# Smoothing Error Dynamics and Their Use in the Solution of Smoothing and Mapping Problems

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*Abstract*—**Martingale decomposition techniques are used to derive Markovian models for the error in smoothed estimates of processes described by linear models driven by white noise. These models, together with some simple Hilbert space decomposition ideas, provide a simple unified framework for examining a variety of problems involving the efficient assimilation of spatial data, which we refer to as mapping problems. Algorithms for several different mapping problems are derived. A specific example of map updating for a two-dimensional random field is included.**

## I. Introduction

IN THIS PAPER we consider several estimation problems motivated by the subject of mapping. Our work is directed toward problems in which the objective is to obtain an efficient procedure for producing a map of a random field which combines the information contained in several other maps and/or sets of measurements. Problems of this type arise in a variety of disciplines including geodesy and meteorology [2], [3].

In a previous paper [1] we presented derivations of algorithms for several of the problems we consider here. Unfortunately, the approach in [1] consisted of tedious manipulations of filtering and smoothing equations which shed no light on the fundamental nature of the problems under investigation. The final forms of the solutions obtained in [1] were simple, suggesting that a more elegant approach must exist which would provide greater insight into problems of mapping and which would generalize more readily to problems outside the class considered in [1]. In this paper we present such an approach and use it to derive solutions to several problems.

A variety of different mapping problems is of practical interest. The first problem we will consider is that of *map updating*, in which one wishes to update an existing map (based on previously available measurements) with information contained in a new set of data. We also consider two other problems: the *map combining* problem, in which we wish to combine two maps over a given region each of which is based on a different set of data; and the *map centralization* problem, in which we are to produce a single map over a given region given several individual maps of subregions. All three of these problems arise in a variety of applications, including mapping of gravitational fields, topographical mapping, and the production and updating of meteorological maps. Given the sizes of the regions being mapped and the large volumes of data to be used to produce the maps, a critical issue in these applications is the development of *efficient* methods for assimilating new information to produce up-to-date maps incorporating all available data sets. It is the need for efficiency that motivated our research, which had as its goals the development of recursive procedures for updating, combining, and centralization.

The basis for our approach comes from viewing a map as an estimate, that is, as the projection of a random quantity onto the space spanned by a set of measurements. At this abstract level the solutions to our mapping problems are relatively clear. For example, in the map updating problem our objective is to compute the projection onto the space spanned by the old and new measurements. What we would like to do, however, is to compute this estimate explicitly in terms of the projection onto the old data and the new measurements. As we will explain more precisely in Section III, we can achieve our goal by projecting the error in our old map onto the space spanned by the new information available from the more recent measurement survey. In a similar fashion one can view the other two estimation problems in terms of appropriate projections. The crucial problem then is to find efficient methods for computing these projections. As we will discuss, the key to solving this problem is the construction of a model for the error in a given map.

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While the motivation for the problems we have described involves random fields, that is, random functions of several independent variables, the solution of these problems has not even been considered (except in [1]) in the case of a single independent variable. Consequently, in this paper we concentrate on this latter case in order to develop an understanding of mapping problems in a context about which a great deal is known already. In addition, as we illustrate in Section IV, there are two-dimensional problems of practical importance that can be solved using the results presented in this paper. Furthermore, as we develop in a companion paper [13], the results developed here can be extended to allow us to solve problems with far less restrictive geometries than that considered in Section IV.

In one dimension the problem of mapping is simply one of computing smoothed estimates. From the geometric perspective described previously, we see that in this context what is important is the construction of dynamic models for smoothing errors. Several examples of such models can be found in the literature. Perhaps the earliest of these is in [15, pp. 221–226], in which a discrete-time formulation is considered and a reversed-time model of dimension $2n$ (where $n$ is the dimension of the state) is constructed for the smoothing error process as an intermediate step in determining expressions for smoothing error covariances. A reversed-time model of dimension $n$ was constructed in [4] and [16] as part of a study of the relationship between the theories of smoothing and stochastic realization, while a forward model of dimension $n$ for smoothing errors can be obtained from results in [14] that are used in the development of new fixed-interval smoothing formulas. In Section II we present a simple direct derivation of an $n$-dimensional Markovian model for smoothing errors. An alternate derivation based on innovations is presented in the Appendix. Both of these are new. In Section III, we solve the one-dimensional map updating problem in a simpler and more illuminating way than in [1]. In particular, using our smoothing error model, we are able to reduce the map updating problem to a smoothing problem. This immediately provides numerous alternative algorithms for map updating of which that in [1] is but one example. In Section IV we apply the result of the previous section to solve a map updating problem for a two-dimensional random field given parallel track survey data, and in Section V we present the solutions to the map combining and centralization problems. Section VI contains conclusions and some comments on more general mapping problems.

## II. Smoothing Formulas and a Markov Model for the Smoothing Error Process

The starting point for our investigation is a conventional model for a finite-dimensional zero-mean Gauss–Markov process in one dimension:

$$dx(t) = A(t)x(t)\,dt + dw(t), \qquad 0 \le t \le T, \quad (2.1)$$

where $w(t)$ is a vector Wiener process independent of $x(0)$, with

$$E[dw(t)\,dw'(t)] = Q(t)\,dt. \quad (2.2)$$

We denote by $\pi(0)$ the covariance of $x(0)$. We assume that we have available a set of measurements

$$dy(t) = H(t)x(t)\,dt + dv(t), \qquad 0 \le t \le T, \quad (2.3)$$

where $v(t)$ is a Wiener process, independent of $x(\cdot)$, with

$$E[dv(t)\,dv'(t)] = R(t)\,dt \quad (2.4)$$

and $R(t) > 0$ for all $t$. In the one-dimensional map updating problem considered in Section III we will have two sets of measurements as in (2.3), and we are interested in updating our estimate of $x(\cdot)$ based on the first set, given the new information in the second set. As a prelude to this, we first review some basic smoothing formulas for the model (2.1)–(2.4) for future reference, and we then derive forward- and reverse-time models for the error in the smoothed estimate of $x(\cdot)$, given the measurements (2.3).

### A. Basic Fixed-Interval Smoothing Formulas

In this subsection we establish some notation and briefly summarize formulas from [8], [10], [17], [18]. The covariance $\pi(t)$ of $x(t)$ in (2.2) satisfies the equation[1]

$$\dot{\pi} = A\pi + \pi A' + Q. \quad (2.5)$$

Define the three $\sigma$ fields

$$Y_t^- = \sigma\{dy(\tau), \quad 0 \le \tau \le t\} \quad (2.6a)$$
$$Y_t^+ = \sigma\{dy(\tau), \quad t \le \tau \le T\} \quad (2.6b)$$
$$Y = \sigma\{dy(\tau), \quad 0 \le \tau \le T\}. \quad (2.6c)$$

The forward-filtered, reverse-filtered, and smoothed estimates of $x(t)$ are then, respectively,

$$\hat{x}_f(t) = E[x(t)|Y_t^-] \quad (2.7a)$$
$$\hat{x}_r(t) = E[x(t)|Y_t^+] \quad (2.7b)$$
$$\hat{x}_s(t) = E[x(t)|Y]. \quad (2.7c)$$

The covariances of the errors (denoted by $\tilde{x}_f(t)$, $\tilde{x}_r(t)$, and $\tilde{x}_s(t)$) in these estimates are denoted by $P_f(t)$, $P_r(t)$, and $P_s(t)$, respectively.

The calculation of $\hat{x}_f$ and $P_f$ may be done by standard Kalman filtering equations:

$$d\hat{x}_f = A\hat{x}_f\,dt + P_f H'R^{-1}(dy - H\hat{x}_f\,dt)$$
$$\hat{x}_f(0) = 0 \quad (2.8)$$
$$\dot{P}_f = AP_f + P_f A' + Q - P_f H'R^{-1}HP_f$$
$$P_f(0) = \pi(0). \quad (2.9)$$

Similarly, $\hat{x}_r$ and $P_r$ satisfy analogous reverse-time Kal-

[1] Whenever possible we will suppress the time dependence of system matrices, state matrices, and error covariance matrices.

man filter equations

$$-d\hat{x}_r = -(A + Q\pi^{-1})\hat{x}_r\,dt + P_rH'R^{-1}(dy - H\hat{x}_r\,dt)$$
$$\hat{x}_r(T) = 0 \tag{2.10}$$
$$-\dot{P}_r = -(A + Q\pi^{-1})P_r - P_r(A' + \pi^{-1}Q) + Q - P_rH'R^{-1}HP_r$$
$$P_r(T) = \pi(T). \tag{2.11}$$

The smoothed estimate and its error covariance can be expressed as follows in terms of the quantities defined earlier:

$$\hat{x}_s(t) = P_s(t)\big(P_f^{-1}(t)\hat{x}_f(t) + P_r^{-1}(t)\hat{x}_r(t)\big) \tag{2.12}$$
$$P_s^{-1}(t) = P_f^{-1}(t) + P_r^{-1}(t) - \pi^{-1}(t). \tag{2.13}$$

