

Conditions for Scale-Based Decompositions in Singularly Perturbed Systems

Sheldon X-C. Lou, George C. Verghese, and Alan S. Willsky
*Department of Electrical Engineering and Computer Science
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139*

and

Pamela G. Coxson
*Department of Mathematics
Ohio State University
Columbus, Ohio 43210*

Submitted by M. Vidyasagar

ABSTRACT

Singularly perturbed models of the form $\dot{x}(t) = A(\epsilon)x(t)$, with $A(\epsilon)$ analytic at 0, nonsingular for $\epsilon \in (0, \epsilon_0]$, and singular at $\epsilon = 0$, arise naturally in various problems of systems and control theory. Under a so-called *multiple semisimple null structure* (MSSNS) condition on $A(\epsilon)$, the eigenstructure of this matrix has a multiple-scale property that allows the asymptotic eigenstructure of the matrix to be studied via reduced-order matrices associated with the separate scales. Under a stronger *multiple semistability* or MSST condition, this eigenstructure decomposition translates into a time-scale decomposition of the solution $x(t)$ of the system. This paper is aimed at illuminating the MSSNS and MSST conditions. Using ideas from an algebraic approach that we have developed for the study of singularly perturbed systems, we show (among other results): that the Smith decomposition of $A(\epsilon)$ permits transformation to a form in which the scales become explicit and the computations become transparent and simple; that this form allows us to identify perturbations of $A(\epsilon)$ that all have the same scale-based decompositions; that the eigenstructure of $A(\epsilon)$ does indeed display multiple scales under MSSNS, and that the eigenvalues in this case can be approximated using reduced-order calculations; that $A(\epsilon)$ has MSSNS if and only if the orders of its invariant factors equal the orders of its eigenvalues; that this happens if and only if the orders of its invariant factors and principal minors are related in a specified way; and that $A(\epsilon)$ has MSST if and only if it is Hurwitz for

$\epsilon \in (0, \epsilon_0]$ and has MSSNS and satisfies the requirement that the order in ϵ of the real part of every eigenvalue is not greater than the order of its imaginary part.

1. INTRODUCTION

Various problems in systems and control theory give rise to singularly perturbed, linear, time-invariant, n th-order models of the form

$$\dot{x}(t) = A(\epsilon)x(t) \quad (1)$$

with the matrix $A(\epsilon)$ being $n \times n$, analytic at 0, nonsingular¹ for $\epsilon \in (0, \epsilon_0]$ and singular at $\epsilon = 0$; see the recent books [1], [2] and survey [3]. Such models are found, for example, in studies of root-locus behavior under high-gain feedback [4–6], in the analysis of power systems [7], and in the context of Markov models [8, 9]. The small positive parameter ϵ in these respective contexts may, for example, represent the reciprocal of a high gain in a feedback controller, the reciprocal of the large rotational inertia of some generator, or a small failure rate or inspection rate in a reliability model.

A key objective in the study of singularly perturbed models is to express the behavior of (1) as a perturbation of its behavior for $\epsilon = 0$. In particular, since $A(\epsilon)$ becomes singular at $\epsilon = 0$, attention focuses on behavior associated with those eigenvalues of $A(\epsilon)$ that tend to 0 as ϵ tends to 0, i.e. on the “zero group” of eigenvalues of $A(\epsilon)$ [10]. A prototype for results on singularly perturbed systems is provided by the following example.

EXAMPLE 1. Suppose

$$A(\epsilon) = \begin{pmatrix} A_{11} & A_{12} \\ \epsilon A_{21} & \epsilon A_{22} \end{pmatrix}, \quad (2)$$

where the A_{ij} are constant matrices. If A_{11} is nonsingular, then it is not hard to show that the eigenvalues of $A(\epsilon)$ fall into two separate groups: a *fast group* that is asymptotically dominated by the eigenvalues of A_{11} , and a *slow group*—the zero group—that is asymptotically dominated by the eigenvalues

¹Our results can be extended quite easily (see [14], [15]) to the case where $A(\epsilon)$ has semisimple null structure, i.e. where the eigenvalue at 0 has geometric multiplicity equal to its algebraic multiplicity, but for simplicity we avoid this case here.

of $\epsilon \tilde{A}_{22}$, where \tilde{A}_{22} is the Schur complement of A_{11} in the matrix

$$\bar{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \tag{3}$$

i.e.,

$$\tilde{A}_{22} = A_{22} - A_{21}A_{11}^{-1}A_{12}. \tag{4}$$

Note that our standing assumption on the nonsingularity of $A(\epsilon)$ away from 0 guarantees that \bar{A} in (3) is nonsingular. With this assumption, it is easy to see that the assumed nonsingularity of A_{11} implies the nonsingularity of \tilde{A}_{22} .

The role of A_{11} in approximating the fast group of $A(\epsilon)$ is easy to understand on the basis of simple continuity arguments, since the nonzero eigenvalues of $A(0)$ are those of A_{11} . The appearance of \tilde{A}_{22} requires a little more motivation. For this, apply to $A(\epsilon)$ a similarity transformation defined by the matrix

$$T(\epsilon) = \begin{pmatrix} I & A_{11}^{-1}A_{12} \\ 0 & I \end{pmatrix}, \tag{5}$$

thereby obtaining

$$T(\epsilon)A(\epsilon)T^{-1}(\epsilon) = \begin{pmatrix} A_{11} + \epsilon * & \epsilon * \\ \epsilon * & \epsilon \tilde{A}_{22} \end{pmatrix} \tag{6}$$

where the *'s denote constant matrices. Since the coupling between the nonsingular diagonal blocks that is induced by the off-diagonal blocks appears smaller in (6) than in (1), one might now be more inclined to believe the above claim regarding approximation of the fast and slow groups of eigenvalues of $A(\epsilon)$. A rigorous proof can be obtained by applying well-established arguments to the matrix in (6), for example a block version of Gerschgorin's theorem [11], or the arguments in [12], which we follow in Section 3.

Further intuition for this separation result will be obtained when we continue the example later in this section. The conclusion for now is that knowing the eigenvalues of A_{11} and \tilde{A}_{22} allows one to determine the *dominant behavior* of the eigenvalues of $A(\epsilon)$, and hence to obtain asymptotically good approximations of them via reduced-order calculations. It should already be evident from the sketch above that the separation results are

unchanged if the A_{ij} in (2) have higher-order terms appended to them. This fact will be elaborated on in Section 2.

The special form of $A(\epsilon)$ in (2) made the computations above quite transparent, while the condition that A_{11} is invertible made the computations possible. The invertibility condition is equivalent to this special $A(\epsilon)$ having what is termed *multiple semisimple null structure*, or MSSNS [8, 13]. The MSSNS condition for a general $A(\epsilon)$ plays a fundamental role in studies of its asymptotic eigenstructure, and a primary objective of the present paper is to illuminate this condition by presenting certain equivalent forms of this condition and by examining certain consequences of MSSNS. We begin in Section 2 with a review of results in [14], [15] that show how the Smith decomposition of a general $A(\epsilon)$ yields a similarity transformation that brings this matrix to a special form—which we term the (reduced) *explicit form*—in which MSSNS is easily checked and scale-based decomposition easily carried out. This explicit form is an extension of the one in Example 1 above, and checking it for MSSNS reduces to checking invertibility of a sequence of Schur complements. The explicit form also allows us to identify perturbations of $A(\epsilon)$ that have the same scale-based decompositions.

