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Discrete-time markovian-jump linear quadratic optimal control

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This paper is concerned with the optimal control of discrete-time linear systems that possess randomly jumping parameters described by finite-state Markov processes. For problems having quadratic costs and perfect observations, the optimal control laws and expected costs-to-go can be precomputed from a set of coupled Riccati-like matrix difference equations. Necessary and sufficient conditions are derived for the existence of optimal constant control laws which stabilize the controlled system as the time horizon becomes infinite, with finite optimal expected cost.

1. Introduction and problem formulation

Consider the discrete-time *jump linear system*

$$x_{k+1} = A_k(r_k)x_k + B_k(r_k)u_k, \quad k = k_0, \ldots, N. \tag{1}$$

$$\Pr\{r_{k+1} = j \mid r_k = i\} = p_{k+1}(i, j) \tag{2}$$

where the initial state is

$$x(k_0) = x_0, \quad r(k_0) = r_0$$

Here the x-process is n-dimensional, the control $u \in R^m$ and the *form process* $\{r_k : k = k_0, \ldots, N\}$ is a finite-state Markov chain taking values in $\mathbf{M} = \{1, 2, \ldots, M\}$, with transition probabilities $p_k(i, j)$.

The cost criterion to be minimized is

$$J_k(x_0, r_0) = E\left[\sum_{k=k_0}^{N-1} [u_k' R_k(r_k)u_k + x_{k+1}' Q_{k+1}(r_{k+1})x_{k+1}] + x_N' K_T(r_N)x_N\right] \tag{3}$$

The matrices $R_k(j)$, $Q_{k+1}(j)$ and $K_T(j)$ are positive-semidefinite for each j and k. In addition, we assume that

$$R_k(j) + B_k'(j)\left[\sum_{i=1}^{M} p_{k+1}(j, i)Q_{k+1}(i)\right] B_k(j) > 0 \tag{4}$$

The role of this condition will become clear in the sequel. Note in particular that (4) is satisfied if $R_k(j) > 0$ and $Q_k(j) \geqslant 0$ for all $j \in \mathbf{M}$ at all times k.

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This kind of problem formulation can be used to represent the control of systems subject to abrupt phenomena such as component and interconnection failures. We call this the *jump linear quadratic* (JLQ) control problem. The continuous-time version of this problem was apparently first formulated and solved by Krasovskii and Lidskii (1961). The problem was studied later by Wonham (1970). He obtained sufficient conditions for the existence and uniqueness of solutions in the JLQ case, and also derived a separation theorem under gaussian noise assumptions for JLQ control problems with markovian forms and noisy x (but perfect r) observations. Sworder (1969) obtained similar results using a stochastic maximum principle, and has published a number of extensions with his co-workers (Sworder 1970, 1972 a, b, 1977, Pierce and Sworder 1971). Stochastic minimum principle formulations for continuous-time problems involving jump process have also been considered by Rishel (1975), Kushner (1971) and others. Robinson and Sworder (1974) and Sworder and Robinson (1974) derived the appropriate non-linear partial differential equation for continuous-time jump parameter systems having state- and control-dependent rates. A similar result appears in the work of Kushner, and an approximation method for the solution of such problems was developed by Kushner and DiMasi (1978).

Discrete-time versions of the JLQ-control problem have not been thoroughly investigated in the literature. A special case of the x-independent JLQ discrete-time problem is considered by Birdwell *et al.* (1979), and the finite-time horizon x-independent problem was solved by Blair and Sworder (1975). Minor extensions are discussed by Chizeck and Willsky (1980). In this paper we develop necessary and sufficient conditions for the existence of steady-state optimal controllers for the discrete-time JLQ problem. These conditions are much more complicated than in the usual discrete-time linear quadratic regulator problem. Specifically, these conditions must account for the difference in the stability properties of the closed-loop system for different values of r_k. For example, it is possible for a particular component of x to diverge when r_k takes on a particular value, if r_k takes on this value rarely enough and if this component of x is stabilized sufficiently when the system is in other structural forms. Thus one finds that

stable closed-loop dynamics in each or all of the structural forms is neither necessary nor sufficient;
stabilizability of the dynamics in each form is neither necessary nor sufficient;
controllability of the dynamics in each form is neither necessary nor sufficient

for the existence of steady-state optimal controllers yielding finite expected cost.

In §2 we review the basic form of the solution to the discrete-time JLQ problem over a finite time horizon, and in §3 we present examples that illustrate several qualitative features of the solution. In §4 we present the rather complicated necessary and sufficient conditions for the existence of a steady-state solution for time-invariant JLQ problems over infinite horizons, and in §5 we present an example illustrating this condition and several other examples which serve to show that simpler conditions such as stabilizability or controllability are neither necessary nor sufficient. Section 6 contains simpler sufficient conditions for the existence of solutions in the infinite-horizon case, and §7 contains a brief summary.

2. Problem solution

The optimal control law can be derived using dynamic programming. Let $V_k(x_k, r_k)$ be the expected cost-to-go from state (x_k, r_k) at the time k (after $x_k' Q(r_k) x_k$ is charged):

$$V_N[x_N, r_N] = x'_N K_{\mathrm{T}}(r_N) x_N$$

$$V_k[x_k, r_k] = \min_{u_k} E\,[u'_k R_k(r_k) u_k + x'_{k+1} Q_{k+1}(r_{k+1}) x_{k+1} + V_{k+1}(r_{k+1}, x_{k+1})] \quad (5)$$