In the sequel several identities such as (2.13) will arise. Thus it is worthwhile to provide some intuition which explains (2.13) rather simply and which also can be used in interpreting similar identities in this paper. Specifically, we interpret the inverse of an error covariance as the information contained in the corresponding estimate. Thus $\pi^{-1}$ is a measure of our *a priori* information concerning $x(t)$ (here the *a priori* estimate is the *a priori* mean = 0), while $P_f^{-1}$, $P_r^{-1}$, and $P_s^{-1}$ are measures of the *a priori* information *together* with the information provided, respectively, by the past, future, and entire history of the observations. Thus (2.13) says only that the total information available concerning $x(t)$ consists of the *a priori* information plus the past ($P_f^{-1}$) plus the *a priori* plus the future ($P_r^{-1}$) minus the *a priori* ($\pi^{-1}$). Alternatively, if we define the information matrix $0_0$

$$0_0(t) = P_r^{-1}(t) - \pi^{-1}(t), \tag{2.14}$$

which represents the information in the future measurements excluding *a priori* information, then

$$P_s^{-1}(t) = P_f(t)^{-1} + 0_0(t). \tag{2.15}$$

### *B. Markov Model for the Smoothing Error Process*

In this subsection we derive a Markov model for the evolution of $\tilde{x}_s(t) = x(t) - \hat{x}_s(t)$. Our approach uses martingale decompositions, and its appeal is in its simplicity and in the straightforward interpretation that can be given to each term in the resulting realization. To highlight these features, we will present our derivation informally. The validity of our calculations is easily verified using elementary properties of martingales and their decompositions [5], [6].

To begin, if we formally take the conditional expectation $E[\cdot|Y]$ on both sides of (2.1), we obtain an equation for the evolution of $\hat{x}_s(\cdot)$, which when subtracted from (2.1) yields the following evolution equation for $\tilde{x}_s(\cdot)$:

$$d\tilde{x}_s(t) = A(t)\tilde{x}_s(t)\,dt + d\tilde{w}(t) \tag{2.16}$$

where

$$d\tilde{w}(t) = dw(t) - E[dw(t)|Y]. \tag{2.17}$$

The relation (2.16) does not represent a Markovian realization of the smoothing error process, since $\tilde{w}(\cdot)$ is not a Wiener process and since $\tilde{w}(\cdot)$ is not independent of the initial condition of $\tilde{x}_s(\cdot)$, that is, $\sigma\{d\tilde{w}(\tau)\ 0 \le t \le T\}$ is not independent of either $\sigma\{\tilde{x}_s(0)\}$ or $\sigma\{\tilde{x}_s(T)\}$. Our method for deriving forward and reverse Markovian realizations of the smoothing error process is to decompose $dw(t)$ with respect to $\sigma$ fields associated with future and past values of the smoothing error process. Specifically, let

$$\tilde{X}_t^- = \sigma\{\tilde{x}_s(\tau)\ 0 \le \tau \le t\} \tag{2.18}$$

and

$$\tilde{X}_t^+ = \sigma\{\tilde{x}_s(\tau)\ t \le \tau \le T\}. \tag{2.19}$$

Now, we define two families of $\sigma$ fields, $F_t$ (forward) and $B_t$ (backward), by

$$F_t = \sigma\{\tilde{X}_t^-, Y\} \tag{2.20}$$

and

$$B_t = \sigma\{\tilde{X}_t^+, Y\}. \tag{2.21}$$

We may easily verify that the past increments of $w(t)$, that is, $\{dw(\tau), \tau < t\}$, are measurable with respect to $F_t$ and that the future increments $\{dw(\tau), \tau > t\}$ are measurable with respect to $B_t$. Therefore, we can perform two martingale decompositions of $dw(t)$, one going forward and one going backward, as follows:[2]

$$dw(t) = E[dw(t)|F_t] + d\tilde{w}_f(t) \tag{2.22}$$
$$dw(t) = E[dw(t)|B_t] + d\tilde{w}_r(t) \tag{2.23}$$

where the forward increment $d\tilde{w}_f(t)$ is independent of $F_t$ and the reverse-time increment $d\tilde{w}_r(t)$ is independent of $B_t$.

Substituting (2.22) and (2.23) into (2.17) and (2.16), we immediately obtain reverse and forward Markovian realizations for the smoothing error process in which $\tilde{w}_f(\cdot)$ and $\tilde{w}_r(\cdot)$ are the input processes for the forward- and reverse-time realizations of $\tilde{x}_s(\cdot)$. (Note also that by construction these noises will be independent, respectively, of $\tilde{x}_s(0)$ and $\tilde{x}_s(T)$.)

Since $Y$ is independent of both $\tilde{X}_t^-$ and $\tilde{X}_t^+$, the identities (2.22) and (2.23) may be expressed as

$$dw(t) = E[dw(t)|\tilde{X}_t^-] + E[dw(t)|Y] + d\tilde{w}_f(t) \tag{2.24}$$

and

$$dw(t) = E[dw(t)|\tilde{X}_t^+] + E[dw(t)|Y] + d\tilde{w}_r(t). \tag{2.25}$$

Note that in (2.24) and (2.25) we have made use of the fact that if $z_1, z_2, z_3$ are jointly Gaussian, *zero-mean* random vectors such that $z_2$ and $z_3$ are uncorrelated, then

[2] To be precise, what we must verify is that the integrated increments from 0 to $t$ and from $t$ to $T$ are quasi-martingales with respect to $F_t$ and $B_t$, respectively. According to [6], to verify this we need only check that they are adapted to these $\sigma$ fields. This is guaranteed if we note that

$$dw(t) = d\tilde{x}_s(t) - A(t)\tilde{x}_s(t)\,dt + E[dw(t)|Y],$$

since this shows that these integrated increments depend only on either $[\tilde{X}_t^-, Y]$ or $[\tilde{X}_t^+, Y]$, respectively.

$E(z_1|z_2, z_3) = E(z_1|z_2) + E(z_1|z_3)$. Using the fact [5] that the martingale parts of a given quasi-martingale decomposed with respect to different $\sigma$ fields have the same quadratic variation, we conclude that $d\tilde{w}_f(t)$ and $d\tilde{w}_r(t)$ are Wiener processes with

$$E\left[d\tilde{w}_f(t)\, d\tilde{w}_f'(t)\right] = E\left[d\tilde{w}_r(t)\, d\tilde{w}_r'(t)\right] = Q(t)\, dt. \tag{2.26}$$

Using (2.16), (2.17), (2.24), and (2.25), we obtain the following equations for the evolution of the smoothing error process:

$$d\tilde{x}_s(t) = A(t)\tilde{x}_s(t)\, dt + E\left[dw(t)|\tilde{X}_t^-\right] + d\tilde{w}_f(t) \tag{2.27}$$

and

$$-d\tilde{x}_s(t) = -A(t)\tilde{x}_s(t)\, dt - E\left[dw(t)|\tilde{X}_t^+\right] - d\tilde{w}_r(t) \tag{2.28}$$

where (2.27) is to be interpreted as a forward model and (2.28) as a reverse-time model. From (2.27) and (2.28) it is apparent that

$$d\tilde{w}_f(t) = d\tilde{x}_s(t) - E\left[d\tilde{x}_s(t)|\tilde{X}_t^-\right] \tag{2.29}$$

and that

$$-d\tilde{w}_r(t) = -d\tilde{x}_s(t) - E\left[-d\tilde{x}_s(t)|\tilde{X}_t^+\right]. \tag{2.30}$$

Equations (2.29) and (2.30) can be interpreted to mean that $d\tilde{w}_f(t)$, $(-d\tilde{w}_r(t))$ represent the components of $d\tilde{x}_s(t)$, $(-d\tilde{x}_s(t))$, which are *not predictable* from $\tilde{X}_t^-$, $(\tilde{X}_t^+)$, respectively.