Building on the explicit form, Section 3 deduces certain consequences of MSSNS and certain equivalent tests for it. In particular, it is shown that the eigenstructure of $A(\epsilon)$ displays multiple scales under MSSNS, and that the eigenvalues in this case can be approximated using reduced-order calculations. It is also demonstrated that $A(\epsilon)$ has MSSNS if and only if the orders of its invariant factors equal the orders of its eigenvalues.

Since eigenvalues are determined by principal minors while invariant factors are determined by all the minors, it is evident that MSSNS must imply some relation between these two sets of minors. We shown in Section 4 that in fact the MSSNS condition is satisfied if and only if the orders of the invariant factors and principal minors of $A(\epsilon)$ are related in a specified way. The insights provided by these results can be applied, as outlined in that section, to the problem of scaling a matrix $A(\epsilon)$ via a diagonal similarity transformation to induce MSSNS in it.

Many of the results on MSSNS in Sections 3 and 4 turn out to echo results in [5]. The development in [5] is in the context of asymptotic root loci of systems under high-gain feedback, and in principle a mapping can be made between the formulation there and the one here. Though our route to results of interest for the system (1) is more direct, the treatment in [5] is of value in rounding out our treatment, and the interested reader is encouraged to examine that paper as well.

The illustration in Example 1 involved *eigenstructure* decomposition. One can, with further assumptions, go beyond this to a *time-scale* decomposition of the solution $x(t)$ of (1), as shown next.

EXAMPLE 1 (Continued). Consider the system (1) with $A(\epsilon)$ defined as in (2), so that

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ \epsilon A_{21} & \epsilon A_{22} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \tag{7}$$

Much of the systems and control literature (see [1]–[3]) focuses on this prototype example (or one that can be obtained from it by simply changing the independent variable from t to $\tau = \epsilon t$). Take A_{11} to be nonsingular, as before. If $\epsilon = 0$, then $x_2(t)$ evidently remains constant at its initial value $x_2(0)$, while $x_1(t)$ is governed by the fast group of eigenvalues of $A(0)$, namely the eigenvalues of A_{11} . Under the assumption that A_{11} is Hurwitz and not just nonsingular, i.e. that all its eigenvalues lie in the open left half of the complex plane, $x_1(t)$ settles down after a fast transient to a steady-state value that is readily calculated by setting $\dot{x}_1(t) = 0$ in (7): $x_1(\infty) = -A_{11}^{-1}A_{12}x_2(\infty) = -A_{11}^{-1}A_{12}x_2(0)$.

Under the same assumption of Hurwitz A_{11} , the behavior for small nonzero ϵ is only slightly different from this: $x_2(t)$ now varies slowly instead of being constant, so that the behavior of $x_1(t)$, after a fast transient governed approximately by the eigenvalues of A_{11} , is well approximated by $-A_{11}^{-1}A_{12}x_2(t)$. Using this latter approximation for $x_1(t)$ in the equation for $\dot{x}_2(t)$ yields the following approximate governing equation for $x_2(t)$:

$$\dot{x}_2(t) \approx \epsilon (A_{22} - A_{21}A_{11}^{-1}A_{12})x_2(t) = \epsilon \tilde{A}_{22}x_2(t) \tag{8}$$

The appearance of the matrix $\epsilon \tilde{A}_{22}$ here is consistent with what was claimed earlier, namely that the slow group of eigenvalues of $A(\epsilon)$ is well approximated by the eigenvalues of $\epsilon \tilde{A}_{22}$. If we assume that \tilde{A}_{22} is also Hurwitz, (8) yields a uniformly good approximation to the solution over an infinite interval. To summarize all this more precisely, what can be shown under the above assumptions is that

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} x_{1f}(t) + x_{1s}(\epsilon t) + O(\epsilon) \\ x_{2s}(\epsilon t) + O(\epsilon) \end{pmatrix}, \quad t \geq 0, \tag{9a}$$

where

$$\dot{x}_{1f}(t) = A_{11}x_{1f}(t), \quad x_{1f}(0) = x_1(0) - x_{1s}(0), \tag{9b}$$

$$x_{1s}(t) = -A_{11}^{-1}A_{12}x_{2s}(t), \tag{9c}$$

$$\dot{x}_{2s}(t) = \tilde{A}_{22}x_{2s}(t), \quad x_{2s}(0) = x_2(0), \tag{9d}$$

and the subscripts f and s respectively denote “fast” and “slow” parts of the solutions.

The Hurwitz conditions above are equivalent to having the special $A(\epsilon)$ in the example satisfy a so-called *multiple semistability*, or MSST, condition [8, 13]. The MSST condition for a general $A(\epsilon)$ plays a fundamental role in the time-scale decomposition of singularly perturbed systems; see [8] and [13]–[15]. The last reference, [15], contains particularly strong results on the connection between MSST and time-scale decompositions. In the present paper, our focus is not on the time-scale decomposition itself but on features of the MSST condition and on its relation to other system properties. The MSST condition is introduced along with MSSNS in Section 2, where the test for MSST for systems in explicit form is presented. We then show in Section 3 that $A(\epsilon)$ has MSST if and only if it is Hurwitz for $\epsilon \in (0, \epsilon_0]$ and has MSSNS and satisfies the requirement that the order of the real part of every eigenvalue is not greater than the order of its imaginary part. This result suggests the role of MSST in obtaining time-scale decompositions.

2. DECOMPOSITION CONDITIONS FOR $A(\epsilon)$ IN EXPLICIT FORM

2.1. Definitions of MSSNS and MSST

Before presenting our algebraic approach to scale-based decompositions and the conditions that enable these decompositions, we shall review the definitions of MSSNS and MSST presented in [8], [13]. First note that a matrix is said to have *semisimple null structure*, or SSNS, if the geometric multiplicity of the eigenvalue at 0 (when it is present) equals its algebraic multiplicity; it is said to be *semistable* if it has SSNS and all its nonzero eigenvalues have negative real parts.

Since $A(\epsilon)$ is analytic at 0, it has a Taylor expansion:

$$A(\epsilon) = \sum_{p=0}^{\infty} \epsilon^p F_{1p}. \quad (10)$$

Let $M_1(\epsilon)$ be the *total projection* for the zero group of eigenvalues of $A(\epsilon)$, i.e. the projection onto the corresponding eigenspace [10]. If F_{10} in (10) has SSNS, then it can be shown that

$$A_2(\epsilon) = \frac{M_1(\epsilon)A(\epsilon)}{\epsilon} \quad (11a)$$

also has a Taylor expansion, of the form

$$A_2(\epsilon) = \sum_{p=0}^{\infty} \epsilon^p F_{2p}. \tag{11b}$$

If F_{20} also has SSNS, we can similarly define

$$A_3(\epsilon) = \frac{M_2(\epsilon)A_2(\epsilon)}{\epsilon} = \sum_{p=0}^{\infty} \epsilon^p F_{3p}, \tag{12}$$

where $M_2(\epsilon)$ is the total projection for the zero group of eigenvalues of $A_2(\epsilon)$. This process can be continued, terminating when F_{m0} does not have SSNS or when

$$\rho(F_{10}) + \dots + \rho(F_{m0}) = n, \tag{13}$$

where $\rho(F)$ denotes the rank of F , and n is the order of the system (1), i.e. the dimension of $A(\epsilon)$.