Proposition 1

Consider the discrete-time noiseless markovian-form jump linear quadratic optimal control problem (1)–(4). The optimal control law is given by

$$u_{k-1} = -L_{k-1}(j) x_{k-1} \quad \text{for } r_{k-1} = j \in \mathbf{M}$$
$$k = k_0, k_0 + 1, \ldots, N$$

where for each possible form j the optimal gain is given by

$$L_{k-1}(j) = [R_{k-1}(j) + B'_{k-1} Q^*_k(j) B_{k-1}(j)]^{-1}\, B'_{k-1}(j) Q^*_k(j) A_{k-1}(j) \quad (6)$$

where

$$Q^*_k(j) = \sum_{i=1}^{M} p_k(j, i)\,[Q_k(i) + K_k(i)] \quad (7)$$

Hence the sequence of sets of positive-semidefinite symmetric matrices $\{K_{k-1}(j) : j \in \mathbf{M}\}$ satisfies the set of M coupled matrix difference equations

$$K_{k-1}(j) = A'_{k-1}(j) Q^*_k(j)\,[A_{k-1}(j) - B_{k-1}(j) L_{k-1}(j)] \quad (8)$$

with terminal conditions

$$K_N(j) = K_{\mathrm{T}}(j)$$

The value of the optimal expected cost (3) that is achieved with this control law is given by

$$x'_0 K_{k0}(r_0) x_0$$

The proof of this result appears in Chizeck (1982) and is sketched in the Appendix. An earlier and essentially identical result was established by Blair and Sworder (1975).

Note that the $\{K_k(j) : j \in \mathbf{M}\}$ and optimal gains $\{L_k(j) : j \in \mathbf{M}\}$ can be recursively computed off-line, using the M coupled difference equations (6)–(8). The M coupled Riccati-like matrix difference equations cannot be written as a single nM-dimensional Riccati equation.

3. Examples

In this section some qualitative aspects of the JLQ controller given in Proposition 1 are illustrated via a time-invariant scalar example with $M = 2$ forms. This example serves to point out issues that arise in the consideration of steady-state JLQ controllers in the following sections. We take

$$x_{k+1} = a_1 x_k + b_1 u_k \quad \text{if } r_k = 1$$
$$x_{k+1} = a_2 x_k + b_2 u_k \quad \text{if } r_k = 2$$
$$p(i, j) = p_{ij} \quad (9)$$
$$\min E\left[\sum_{k=0}^{N-1} [x^2_{k+1} Q(r_k) + u^2_k R(r_k)] + x^2_N K_{\mathrm{T}}(r_N)\right]$$

In this case the cost matrix sequences $\{K_k(j): j \in \mathbf{M}\}$ may or may not converge as k decreases from N, and furthermore x_k may or may not be driven to zero, as shown in the following.

Example 1

Consider the following choice of parameters for (9):

$$x_{k+1} = x_k + u_k \quad \text{if } r_k = 1$$

$$x_{k+1} = 2x_k + 2u_k \quad \text{if } r_k = 2$$

$$p_{ij} = 0{\cdot}5, \quad K_T(j) = 0, \quad Q(j) = 1, \quad R(j) = 1 \quad \text{for } j = 1, 2$$

The optimal costs, control gains and closed-loop dynamics are given in Table 1, for four iterations.

As the table indicates, in this case the optimal costs and gains converge quickly. Furthermore, note that in the 'worst case' of $r_k = 2$ for all k,

$$\lim_{N\to\infty} |x_N| \geqslant \lim_{N\to\infty} (0{\cdot}5)^{N-1} |x_0| = 0$$

Thus x is driven to zero by the optimal controller.

This example demonstrates the 'passive-hedging' behaviour of the optimal controller. That is, possible future form changes and their associated costs are taken into account. To see this, consider the usual LQ regulator gains and cost parameters (as if $p_{11} = p_{22} = 1$ and $p_{12} = p_{21} = 0$), which are listed in Table 2.

Comparing Tables 1 and 2, we note that for $k \leqslant N - 2$ the gains of the Proposition 1 JLQ controller are modified (relative to the LQ controller) to reflect future form changes and costs. The JLQ controller has higher $r = 1$ gains to compensate for the possibility that the system might shift to the more expensive form $r = 2$. Similarly, the $r = 2$ gains are lower in the JLQ controller, reflecting the likelihood of future shifts to $r_k = 1$.

	$K_k(1) = L_k(1)$	$K_k(2) = L_k(2)$	$a_1 - b_1 L_k(1)$	$a_2 - b_2 L_k(2)$
$k = N - 1$	0·5	0·8	0·5	0·4
$N - 2$	0·623	0·868	0·377	0·263
$N - 3$	0·636	0·875	0·364	0·251
$N - 4$	0·637	0·875	0·363	0·249

Table 1. Optimal cost and controller parameters, and closed-loop dynamics for Example 1.

	$K_k(1) = L_k(1)$ (with $p_{11} = 1$)	$K_k(2) = L_k(2)$ (with $p_{22} = 1$)
$k = N - 1$	0·5	0·8
$N - 2$	0·6	0·878
$N - 3$	0·615	0·883
$N - 4$	0·618	0·883

Table 2. Standard LQ solution for Example 1.

Example 2

Here we choose the parameters of (9) so that the optimal closed-loop systems in different forms are *not all stable*, although the expected value of x is driven to zero.

Let

$$x_{k+1} = x_k + u_k \quad \text{if } r_k = 1$$

$$x_{k=1} = 2x_k + u_k \quad \text{if } r_k = 2$$

$$p_{11} = p_{21} = 0{\cdot}9 \quad p_{12} = p_{22} = 0{\cdot}1$$

where

$$K_{\mathrm{T}}(j) = 0, \quad Q(j) = 1, \quad j = 1, 2$$
$$R(1) = 1, \quad R(2) = 1000$$

Thus there is a high penalty on control in form 2.

This system is much more likely to be in $r = 1$ than in $r = 2$ at any time. We might expect that the optimal control strategy may tolerate instability while in the expensive-to-control form $r = 2$, since the system is likely to return soon to the form $r = 1$, where control costs are much less. Computation for four iterations demonstrates this, as shown in Tables 3 and 4.

As our analysis in subsequent sections will confirm, these quantities converge as $N - k \to \infty$. Note that the closed-loop system is *unstable* while in $r = 2$.

Direct calculation of the expected value of x_k, given x_0 and r_0, shows that $|E(x_k)|$ decreases as k increases. This is shown in Table 5. In four time steps, $E\{x\}$ is reduced by over 95% if initially the system is in form 1 and 68% if it starts in form 2. Note that if the system starts in the expensive-to-control form $r = 2$, x is allowed to increase for one time step (until control while in $r = 1$ is likely to reduce it).