From (2.27) and (2.28) we see that to complete the specification of our forward- and reverse-time Markovian models for $\tilde{x}_s(\cdot)$, we need to compute the means for the process noise, $dw(t)$, conditioned on the $\sigma$ fields spanned by past and future values of the smoothing error process, respectively. To do this, we rely on results in the literature (see, for example, [10] and [17]) to write an explicit representation for the smoothing error process. Specifically,

$$\tilde{x}_s(t) = \tilde{x}_f(t) - P_f(t)\int_{s=t}^{T} \Phi_\Gamma'(s,t)H'(s)R^{-1}(s)\, d\nu(s) \tag{2.31}$$

where $d\nu(t)$ is the forward innovations process

$$d\nu(t) = dy(t) - H(t)\hat{x}_f(t)\, dt \tag{2.32}$$

and $\Phi_\Gamma(\cdot,\cdot)$ is the transition matrix associated with the forward filtering dynamics matrix

$$\Gamma(t) = A(t) - P_f(t)H'(t)R^{-1}(t)H(t). \tag{2.33}$$

We may verify from (2.31) that

$$E\left[dw(t)|\tilde{X}_t^-\right] = Q(t)\left(P_f^{-1}(t) - P_s^{-1}(t)\right)\tilde{x}_s(t)\, dt \tag{2.34}$$

and

$$E\left[dw(t)|\tilde{X}_t^+\right] = Q(t)P_f^{-1}(t)\tilde{x}_s(t)\, dt, \tag{2.35}$$

where $P_s(t)$ denotes the smoothing error covariance matrix. We do this by checking the orthogonality conditions

$$E\left[\left(dw(t) - Q(t)\left(P_f^{-1}(t) - P_s^{-1}(t)\right)\tilde{x}_s(t)\, dt\right)\tilde{x}_s'(\tau)\right] = 0 \tag{2.36}$$

for $0 \le \tau \le t$ and

$$E\left[\left(dw(t) - Q(t)P_f^{-1}(t)\tilde{x}_s(t)\, dt\right)\tilde{x}_s'(\tau)\right] = 0 \tag{2.37}$$

for $t \le \tau \le T$. Therefore, by substituting (2.34) and (2.35) into (2.27) and (2.28), respectively, we obtain the following forward- and reverse-time models for the smoothing error process:

$$d\tilde{x}_s(t) = \left[A(t) + Q(t)\left(P_f^{-1}(t) - P_s^{-1}(t)\right)\right] \cdot \tilde{x}_s(t)\, dt + d\tilde{w}_f(t) \tag{2.38}$$

and

$$-d\tilde{x}_s(t) = -\left[A(t) + Q(t)P_f^{-1}(t)\right]\tilde{x}_s(t)\, dt - d\tilde{w}_r(t). \tag{2.39}$$

In the Appendix we provide an alternate derivation of the reverse-time model (2.39) starting from the representation of $\tilde{x}_s(t)$ in (2.29). The derivation in the Appendix is related to that of Badawi and Lindquist in [4], in which they obtain a reverse-time realization for $P_f(t)\tilde{x}_s(t)$. In addition, the forward model (2.38) is equivalent to that of Weinert and Desai in [14] (see [7] for some additional discussion), although, as in the case of (2.39), the derivation presented in this section is new.

## III. The Map Updating Problem

### A. Formulation and Preliminary Calculations

In this section we examine the problem of map updating for a class of processes with a single independent variable. Specifically, we consider the problem of computing the smoothed estimate of such a process over a fixed interval $(0, T)$, given two data "passes," that is, two sets of measurements. The term updating signifies that we are looking for algorithms which compute such an estimate in terms of the second pass of data and the estimate of the process based on the first pass only. That is, we wish to produce a new map based upon the new data and the old map. Our solution technique makes use of Hilbert space decompositions of the space spanned by the two passes of data, as well as results on Markovian realizations of the smoothing error process derived in Section II. As mentioned in the Introduction, this approach allows us to expose the nature of the map updating problem, and this in turn allows us to derive a variety of alternative algorithms with relative ease. In this section we present two alternative two-filter algorithms for solving the map updating problem, which we call the *smoothing error filter algorithm* and *information filter algorithm*, respectively. The second of these is derived directly from the former and is included explicitly, as it ultimately serves to facilitate the solution of the map combining problem in Section V.

Consider the model (2.1), (2.2), and suppose that two measurement passes have been made, with mutually independent measurement noises, and are modeled by

$$dy_1(t) = H_1 x(t)\, dt + dv_1(t) \tag{3.1}$$

and

$$dy_2(t) = H_2 x(t)\, dt + dv_2(t) \tag{3.2}$$

where

$$E[dv_1(t)\, dv_1'(t)] = R_1\, dt \tag{3.3}$$

and

$$E[dv_2(t)\, dv_2'(t)] = R_2\, dt. \tag{3.4}$$

We assume that for all $t$, $R_1(t)$, $R_2(t) > 0$. Let $Y_i$, $i = 1, 2$, be defined as the Hilbert spaces of zero-mean finite variance random variables spanned by the individual measurement passes, that is,

$$\begin{aligned} Y_i &\triangleq H(dy_i(\tau)\, 0 \le \tau \le T) \\ &\equiv H(y_i(\tau_1) - y_i(\tau_2)\colon \tau_1, \tau_2 \in [0, T]). \end{aligned} \tag{3.5}$$

In addition, let $Y$ denote the space spanned by both data passes, denoted

$$Y = Y_1 \vee Y_2, \tag{3.6}$$

and define the smoothed estimates $\hat{x}_{is}(t)$ corresponding to the maps derived from each data pass separately as

$$\hat{x}_{is}(t) = E[x(t)|Y_i] \tag{3.7}$$

and the smoothed estimate $\hat{x}_s(t)$ corresponding to the aggregate two-data-pass map as

$$\hat{x}_s(t) = E[x(t)|Y]. \tag{3.8}$$

Let $\tilde{x}_{is}(t)$ and $\tilde{x}_s(t)$ denote the smoothing errors corresponding to the estimates (3.7) and (3.8). Let $P_{is}$, $P_s$ denote the corresponding smoothing error covariance matrices. The map updating problem is specified as that of computing $\hat{x}_s(\cdot)$ as a linear functional of $\hat{x}_{1s}(\cdot)$ and the second pass data $y_2(\cdot)$.

What makes map updating somewhat complicated (and interesting) is that the two data passes are not independent, that is, $Y_1$ and $Y_2$ are not orthogonal. For this reason, the projection of $x(t)$ onto $Y$, that is, $\hat{x}_s(t)$, is not simply obtained by adding $\hat{x}_{1s}(t)$ to $\hat{x}_{2s}(t)$. To determine what *should* be added to $\hat{x}_{1s}(t)$ to update the map, consider the following orthogonal decomposition of $Y$:

$$Y = Y_1 \oplus \tilde{Y}_2 \tag{3.9}$$

where

$$\tilde{Y}_2 \triangleq H(d\tilde{y}_2(\tau)\, 0 \le \tau \le T) \tag{3.10}$$

and

$$d\tilde{y}_2(\tau) = dy_2(\tau) - E[dy_2(\tau)|Y_1]. \tag{3.11}$$

The space $\tilde{Y}_2$ denotes the part of the second-pass data space $Y_2$ that is not predictable from the first-pass data $Y_1$. By using the independence of the measurement noises, we can express $d\tilde{y}_2(\tau)$ as

$$d\tilde{y}_2(\tau) = dy_2(\tau) - H_2 \hat{x}_{1s}(\tau)\, d\tau \tag{3.12}$$

or, alternatively, as

$$d\tilde{y}_2(\tau) = H_2 \tilde{x}_{1s}(\tau)\, d\tau + dv_2(\tau) \tag{3.13}$$

where it should be noted that $\tilde{x}_{1s}(\cdot)$ and $dv_2(\cdot)$ are independent. Using the orthogonality of $Y_1$ and $\tilde{Y}_2$, we immediately have that

$$\hat{x}_s(t) = \hat{x}_{1s}(t) + E[x(t)|\tilde{Y}_2]. \tag{3.14}$$

Note, however, that $x(t) = \hat{x}_{1s}(t) + \tilde{x}_{1s}(t)$. From (3.13) and the fact that $\hat{x}_{1s}(\cdot)$ is orthogonal to $\tilde{x}_{1s}(\cdot)$ and $v_2(\cdot)$, we find that

$$\hat{x}_s(t) = \hat{x}_{1s}(t) + E[\tilde{x}_{1s}(t)|\tilde{Y}_2]. \tag{3.15}$$

Equation (3.15) represents the two-pass map as the sum of the first-pass map plus a correction term corresponding to an estimate of the first-pass map error based on new information in the second-pass data. Since from (3.12), $d\tilde{y}_2(\cdot)$ can be expressed in terms of $\hat{x}_{1s}(\cdot)$ and $dy_2(\cdot)$, (3.15) represents the solution to the map updating problem in a still somewhat abstract form. In the next two subsections we present algorithms which implement the solution suggested by (3.15) and which utilize the smoothing error Markov model results of Section II.

### B. The Smoothing Error Filter Algorithm for Map Updating

In the preceding subsection we saw that the map updating problem reduces to the problem of computing the estimate of $\tilde{x}_{1s}(t)$ based on $\tilde{Y}_2$. From the results of Section II, however, we know that it is possible to write forward and reverse Markovian realizations of $\tilde{x}_{1s}(t)$ as in (2.38) and (2.39). In these realizations $\tilde{x}_s$, $P_f$, $P_s$, $\tilde{w}_f$, and $\tilde{w}_r$ are replaced by $\tilde{x}_{1s}$, $P_{1f}$, $P_{1s}$, $\tilde{w}_{1f}$, and $\tilde{w}_{1r}$, respectively, where the "1" denotes that these are quantities based on considering the first pass of data alone. For example, $P_{1f}$ and $P_{1r}$ satisfy an equation like (2.9) and (2.11) with $H$ replaced by $H_1$ and $R$ by $R_1$.