DEFINITION. $A(\epsilon)$ is said to *satisfy MSSNS* if the above construction can proceed all the way to the stage m at which (13) is satisfied, with F_{10}, \dots, F_{m0} all having SSNS. If in addition F_{10}, \dots, F_{m0} are all semistable, then $A(\epsilon)$ is said to *satisfy MSST*.

The significance of these two conditions emerges from the results in [8], [13]–[15]. In particular, [15] shows that (1) has what is termed a *strong time-scale decomposition* if and only if it satisfies MSST. If the system satisfies only MSSNS but is stable for positive ϵ , then it may still be possible to obtain what [14] and [15] term an extended time-scale decomposition. In any case, it is always possible under MSSNS to obtain an eigenstructure decomposition, as will be shown in Section 3.

2.2. Transformation to Explicit Form Using the Smith Decomposition of $A(\epsilon)$

The starting point for our results is the *Smith decomposition* of $A(\epsilon)$. Note that the entries of $A(\epsilon)$ are elements of the (local) *ring* of functions analytic at $\epsilon = 0$, i.e. functions expressible as Taylor series in ϵ . This ring, which we shall denote by \mathscr{R} , is a Euclidean domain [16], with the degree of an element being the order of the first nonzero term in its Taylor expansion—the degree of $a_i \epsilon^i + a_{i+1} \epsilon^{i+1} + \dots$, $a_i \neq 0$, is thus i . The units

in this ring are precisely those elements that have degree 0, i.e. those that are nonzero at $\epsilon = 0$. A square matrix $U(\epsilon)$ over this ring, i.e. one with entries from this ring, is termed *unimodular* if $\det U(\epsilon)$ is a unit, or equivalently if $\det U(0) \neq 0$, or equivalently if $U^{-1}(\epsilon)$ is also over this ring.

It now follows from well-known results on Smith forms of matrices over Euclidean domains [16] that the $n \times n$ matrix $A(\epsilon)$ has the decomposition

$$A(\epsilon) = P(\epsilon)D(\epsilon)Q(\epsilon) \tag{14a}$$

where $P(\epsilon)$ and $Q(\epsilon)$ are unimodular matrices over \mathscr{R} and

$$D(\epsilon) = \text{block diagonal}(I_{k_1}, \epsilon I_{k_2}, \dots, \epsilon^{m-1} I_{k_m}) \tag{14b}$$

with I_{k_j} denoting an identity matrix of dimension $k_j \times k_j$, with $k_j = 0$ corresponding to absence of the j th block, and with $k_m \neq 0$. The indices k_j , and hence $D(\epsilon)$, are unique, though $P(\epsilon)$ and $Q(\epsilon)$ are not.

The above indices actually capture the *invariant factor* structure of $A(\epsilon)$; in the pole-zero language of [17], which is devoted to rational matrix structure but has much that applies to local rings, the indices represent the zero structure of $A(\epsilon)$ at $\epsilon = 0$. The i th invariant factor, by definition, is the ratio of the greatest common divisor (gcd) of all $i \times i$ minors of $A(\epsilon)$ divided by the gcd of all $(i - 1) \times (i - 1)$ minors, with the first invariant factor being defined as the gcd of all the 1×1 entries. Since the gcd is only defined up to a unit, we can always represent it in the ring \mathscr{R} by an element of the form ϵ^r for some nonnegative integer r , so the invariant factors are also of this form. Now what the Smith decomposition in (14) captures is the fact that $A(\epsilon)$ has k_j invariant factors of the form ϵ^{j-1} , i.e. k_j invariant factors of (degree or) order $j - 1$.

The decomposition (14) allows us to similarity transform the given model (1) to an equivalent model in which the potential for scale-based decomposition is considerably more transparent, and the analysis is as direct as in Example 1. For this, define

$$y(t) = P^{-1}(\epsilon)x(t) \tag{15}$$

so that, combining (1) and (14), one gets the *explicit form* model

$$\dot{y}(t) = D(\epsilon)\bar{A}(\epsilon)y(t) \tag{16a}$$

or

$$\begin{pmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \\ \vdots \\ \dot{y}_m(t) \end{pmatrix} = \begin{pmatrix} I_{k_1} & 0 & \cdots & 0 \\ 0 & \epsilon I_{k_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \epsilon^{m-1} I_{k_m} \end{pmatrix} \times \begin{pmatrix} A_{11}(\epsilon) & A_{12}(\epsilon) & \cdots & A_{1m}(\epsilon) \\ A_{21}(\epsilon) & A_{22}(\epsilon) & \cdots & A_{2m}(\epsilon) \\ \vdots & \vdots & \cdots & \vdots \\ A_{m1}(\epsilon) & A_{m2}(\epsilon) & \cdots & A_{mm}(\epsilon) \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_m(t) \end{pmatrix}, \quad (16b)$$

where the matrix $\bar{A}(\epsilon)$ is unimodular and given by

$$\bar{A}(\epsilon) = Q(\epsilon)P(\epsilon) \tag{16c}$$

[We have dispensed with overbars for the entries of $\bar{A}(\epsilon)$ in (16b) so as to simplify the notation.] Since the unimodular transformation matrix $P(\epsilon)$ in (15) is finite and nonsingular at $\epsilon = 0$, the asymptotic behavior of (1) may be retrieved without difficulty from that of the equivalent system (16). It is also easy to see from Section 2.1 that such a unimodular similarity transformation does not alter MSSNS or MSST. (For more on such “analytic similarity” of matrices, see [18].)

We can go still further. Examination of (16a, b) might suggest that the potential for scale-based decomposition is, at least under appropriate conditions, completely displayed by $D(\epsilon)$, with the unimodular matrix $\bar{A}(\epsilon)$ not contributing anything. This thought would lead one to examine a model in which $\bar{A}(\epsilon)$ is replaced by $\bar{A}(0)$:

$$\dot{z}(t) = D(\epsilon)\bar{A}(0)z(t). \tag{17}$$

We shall term this the *reduced explicit form* of (1). A key result in [14] (see also [15]), obtained by carrying out the computations of Section 2.1 in detail for $A(\epsilon)$ given by (14), is that the models in (1), (16) satisfy MSSNS (respectively MSST) if and only if the above reduced explicit model satisfies MSSNS (respectively MSST). Furthermore, the eigenstructure decomposition of (17) obtained under the MSSNS condition is an eigenstructure decomposition of (16) and (1); this will be discussed in Section 3. Similarly, if MSST is satisfied, then a time-scale decomposition of (17) is shown in [14], [15] to be a

time-scale decomposition of (16) as well, and to yield a time-scale decomposition of (1) on transformation by $P(0)$.

These results therefore justify our discarding all ϵ -dependent terms in $\bar{A}(\epsilon)$, if our interest is only in testing for MSSNS or MSST and effecting the scale-based decompositions above. This is a major simplification, since tests of the MSSNS/MSST conditions in (17), and computations associated with scale-based decompositions of it, are as transparent and direct as those associated with the model in Example 1 (which is already in reduced explicit form). The existing literature on systems in the special form (7) can therefore be easily applied, with straightforward extensions, to the much more general systems described by (1), once the Smith decomposition (14) of $A(\epsilon)$ has been determined.

