that correspond to $\mathcal{V}_a$ are both at $-1/\epsilon$. They are stable and approach $-\infty$ as $\epsilon \to 0$. $\square$

B. $(A(\epsilon), B(\epsilon))$-Controllability and Almost (A, B)-Invariance

We now proceed with the notion of $(A(\epsilon), B(\epsilon))$-controllability $L(\epsilon)$-subspaces, adopting Wonham's definition [24] of (A, B)-controllability subspaces. The notation $\langle A(\epsilon)|\mathcal{B}\rangle$ will be used to denote $\mathcal{B} + A(\epsilon)\mathcal{B} + A^2(\epsilon)\mathcal{B} \cdots$.

Definition 4.8: $\mathcal{R}_\epsilon \subset L^n(\epsilon)$ is an $(A(\epsilon), B(\epsilon))$-*controllability subspace* if there exist maps $F(\epsilon):L^n(\epsilon) \to L^m(\epsilon)$ and $G(\epsilon):L^m(\epsilon) \to L^m(\epsilon)$ such that $\mathcal{R}_\epsilon = \langle A(\epsilon) + B(\epsilon)F(\epsilon)|\text{Im }(B(\epsilon)G(\epsilon))\rangle$. We denote the family of $(A(\epsilon), B(\epsilon))$-controllability $L(\epsilon)$-subspaces by R_ϵ. $\square$

To put the above definition into a more usable form, consider the following proposition, which simply restates results of Wonham [24] in the present framework.

Proposition 4.9: a) $\mathcal{R}_\epsilon \in R_\epsilon$ iff there exists a map $F(\epsilon):L^n(\epsilon) \to L^m(\epsilon)$ such that $\mathcal{R}_\epsilon = \langle A(\epsilon) + B(\epsilon)F(\epsilon)|\mathcal{B} \cap \mathcal{R}_\epsilon\rangle$ where $\mathcal{B}$ represents the range of $B(\epsilon)$ over $L(\epsilon)$. b) $\mathcal{R}_\epsilon = \langle A_F(\epsilon)|\mathcal{B} \cap \mathcal{R}_\epsilon\rangle$ for every map $F(\epsilon) \in F(\mathcal{R}_\epsilon)$, where $F(\mathcal{R}_\epsilon)$ represents the family of feedback matrices $F(\epsilon)$ such that $\mathcal{R}_\epsilon$ is $A_F(\epsilon)$-invariant. $\square$

Let $\mathcal{R}_\epsilon \in R_\epsilon$ and $\mathcal{R}_\epsilon \xrightarrow[\epsilon \to 0]{} \mathcal{R}_n$. Then, it turns out that $\mathcal{R}_n$

is almost (A, B)-invariant. Finding inputs for steering trajectories arbitrarily close to $\Re_n$ is done by calculating an $F(\epsilon)$ such that $\Re_\epsilon$ is $A_F(\epsilon)$-invariant and the eigenvalues corresponding to $\Re_\epsilon$ are continuous at $\epsilon = 0$ and asymptotically stable. The following lemma and proposition show this.

Lemma 4.10: Given a pair (A, B), let $\Re_\epsilon \in R_\epsilon$ and $\Re_\epsilon \xrightarrow[\epsilon \to 0]{} \Re_n$, then $\forall 0(1)x_0$ s.t. $d(x_0, \Re_n)^1$ is $0(\epsilon)$ and $\forall \tau > 0$, $\exists$ an input function $u(t)$ s.t. $d(x_0(t, \epsilon), \Re_n)$ is $0(\epsilon)$ for $0 < t < \tau$, where $x_0(t, \epsilon)$ is the trajectory defined by $u(t)$ and the initial condition x_0.