Given these Markov models for $\tilde{x}_{1s}$ and the fact that from (3.13) $d\tilde{y}_2$ can be viewed as a noise-corrupted measurement of $\tilde{x}_{1s}$, we see that the computation of $E[\tilde{x}_{1s}(t)|\tilde{Y}_2]$ is a standard smoothing problem, and consequently any of the existing solutions of the linear fixed interval smoothing problem may be used. In particular, let us make use of the two-filter smoothing algorithm described in Section II-A. Let $\hat{\tilde{x}}_f(t)$ and $\hat{\tilde{x}}_r(t)$ denote, respectively, the forward and reverse filtered estimates of $\tilde{x}_{1s}(t)$ based on $\tilde{y}_2$, and let $P_{fs}$ and $P_{rs}$ denote the error covariances matrices corresponding to these estimates. Then, by using the results in Section II-A applied to the models for $\tilde{x}_{1s}$ in (2.38) and (2.39), we obtain the following equations for the evolution of $\hat{\tilde{x}}_f$, $P_{fs}$, $\hat{\tilde{x}}_r$, and $P_{rs}$:

$$\boxed{\begin{aligned} d\hat{\tilde{x}}_f(t) &= \left(A + Q\left(P_{1f}^{-1} - P_{1s}^{-1}\right) - P_{fs} H_2' R_2^{-2} H_2\right)\hat{\tilde{x}}_f(t)\, dt \\ &\quad + P_{fs} H_2' R_2^{-1}\, d\tilde{y}_2(t) \end{aligned}} \tag{3.16}$$ [3]

[3] We have placed boxes around those equations which together form the required *on-line* computations.

with

$$\boxed{\hat{\tilde{x}}_f(0) = 0} \tag{3.17}$$

$$\dot{P}_{fs} = \left(A + Q\left(P_{1f}^{-1} - P_{1s}^{-1}\right)\right)P_{fs} + P_{fs}\left(A + Q\left(P_{1f}^{-1} - P_{1s}^{-1}\right)\right)' + Q - P_{fs}H_2'R_2^{-1}H_2P_{fs} \tag{3.18}$$

with

$$P_{fs}(0) = P_{1s}(0) \tag{3.19}$$

$$\boxed{-d\hat{\tilde{x}}_r(t) = \left(-A - QP_{1f}^{-1} - P_{rs}H_2'R_2^{-1}H_2\right)\hat{\tilde{x}}_r(t)\,dt + P_{rs}H_2'R_2^{-1}\,d\tilde{y}_2(t)} \tag{3.20}$$

with

$$\boxed{\hat{\tilde{x}}_r(T) = 0} \tag{3.21}$$

and

$$-\dot{P}_{rs} = \left(-A - QP_{1f}^{-1}\right)P_{rs} + P_{rs}\left(-A - QP_{1f}^{-1}\right)' + Q - P_{rs}H_2'R_2^{-1}H_2P_{rs} \tag{3.22}$$

with

$$P_{rs}(T) = P_{1s}(T). \tag{3.23}$$

Specifically, the forward filtering equations (3.16), and (3.18) for the estimate $\hat{\tilde{x}}_f(t)$ and error covariance $P_{fs}$, are obtained by applying the standard forward Kalman filtering equations (2.8) and (2.9) to the forward first-pass smoothing error model (2.38), with the measurements (3.13). In a similar manner, the backward filtering equations (3.20), and (3.22) for $\hat{\tilde{x}}_r(t)$ and $P_{rs}$, follow from applying the backward Kalman filtering equations (2.10) and (2.11) to the backward first-pass smoothing error model (2.39), with measurements (3.13).

The estimates computed in (3.16) and (3.20) can be combined to produce the smoothed estimate of $\tilde{x}_{1s}(\cdot)$, given $\tilde{y}_2(\cdot)$. The only additional quantity needed to do this is the covariance of the error in this smoothed estimate. Note that this error represents the only remaining uncertainty in $x(t)$, given the two-data-pass space $Y$. It is given by

$$\tilde{x}_s(t) = \tilde{x}_{1s}(t) - E\left[\tilde{x}_{1s}(t)|\tilde{Y}_2\right], \tag{3.24}$$

as can be derived from (3.14). Thus the covariance of the right side of (3.24) is simply what we have referred to earlier in this section as $P_s$, and adapting (2.13) to our current problem, we can express $P_s^{-1}$ as

$$P_s^{-1} = P_{fs}^{-1} + P_{rs}^{-1} - P_{1s}^{-1}. \tag{3.25}$$

Finally, $E[\tilde{x}_{1s}(t)|\tilde{Y}_2]$ can be expressed as

$$\boxed{E\left(\tilde{x}_{1s}(t)|\tilde{Y}_2\right) = P_s\left(P_{fs}^{-1}\hat{\tilde{x}}_f(t) + P_{rs}^{-1}\hat{\tilde{x}}_r(t)\right).} \tag{3.26}$$

In summary, we have the following map updating algorithm for computing $\hat{x}_s$ in terms of $\hat{x}_{1s}$ and $y_2$. a) We first compute $\tilde{y}_2$, the new information in the second pass, from (3.12). b) Then $\hat{\tilde{x}}_f$ and $\hat{\tilde{x}}_r$ are computed from (3.15) and (3.20) (with initial conditions (3.17) and (3.21)). c) $\hat{x}_s$ is computed from (3.15) by adding (3.26) to $\hat{x}_{1s}$. The quantities that must be calculated off-line from the original model of (2.1), (2.2), and (3.1)–(3.4) are the covariance $P_s$ of the error in $\hat{x}_s$ (from (3.25)), $P_{fs}$ (from (3.18), (3.19)), $P_{rs}$ (from (3.22), (3.23)), $P_{1f}$, and $P_{1s}$ (these last two quantities would have been needed previously in the original processing of the first pass to compute $\hat{x}_{1s}$).

### C. Information-Filter Algorithm for Map Updating

In this section we describe an alternative map updating algorithm that will aid us later in deriving a solution to the map combining problem. We begin by defining $\gamma_f(t)$ and $\gamma_r(t)$ as

$$\gamma_f(t) = P_{fs}^{-1}\hat{\tilde{x}}_f(t) \tag{3.27}$$

$$\gamma_r(t) = P_{rs}^{-1}\hat{\tilde{x}}_r(t). \tag{3.28}$$

By using (3.16)–(3.23), we can derive the following stochastic differential equations for the evolution of $\gamma_f(\cdot)$ and $\gamma_r(\cdot)$:

$$d\gamma_f(t) = \left(-A' - P_{1f}^{-1} + P_{fs}^{-1} - P_{1s}^{-1}Q\gamma_f(t)\right)dt + H_2'R_2^{-1}\,d\tilde{y}_2(t) \tag{3.29}$$

with

$$\gamma_f(0) = 0 \tag{3.30}$$

and

$$-d\gamma_r(t) = \left(A' + P_{1f}^{-1} - P_{rs}^{-1}Q\gamma_r(t)\right)dt + H_2'R_2^{-1}\,d\tilde{y}_2(t) \tag{3.31}$$

with

$$\gamma_r(T) = 0. \tag{3.32}$$

Then, by substituting (3.27) and (3.28) into (3.26), we obtain the following relation:

$$\boxed{E\left[\tilde{x}_{1s}(t)|\tilde{Y}_2\right] = P_s\left(\gamma_f(t) + \gamma_r(t)\right).} \tag{3.33}$$

As a final note, we point out that relations (3.29) and (3.30) can be expressed more simply as

$$d\gamma_f(t) = \left(-A' - P_f^{-1}Q\right)\gamma_f(t)\,dt + H_2'R_2^{-1}\,d\tilde{y}_2(t) \tag{3.34}$$

$$-d\gamma_r(t) = \left(A' - 0_0Q\right)\gamma_r(t)\,dt + H_2'R_2^{-1}\,d\tilde{y}_2(t) \tag{3.35}$$

where $P_f$ is the forward filtered error covariance based on both passes and $0_0(t)$ is obtained from (2.14) in which $P_r(t)$ is the reverse filtered error covariance based on both passes. Note that in this case $P_f$ and $P_r$ satisfy (2.9) and

(2.11) with

$$H = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} \qquad R = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix}. \tag{3.36}$$

Furthermore, from (2.5), (2.11), (2.14), and a bit of algebra we can obtain the following equation for the direct computation of $0_0(t)$:

$$-\frac{d}{dt}0_0 = A'0_0 + 0_0A - 0_0Q0_0 + H'R^{-1}H$$
$$0_0(T) = 0 \tag{3.37}$$

where $H$ and $R$ are as in (3.36).

Equations (3.34) and (3.35) are immediate consequences of the following two identities:

$$P_f^{-1} = P_{1f}^{-1} + P_{fs}^{-1} - P_{1s}^{-1} \tag{3.38}$$

and

$$0_0 = P_{rs}^{-1} - P_{1f}^{-1}. \tag{3.39}$$

The first of these identities may be interpreted in the following manner. The quantity $P_f^{-1}$ represents the information (concerning $x(t)$) contained in the past of both data passes, together with *a priori* information. The terms on the right side represent a) the information contained in the past of the first pass and the *a priori* information, b) the information in the entire first pass together with that in the past of the second pass and the *a priori* information, c) the information in the entire first pass together with the *a priori* information. Equation (3.39) can be interpreted in a similar manner.