Another important consequence of the results above is that inferences made regarding (1) on the basis of computations with (17) also hold for any other system that has the same reduced explicit form, (17). The set of matrices $A_\Delta(\epsilon)$ that give rise to the same reduced explicit form (17) as $A(\epsilon)$ is precisely given by

$$A_{R,\Delta}(\epsilon) = R(\epsilon)P(\epsilon)D(\epsilon)[\bar{A}(0) + \Delta(\epsilon)]P^{-1}(\epsilon)R^{-1}(\epsilon), \quad (18a)$$

where $R(\epsilon)$ is any unimodular matrix and $\Delta(\epsilon)$ is any matrix over \mathscr{W} such that $\Delta(0) = 0$. This can be rewritten as

$$\begin{aligned} A_{R,\Delta}(\epsilon) &= R(\epsilon)A(\epsilon)[Q^{-1}(\epsilon)[Q(0)P(0) + \Delta(\epsilon)]P^{-1}(\epsilon)R^{-1}(\epsilon) \\ &= R(\epsilon)A(\epsilon)U_\Delta(\epsilon)R^{-1}(\epsilon), \end{aligned} \quad (18b)$$

where $U_\Delta(\epsilon)$ is a unimodular matrix such that $U_\Delta(0) = I$. Each $\Delta(\epsilon)$ gives rise to a unique unimodular matrix satisfying this condition, and conversely. The set of interest is thus precisely characterized by (18b).

The key to the above development is the Smith decomposition of $A(\epsilon)$. Computation of the Smith form is nontrivial, of course, and involves operations comparable to those required by [8], [13] to compute the F_{ip} in (10)–(13) (though we have found, for examples that are small enough to work out by hand, that computation of the Smith form is decidedly simpler than the necessary operations on the matrix coefficients of Taylor series). However, our approach here permits the analysis of (1) to be separated into a transformation step, which involves determination of the Smith form and produces the reduced explicit form, and a greatly simplified decomposition step. In contrast, the operations of transformation and decomposition are interleaved in the approach of [8] and [13].

Partly because of its two-step nature, and partly because of the algebraic connections, our approach has also yielded several new insights and produced

new results, for example on feedback assignment of closed-loop time scales, on perturbations that preserve time-scale decompositions, on ϵ -dependent amplitude scaling of state variables to induce MSSNS, and so on; see [14], [15], and results in this paper.

2.3. *Checking MSSNS and MSST in the Reduced Explicit Form*

To simplify notation for the reduced explicit form, denote $\bar{A}(0)$ of (17) simply by A , and its entries $A_{ij}(0)$ by A_{ij} . The description of interest to us is then

$$\dot{z}(t) = D(\epsilon)Az(t) \tag{19a}$$

or

$$\begin{pmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \\ \vdots \\ \dot{z}_m(t) \end{pmatrix} = \begin{pmatrix} I_{k_1} & 0 & \cdots & 0 \\ 0 & \epsilon I_{k_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \epsilon^{m-1} I_{k_m} \end{pmatrix} \times \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \cdots & A_{2m} \\ \vdots & \vdots & & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mm} \end{pmatrix} \begin{pmatrix} z_1(t) \\ z_2(t) \\ \vdots \\ z_m(t) \end{pmatrix} \tag{19b}$$

The objective of this subsection is to present the tests of MSSNS and MSST for the reduced explicit form above. It is easy in principle, though notationally a little cumbersome, to apply the operations specified in Section 2.1 to the system (19) and thereby deduce the tests. We shall simply state the results obtained in [14]. One might anticipate these results in view of the similarity of (19) to (7); the conditions for MSSNS (respectively, MSST) are those that one would naturally impose in order to get an eigenvalue decomposition (respectively, time-scale decomposition) of the type in Example 1.

To begin, suppose the $k_1 \times k_1$ matrix A_{11} is nonsingular. Let A_2 denote the matrix obtained as the Schur complement of A_{11} in A [$= \bar{A}(0)$], and let \tilde{A}_{22} denote its leading $k_2 \times k_2$ submatrix, so

$$A_2 = \begin{pmatrix} A_{22} & \cdots & A_{2m} \\ \vdots & \cdots & \vdots \\ A_{m2} & \cdots & A_{mm} \end{pmatrix} - \begin{pmatrix} A_{21} \\ \vdots \\ A_{m1} \end{pmatrix} A_{11}^{-1} (A_{12} \quad \cdots \quad A_{1m}) \tag{20a}$$

and

$$\tilde{A}_{22} = A_{22} - A_{21}A_{11}^{-1}A_{12}. \tag{20b}$$

Similarly, if \tilde{A}_{22} is nonsingular, let A_3 denote the Schur complement of \tilde{A}_{22} in A_2 , and let \tilde{A}_{33} denote its leading $k_3 \times k_3$ submatrix. The pattern in this construction is now evident. If at some stage j we have $k_j = 0$, then the corresponding stage is skipped; we then relabel A_j as A_{j+1} , denote its leading $k_{j+1} \times k_{j+1}$ submatrix by $\tilde{A}_{j+1, j+1}$, and proceed. Note that A_1 is to be taken as A and \tilde{A}_{11} as A_{11} . The result now is the following:

The matrix $D(\epsilon)A$ in (19), and therefore $A(\epsilon)$ as well, satisfies MSSNS (respectively, MSST) if and only if every \tilde{A}_{jj} for which $k_j \neq 0$, $j = 1, \dots, m$, is nonsingular (respectively, Hurwitz).

(Note that, for this paper, a Hurwitz matrix is one whose eigenvalues are in the open left half plane.) Since A is nonsingular—a consequence of our assumption that $A(\epsilon)$ is nonsingular away from 0—the nonsingularity of \tilde{A}_{mm} is guaranteed if those preceding \tilde{A}_{jj} for which $k_j \neq 0$ are all nonsingular. The index m here can be seen to be the same one that appears in Section 2.1.

It was mentioned in Section 2.2 that $P(\epsilon)$ and $Q(\epsilon)$ in the Smith decomposition (14) are not unique. A consequence of this is that the matrix $A = Q(0)P(0)$ is not uniquely determined by the decomposition. However, the results here do not depend on which decomposition is chosen. In particular, [14] shows that the above Schur complements obtained from different decompositions are related by similarity transformations.

3. MSSNS/MSST, EIGENVALUES AND INVARIANT FACTORS

3.1. MSSNS and the Orders of Eigenvalues and Invariant Factors

This subsection will firstly establish the following characterization of MSSNS:

RESULT 1. *$A(\epsilon)$ has MSSNS if and only if the orders of its eigenvalues equal the orders of its invariant factors, i.e. if and only if it has precisely k_j eigenvalues of order $j - 1$ for $j = 1, \dots, m$.*

An analogous result is presented as the central theorem of [5], but mapping the (high-gain feedback root locus) formulation and proof there to our setting here would lead to a considerably more cumbersome proof than the direct one below.

In the process of proving Result 1, we shall also be demonstrating the earlier claim that under MSSNS the eigenvalues of $A(\epsilon)$ fall into separate groups, with k_j eigenvalues of order $j - 1$ in the j th group, $j = 1, \dots, m$, and that the eigenvalues in these groups can be approximated via the eigenvalues of reduced-order matrices. The precise result will be stated after proving Result 1.