	$K_k(1)$	$K_k(2)$	$L_k(1)$	$L_k(2)$
$k = N$	0	0	–	–
$N - 1$	0·5	3·996	0·5	$1{\cdot}998 \times 10^{-3}$
$N - 2$	0·649	7·385	0·649	$3{\cdot}672 \times 10^{-3}$
$N - 3$	0·699	9·269	0·699	$4{\cdot}603 \times 10^{-3}$
$N - 4$	0·719	10·198	0·718	$5{\cdot}060 \times 10^{-3}$

Table 3. Optimal gains and costs for Example 2.

	$a_1 - b_1 L_k(1)$	$a_2 - b_2 L_k(2)$
$k = N - 1$	0·5	1·998
$N - 2$	0·359	1·996
$N - 3$	0·301	1·995
$N - 4$	0·281	1·995

Table 4. Closed-loop optimal dynamics for Example 2.

	$r_0 = 1$	$r_0 = 2$
x_0	1·0	1·0
x_1	0·281	1·995
$E\{x_2\}$	0·132	0·938
$E\{x_3\}$	0·069	0·491
$E\{x_4\}$	0·045	0·319

Table 5. $E\{x_k\}$ for Example 2.

4. The steady-state problem

We now consider the control problem in the time-invariant case as the time horizon $N - k_0$ becomes infinite. Specifically consider the model (1), (2) with $A_k(r_k) = A(r_k)$, $B_k(r_k) = B(r_k)$ and $p_{k+1}(i, j) = p_{ij}$. We wish to determine the feedback control law to minimize

$$\lim_{N-k_0\to\infty} E\left[\sum_{k=k_0}^{N-1} [u_k' R(r_k)u_k + x_{k+1}' Q(r_{k+1})x_{k+1}] + x_N' K_T(r_N)x_N \,|\, x_0, r_0\right] \quad (10)$$

For future reference, from Proposition 1 the optimal closed-loop dynamics in each form $j \in \mathbf{M}$ are

$$x_{k+1} = D_k(r_k)x_k$$

where

$$D_k(j) = \{I - B(j)[R(j) + B'(j)Q_{k+1}^*(j)B(j)]^{-1}B'(j)Q_{k+1}^*(j)\}A(j) \quad (11)$$

where $Q_k^*(j)$ is defined in (7) (in the time-invariant case, of course, only $K_k(j)$ in (7) may vary with k).

Before stating the main result of this section, we recall the following terminology pertaining to finite-state Markov chains:

a state is *transient* if a return to it is not guaranteed;
a state i is *recurrent* if an eventual return to i is guaranteed;
state i is *accessible* from state j if it is possible to begin in j and arrive in i in some finite number of steps;
states i and j are said to *communicate* if each is accessible from the other;
a *communicating class* is *closed* if there are no possible transitions from inside the class to any state outside of it;
a closed communicating class containing only one member j is an *absorbing state*; that is, $p_{jj} = 1$;
a Markov-chain state set can be divided into disjoint sets $\mathbf{T}, C_1, \ldots, C_s$ where all of the states in $\mathbf{T}$ are transient, and each C_j is a closed communicating class of recurrent states.

Define the *cover* C_j^* of a form $j \in \mathbf{M}$ to be the set of all forms accessible from j in one time step. That is,

$$C_j^* = \{i \in \mathbf{M} : p(j, i) \neq 0\}$$

The main result of this section is the following.

Proposition 2

For the time-invariant markovian JLQ problem the conditions described below are *necessary* and *sufficient* for the solution of the set of coupled matrix difference equations (6)–(8) to converge to a constant steady-state set

$$\{K(j) \geqslant 0 : j \in \mathbf{M}\}$$

as $N - k_0 \to \infty$. In this case the $K(j)$ are given by the M coupled equations

$$K(j) = A'(j)Q^*(j)D(j) \quad (12)$$

where $D(j)$ is defined as in (11) with $Q_k^*(j)$ replaced by $Q^*(j)$. In turn, $Q^*(j)$ is defined in (7) with $K_k(j)$ replaced by $K(j)$; that is,

$$Q^*(j) = \sum_{i=0}^{M} p_{ji}[Q(i) + K(i)] \tag{13}$$

Furthermore, the steady gains $L(j)$ in the steady-state optimal-control law

$$u(r_k, x_k) = - L(r_k)x_k \tag{14}$$

are given by

$$L_j = [R(j) + B'(j)Q^*(j)B(j)]^{-1}B'(j)Q^*(j)A(j) \tag{15}$$

Thus under the conditions described below the optimal infinite-horizon cost is

$$V(x_0, r_0) = x_0'K(r_0)x_0$$

The conditions to be satisfied are as follows. There exists a set of constant control laws

$$u_k = -F(j)x_k, \quad j = 1, \ldots, M \tag{16}$$

such that the following holds.

Condition 1

For each *closed communicating class* C_i the expected cost-to-go from $(x_k = x, r_k = j \in C_i)$ at time k remains finite as $N - k \to \infty$.

This will be true if and only if for each closed communicating class C_i, for all forms $j \in C_i$, there exists a set of finite positive-semidefinite $n \times n$ matrices $\{Z_1, Z_2, \ldots, Z_{|C_i|}\}$ satisfying the $|C_i|$ coupled equations

$$\begin{aligned} Z_j &= \sum_{t=0}^{\infty} p_{jj}^t [A_j - B_jF_j]'^t \{Q_j + F_j'R_jF_j\} [A_j - B_jF_j]^t \\ &+ \sum_{t=1}^{\infty} p_{jj}^{t-1} [A_j - B_jF_j]'^t \left[\sum_{\substack{q \in C_i \\ q \neq j}} p_{jq} Z_q \right] [A_j - B_jF_j]^t \end{aligned} \tag{17}$$

Note that in the case of an absorbing form j (i.e. a singleton communicating class) Z_j reverts to the quantity

$$Z_j = \sum_{t=0}^{\infty} [A_j - B_jF_j]'^t \{Q_j + F_j'R_jF_j\} [A_j - B_jF_j]^t$$

Once we are in an absorbing form, our problem reduces to a standard LQ problem, and Condition 1 in effect states that unstable modes in such a form that lead to non-zero costs must be controllable.