Proof: Here we first need to find a trajectory in $\Re_\epsilon$ which is $0(1)$ for $0 < t < \tau$. Find $F(\epsilon)$ s.t. $\Re_\epsilon$ is $A_F(\epsilon)$-invariant and the eigenvalues of $A_F(\epsilon)$ corresponding to $\Re_\epsilon$ are all continuous at $\epsilon = 0$ and asymptotically stable. Then $\forall 0(1)x_1 \in \Re_\epsilon$, $x_1(t, \epsilon) \in \Re_\epsilon$ $\forall t > 0$ where $x_1(t, \epsilon)$ is the trajectory defined by the initial condition x_1 and the input specified by $F(\epsilon)x(t)$. Since the eigenvalues of $A_F(\epsilon)$ corresponding to $\Re_\epsilon$ are all continuous at $\epsilon = 0$ and stable, $x_1(t, \epsilon)$ is $0(1)$. Therefore, $d(x_1(t, \epsilon), \Re_n)$ is $0(\epsilon)$, since $\Re_\epsilon \xrightarrow[\epsilon \to 0]{} \Re_n$. Consider $x_2(t, \epsilon)$, the trajectory defined by the initial condition $x_2 = x_0 - x_1$, with $x_1 \in \Re_\epsilon$ chosen such that x_2 is $0(\epsilon)$. Since the eigenvalues of $A_F(\epsilon)$ are continuous at $\epsilon = 0$, $\forall \tau > 0$ $x_2(t, \epsilon)$ is $0(\epsilon)$ for $0 < t \leq \tau$. Thus, $d(x_0(t, \epsilon), \Re_n)$ is $0(\epsilon)$ for $0 < t \leq \tau$. $\square$

Proposition 4.11: Given a pair (A, B), let $\Re_\epsilon \in R_\epsilon$ and $\Re_\epsilon \xrightarrow[\epsilon \to 0]{} \Re_n$, then $\Re_n \in V_a$.

Proof: Pick some $\tau > 0$ and apply Lemma 4.10. Thus, $\exists u(t)$ s.t. $d(x(t, \epsilon), \Re_n)$ is $0(\epsilon)$ for $0 < t \leq \tau$. Then $\exists \epsilon_o > 0$ s.t. $d(x(t, \epsilon), \Re_n) < \delta$ for $0 < t \leq \tau$ and $\forall \epsilon \leq \epsilon_o$. Use $x(\tau, \epsilon)$ as the initial condition to reapply Lemma 4.10 for the interval $\tau < t \leq 2\tau$. Find $\epsilon_1 > 0$ s.t. $\epsilon_1 \leq \epsilon_o$ and $d(x(t, \epsilon_1), \Re_n) < \delta$ for $\tau < t \leq 2\tau$. Repeated use of Lemma 4.10 achieves the desired result. $\square$

To illustrate these, consider the following example.

Example 4.12: Let

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and

$$\Re_\epsilon = \text{Im} \begin{bmatrix} 1 \\ \epsilon \\ 0 \end{bmatrix} + \text{Im} \begin{bmatrix} 0 \\ 0 \\ \epsilon \end{bmatrix}.$$

Note that $\Re_n = \text{Im} [1\ 0\ 0]' + \text{Im} [0\ 0\ 0]'$ and it is an almost (A, B)-invariant subspace. Let $F(\epsilon) = [-3\ 0\ -2/\epsilon]$, then $\Re_\epsilon$ is $A_F(\epsilon)$-invariant and the eigenvalues corresponding to $\Re_\epsilon$ are at -2, -4, asymptotically stable and $0(1)$. Pick the initial state x_0 of Lemma 4.10 as $x_0 = [1\ 0\ 0]'$. Let $x_1 = [1\ \epsilon\ 0]' \in \Re_\epsilon$. Then, $x_1(t, \epsilon) = [-e^{-t} + 2e^{-2t}, -\epsilon e^{-t} + 2\epsilon e^{-2t}, -\epsilon e^{-t} - \epsilon e^{-2t}]' \in \Re_\epsilon$, and $d(x_1(t, \epsilon), \Re_n)$ is clearly $0(\epsilon)$ for any finite τ. On the other hand, $x_2 = [0\ -\epsilon\ 0]'$ and $x_2(t, \epsilon) = [2\epsilon e^{-2t}, 2\epsilon^2 e^{-2t}, -\epsilon^2 e^{-2t}]'$. Thus, $d(x_0(t), \Re_n)$ is $0(\epsilon)$. So, in the spirit of Proposition 4.11, this may be bounded by any δ for any given τ by picking an appropriate $\epsilon = \epsilon_o$. Then, using $x(\tau, \epsilon)$ as the new initial state and repeated use of this procedure achieves the desired result. $\square$

In this section, we examined the notions of almost (A, B)-invariant and almost (A, B)-controllability subspaces in the framework that we have developed in this paper and [18]. We outlined a method for calculating inputs that steer trajectories arbitrarily close to almost (A, B)-invariant subspaces or equivalently force the eigenvalues corresponding to sliding parts of almost (A, B)-controllability subspaces to approach $-\infty$. We also analyzed the properties of limits of elements in V_ϵ and R_ϵ as $\epsilon \to 0$ from a trajectory point of view.