Proof. Since the unimodular similarity transformation $P(\epsilon)$ described in Section 2.2 preserves MSSNS/MSST, as well as eigenvalues and invariant factors, it suffices to consider a matrix $A(\epsilon)$ that is already in explicit form:

$$A(\epsilon) = D(\epsilon)\bar{A}(\epsilon) \tag{21a}$$

$$= \begin{pmatrix} I_{k_1} & 0 & \cdots & 0 \\ 0 & \epsilon I_{k_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \epsilon^{m-1} I_{k_m} \end{pmatrix} \times \begin{pmatrix} A_{11}(\epsilon) & A_{12}(\epsilon) & \cdots & A_{1m}(\epsilon) \\ A_{21}(\epsilon) & A_{22}(\epsilon) & \cdots & A_{2m}(\epsilon) \\ \vdots & \vdots & & \vdots \\ A_{m1}(\epsilon) & A_{m2}(\epsilon) & \cdots & A_{mm}(\epsilon) \end{pmatrix} \tag{21b}$$

with $\bar{A}(\epsilon)$ unimodular.

To show the “only if” part first, assume that $A(\epsilon)$ has MSSNS. Following arguments in [12], let

$$d_j(\epsilon, \lambda) = \det[\epsilon^{-(j-1)}A(\epsilon) - \lambda I] \tag{22}$$

and

$$f_j(\epsilon, \lambda) = \det F(\epsilon, \lambda), \tag{23a}$$

where

$$F(\epsilon, \lambda) =$$

$$\begin{pmatrix} \epsilon^{j-1}I_{k_1} & 0 & \cdots & 0 & 0 \\ 0 & \epsilon^{j-2}I_{k_2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \epsilon I_{k_{j-1}} & 0 \\ 0 & 0 & \cdots & 0 & I_{k_j+k_{j+1}+\cdots+k_m} \end{pmatrix} [\epsilon^{-(j-1)}A(\epsilon) - \lambda I], \tag{23b}$$

so that

$$f_j(\epsilon, \lambda) = \epsilon^v d_j(\epsilon, \lambda), \quad v = \sum_{i=1}^{j-1} (j-i)k_i. \tag{24}$$

For $\epsilon \neq 0$, $d_j(\epsilon, \lambda)$, regarded as polynomials in λ , will have the same roots. Also, $f_j(\epsilon, \lambda)$ is a continuous function of ϵ at $\epsilon = 0$, with

$$f_j(0, \lambda) = \det \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1j} & \cdots & A_{1m} \\ A_{21} & A_{22} & \cdots & A_{2j} & \cdots & A_{2m} \\ \vdots & \vdots & & \vdots & & \vdots \\ A_{j1} & A_{j2} & \cdots & A_{jj} - \lambda I & \cdots & A_{jm} \\ 0 & 0 & \cdots & 0 & \cdots & \lambda I_{k_{j+1}+\cdots+k_m} \end{pmatrix} \tag{25}$$

The A_{ij} here are the same as in (19), i.e. submatrices of the reduced explicit form. The latter determinant is easily evaluated by iteratively using the fact that

$$\det \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \det A_{11} \det \tilde{A}_{22} \tag{26}$$

if A_{11} is nonsingular. The result of the evaluation, invoking the characterization of MSSNS in Section 2.3 to ensure that the Schur complements \tilde{A}_{ii} below (which are the same as those in Section 2.3) are nonsingular for $i = 1, \dots, j - 1$, is

$$f_j(0, \lambda) = \left(\prod_{i=1}^{j-1} \det \tilde{A}_{ii} \right) (\det [\tilde{A}_{jj} - \lambda I]) \lambda^w, \quad w = \sum_{j+1}^m k_i. \tag{27}$$

Thus $f_j(0, \lambda)$ has k_j roots that are identical to the k_j eigenvalues of \tilde{A}_{jj} , which we shall denote by $\lambda_i(\tilde{A}_{jj})$, $i = 1, \dots, k_j$.

It follows that $f_j(\epsilon, \lambda)$ and hence $d_j(\epsilon, \lambda)$ have k_j roots that, after appropriate matching, can be made arbitrarily close to these respective eigenvalues by choosing ϵ small enough. Now (22) shows that the roots of $d_j(\epsilon, \lambda)$ are the eigenvalues of $\epsilon^{-(j-1)}A(\epsilon)$. Hence there are k_j eigenvalues of $A(\epsilon)$, which we shall denote by

$$\lambda_{s_j+i}, \quad s_j = \sum_{q=1}^{j-1} k_q, \quad i = 1, \dots, k_j, \tag{28}$$

such that $\lambda_{s_j+i}/\epsilon^{j-1}$ can be brought arbitrarily close to $\lambda_i(\tilde{A}_{jj})$ by choosing ϵ small enough (we are assuming for notational simplicity that the indexing follows the matching). The conclusion, since all the eigenvalues of \tilde{A}_{jj} are nonzero by the assumption of MSSNS, is that $A(\epsilon)$ has precisely k_j eigenvalues of order $j - 1$.

For the “if” part, suppose that \tilde{A}_{jj} is singular, but \tilde{A}_{ii} for $i = 1, \dots, j - 1$ are nonsingular. Then $f_j(0, \lambda)$ in (27) has at least $w + 1$ roots at 0, so $\epsilon^{-(j-1)}A(\epsilon)$ has at least $w + 1$ eigenvalues that can be made arbitrarily close to 0 by choosing ϵ small enough. It follows that $A(\epsilon)$ has at least $w + 1$ eigenvalues of order higher than $j - 1$. However, it has only w invariant factors of order higher than $j - 1$. ■

This result is of intrinsic interest as a characterization of MSSNS. It also leads naturally to the results of Section 4 and provides approaches to the problem of inducing MSSNS by amplitude scaling of state variables via (nonunimodular) ϵ -dependent diagonal similarity transformations. The proof of Result 1 also establishes:

RESULT 2. *Under MSSNS, the eigenvalues of $A(\epsilon)$ converge asymptotically to the eigenvalues of $\epsilon^{j-1}\tilde{A}_{jj}$, $j = 1, \dots, m$.*

3.2 Relating MSST to MSSNS and the Eigenvalues of $A(\epsilon)$

It is of interest to see precisely what sorts of stability conditions on the original system (1) will make MSST a consequence of MSSNS. The new result to be proved in this subsection is the following:

RESULT 3. *$A(\epsilon)$ has MSST if and only if (i) it has MSSNS, and (ii) it is Hurwitz for $\epsilon \in (0, \epsilon_0]$, and (iii) the order of the real part of every eigenvalue is not greater than the order of its imaginary part.*

The third condition requires that the order of the damping be at least as significant as the order of the oscillation frequency.

Some elementary examples may help to illustrate the result before we present the proof. The first condition rules out the matrix

$$\begin{pmatrix} -\epsilon & 1 \\ 0 & -\epsilon \end{pmatrix} \tag{29}$$

from having MSST, even though it satisfies the other two conditions, but

$$\begin{pmatrix} -\epsilon & \epsilon \\ 0 & -\epsilon \end{pmatrix} \tag{30}$$

satisfies all three conditions and has MSST. The third condition rules out

$$\begin{pmatrix} -\epsilon & 1 \\ -1 & -\epsilon \end{pmatrix}. \tag{31}$$

Proof. To prove the “only if” part, note from Section 2.3 that MSST implies that all the \tilde{A}_{jj} , $j = 1, \dots, m$, are Hurwitz. They are hence *a fortiori* nonsingular, so $A(\epsilon)$ has MSSNS and (i) is established. It then follows from Result 2 that the eigenvalues of $A(\epsilon)$ converge asymptotically to those of $\epsilon^{j-1}\tilde{A}_{jj}$, so (ii) and (iii) follow.