Condition 2

For each transient form $j \in \mathbf{T} \subset \mathbf{M}$, the expected cost-to-go is finite. This is true if and only if set of finite positive-semidefinite $n \times n$ matrices $\{G_1, G_2, \ldots, G_{|\mathbf{T}|}\}$ satisfying the $|\mathbf{T}|$ coupled equations

$$G_j = \sum_{t=0}^{\infty} p_{jj}^t [A_j - B_jF_j]'^t \{Q_j + F_j'R_jF_j\} [A_j - B_jF_j]^t$$

$$+ \sum_{t=1}^{\infty} p_{jj}^{t-1} [A_j - B_jF_j]'^t \left[\sum_{\substack{q \in \mathbf{T} \\ q \neq j}} p_{jq} G_q + \sum_{\substack{q \in \mathbf{M} - \mathbf{T} \\ q \neq j}} p_{jq} Z_q \right] [A_j - B_jF_j]^t \tag{18}$$

Condition 1 states that it is possible to achieve finite expected cost after the form process leaves the set of transient states and enters one of the closed communicating classes. Note that for absorbing states (i.e. ($|C_i| = 1$), Condition 1 reduces to the usual LQ condition. Condition 2 states that the expected cost from any transient form is finite. This precludes the possibility of an unstable mode of x_k growing without bound in mean square, either leading to infinite accrued cost while the form resides in the transient state set (this occurs if the x_k mode is observable through the cost in transient forms) or to infinite cost once the form jumps into a closed communicating class (if this mode becomes observable after the transition).

The proof of the Proposition, which is given in Chizeck (1982), is quite straightforward, and we confine ourselves here to sketching the basic idea. Necessity is clear, since if conditions 1 and 2 are not satisfied for any control law of the type (16) then the finite-horizon optimal control laws cannot converge to one with finite cost as $N - k_0 \to \infty$. To show sufficiency, one first shows that if one applies the control law (16), then, under conditions 1 and 2, the expected cost is finite as $N - k_0 \to \infty$. In fact it is given by

$$x'(k_0)[Z(r(k_0)) - Q(r(k_0))]x(k_0) \quad \text{if } r(k_0) \in \mathbf{M} - \mathbf{T}$$

$$x'(k_0)[G(r(k_0)) - Q(r(k_0))]x(k_0) \quad \text{if } r(k_0) \in \mathbf{T}$$

This establishes an upper bound on the optimal cost matrices $K_{k0}(j)$ for the finite-time-horizon problem for the particular case when the terminal costs $K_N(j) = 0$. Furthermore, in this case the $K_{k0}(j)$ are monotone-increasing as $N - k_0$ increases, and thus they converge. It is then immediate that the limits

$$\lim_{N - k_0 \to \infty} K_{k0}(j) = K(j)$$

satisfy (16). Straightforward adaptations of standard LQ arguments then allow us first to extend the convergence result to the case of arbitrary terminal cost matrices for the finite-horizon problem, and, secondly, to show that there is a unique set of positive-definite solutions of (16). Conditions 1 and 2 of Proposition 2 take into account the probability of being in forms that have unstable closed-loop dynamics, and the relative expansion and contraction effects of unstable and stable form dynamics and how the closed-loop eigenvectors of accessible forms are 'aligned'. That is, it is not necessary or sufficient for all (or even *any*) of closed-loop dynamics corresponding to sufficient forms to be stable, since the *interaction* of different form dynamics determines the behavior of $E\{x_k'x_k\}$.

These various characteristics will be illustrated in the examples in §5. The Conditions in Proposition 2 differ from those of the usual discrete-time linear quadratic regulator problem in that necessary and sufficient Conditions 1, 2 replace the sufficient condition that the (single-form) system is *stabilizable*. Unfortunately these conditions are not easily verified. There is no evident algebraic test for (17), (19) like the controllability and observability tests in the LQ problem. The use of the conditions in Proposition 2 will be demonstrated in the examples that follow.

It is important to note that even if the conditions of Proposition 2 are satisfied, we are not guaranteed that $x_k \to 0$ in mean square. One obvious reason for this is that Conditions 1 and 2 are trivially satisfied (with $F(j)$, $Z(j)$, $G(j)$ all zero) if $Q(j) = 0$ in all forms. Of course, the same comment applies in the usual linear–quadratic problem. In that case, a set of conditions that guaranteee that $x_k \to 0$ in mean square are the stabilizability condition mentioned previously and the requirement that $(A, Q^{1/2})$ be detectable.

Example 3

One might conjecture, given the LQ result, that Conditions 1 and 2 together with the requirement that $(A(j), Q^{1/2}(j))$ be detectable for each j might be sufficient for the JLQ problem. This is not the case, however, as one can certainly construct deterministically jumping systems (i.e. time-varying linear systems) that are counterexamples, such as the following:

$$A(1) = \begin{pmatrix} 0 & 2 \\ \frac{1}{2} & 0 \end{pmatrix}, \quad Q(1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B(1) = 0$$

$$A(2) = \begin{pmatrix} 0 & \frac{1}{2} \\ 2 & 0 \end{pmatrix}, \quad Q(2) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad B(2) = 0$$

$$p_{12} = p_{21} = 1$$

The following corollary presents one sufficient condition that guarantees that $x_k \to 0$ in mean square.

Corollary 1

Consider the time-invariant JLQ problem, and suppose that Conditions 1 and 2 of Proposition 2 are satisfied. Suppose also that the closed-loop transition matrix $A(j) - B(j)L(j)$ is invertible for all j. Then $E\{x_k' x_k\} \to 0$ if the matrix $Q(j) + L(j)R(j)L(j)$ is positive-definite for at least one form in each closed communicating class.