At this point, we note that while we have explicitly defined two algorithms for the solution of the map updating problem, the most important aspect of our approach is that, by recasting the problem as one of computing the smoothed estimate of a process, $\tilde{x}_{1s}(t)$, described by a causal state-space model, we immediately have at our disposal the plethora of algorithms that have been developed for such problems. Displaying this fundamental structure of the map updating problem is a major improvement over the previous algebraic manipulations of [1].

## IV. Updating of Two-Dimensional Maps Using Data Obtained Along Parallel Tracks

In this section we illustrate the applicability of the results of the preceding section to a problem of updating the map of a two-dimensional random field, given measurements along sets of parallel tracks (we will also use this same example in Section V-B). Such a problem can arise in the production of maps of gravitational anomalies, given measurements taken on survey ships traversing sets of straight-line paths.

Let $F(t, s)$ be a stationary zero-mean two-dimensional scalar random field with separable correlation function

$$R(\tau_1, \tau_2) = E[F(\tau_1, \tau_2)F(0,0)] = \phi(\tau_1)\psi(\tau_2), \tag{4.1}$$

where $\phi(\tau)$ and $\psi(\tau)$ are assumed to be one-dimensional correlation functions for the outputs of finite-dimensional linear systems. Such models have been used by several authors. For example, Powell and Silverman [11] have assumed correlation models of the form in (4.1) to model scalar image intensity random fields. Let us assume that it is desired to map the field along a set of trajectories $z_i(t)$ defined by[4]

$$z_i(t) = F(t, s_i), \tag{4.2}$$

for $0 \le t \le T$ and $i = 1 \cdots M$, and that for each survey, measurements are obtained along some subset of the $M$ tracks. Defining the $M$-track field vector $Z(t)$ by

$$Z(t) = \begin{pmatrix} z_1(t) \\ \vdots \\ z_M(t) \end{pmatrix}, \tag{4.3}$$

we show in the following that $Z(\cdot)$ has a finite-dimensional Markovian representation. We then formulate the map updating problem, relying on the results of Section III for its solution.

From relations (4.2) and (4.3) we can show that the one-dimensional correlation function associated with the aggregate $M$-track field vector process $Z(t)$ assumes the form

$$R_Z(\tau) = \begin{pmatrix} \phi(\tau) & & 0 \\ & \ddots & \\ 0 & & \phi(\tau) \end{pmatrix} \Psi, \tag{4.4}$$

where $\Psi$ is an $M \times M$ matrix with the $i, j$th element $\Psi_{ij}$ defined by

$$\Psi_{ij} \triangleq \psi(s_i - s_j). \tag{4.5}$$

Since $\Psi$ is a symmetric positive semidefinite matrix, it may be represented as a product of lower and upper triangular factors as

$$\Psi = LL', \tag{4.6}$$

where

$$L \triangleq \begin{pmatrix} l_{1,1} & & 0 \\ \vdots & \ddots & \\ l_{M,1} & \cdots & l_{M,M} \end{pmatrix}. \tag{4.7}$$

Hence by employing (4.5) and noting that $L$ commutes with the diagonal matrix whose entries are specified by $\phi(\tau)$, we may express the correlation function $R_Z(\tau)$ for the aggregate $M$-track field vector process $Z(t)$ as

$$R_Z(\tau) = L \begin{pmatrix} \phi(\tau) & & 0 \\ & \ddots & \\ 0 & & \phi(\tau) \end{pmatrix} L'. \tag{4.8}$$

[4] This amounts to the assumption of constant velocity as the survey sensors traverse the field. This is not an essential assumption and can be removed using the results in [12].

We next use our assumption that $\phi(\cdot)$ corresponds to a correlation function generated by a finite-dimensional system, and we use (4.8) to determine our desired Markovian representation for $Z(\cdot)$. Letting $\mathscr{L}\{\cdot\}$ denote the bilateral Laplace transform, we define $\Phi(s)$ as

$$\Phi(s) = \mathscr{L}\{\phi(\cdot)\}. \tag{4.9}$$

Assuming that $\Phi(s)$ is strictly proper and rational, that is, that $\Phi(\infty) = 0$, we can determine a strictly proper rational spectral factor $h(s)$ with no poles in the right half-plane [19, p. 173] such that

$$\Phi(s) = h(s)h(-s). \tag{4.10}$$

For the purpose of realizing $Z(t)$ we will need $M$ copies of an irreducible state-space realization for $h(s)$ [19, pp. 105–114] of the form

$$d\xi_i(t) = F\xi_i(t)\,dt + G\,dw_i(t) \tag{4.11}$$

$$r_i(t) = h\xi_i(t), \tag{4.12}$$

for $i = 1, \cdots, M$, where $w_i(\cdot)$ is a standard scalar Brownian motion process and

$$E[\xi_i(0)\xi_i'(0)] = \Xi \tag{4.13}$$

with $\Xi$ being the unique positive definite solution to the equation

$$F\Xi + \Xi F' + GG' = 0. \tag{4.14}$$

Finally, we employ (4.11)–(4.14) and (4.8) to obtain a Markovian representation for the aggregate field vector process $Z(t)$ of the form

$$Z(t) = Hx(t) \tag{4.15}$$

with

$$dx(t) = Ax(t)\,dt + B\,du(t) \tag{4.16}$$

where

$$x(t) \triangleq \begin{pmatrix} \xi_1(t) \\ \vdots \\ \xi_M(t) \end{pmatrix} \tag{4.17}$$

$$du(t) \triangleq \begin{pmatrix} dw_1(t) \\ \vdots \\ dw_M(t) \end{pmatrix} \tag{4.18}$$

$$A \triangleq \begin{pmatrix} F & & 0 \\ & \ddots & \\ 0 & & F \end{pmatrix} \tag{4.19}$$

$$B \triangleq \begin{pmatrix} G & & 0 \\ & \ddots & \\ 0 & & G \end{pmatrix} \tag{4.20}$$

$$H = \begin{pmatrix} l_{1,1}h & & & \\ \vdots & \ddots & & \\ l_{M,1}h & \cdots & & l_{M,M}h \end{pmatrix} \tag{4.21}$$

and

$$E[x(0)x'(0)] = \pi(0) = \begin{pmatrix} \Xi & & & 0 \\ & \Xi & & \\ & & \ddots & \\ 0 & & & \Xi \end{pmatrix}. \tag{4.22}$$

Given the foregoing finite-dimensional Markovian representation for the aggregate field vector process $Z(t)$, we can model the physical field variables of the $i$th survey as

$$z_i(t) = M_iHx(t) \tag{4.23}$$

where

$$(M_i)_{j,k} \begin{cases} 1, & \text{if the } k\text{th track in (4.3) is the } j\text{th track associated with the } i\text{th survey} \\ 0, & \text{otherwise.} \end{cases} \tag{4.24}$$

Letting $H_i = M_iH$ and assuming an additive measurement noise model, we obtain the equation for the $i$th survey measurements

$$dy_i(t) = H_ix(t)\,dt + dv_i(t), \tag{4.25}$$

for $0 \le t \le T$, where the $v_i(\cdot)$'s are assumed to be independent Wiener processes with

$$E[dv_i(t)\,dv_i'(t)] = R_i\,dt \tag{4.26}$$

and $R_i > 0$. Defining the $l$th pass smoothed estimate of the global state, $\hat{x}_s^{(l)}(t)$, as

$$\hat{x}_s^{(l)}(t) \triangleq E\left[x(t)|\bigvee_{i=1}^{l} Y_i\right] \tag{4.27}$$

where $Y_i$ denotes the Hilbert space spanned by the $i$th survey measurements, we can use the map updating results of Section III to express $\hat{x}_s^{(l)}(t)$ as a linear functional of $\hat{x}_s^{(l-1)}(\cdot)$ and $dy_l(\cdot)$.

As a final remark, let us note that the results presented in this section and in Section V-B extend simply to the mapping of a general stationary field, whose correlation function is approximated as a weighted sum of separable correlation functions. Furthermore, these results also extend, in principle, to the case of general survey data over nonparallel tracks, although in such a case the realization obtained corresponding to (4.15), (4.16) is infinite-dimensional (see [7]). However, as we will discuss in a future paper, the discrete-time counterparts of the results in this paper, when applied to survey data of *arbitrary* geometry, result in finite dimensional mapping algorithms.

## V. Map Combining and Map Centralization

In Section III we showed how algorithms for map updating followed from an orthogonal decomposition of the two-data-pass space $Y$, and we showed the use of the smoothing error models obtained in Section II. In this section we use these results to derive a solution to the map combining problem—the problem of forming the two-pass smoothed estimate $\hat{x}_s(t)$ as a linear functional of the smoothed estimates corresponding to each data pass, $\hat{x}_{1s}(\cdot)$ and $\hat{x}_{2s}(\cdot)$. We then formulate and use the map combining results to solve a special case of the map centralization problem, that is, the problem of forming a map over some global region of interest by combining local maps formed over subregions. In what follows, we again display inside boxes the essential relations defining the map combining and map centralization procedures.