For the “if” part, it follows from (i) and Result 2 that the eigenvalues λ_{s_j+i} , $i = 1, \dots, k_j$, $j = 1, \dots, m$, of $A(\epsilon)$ converge asymptotically to those of $\epsilon^{j-1}\tilde{A}_{jj}$. From the nonsingularity of the Schur complements \tilde{A}_{jj} , we see that

$$\lim_{\epsilon \rightarrow 0} \frac{\lambda_{s_j+i}}{\epsilon^{j-1}} = \lambda_i(\tilde{A}_{jj}) \neq 0. \tag{32a}$$

It follows from (iii) that

$$\lim_{\epsilon \rightarrow 0} \operatorname{Re} \frac{\lambda_{s_j+i}}{\epsilon^{j-1}} = \operatorname{Re} \lambda_i(\tilde{A}_{jj}) \neq 0 \tag{32b}$$

Now (ii) implies that the limit above is negative, and it follows that the Schur complements are Hurwitz. The results of Section 2.3 then show that $A(\epsilon)$ has MSST. ■

The assumption of MSST has dominated studies of time-scale decompositions. However, systems such as (31) that satisfy conditions (i) and (ii) of Result 3, but *not* condition (iii), have been considered in [14], [15], where it is shown that so-called *extended* time-scale decompositions can be obtained under appropriate conditions. The basic idea here is to retain some critical ϵ -dependent terms in the scale-based decomposition, rather than simplifying all the way to the constant matrices \tilde{A}_{jj} .

4. MSSNS, PRINCIPAL MINORS, AND AMPLITUDE SCALING

4.1. MSSNS and Principal Minors

Denote the eigenvalues of $A(\epsilon)$ by $\lambda_i, i = 1, \dots, n$, and its $j \times j$ principal minors by $M_i^{(j)}, i = 1, \dots, \binom{j}{n}, j = 1, \dots, n$. It is then well known that

$$\det[\lambda I - A(\epsilon)] = \lambda^n - \lambda^{n-1} \left(\sum \lambda_i \right) + \lambda^{n-2} \left(\sum_{i \neq k} \lambda_i \lambda_k \right) - \dots + (-1)^n (\lambda_1 \dots \lambda_n) \tag{33a}$$

$$= \lambda^n - \lambda^{n-1} \left(\sum M_i^{(1)} \right) + \lambda^{n-2} \left(\sum M_i^{(2)} \right) - \dots + (-1)^n M^{(n)}. \tag{33b}$$

Evidently the characteristic polynomial and hence the eigenvalues of $A(\epsilon)$ are determined by its principal minors. In addition, the *orders* of the coefficients $\sum M_i^{(j)}$ of the characteristic polynomial suffice to determine the (fractional or integer) *orders* of the eigenvalues, via the classical “Newton polygon” construction reviewed below; see also [5], [19].

The invariant factors of $A(\epsilon)$ are determined by the gcd’s of all its minors of each dimension. However, Result 1 and (33) lead one to conclude that, under MSSNS, the orders of the $\sum M_i^{(j)}$ suffice to compute the invariant factors. We show in this subsection that *under MSSNS the gcd’s of the principal minors of each dimension suffice to determine the invariant factors*. In the process, we clarify the role that MSSNS plays in this determination, obtain an alternate characterization of MSSNS, and develop insights that will be useful in the amplitude scaling problem considered in the next subsection. Again, there are connections with results in [5], and the reader may wish to explore these (see especially Appendix C in [5]).

In order to state our results, some additional notation is needed:

- a_i will denote the order of the i th invariant factor, so $a_i = j - 1$ if j is the smallest integer for which $i \leq k_1 + \dots + k_j$, with the k_j defined as in Section 2.2;
- b_i will denote the (possibly fractional) order of the i th eigenvalue, with the eigenvalues assumed to be numbered such that $b_1 \leq \dots \leq b_n$;
- p_j will denote the order of the gcd of all $j \times j$ principal minors $M_i^{(j)}$ of $A(\epsilon)$; and
- r_j will denote the order of the sum $\sum M_i^{(j)}$ of these principal minors.

As mentioned above, the standard Newton polygon construction [5, 19] can be applied to the relationships embodied in (33a, b) to compute the b_i from the r_j . The basis for this lies in the following lemma.

LEMMA 1. *Given a set of real numbers $x_j, j = 1, \dots, n$, there is a unique set of real numbers $y_i, i = 1, \dots, n$, such that the following hold:*

$$y_1 \leq \dots \leq y_n, \tag{L1}$$

$$\sum_{i=1}^j y_i \leq x_j, \quad \text{all } j, \tag{L2}$$

$$\sum_{i=1}^j y_i = x_j \quad \text{when } y_j \neq y_{j+1} \text{ or } j = n. \tag{L3}$$

The numbers y_i may be obtained as the slopes of the segments of the “lower hull” in the (Newton diagram) plot of x_j versus j (with the origin included, i.e. $x_0 = 0$); see Figure 1.

The values of j that appear in (L3), i.e. values where $y_j \neq y_{j+1}$ or $j = n$, will be termed *corner points*.

Though this lemma is not explicitly articulated in the usual approaches to the Newton polygon, it is easily seen to underlie them. Both for this reason and because a proof follows easily from the construction described in the lemma and Figure 1, we omit the proof here; see [14] for details. The lemma is of potential value in other settings as well, which is why we have isolated it.

What makes Lemma 1 applicable to computing the b_i from the r_j is the fact that these two sets of numbers satisfy the above inequalities, with the substitutions $x_j \leftarrow r_j$ and $y_i \leftarrow b_i$. To see this, note first that (L1) is trivially satisfied, by definition. Also, (33) shows that the sum of the $j \times j$ principal minors equals the sum of all j -fold products of the eigenvalues, so that (L2) is

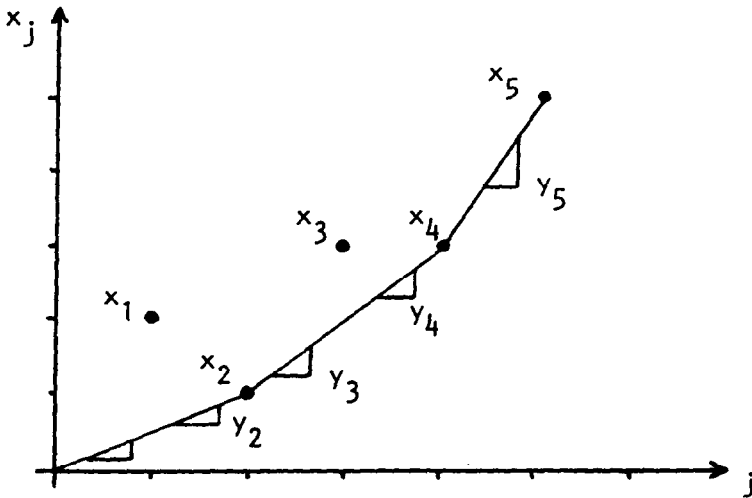


FIG. 1. Newton polygon construction.

satisfied. Equality fails to hold in (L2) precisely when there are cancellations of the lowest-order terms when summing the j -fold products of eigenvalues. For example, if $n = 2$, $\lambda_1 = -1$, and $\lambda_2 = 1 + \epsilon$, then $b_1 = 0$ (and $b_2 = 0$) but $r_1 = 1$. However, if $b_j \neq b_{j+1}$, then there will be only one j -fold product of eigenvalues that has order $b_1 + \dots + b_j$, namely the product $\lambda_1 \dots \lambda_j$, and all other j -fold products will have higher order, so equality holds in (L2). Also, if $j = n$, then there is only one term in the sum, namely $\lambda_1 \dots \lambda_n$, so equality holds again. Thus (L3) is established. The conclusion is that the b_i are given by the slopes of the segments of the lower hull in the Newton diagram for the points r_j .