$^1 d(x, L) = \inf_{x' \in L} \|x - x'\|$.

V. Conclusions

In this paper, we have developed an algebraic approach to high gain controls for linear dynamic systems with varying orders of reachability. Based on this approach, we addressed the issues of high gain inputs for reaching target states, high gain feedback for pole placement, and high gain inputs for steering trajectories arbitrarily close to almost (A, B)-invariant subspaces and almost (A, B)-controllability subspaces.

The results presented here suggest several directions for further research. It is of interest to analyze the orders of feedback gains for shifting the limiting eigenvalues as $\epsilon \to 0$ in the more general case of proper systems, rather than just systems over $T(\epsilon)$. Intuitively, if a mode is ϵ-reachable but "$1/\epsilon$-observable," in that it has a $1/\epsilon$ coupling to other states, then it should be possible to shift its eigenvalue by $0(1)$ using $0(1)$ feedback gain. A related problem is that of changing the dynamics of a given continuous-time system that has multiple time scales [1, 2] without changing its time scale structure. This would involve shifting an eigenvalue λ, where λ/ϵ^j is continuous at $\epsilon = 0$, by some $a\epsilon^j$, $a \neq 0$.

A key problem that bears attention is that of parametrizing systems over $\mathbb{R}$. Two heuristic methods could be suggested for this. One is to recognize small entries in the matrix, either isolated or added to another entry, and replace these with powers of ϵ. Another method for parametrization could come from numerical reachability tests [3], where for example small singular values at different stages of a test may be replaced by (appropriate powers of) ϵ.

It will be important to develop dual results for systems with observations $y[k] = C(\epsilon)x[k]$ or $y(t) = C(\epsilon)x(t)$. This could then lead to research on connections to optimal control [25], [26], realization theory, balanced realizations [15] and so on.

Very interesting and important generalizations may be expected from more explicit connection to and exploitation of the rather large literature on systems over rings, as represented in [6]-[9] for example. In particular, extensions to problems involving outputs will undoubtedly emerge from this.

Appendix

Here we develop an algorithm to recover a standard form without forming the reachability matrix and computing its Smith decomposition. The proofs and details on the algorithm are presented in [18]. Our algorithm can only deal with a pair $(A(\epsilon), B(\epsilon))$ over $T(\epsilon)$, so this restriction is assumed here. The structure of a pair $(A(\epsilon), B(\epsilon))$ in standard form is then as follows:

$$A(\epsilon) = \begin{bmatrix} A_{0,0}(\epsilon) & A_{0,1}(\epsilon) & \cdots & A_{0,k}(\epsilon) \\ \epsilon A_{1,0}(\epsilon) & A_{1,1}(\epsilon) & \cdots & A_{1,k}(\epsilon) \\ & & \ddots & \vdots \\ \epsilon^k A_{k,0}(\epsilon) & \epsilon^{k-1} A_{k,1}(\epsilon) & \cdots & A_{k,k}(\epsilon) \end{bmatrix} \begin{matrix} \}p_0 \\ \}p_1 \\ \vdots \\ \}p_k \end{matrix}$$

(A.1a)

$$B(\epsilon) = \begin{bmatrix} B_0(\epsilon) \\ \epsilon B_1(\epsilon) \\ \vdots \\ \epsilon^k B_k(\epsilon) \end{bmatrix} \begin{matrix} \}p_0 \\ \}p_1 \\ \vdots \\ \}p_k \end{matrix}.$$

(A.1b)

Proposition A.1: An ϵ^k-reachable pair $(A(\epsilon), B(\epsilon))$ over $T(\epsilon)$ is in proper standard form with indexes $n_0, \cdots, n_k$ iff $A(\epsilon)$ and $B(\epsilon)$ satisfy the following condition. Let $F_i(\epsilon) = D_i^{-1}(\epsilon)A(\epsilon)D_i(\epsilon)$, $G_i(\epsilon) = D_i^{-1}(\epsilon)B(\epsilon)$ where $D_i(\epsilon) = \text{diag} \{I_{p_0}, \cdots, \epsilon^i I_{p_i + \cdots + p_k}\}$ then the reachable subspace of $(F_i(0), G_i(0))$ is

$$\mathcal{E}_i = \text{Im} \begin{pmatrix} I_{n_i} \\ 0 \end{pmatrix}, \quad \text{for } \forall i \in [0 \cdots k]. \qquad \square$$