Before sketching the proof of the corollary it is worth providing an example that illustrates the types of situations that motivated the inclusion of the assumption that $A(j) - B(j)L(j)$ is invertible for all j.

Example 4

Consider a scalar system with form dynamics illustrated in Fig. 1, where

$$A(1) = 2, \quad A(2) = 0, \quad A(3) = 1$$

$$B(1) = B(2) = B(3) = 0$$

$$Q(1) = Q(2) = 0, \quad Q(3) = 1$$

In this case, assuming that the initial form is *not* 2, it is not difficult to show that $E[x_k^2] \to \infty$, while the cost incurred over the infinite horizon is zero, even though

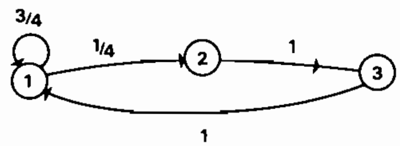

Figure 1. Form structure for Example 4.

$Q(3) = 1$. The reason for this is that the form process is likely to remain in form 1 for too long a time, but this large value of the state is not penalized because of the nulling of the state at the time of the first transition to form 2. Note also that in this case, although $E[x_k^2]$ diverges, $x_k \to \infty$ with probability 1.

For simplicity in our proof of the corollary, let us assume that there is a single closed communicating class. The extension to several classes is straightforward. First let us denote by j^* the form specified in the Corollary; i.e. j^* is in the closed communicating class, and

$$\sigma_{\min}[Q(j^*) + L(j^*)'R(j^*)L(j^*)] = \gamma > 0 \tag{19}$$

where $\sigma_{\min}(A)$ is the smallest singular value of A. Note next that if we apply the optimal steady-state control law as specified in Proposition 2, and if $r_k = j$, then the cost accrued at time k is

$$x_k'[Q(j) + L(j)'R(j)L(j)]x_k$$

Suppose that $\{t_i\}$ is any sequence of strictly increasing stopping times so that $r_{t_i} = j^*$. Then under the conditions of Proposition 2, the optimal cost J^* is finite, and in fact

$$\begin{aligned}\infty > J^* &= E\sum_{k=0}^{\infty} x_k'[Q(r_k) + L'(r_k)R(r_k)L(r_k)]x_k \\ &\geqslant E\sum_{i=0}^{\infty} x_{t_i}'[Q(j^*) + L'(j^*)R(j^*)L(j^*)]x_{t_i} \\ &\geqslant \gamma\sum_{i=0}^{\infty} E[\|x_{t_i}\|^2]\end{aligned} \tag{20}$$

From this we can immediately conclude that

$$\lim_{i\to\infty} E[\|x_{t_i}\|^2] = 0 \tag{21}$$

What we wish to show is that

$$\lim_{k\to\infty} E[\|x_k\|^2] = 0 \tag{22}$$

and we do this by contradiction. Specifically, suppose that (22) is not true; that is, we can find an ε so that for any positive integer m there exists another integer $K(\varepsilon, m) \geqslant m$ so that

$$E[\|x_{K(\varepsilon,m)}\|^2] > \varepsilon \tag{23}$$

We will show that this supposition contradicts (21) by constructing a sequence of stopping times for which (21) does not hold if (23) does. Let

t_0 = the earliest time after $K(\varepsilon, 0)$ that the form process is in state j^*
t_k = the earliest time after both $K(\varepsilon, k)$ and $t(\varepsilon, k-1)$ that the process is in state j^*

Denote by U_m the set of form trajectories that begin in state m and end in state j^* without any intermediate visits to j^*. For any $u \in U_m$ let $\phi(u)$ denote the closed-loop state transition matrix along the trajectory u. Then

$$\begin{aligned} E[\|x_{tk}\|^2] &= E[E[\|x_{tk}\|^2 \mid x_{K(\varepsilon,k)}, r_k = m]] \\ &= E[E[\|\phi(u)x_{K(\varepsilon,k)}\|^2 \mid x_{K(\varepsilon,k)}, r_k = m]] \\ &= E[x'_{K(\varepsilon,k)} E[\phi'(u_k)\phi(u_k) \mid r_k = m] x_{K(\varepsilon,k)}] \end{aligned} \tag{24}$$

where u_k denotes the form trajectory from $K(\varepsilon, k)$ to t_k. Note that the invertibility assumption immediately implies that

$$\gamma_m = \sigma_{min}\{E[\phi'(u_k)\phi(u_k) \mid r_k = m]\} > 0$$

Letting

$$\gamma = \min_m \gamma_m$$

we see that (23) and (24) together imply that

$$E[\|x_{tk}\|^2] > \gamma\varepsilon$$

5. Examples

The following simple scalar example illustrates the conditions of Proposition 2.

Example 5

Consider the form dynamics depicted in Fig. 2, where the x-process dynamics are autonomous in all forms:

$$x_{k+1} = a(r_k)x_k, \quad r_k \in \{1, 2, 3, 4, 5, 6, 7\}$$

and $Q(j) > 0 \ \forall j$. Here 6 is an absorbing form, $\{3, 4\}$ is a closed communicating class, and $\mathbf{T} = \{1, 2, 5, 7\}$ is the set of transient forms. For the absorbing form $r = 6$ Condition 1 yields

(i) $a^2(6) < 1$

and in this case

$$Z(6) = \frac{Q(6)}{1 - a^2(6)}$$

For the closed communicating class $\{3, 4\}$, (17) gives the coupled equations

$$\begin{aligned} Z(3) &= Q(3) + a^2(3)Z(4) \\ Z(4) &= Q(4) + a^2(4)Z(3) \end{aligned}$$

Consequently

$$Z(3) = \frac{1}{1 - a^2(3)a^2(4)}[Q(3) + a^2(4)Q(4)]$$

$$Z(4) = \frac{1}{1 - a^2(3)a^2(4)}[Q(4) + a^2(3)Q(3)]$$

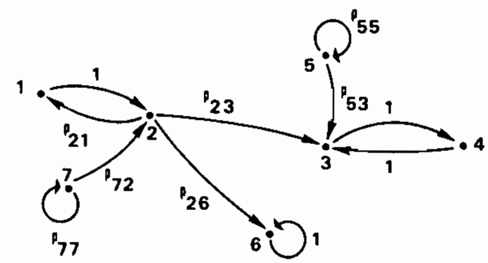

Figure 2. Form structure for Example 5.