Some further connections between the numbers defined above should be noted. In general it will be the case that

$$p_j \leq r_j. \tag{34}$$

Equality fails to hold precisely when the leading (lowest-order) terms cancel out upon summing the $j \times j$ principal minors. When equality holds in (34), we shall say that the *no-cancellation condition* holds. It is also evident from the definition of the a_i that

$$a_1 \leq \dots \leq a_n \tag{35}$$

Furthermore, the definition of invariant factors in Section 2.2 shows that $a_1 + \cdots + a_j$ is the order of the gcd of all $j \times j$ minors of $A(\epsilon)$, so in particular

$$\sum_{i=1}^j a_i \leq p_j. \quad (36)$$

What (35) and (36) demonstrate is that, with the substitutions $x_j \leftarrow p_j$ and $y_i \leftarrow a_j$, the integers p_j and a_i satisfy (L1) and (L2). If (L3) was also satisfied, then Lemma 1 would imply that the a_i can be uniquely determined from the p_j , as the slopes of the segments of the lower hull in the Newton diagram for the points p_j .

The next result shows that MSSNS guarantees (L3) with the above substitutions, so under MSSNS the invariant factors can indeed be determined from the gcd's of the principal minors of each dimension, using the Newton polygon construction associated with Lemma 1. The result below also shows that under MSSNS (34) holds with equality (i.e., the no-cancellation condition holds) at the corner points. Conversely, if (L3) holds with the preceding substitutions, and if the no-cancellation condition holds at the corner points, then $A(\epsilon)$ satisfies MSSNS. When $A(\epsilon)$ is in explicit form, then the no-cancellation condition need not be checked: $A(\epsilon)$ in explicit form has MSSNS if and only if (L3) holds with the preceding substitutions.

RESULT 4. $A(\epsilon)$ has MSSNS if and only if

$$\sum_{i=1}^j a_i = p_j = r_j \quad \text{when } a_j \neq a_{j+1}. \quad (37)$$

Furthermore, when $A(\epsilon)$ is in explicit form, only the first equality in (37) needs to be checked, because it implies the second.

Proof. To prove the “only if” part first, assume $A(\epsilon)$ satisfies MSSNS. Then from Result 1 we have that $b_i = a_i$, $i = 1, \dots, n$. Now suppose $a_j \neq a_{j+1}$. Then $b_j \neq b_{j+1}$, so by our earlier arguments $b_1 + \cdots + b_j = r_j$. Hence $a_1 + \cdots + a_j = r_j$. Combining this with the inequalities in (34) and (36) demonstrates the result.

For the “if” part, note from (34) that the lower hull for the points r_j in the Newton polygon construction cannot lie below that for the p_j . Furthermore, since (37) shows that the corner points for the r_j hull coincide with those for the p_j hull, the two hulls must be the same. Hence the numbers b_i

obtained by applying Lemma 1 to the r_j must be the same as the numbers a_i obtained by applying Lemma 1 to the p_j . In other words, $b_i = a_i$, so by Result 1 the matrix $A(\epsilon)$ has MSSNS.

Suppose now that $A(\epsilon)$ is in the explicit form (16) and that the first equality in (37) holds, so $a_j \neq a_{j+1}$ and $a_1 + \dots + a_j = p_j$. Now examination of the explicit form shows that there is only *one* principal minor of order p_j , namely the *leading* principal minor of appropriate size (its size is actually $k_1 + \dots + k_q$, where q is the smallest integer for which $k_2 + 2k_3 + \dots + qk_{q+1} > p_j$). Hence $r_j = p_j$, i.e., the second equality in (37) holds and the no-cancellation condition is satisfied at the corner points. ■

The need to check the second equality in (37) when $A(\epsilon)$ is not in explicit form can be illustrated by the case of the matrix

$$\begin{pmatrix} 1 & -1 \\ 1 & -1 + \epsilon^2 \end{pmatrix}, \tag{38}$$

which has $p_1 = 0$, $p_2 = 2$ and $a_1 = 0$, $a_2 = 2$, so (L1)–(L3) are satisfied with the substitutions above; but $r_1 = 2$, $r_2 = 2$ and $b_1 = 1$, $b_2 = 1$, so indeed the matrix does not have MSSNS. For an example to illustrate how we might use all the above results, consider the following.

EXAMPLE 2. Let

$$A(\epsilon) = \begin{pmatrix} \epsilon & 1 & 1 & 1 \\ 1 & \epsilon^7 & \epsilon & \epsilon \\ \epsilon^2 & \epsilon^3 & 0 & \epsilon^3 \\ \epsilon^6 & \epsilon^{11} & \epsilon^{11} & \epsilon^{11} \end{pmatrix}. \tag{39}$$

It is easily seen that this is not in explicit form. To determine the a_i , we use the definition of invariant factors in Section 2.2, which involves examining all minors of each dimension. A quick inspection shows that there are 1×1 and 2×2 minors that are of order 0, so $a_1 = 0$, $a_2 = 0$. There is a 3×3 minor (which happens to be the leading principal minor) of order 3, and none of order 2; hence $a_3 = 3$. Finally, the determinant is of order 10, so $a_4 = 7$. Examining the principal minors, we see that $p_1 = 1$, $p_2 = 0$, $p_3 = 3$, $p_4 = 10$. Hence the a_i and p_j satisfy (L1)–(L3), with the former being the slopes of the lower hull of the latter set of points in the Newton diagram. Now if the no-cancellation conditions hold at the corner points, i.e. if $r_2 = p_2$ and $r_3 = p_3$, then we shall be able to conclude from Result 4 that $A(\epsilon)$ does indeed satisfy MSSNS. Since there is only one 2×2 principal minor of order

0 and only one 3×3 principal minor of order 3, these conditions are indeed satisfied. With the assurance that $A(\epsilon)$ satisfies MSSNS, one can now proceed with the additional work required to transform it to explicit form and carry out scale-based decompositions of it.

4.2. *Amplitude Scaling to Induce MSSNS*

While unimodular similarity transformations do not affect MSSNS (or MSST), nonunimodular similarity transformations may do so, by modifying the invariant factors of the matrix (the eigenvalues are of course preserved). In particular, such transformations may be used to induce MSSNS in a matrix that does not satisfy it, after which scale-based decompositions may be carried out as earlier. Our aim in this section is only to illustrate what is possible.

We restrict ourselves to nonunimodular transformations that are diagonal, which correspond to ϵ -dependent *amplitude scaling* of the individual state variables. Such transformations, in addition to preserving eigenvalues, also preserve all principal minors of the matrix they act on. The results of the previous subsection then show that a necessary condition for such a transformation to induce MSSNS is that the no-cancellation condition $p_j = r_j$ is satisfied at the corner points of the Newton diagram for either of these sets of indices.