### *A. The Map Combining Problem*

Recall that (3.15), which yields the solution to the map updating problem, follows from decomposing $Y$ as

$$Y = Y_1 \oplus \tilde{Y}_2. \tag{5.1}$$

We could similarly imagine decomposing $Y$ as

$$Y = Y_2 \oplus \tilde{Y}_1 \tag{5.2}$$

where

$$\tilde{Y}_1 \triangleq H(d\tilde{y}_1(\tau), \quad 0 \le \tau \le T) \tag{5.3}$$

and

$$\begin{aligned} d\tilde{y}_1(t) &= dy_1(t) - H_1\hat{x}_{2s}(t)\,dt \\ &\equiv H_1\tilde{x}_{2s}(t)\,dt + dv_1(t). \end{aligned} \tag{5.4}$$

Hence, by employing the decomposition expressed by (5.2), we can derive the following formula for $\hat{x}_s(t)$, analogous to (3.15):

$$\hat{x}_s(t) = \hat{x}_{2s}(t) + E\left[\tilde{x}_{2s}(t)|\tilde{Y}_1\right]. \tag{5.5}$$

By adding (5.5) and (3.15) and by subtracting $\hat{x}_s(t)$ from both sides of the resulting identity, we obtain the following relation:

$$\begin{aligned} \hat{x}_s(t) &= \hat{x}_{1s}(t) + \hat{x}_{2s}(t) \\ &\quad + \left[E\left[\tilde{x}_{1s}(t)|\tilde{Y}_2\right] + E\left[\tilde{x}_{2s}(t)|\tilde{Y}_1\right] - \hat{x}_s(t)\right]. \end{aligned} \tag{5.6}$$

In what follows we show that (5.6) represents the desired map combining algorithm by demonstrating that the term inside the brackets may be expressed as a linear functional of $\hat{x}_{1s}(\cdot)$ and $\hat{x}_{2s}(\cdot)$.

First, note that the two-pass smoothed estimate $\hat{x}_s(t)$ defined by (2.8), (2.10), and (2.12), with $y' = (y_1', y_2')$ and $H$, $R$ as in (3.36), could also be computed by the information filter algorithm obtained by setting

$$\begin{aligned} \beta_f(t) &\triangleq P_f^{-1}\hat{x}_f(t) \\ \beta_r(t) &\triangleq P_r^{-1}\hat{x}_r(t), \end{aligned} \tag{5.8}$$

in which case

$$\hat{x}_s(t) = P_s(t)\big(\beta_f(t) + \beta_r(t)\big) \tag{5.9}$$

where

$$\begin{aligned} d\beta_f(t) &= \left(-A' - P_f^{-1}Q\right)\beta_f(t)\,dt \\ &\quad + H_2'R_2^{-1}\,dy_2(t) + H_1'R_1^{-1}\,dy_1(t) \end{aligned} \tag{5.10}$$

and

$$\begin{aligned} -d\beta_r(t) &= (-A' - 0_0Q)\beta_r(t)\,dt \\ &\quad + H_2'R_2^{-1}\,dy_2(t) + H_1'R_1^{-1}\,dy_2(t) \end{aligned} \tag{5.11}$$

with

$$\beta_f(0) = \beta_r(T) = 0. \tag{5.12}$$

We will employ the notations $\mathscr{C}_t(dy_1(\cdot), dy_2(\cdot))$ and $\mathscr{A}_t(dy_1(\cdot), dy_2(\cdot))$ to denote the Wiener integrals defined over the past and future increments of $(dy_1(\cdot), dy_2(\cdot))$, respectively, that represent the causal and anticausal contributions to the smoothed estimate $\hat{x}_s(t)$ determined by (5.9)–(5.12), that is, we let

$$\hat{x}_s(t) = \mathscr{C}_t(dy_1(\cdot), dy_2(\cdot)) + \mathscr{A}_t(dy_1(\cdot), dy_2(\cdot)). \tag{5.13}$$

Then, from examining the information-filter algorithm for computing $E[\tilde{x}_{1s}(t)|\tilde{Y}_2]$, defined by (3.33)–(3.35) and the similar information-filter algorithm for computing $E[\tilde{x}_{2s}(t)|\tilde{Y}_1]$, it can be concluded that these algorithms may be restated using the formalism of (5.13) as

$$E\left[\tilde{x}_{1s}(t)|\tilde{Y}_2\right] = \mathscr{C}_t(0, d\tilde{y}_2(\cdot)) + \mathscr{A}_t(0, d\tilde{y}_2(\cdot)) \tag{5.14}$$

and

$$E\left[\tilde{x}_{2s}(t)|\tilde{Y}_1\right] = \mathscr{C}_t(d\tilde{y}_1(\cdot), 0) + \mathscr{A}_t(d\tilde{y}_1(\cdot), 0). \tag{5.15}$$

Now, by adding relations (5.14) and (5.15) and then subtracting (5.13), making use of the linearity of the Wiener integral, and employing the definitions of $d\tilde{y}_1(\cdot)$, $d\tilde{y}_2(\cdot)$ in (5.4) and (3.12), the following relation is obtained:

$$\begin{aligned} &E\left[\tilde{x}_{1s}(t)|\tilde{Y}_2\right] + E\left[\tilde{x}_{2s}(t)|\tilde{Y}_1\right] - \hat{x}_s(t) \\ &\quad = \mathscr{C}_t(-H_1\hat{x}_{2s}(t)\,dt, -H_2\hat{x}_{1s}(t)\,dt) \\ &\qquad + \mathscr{A}_t(-H_1\hat{x}_{2s}(t)\,dt, -H_2\hat{x}_{1s}(t)\,dt). \end{aligned} \tag{5.16}$$

Relation (5.16) represents the term in brackets of relation (5.6) as a linear functional of $\hat{x}_{1s}(\cdot)$, $\hat{x}_{2s}(\cdot)$. From the form of $\mathscr{C}_t(\cdot,\cdot)$ and $\mathscr{A}_t(\cdot,\cdot)$ determined by (5.9)–(5.12), we may express (5.16) more explicitly as

$$\begin{aligned} &E\left[\tilde{x}_{1s}(t)|\tilde{Y}_2\right] + E\left[\tilde{x}_{2s}(t)|\tilde{Y}_1\right] - \hat{x}_s(t) \\ &\quad = P_s(t)\big(\eta_f(t) + \eta_r(t)\big) \end{aligned} \tag{5.17}$$

where

$$\boxed{\begin{aligned} d\eta_f(t) &= \left(-A' - P_f^{-1}Q\right)\eta_f(t)\,dt - H_2'R_2^{-1}H_2\hat{x}_{1s}(t)\,dt \\ &\quad - H_1'R_1^{-1}H_1\hat{x}_{2s}(t)\,dt \end{aligned}} \tag{5.18}$$

$$\boxed{\begin{aligned} -d\eta_r(t) &= (-A' - 0_0A)\eta_r(t)\,dt - H_2'R_s^{-1}H_2\hat{x}_{1s}(t)\,dt \\ &\quad - H_1'R_1^{-1}H_1\hat{x}_{2s}(t)\,dt \end{aligned}} \tag{5.19}$$

with

$$\boxed{\eta_f(0) = \eta_r(T) = 0.} \tag{5.20}$$

Consequently, from (5.6) we see that the solution of the map combining problem consists of the computation of $\eta_f$ and $\eta_r$ according to (5.18)–(5.20) followed by the calculation

$$\hat{x}_s(t) = \hat{x}_{1s}(t) + \hat{x}_{2s}(t) + P_s(t)(\eta_f(t) + \eta_r(t)). \quad (5.21)$$

*B. The Map Centralization Problem*

While the map updating and map combining problems were motivated by situations where a centralized computing facility produces a map of the random field over a given region based on either new data and an old map, or different maps constructed from different data sources, in the map centralization problem we have several maps, produced from different surveys, where the maps *may not be over identical regions*. Thus local surveys might be used to produce local maps. The map centralization problem is one of combining these local maps to produce an overall global map of the entire region of interest. In this section we will discuss the solution of this problem when all of the processing done for the local maps is "consistent," that is, where the random field models used to do the local processing are exactly interpretable as the restriction of the global field model. The case for which local models are inconsistent will not be discussed here. This case typically happens when the local processing is based on a simplified approximate model obtained by neglecting or approximating some of the correlations that exist in the actual global model.

Consider the problem formulation developed in Section IV. We obtain several sets of parallel track survey measurements of the scalar random field $F(\cdot,\cdot)$ and would like to use all of these data to obtain a map of $F(\cdot,\cdot)$ along a specified set of tracks (4.2) (which include all of the survey tracks). In Section IV we focused on the updating problem, that is, on the computation of the map based on the first $l$ surveys in terms of the map based on the first $(l-1)$ surveys and the $l$th survey data. The map centralization problem is somewhat different. Each of the $l$ sets of survey data has been processed to produce a local map of $F(\cdot,\cdot)$, that is, an estimate of $F(\cdot,\cdot)$ over a subset of the set of tracks in (4.2) (where the particular regions mapped by the surveys may differ from survey to survey). The objective, then, is to combine these local maps to produce the overall global estimate of $F(\cdot,\cdot)$ over the full region of interest based on all $l$ surveys. For the sake of simplicity, we restrict our development here to the case of two surveys $(l = 2)$, as the generalization to $l > 2$ is immediate but notationally cumbersome.