Thus for $Z(3)$, $Z(4)$ to be positive (as in Condition 1) we must have

(ii) $a^2(3)a^2(4) < 1$

(i.e. the two-step dynamics corresponding to the form transitions 3–4–3 or 4–3–4 must be stable). For the transient forms $\{1, 2, 5, 7\}$, (18) yields

$$G(1) = Q(1) + a^2(1)G(2)$$
$$G(2) = Q(2) + a^2(2)\,[p_{21}G(1)] + p_{23}Z(3) + p_{26}Z(6)$$
$$G(5) = Q(5) + \sum_{t=1}^{\infty} p_{55}^{t-1}\, a^{2t}(5)\,[Q(5)p_{55} + p_{53}Z(3)]$$
$$G(7) = Q(7) + \sum_{t=1}^{\infty} p_{77}^{t-1} a^{2t}(7)\,[Q(7)p_{77} + p_{72}G(2)]$$

From the equations for $G(1)$ and $G(2)$,

$$G(1) = \frac{Q(1) + a^2(1)Q(2) + [p_{23}Z(3) + p_{26}Z(6)]a^2(1)}{1 - a^2(1)a^2(2)p_{21}}$$
$$G(2) = \frac{Q(2) + a^2(2)Q(1)p_{21} + p_{23}Z(3) + p_{26}Z(6)}{1 - a^2(1)a^2(2)p_{21}}$$

So for $0 < G(1), G(2) < \infty$ we have

(iii) $a^2(1)a^2(2)p_{21} < 1$

From the expression for $G(5)$ we see that for $0 < G(5) < \infty$ we have

(iv) $p_{55}a^2(5) < 1$

with the resulting

$$G(5) = \frac{Q(5) + p_{53}a^2(5)Z(3)}{1 - p_{55}a^2(5)}$$

From the expression for $G(7)$ we see that for $0 < G(7) < \infty$ we have

(v) $p_{77}a^2(7) < 1$

with

$$G(7) = \frac{Q(7) + p_{72}a^2(7)G(2)}{1 - p_{77}a^2(7)}.$$

The conditions (i)–(v) above result from the necessary and sufficient conditions of Proposition 2, applied to this problem. For this example we see that

the absorbing form ($r = 6$) must have stable dynamics; (i);
one of the forms in the closed communicating class $\{3, 4\}$ can be unstable as long as the other form's dynamics make up for the instability; (ii);
transient forms $r = 5, 7$ can have unstable dynamics as long as the probability of staying in them for any length of time is low enough; (iii), (iv);
some instability of the dynamics of forms $r = 1, 2$ is okay so long as the probability of repeating a $2 \to 1 \to 2$ cycle is low enough; (iv).

In the proof of the LQ problem, the existence of an upper bound can be guaranteed by assuming the stabilizability of the system. This does not suffice here (except for scalar x), as shown in the following example.

Example 6: stabilizability not sufficient for finite cost
Let $M = 2$, where

$$A_1 = \begin{pmatrix} \frac{1}{2} & 10 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} \frac{1}{2} & 0 \\ 10 & \frac{1}{2} \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

with $p_{12} = p_{21} = 1$ and $p_{11} = p_{22} = 0$ (a 'flip–flop' system as in Fig. 3). Both forms have stable dynamics (eigenvalues $\frac{1}{2}, \frac{1}{2}$) and hence are trivially stabilizable. However,

$$x_{k+2} = \begin{pmatrix} 100{\cdot}25 & 5 \\ 5 & 0{\cdot}25 \end{pmatrix} x_k \quad \text{if } r_k = 1$$

$$x_{k+2} = \begin{pmatrix} 0{\cdot}25 & 5 \\ 5 & 100{\cdot}25 \end{pmatrix} x_k \quad \text{if } r_k = 2$$

which is clearly unstable. Thus x_k and the expected cost-to-go become infinite as $N - k_0$ goes to infinity.

In fact, controllability in each form is not sufficient for finite cost, as demonstrated below.

Example 7: controllability not sufficient for finite cost
Let $M = 2$, where

$$A_1 = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

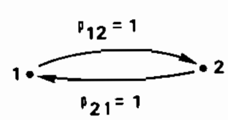

Figure 3. From structure for Examples 6, 7 and 8.

$$A_2 = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Thus in each form ($r = 1, 2$) the system is controllable, and the closed-loop systems have dynamics

$$x_{k+1} = D(r_k)x_k$$

with

$$D(1) = \begin{pmatrix} 0 & 2 \\ f_1 & f_2 \end{pmatrix}, \quad D(2) = \begin{pmatrix} f_3 & f_4 \\ 2 & 0 \end{pmatrix}$$

where f_1, f_2, f_3, f_4 are determined by the feedback laws chosen. Now suppose that we have a "flip-flop" system as in Figure 3. Then

$$x_{2k} = \begin{cases} [D(2)\ D(1)]^k x_0 & \text{if } r_0 = 1 \\ [D(1)\ D(2)]^k x_0 & \text{if } r_0 = 2 \end{cases}$$

where

$$D(2)D(1) = \begin{pmatrix} f_1 f_4 & 2f_3 + f_2 f_4 \\ 0 & 4 \end{pmatrix}, \quad D(1)D(2) = \begin{pmatrix} 4 & 0 \\ f_1 f_3 + 2f_2 & f_1 f_4 \end{pmatrix}$$

Both $D(1)D(2)$ and $D(2)D(1)$ have 4 as an eigenvalue. Thus x_k grows without bound for $x_0 \neq 0$ as k increases. Controllability in each form allows us to place the *eigenvalues* of each form's closed-loop dynamics matrix $D(i)$ as we choose, but we *cannot place the eigenvectors* arbitrarily. In this example there is no choice of feedback laws that can align the eigenstructures of each of the closed loop systems so that the overall dynamics are stable.