Definition A.2: Let

$$\bar{A}_i(\epsilon) = \begin{bmatrix} A_{0,0}(\epsilon) & A_{0,1}(\epsilon) & \cdots & A_{0,i}(\epsilon) \\ \epsilon A_{1,0}(\epsilon) & A_{1,1}(\epsilon) & \cdots & A_{1,i}(\epsilon) \\ & & \ddots & \vdots \\ \epsilon^i A_{i,0}(\epsilon) & \epsilon^{i-1} A_{i,1}(\epsilon) & \cdots & A_{i,i}(\epsilon) \end{bmatrix} \begin{matrix} \}p_0 \\ \}p_1 \\ \vdots \\ \}p_i \end{matrix}$$

(A.2a)

$$\bar{B}_i(\epsilon) = \begin{bmatrix} B_0(\epsilon) \\ \epsilon B_1(\epsilon) \\ \vdots \\ \epsilon^i B_i(\epsilon) \end{bmatrix} \begin{matrix} \}p_0 \\ \}p_1 \\ \vdots \\ \}p_i \end{matrix} .$$

(A.2b)

then $(\bar{A}_i(\epsilon), \bar{B}_i(\epsilon))$ is the ϵ^i-*reachable subsystem of* $(A(\epsilon), B(\epsilon))$ *with indexes* $n_0, \cdots, n_i$.

As with the submodule structure, the ϵ^i-reachable subsystem contains all ϵ^j-reachable subsystems for $j = 0, \cdots, i - 1$. The subsystems are layered with weak couplings of different orders of ϵ between each component. Also,

$$\mathcal{C}_i(\epsilon) T^{mn_i}(\epsilon) \oplus \epsilon^{i+1} T^{n-n_i}(\epsilon) \supset \mathcal{Y}^0 \qquad (A.3)$$

and the sequence $\{\mathcal{C}_i(\epsilon) T^{mn_i}(\epsilon) \oplus \epsilon^{i+1} T^{n-n_i}(\epsilon)\}$ converges to $\mathcal{Y}^0$ in k steps. In other words, the ϵ^0-reachable submodules of the ϵ^i-reachable subsystems approximate the ϵ^0-reachable submodule of the system in standard form up to ϵ^{i+1} accuracy. We use this in Algorithm A.3 below.

Computation of the reachability matrix is very costly. One has to calculate $A^i(\epsilon)B(\epsilon)$ with all the terms in the expansions of $A(\epsilon)$ and $B(\epsilon)$. Thus, it is desirable to work directly with the pair $(A(\epsilon), B(\epsilon))$. The following algorithm takes advantage of Proposition A.1 to recover the ϵ^j-reachability indexes. At every step, the reachable subspace of a pair, evaluated at $\epsilon = 0$, is computed. Then the pair is updated by an appropriate scaling of the unreachable part by $1/\epsilon$. The algorithm uses the higher order coefficients of the asymptotic expansions only when necessary. Also, it is possible to recover the actual Smith decomposition of the reachability matrix from the algorithm, if the transformations used in the algorithm are restricted to be permutation matrices and lower triangular matrices, although this restriction compromises numerical stability (see [18]).

Algorithm A.3:
Initialize: $A_0(\epsilon) = A(\epsilon)$, $B_0(\epsilon) = B(\epsilon)$, $i = 0$
Step i:
   1. Find U_i such that

$$U_i^{-1} A_i(0) U_i = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} \}n_i, \quad U_i^{-1} B_i(0) = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \}n_i$$

with (A_1, B_1) reachable. This determines n_i.
   2. If $n_i = n$ then go to End, else continue.
   3. Let $A_{i+1}(\epsilon) = D_i^{-1}(\epsilon) U_i^{-1} A_i(\epsilon) U_i D_i(\epsilon)$, $B_{i+1} = D_i^{-1}(\epsilon) U_i^{-1}(\epsilon) B_i(\epsilon)$ where $D_i(\epsilon) = \text{diag }\{I_{n_i}, \epsilon I_{n-n_i}\}$.
   (It is not necessary to carry out the computation for all the coefficients of $A_i(\epsilon)$ and $B_i(\epsilon)$; see [18, Note 1].)
   **4. Increment** i, go to Step i.
   End: $k = i$, the system is ϵ^k-reachable with indexes $n_0, \cdots, n_k$. □

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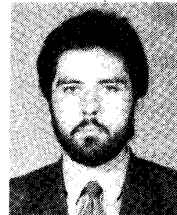
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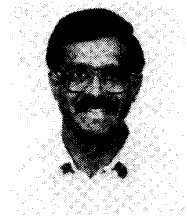
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