To begin, recall that the problem is to estimate $Z(t)$ in (4.3), for $0 \le t \le T$. Since $Z(t)$ can be realized by the finite-dimensional Gauss–Markov model (4.15), (4.16), we can view the goal of the map centralization problem as the estimation of $x(\cdot)$, which we call the *global* state process, since it is employed to represent the field over the global region of interest. As in Section IV, the $i$th survey $(i = 1,2)$ consists of measurements on a subset of the tracks (4.2). The physical variables measured by the $i$th survey are the $z_i(t)$ defined in (4.23), (4.24), and the actual survey measurements are given by (4.25), (4.26). For convenience, we summarize here the global state model and the two-survey measurement equations:

$$dx(t) = Ax(t)\,dt + B\,du(t), \quad (5.21a)$$

$$dy_1(t) = H_1x(t)\,dt + dv_1(t), \quad (5.21b)$$

and

$$dy_2(t) = H_2x(t)\,dt + dv_2(t). \quad (5.21c)$$

Let $Y_1$ and $Y_2$ denote the Hilbert spaces spanned by the first and second survey measurements, respectively, and let $Y = Y_1 \vee Y_2$. Then, the overall objective is to compute the smoothed global state estimate

$$\hat{x}_s(t) \triangleq E[x(t)|Y]. \quad (5.22)$$

However, we are required to perform this computation completely in terms of individual local maps produced based on each of the two surveys. Specifically, we suppose that the $i$th survey data are used (by a *local processor*) to compute a map of $F(\cdot,\cdot)$ along a subset of the full set of tracks (4.2) (the subset, of course, includes the actual $i$th survey tracks). Applying precisely the same realization procedure as that used in Section IV, we obtain a reduced-order model for $i = 1,2$:

$$z_i(t) = C_ix_i(t) \quad (5.23)$$

$$dx_i(t) = A_ix_i(t)\,dt + B_i\,du_i(t) \quad (5.24)$$

where the *local state* $x_i(t)$ is employed to represent the random field over the set of tracks mapped by the local processor. That is, the $i$th local processor uses the model (5.24), and the $i$th survey data are written as

$$dy_i(t) = C_ix_i(t)\,dt + dv_i(t) \quad (5.25)$$

to produce the smoothed estimates

$$\hat{x}_{is}(t) = E[x_i(t)|Y_i]. \quad (5.26)$$

The map centralization problem, then, is the computation of $\hat{x}_s(\cdot)$ in terms of $\hat{x}_{1s}(\cdot)$ and $\hat{x}_{2s}(\cdot)$. Note that, in general, $x_1(\cdot)$ and $x_2(\cdot)$, and hence $\hat{x}_{1s}(\cdot)$ and $\hat{x}_{2s}(\cdot)$, may have different dimensions.

The solution to this problem is obtained in two steps: we first use $\hat{x}_{1s}(\cdot)$ to recover the smoothed estimate of the global state based on the $i$th pass alone, that is,

$$\hat{x}_{g,is}(t) = E[x(t)|Y_i]. \quad (5.27)$$

Then we can use the map combining algorithm of Section V-A to express $\hat{x}_s(\cdot)$ as a linear functional of $\hat{x}_{g,is}(\cdot)$, $i = 1,2$. It is only the first step that remains to be specified, and it is on this that we focus attention in the remainder of this section. As a first point we note that (5.23)–(5.25) represent a reduced-order realization of the $i$th survey data (as compared with (5.21a) and (for $i = 1$) (5.21b) or (for $i = 2$) (5.21c)). In fact, it is not difficult to see from the realization procedure of Section IV, that $x_i(t)$ is a subprocess of $x(t)$, that is, that the Gauss–Markov model (5.24), is a restriction of the model (5.21a). That is, a transformation $D_i$ exists so that

$$x_i(t) = D_ix(t). \quad (5.28)$$

This transformation can be readily determined from the realization procedure in Section IV and the specification of the subset of the tracks in (4.2) which $x_i(\cdot)$ is used to represent. Equation (5.28) implies that $x_i(\cdot)$ and $D_i x(\cdot)$ have the same covariance matrices, and hence

$$D_i \pi(0) D_i' = \pi_i(0) \tag{5.29}$$

and

$$D_i Q D_i' = Q_i \tag{5.30}$$

where $\pi(0)$ and $\pi_i(0)$ are the covariances of $x(0)$ and $x_i(0)$, respectively, and

$$Q = BB' \qquad Q_i = B_i B_i' \tag{5.31}$$

are the strengths of the input noise terms ($B\,du(t)$) in (5.21a) and $B_i\,du_i(t)$ in (5.24)) in the global and local models, respectively. Furthermore, since $x_i$ is a subprocess of $x(t)$, $D_i$ is one-to-one, and we can augment it to form a nonsingular matrix

$$T_i = \begin{bmatrix} D_i \\ E_i \end{bmatrix} \tag{5.32}$$

so that the process $x(t)$ expressed in terms of the change of basis implied by (5.32), that is,

$$\zeta_i(t) \triangleq \begin{pmatrix} x_i(t) \\ \rho_i(t) \end{pmatrix} = T_i x(t) \tag{5.33}$$

obeys equations of the form

$$z_i(t) = (C_i \mid 0)\xi_i(t) \tag{5.34}$$

with

$$d\zeta_i(t) = \left(\begin{array}{c|c} A_i & 0 \\ \hline M_i & N_i \end{array}\right)\zeta_i(t)\,dt + T_i B\,du(t). \tag{5.35}$$

The critical point to observe in (5.35) is the block of zeros in the upper right corner of the block matrix on the right side. That such a transformation can be found follows from the fact that $x_i$ by itself is Gauss–Markov.

Let

$$\hat{\rho}_{is}(t) = E\left[\rho_i(t) \mid Y_i\right]. \tag{5.36}$$

We now must determine how to compute $\hat{\rho}_{is}(\cdot)$ in terms of $\hat{x}_{is}(\cdot)$. As a first step, define the two input-noise processes $dw_{i1}(\cdot)$ and $dw_{i2}(\cdot)$ by

$$dw_{i1}(t) = D_i B\,du(t) \tag{5.36a}$$

and

$$dw_{i2}(t) = E_i B\,du(t). \tag{5.37}$$

We may then express the dynamics for $\rho_i(\cdot)$ and $x_i(\cdot)$ as

$$dx_i(t) = A_i x_i(t)\,dt + dw_{i1}(t) \tag{5.38}$$

and

$$d\rho_i(t) = M_i x_i(t)\,dt + N_i \rho_i(t)\,dt + dw_{i2}(t). \tag{5.39}$$

Equations (5.38) and (5.39) are now employed to derive an algorithm for computing $\hat{\rho}_{is}(\cdot)$ as a linear functional of $\hat{x}_{is}(\cdot)$ through the following two steps:

1) decomposing $dw_{i2}(t)$ with respect to the $\sigma$ field $U_{it}^1 \triangleq \sigma\{dw_{i1}(\tau) 0 \le \tau \le t\}$, and

2) formally taking $E\{\cdot \mid Y_i\}$ of both sides of (5.38) and (5.39) in order to determine effectively equations satisfied by the incremental predictable part of the decomposition of both $x_i(t)$ and $\rho_i(t)$ with respect to the $\sigma$ field spanned by the $i$th pass observations.

Because both $dw_{i1}(t)$ and $dw_{i2}(t)$ are related to the increments of $du(t)$ by (5.36a) and (5.37) and $du(t_1)$, $du(t_2)$ are orthogonal for $t_1 \neq t_2$, some matrix $K_i$ exists so that the decomposition of $dw_{i2}(t)$ with respect to $U_{it}^1$ may be expressed as

$$dw_{i2}(t) = K_i\,dw_{i1}(t) + d\tilde{w}_{i2}(t), \tag{5.40}$$

where $d\tilde{w}_{i2}(t)$ is a Wiener process orthogonal to $U_{it}^1$. By using the orthogonality property

$$E\left[\left(dw_{i2}(t) - K_i\,dw_{i1}(t)\right) dw_{i1}'(\tau)\right] = 0 \tag{5.41}$$

for $0 \le \tau \le t$, it can be shown that $K_i$ has the form

$$K_i \triangleq [E_i Q D_i'] Q^{\#} \tag{5.42}$$

where # denotes the Moore–Penrose pseudoinverse.

Now, substituting (5.40) into (5.39), taking $E[\cdot \mid Y_i]$ of both sides of (5.38) and (5.39), and using (5.38) to express $E[dw_{i1}(t) \mid Y_i]$, the following equation is obtained for the evolution of $\hat{\rho}_{is}(\cdot)$:

$$d\hat{\rho}_{is}(t) = M_i \hat{x}_{is}(t)\,dt + N_i \hat{\rho}_{is}(t)\,dt + K_i\left[d\hat{x}_{is}(t) - A_i \hat{x}_{is}(t)\,dt\right]. \tag{5.43}$$

In deriving (5.43), we have used the fact that $E[d\tilde{w}_{i2}(t) \mid Y_i] = 0$, since $d\tilde{w}_{i2}(t)$ is orthogonal to $dw_{i1}(\tau), 0 \le \tau \le t, \zeta_i(0)$, and the observation noise $dv_i(\cdot)$.