The following example demonstrates that (for $n \geqslant 2$) stabilizability of even one form's dynamics is not necessary for the costs to be bounded.

Example 8: *stabilizability not necessary for finite cost*

Let $M = 2$ with

$$A(1) = \begin{pmatrix} 1 & -1 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad B(1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$A(2) = \begin{pmatrix} \frac{1}{2} & 1 \\ 0 & 1 \end{pmatrix}, \quad B(2) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Both forms are unstable uncontrollable systems, so neither is stabilizable. We again take the form dynamics as in fig. 3. Then

$$x_{2k} = \begin{cases} (A(2)A(1))^k x_0 & \text{if } r_0 = 1 \\ (A(1)A(2))^k x_0 & \text{if } r_0 = 2 \end{cases}$$

where

$$A(1)A(2) = A(2)A(1) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

Thus $x_{2k} \to 0$, and hence the cost is finite. We next show that this example does satisfy Condition 1 of Proposition 2. From (17) with $F(1) = F(2) = 0$ we have

$$\begin{aligned} Z(1) &= Q(1) + A(1)'Z(2)A(1) \\ Z(2) &= Q(2) + A(2)'Z(1)A(2) \end{aligned}$$

Suppose, for convenience, that $Q(1) = Q(2) = 1$. Then we obtain from the first equation above that

$$\begin{pmatrix} Z_{11}(2) & Z_{12}(2) \\ Z_{21}(2) & Z_{22}(2) \end{pmatrix}$$
$$= \begin{pmatrix} 1 + Z_{11}(2) & -Z_{11}(2) + \frac{1}{2}Z_{12}(2) \\ -Z_{11}(2) + \frac{1}{2}Z_{21}(2) & 1 + Z_{11}(2) - Z_{21}(2) + \frac{1}{4}Z_{22}(2) \end{pmatrix}$$

and, putting this into the second equation,

$$\begin{pmatrix} Z_{11}(2) & Z_{12}(2) \\ Z_{21}(2) & Z_{22}(2) \end{pmatrix} = \begin{pmatrix} \frac{5}{4} + \frac{1}{4}Z_{11}(2) & \frac{1}{2} + \frac{1}{4}Z_{12}(2) \\ \frac{1}{2} + \frac{1}{4}Z_{21}(2) & 3 + \frac{1}{4}Z_{22}(2) \end{pmatrix}$$

This yields four equations in four unknowns. Solving, we find

$$\begin{pmatrix} Z_{11}(1) & Z_{21}(1) \\ Z_{21}(1) & Z_{22}(1) \end{pmatrix} = \begin{pmatrix} 6 & -\frac{14}{3} \\ -\frac{14}{3} & \frac{13}{3} \end{pmatrix}$$

and

$$\begin{pmatrix} Z_{11}(2) & Z_{12}(2) \\ Z_{21}(2) & Z_{22}(2) \end{pmatrix} = \begin{pmatrix} 5 & \frac{2}{3} \\ \frac{2}{3} & 4 \end{pmatrix}$$

which are both positive-definite. Thus Z_1 and Z_2 satisfy Condition 2 of Proposition 2.

6. Sufficient conditions for finite expected cost

In this section we examine sufficient conditions for the existence of finite expected costs-to-go that replace the necessary and sufficient Conditions 1–3 in Proposition 2, and are somewhat easier to compute, in terms of the spectral norms of certain matrices. Recall that for any matrix A, the *spectral norm* of A is

$$\|A\| = \max_{\|u\| = u'u = 1} \{\|Au\|\} = [\text{max eigenvalue } (A'A)]^{1/2} \tag{25}$$

Corollary 2

A sufficient condition for the existence of the steady-state control law (and finite expected costs-to-go) for the time-invariant JLQ problem is that there exists a set of feedback control laws

$$u_k(r_k, x_k) = -F(r_k)x_k$$

such that

(i) for each *absorbing form* i ($p_{ii} = 1$), the pair $(A(i), B(i))$ is stabilizable;
(ii) for each *recurrent nonabsorbing form* i and for each *transient form* $i \in \mathbf{T}$ that is accessible from a form $j \in C_i^*$ in its cover ($j \neq i$)

$$E \sum_{t=1}^{\infty} p_{ii}^{t-1} \|(A(i) - B(i)F(i))^t\|^2 < c < 1 \tag{26}$$

(iii) for each *transient form* $i \in \mathbf{T}$ that is not accessible from any form $j \in C_i^*$ in its cover (except itself)

$$E \sum_{t=1}^{\infty} p_{ii}^{t-1} \|(A(i) - B(i)F(i))^t\|^2 < \infty \tag{27}$$

The proof of this Corollary is immediate. A similar result for continuous-time systems was obtained by Wonham (1970, theorem 6.1), except that stabilizability and observability of each form is required, and a condition like (26) is required for all non-absorbing forms.

Condition (2) is motivated as follows. The cost incurred while in a particular transient form is finite *with probability one* since, eventually, the form process leaves the transient class $\mathbf{T}$ and enters a closed communicating class. If a particular transient form $i \in \mathbf{T}$ can be repeatedly re-entered, however, the *expected* cost incurred while in i may be *infinite*; (26) excludes such cases. Note that the sufficient conditions of Corollary 2 are violated in Example 8 (in both forms). This demonstrates that they are restrictive, in that they ignore the relative 'directions' of x growth in the different forms (i.e. the eigenvector structure). We consider next a sufficient condition that is easier to verify than Corollary 2, but is even more conservative.

Corollary 3

Sufficient conditions (1) and (2) in Corollary 2 can be replaced by the following. There exists a set of feedback control laws

$$u(r_k, x_k) = -F(r_k)x_k$$

such that

$$\|(A(i) - B(i)F(i)\| < c < 1 \tag{28}$$

The proof of this corollary is also immediate.