Our amplitude-scaling results are drawn from the thesis [14], which includes results on a systematic amplitude-scaling procedure for matrices $A(\epsilon)$ that satisfy certain conditions. Since these conditions are not only rather strong but also hard to test for, we do not attempt to do more than illustrate a simple case of the procedure here. We begin with a canonical example that serves to illustrate the basic idea behind the procedure.

EXAMPLE 3. Let

$$A(\epsilon) = \begin{pmatrix} 0 & \epsilon^{a_1} & 0 & \dots & 0 \\ 0 & 0 & \epsilon^{a_2} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \epsilon^{a_{n-1}} \\ \epsilon^{a_n} & 0 & 0 & \dots & 0 \end{pmatrix}. \tag{40}$$

The nonnegative integers a_i can be seen to be the orders of the invariant factors of $A(\epsilon)$, though not necessarily in ascending order now. The $j \times j$ principal minors for $j < n$ and hence the corresponding p_j, r_j are all 0, while $a_1 + \dots + a_n = p_n = r_n$. By applying the Newton polygon construction to

the r_j or by directly evaluating the characteristic polynomial of $A(\epsilon)$, it is easily seen that the eigenvalues all have the same order, namely $b = r_n/n$. Thus $A(\epsilon)$ will have MSSNS if and only if all the a_i are equal (and equal to b).

Now defining the similarity transformation

$$S(\epsilon) = \text{diagonal}[\epsilon^{w_1}, \dots, \epsilon^{w_{n-1}}, 1] \tag{41a}$$

with

$$w_i = w_{i+1} + b - a_i, \quad i = 1, \dots, n - 1, \quad w_n = 0, \tag{41b}$$

it is easily verified that

$$S(\epsilon)A(\epsilon)S^{-1}(\epsilon) = \begin{pmatrix} 0 & \epsilon^b & 0 & \dots & 0 \\ 0 & 0 & \epsilon^b & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \epsilon^b \\ \epsilon^b & 0 & 0 & \dots & 0 \end{pmatrix} \tag{42}$$

so that this transformed matrix has MSSNS.

The following procedure is suggested by examples such as the above, and can be guaranteed to work under certain strong conditions [14]. The *first step* is to transform $A(\epsilon)$ to its explicit form. We assume from now on that this has been done, and use $A(\epsilon)$ to denote the matrix in explicit form. In the *second step*, we identify what may be termed a *skeleton* in $A(\epsilon)$. A skeleton consists of n elements, precisely one from each row and column of the matrix, with the order of the element in the i th row being a_i . There has to be at least one skeleton in $A(\epsilon)$, because of our standing assumption of nonsingularity away from 0.

In the *third step*, we similarity-transform $A(\epsilon)$ with a permutation matrix that brings the elements of the skeleton to the locations of the 1's in the block-diagonal canonical circulant matrix, whose diagonal blocks take the form

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}. \tag{43}$$

Though [14] considers the case of multiple blocks, we restrict ourselves here to the case of a single block, i.e. to the case where the elements of the skeleton, after transformation, lie at the locations of the $n \times n$ circulant matrix (43). Let \bar{a}_i now denote the order of the skeleton element in the i th row of the transformed matrix.

The *final step* of the procedure is to transform the matrix with the similarity transformation

$$S(\epsilon) = \text{diagonal}[\epsilon^{w_1}, \dots, \epsilon^{w_{n-1}}, 1] \tag{44a}$$

with

$$w_i = w_{i+1} + b_i - \bar{a}_i, \quad i = 1, \dots, n - 1, \quad w_n = 0, \tag{44b}$$

where the b_i are the orders of the eigenvalues. Under conditions described in [14], the resulting matrix satisfies MSSNS.

We illustrate this procedure with an example.

EXAMPLE 4. Consider the explicit form matrix below, with the skeleton elements enclosed in brackets:

$$A(\epsilon) = \begin{pmatrix} \epsilon^3 & \epsilon^4 & \epsilon^5 & [1] \\ [\epsilon] & \epsilon^3 & \epsilon^3 & \epsilon \\ \epsilon^3 & [\epsilon] & \epsilon^2 & \epsilon^7 \\ \epsilon^6 & \epsilon^8 & [\epsilon^6] & \epsilon^7 \end{pmatrix}. \tag{45}$$

Since the matrix is in explicit form, it is easy to check for MSSNS. The matrix \tilde{A}_{11} referred to in Section 2.3 is simply the (1,1) entry of the $A(\epsilon)$ here evaluated at $\epsilon = 0$, and is 0, so MSSNS does not hold. This fact can also be seen after determining that $a_1 = 0$, $a_2 = 1$, $a_3 = 1$, $a_4 = 6$, and $b_1 = b_2 = b_3 = b_4 = 2$.

The similarity transformation that brings the skeleton elements to the canonical positions mentioned above simply involves interchanging the first and third rows and then the first and third columns. The result is the matrix

$$A_1(\epsilon) = \begin{pmatrix} \epsilon^2 & [\epsilon] & \epsilon^3 & \epsilon^7 \\ \epsilon^3 & \epsilon^3 & [\epsilon] & \epsilon \\ \epsilon^5 & \epsilon^4 & \epsilon^3 & [1] \\ [\epsilon^6] & \epsilon^8 & \epsilon^6 & \epsilon^7 \end{pmatrix}. \tag{46}$$

Now using (44) we find that $w_1 = 4$, $w_2 = 3$, $w_3 = 2$, $w_4 = 0$. Transforming $A_1(\epsilon)$ with the resulting $S(\epsilon)$ gives

$$A_2(\epsilon) = S(\epsilon)A_1(\epsilon)S^{-1}(\epsilon) = \begin{pmatrix} \epsilon^2 & \epsilon^2 & \epsilon^5 & \epsilon^{11} \\ \epsilon^2 & \epsilon^3 & \epsilon^2 & \epsilon^4 \\ \epsilon^3 & \epsilon^3 & \epsilon^2 & \epsilon^2 \\ \epsilon^2 & \epsilon^5 & \epsilon^4 & \epsilon^7 \end{pmatrix}, \quad (47)$$

which is easily seen to have MSSNS (it is in explicit form, with invariant factor orders all equal to 2).

The amplitude-scaling procedure illustrated above is further developed in [14]. It has been used to motivate the amplitude scaling carried out in [20], and to treat some further generalizations.

5. CONCLUSION

The algebraic approach to the analysis of singularly perturbed systems of the form (1), as described here and in [14], [15], has added a useful dimension to what continues to be an active area of study. It has provided novel ways to understand and interpret previous results, and has yielded new results and new tools for analyzing singularly perturbed systems. In particular, the MSSNS and MSST conditions that have been crucial to earlier work in singular perturbations have been examined and illuminated from the algebraic viewpoint in this paper. The connections with (frequency-domain) techniques and results developed in the study of asymptotic root loci are also much more evident in our setting.

Fresh questions have arisen naturally in the context of this algebraic approach. For example, it is of interest to further understand conditions and procedures for the extended time-scale decompositions mentioned briefly at the end of Section 3. Also, Section 4 has only touched on the task of inducing MSSNS by nonunimodular similarity transformations, and has not discussed in any detail the problem of relating properties of the transformed system back to the original system. Another worthwhile task is to understand connections with the Jordan structure of $A(\epsilon)$, perhaps making deeper contact with the results of [10], [21] in the process.

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