Note that if (28) holds then conditions (1)–(3) of Proposition 2 hold. Note also that we are guaranteed that $\|x_k\| \to 0$ with probability one, if (28) holds only for recurrent forms. However, this is not enough to have finite expected cost, as demonstrated in the following examples.

Example 9: Let

$$A(1) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \quad A(1) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad B(1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = B(2)$$

where $a > 1$, and with $Q(1) = \mathrm{I}$, $Q(2) = 0$. Also, let

$$p_{11} = p, \qquad p_{22} = 1$$
$$p_{12} = 1 - p, \quad p_{21} = 0$$

In this case

$$\min_{F(1)} \|A(1) - B(1)F(1)\| = \left\| \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right\| = a > 1$$

$$\min_{F(2)} \|A(2) - B(2)F(2)\| = 0$$

and for $r_0 = 1$

$$E\left[\sum_{k=0}^{\infty} \{x_k' Q(r_k)x_k + u_k' R(r_k)u_k\}\right]$$

$$= \|x_0\|^2 \sum_{k=0}^{\infty} (a^2 p)^k$$

If $a^2 p < 1$ then the expected cost is

$$\frac{\|x_0\|^2}{1 - a^2 p} < \infty$$

However, if $a^2 p \geqslant 1$ then the expected cost-to-go is infinite. This demonstrates that (28) holding only for non-transient forms is not sufficient for finite expected cost-to-go. Specifically, as this example demonstrates, the cost-to-go will be infinite if one is likely to remain sufficiently long in transient forms that are unstable enough.

Example 10

Let

$$x_{k+1} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} x_k \quad \text{if } r_k = 1,3$$

$$x_{k+1} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} x_k \quad \text{if } r_k = 2$$

where the form transition dynamics are given in Fig. 4. We also assume $Q_i > 0$, $i = 1, 2, 3$.

If the system is in form 1 or 3 for three successive times ($r_k = r_{k+1} = r_{k+2} = 1$) then $x_{k+2} = (0\ 0)$ for *any* x_k. In form $r = 2$ the expected cost incurred until the system leaves (at time τ) given that the state at time k is $(x_k, r_k = 2)$ is

$$E\left[\sum_{t=k}^{\tau-1} x_t' Q(2) x_t\right] = x_k'\left[\sum_{t=0}^{\infty} p_{22} A'(2)' q(2) A(2)'\right] x_k$$

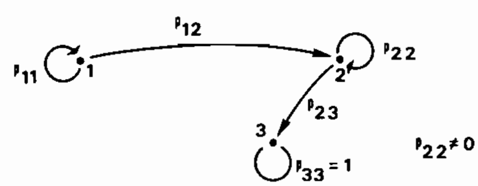

Figure 4. Form transition for Example 10.

For this cost to be finite we must have

$$\sum_{t=0}^{\infty} p_{22}^t (A'(2))^t Q(2) A(2)^t \;=\; Q(2) \sum_{t=0}^{\infty} p_{22}^t a^{2t} \;<\; \infty$$

which is true if and only if

$$a^2 p_{22} \;<\; 1 \tag{29}$$

Thus we would expect that the optimal expected costs-to-go in Proposition 2 will be finite if and only if (29) holds. We next verify that the necessary and sufficient conditions of Proposition 2 say this.

The matrix

$$A(3) \;=\; \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

is nilpotent; hence the absorbing form $r = 3$ is stabilizable (so Condition 2 of Proposition 2 is met). For transient forms $\{1, 2\}$ we must have $0 < G(1), G(2) < \infty$, where

$$G(1) \;=\; \sum_{t=0}^{\infty} p_{11}^t A'(1)^t Q(1) A(1)^t \;+\; \sum_{t=1}^{\infty} p_{11}^{t-1} A'(1)^t p_{12} G(2) A(1)^t$$

$$G(2) \;=\; \sum_{t=0}^{\infty} p_{22}^t A'(2)^t Q(2) A(2)^t$$

$$=\; Q(2) \sum_{t=0}^{\infty} p_{22}^t a^{2t} \;=\; \frac{Q(2)}{1 - p_{22}a^2}$$

Thus for $G(2)$ to be positive-definite we have the condition (29). Finally, since $A(1)^t = 0$ for $t \geqslant 2$, we have

$$G(1) \;=\; Q(1) \;+\; A'(1)[p_{11}Q(1) \;+\; p_{12}G(2)]A(1)$$

$$=\; Q(1) \;+\; A'(1)\left[p_{11}Q(1) \;+\; \frac{p_{12}Q(2)}{1 - p_{22}a^2}\right]A(1)$$

which is positive-definite since $Q(1)$, $Q(2) > 0$. Thus the necessary and sufficient conditions of Proposition 2 here reduce to (29). Note that he sufficient condition (28) of Corollary 3 is *never* met for $r = 1$ and $r = 3$, since $\|A(1)\| = \|A(3)\| = 2$, and to meet (28) for $r = 2$ requires $|a| < 1$. On the other hand, the sufficient conditions for Corollary 2 are met if (29) holds, because forms $\{1, 2\}$ are 'non-re-enterable' transient forms satisfying (27).

7. Summary

In this paper we have formulated and solved the discrete-time linear quadratic control problem with perfect observations when the system and cost parameters jump randomly according to a finite Markov process. The optimal control law is linear in x_k at each time k, and is different (in general) for each possible set of parameter values. Proposition 2 provides necessary and sufficient conditions for existence of the optimal steady-state JLQ controller. These conditions are not easily tested, however, since they require the simultaneous solution of coupled matrix equations containing infinite sums. In Corollaries 2 and 3 sufficient conditions are presented that are more easily tested.

Perhaps the most important contribution of this paper is the set of examples that explore the reasons for the complexity of the conditions of Proposition 2. For example, we have shown that stabilizability of the system in each form is neither necessary *nor* sufficient for the existence of a stable steady-state closed-loop system. Issues such as the amount of time spent in unstable forms, and the differences among the stable and unstable subspaces in different forms have been illustrated.

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