Equation (5.43) now yields the desired relations for computing $\hat{\rho}_{is}(\cdot)$ as a linear functional of $\hat{x}_{is}(\cdot)$. If we define $\eta_i(t)$ as

$$\eta_i(t) \triangleq \hat{\rho}_{is}(t) - K_i \hat{x}_{is}(t), \tag{5.44}$$

a straightforward calculation yields

$$\boxed{d\eta_i(t) = (M_i + N_i K_i - K_i A_i)\hat{x}_{is}(t)\,dt + N_i \eta_i(t)\,dt.} \tag{5.45}$$

Given (5.45) and (5.44), the specification of an algorithm for forming $\hat{\rho}_{is}(\cdot)$ from $\hat{x}_{is}(\cdot)$ will be complete once the calculation of $\hat{\rho}_{is}^{(0)}$ is defined. If $\Sigma_{\rho s}$ denotes the correlation between $\rho_i(0)$ and $x_i(0)$, then $\rho_i(0)$ may be expressed as

$$\rho_i(0) = \tilde{\rho}_i(0) + \Sigma_{\rho x} \pi_i^{-1}(0) x_i(0) \tag{5.46}$$

where $\tilde{\rho}_i(0)$ is orthogonal to $x_i(\tau), \tau \ge 0$. Since $\tilde{\rho}_i(0)$ is orthogonal to both $x_i(\cdot)$ and $dv_i(\cdot)$, taking $E[\cdot \mid Y_i]$ of both sides of (5.46) results in the relation

$$\hat{\rho}_{is}(0) = \Sigma_{\rho x} \pi_i^{-1}(0) \hat{x}_{is}(0), \tag{5.47}$$

and hence from (5.44)

$$\boxed{\eta_i(0) = \left(\Sigma_{\rho x} \pi_i^{-1}(0) - K_i\right) \hat{x}_{is}(0).} \tag{5.48}$$

Finally, we note that by using (5.44) and taking $E(\cdot|Y_i)$ of both sides of (5.33), we may compute $\hat{x}_{g,is}(t)$ as

$$\hat{x}_{g,is}(t) = T_i^{-1}\left(\frac{\hat{x}_{is}(t)}{K_i(t)\hat{x}_{is}(t)+\eta_i(t)}\right). \tag{5.49}$$

Given (5.49) we can invoke to the map combining results of Section V-A in order to express $\hat{x}_s(t)$ as a linear functional of $\hat{x}_{g,is}(\cdot)$, $i = 1,2$, and hence ultimately to express $\hat{x}_s(\cdot)$ as a linear functional of $\hat{x}_{is}(\cdot)$, $i = 1,2$, thus solving the map centralization problem.

## VI. Conclusion

In this paper, by employing Hilbert space and martingale decomposition techniques, we have provided a unifying framework for understanding and deriving the solutions to problems of map updating, map combining, and map centralization. While the particular two-dimensional examples of Sections IV and V-B rely heavily on the parallel nature of the measurement geometries considered, we will show in a future paper how mapping algorithms may be applied to essentially general measurement geometries in the case of *discrete-space* stationary random fields. Even in the case of nonparallel measurement trajectories through a continuous space random field, for which the aggregate field vector process defined in Section IV-C has no finite-dimensional Markovian representation, our results still indicate the structure of the solution to the mapping problems, so that the remaining issues are technical or algorithmic in nature.

## Appendix

In this Appendix we show how the explicit representation of the smoothing error process in (2.31) may be used to derive a backwards Markovian realization of the smoothing errors. We first indicate how the representation (2.31) follows from the innovations form of the smoothed estimate [10]. Consider the state-space model (2.1) with measurements defined by (2.3). The innovations process is defined in (2.32), where $\hat{x}_f$ and its covariance $P_f$ are specified in (2.8) and (2.9), respectively.

If we let $\nu_t^+$ denote the Hilbert space defined by the future increments of the innovations, then the innovations form of the smoothed estimate follows from using the following orthogonal decomposition for the Hilbert space $Y$ spanned by $dy(\tau)$, for $0 \le \tau \le T$:

$$Y = Y_t^- \oplus \nu_t^+. \tag{A.1}$$

Projecting $x(t)$ onto both sides of (A.1), we obtain the equation

$$\hat{x}_s(t) = \hat{x}_f(t) + E\left[x(t)|\nu_t^+\right]. \tag{A.2}$$

Now if $x(t)$ is expressed as

$$x(t) = \hat{x}_f(t) + \tilde{x}_f(t) \tag{A.3}$$

where $\tilde{x}_f(\cdot)$ denotes the filtering error process and (A.3) is substituted into (A.2) (making use of the fact that $\hat{x}_f(t) \perp \nu_t^+$), and finally, if the resulting equation for $\hat{x}_s(t)$ is used to form $\tilde{x}_s(t) = x(t) - \hat{x}_s(t)$, the following representation for $\tilde{x}_s(t)$ is obtained:

$$\tilde{x}_s(t) = \tilde{x}_f(t) - E\left[\tilde{x}_f(t)|\nu_t^+\right]. \tag{A.4}$$

It can be shown by straightforward calculation that (A.4) corresponds directly to (2.31).

A backwards Markovian representation for $\tilde{x}_s(t)$ now follows by employing (A.4) together with backwards representations for $\tilde{x}_f(t)$, and $\phi(t) = E[\tilde{x}_f(t)|\nu_t^+]$. A reverse-time realization of $\tilde{x}_f(\cdot)$ can be derived from the forward realization

$$d\tilde{x}_f(t) = \left(A - P_fH'R^{-1}H\right)\tilde{x}_f(t)\,dt + dw(t) - P_fH'R^{-1}\,dv(t) \tag{A.5}$$

by decomposing the input noises $dw(t)$ and $dv(t)$ with respect to $\sigma\{\tilde{x}_f(\tau)\ t \le \tau \le T\}$ as

$$dw(t) = QP_f^{-1}\tilde{x}_f(t)\,dt + d\tilde{w}_b(t) \tag{A.6}$$

and

$$dv(t) = -H\tilde{x}_f(t)\,dt + d\tilde{v}_b(t). \tag{A.7}$$

By employing (A.6) and (A.7) and noting that

$$d\tilde{v}_b(t) = H\tilde{x}_f(t)\,dt + dv(t) \equiv d\nu(t), \tag{A.8}$$

we obtain the following backward realization for $\tilde{x}_f(\cdot)$:

$$-d\tilde{x}_f(t) = -\left[A + QP_f^{-1}\right]\tilde{x}_f(t)\,dt - d\tilde{w}_b(t) + P_fH'R^{-1}\,d\nu(t). \tag{A.9}$$

Next, we employ the backward realization (A.9) to derive a backwards representation for the process $\phi(t)$. This realization follows directly from (A.9) after first showing that $d\tilde{w}_b(t) \perp Y$, and hence $d\tilde{w}_b(t) \perp \nu_t^+$. The fact that $d\tilde{w}_b(t) \perp Y$ can be proven by demonstrating that

$$d\tilde{w}_b(t) = d\tilde{w}_r(t) \tag{A.10}$$

where $d\tilde{w}_r(t)$ is the input-noise process to the reverse-time realization for $\tilde{x}_s(\cdot)$ derived in Section II and is defined through (2.27) as

$$d\tilde{w}_r(t) \triangleq dw(t) - E\left[dw(t)|\tilde{X}_t^+\right] - E\left[dw(t)|Y\right]. \tag{A.11}$$

Using the representation for $\tilde{x}_s(t)$ of (2.31) or, equivalently, (A.4), we can show that

$$E\left[dw(t)|\tilde{X}_t^+\right] = QP_f^{-1}\tilde{x}_s(t)\,dt. \tag{A.12}$$

In addition, using the fact that $\nu_T^- = y$ to compute $E[dw(t)|Y]$, explicitly and also using the innovations form of the smoothed estimate, it can be shown that

$$E\left[dw(t)|Y\right] = QP_f^{-1}\left(\hat{x}_s(t) - \hat{x}_f(t)\right)dt. \tag{A.13}$$

By employing both (A.12) and (A.13) in (A.11) and noting (A.6), we prove (A.10). Since $d\tilde{w}_r(t) \perp Y$, by the nature of its construction in (A.12), then $d\tilde{w}_b(t) \perp Y$, and hence by formally taking $E[\cdot|\nu_t^+]$ of both sides of (A.9), we obtain the following backward model for $\phi(\cdot)$:

$$-d\phi(t) = -\left[A + QP_f^{-1}\right]\theta(t)\,dt + P_fH'R^{-1}\,d\nu(t) \tag{A.14}$$

with

$$\phi(T) = 0. \tag{A.15}$$

Finally, by employing (A.4) in order to form $d\tilde{x}_s(t) = d\tilde{x}_f(t) - d\phi(t)$, using the backward realization for $\tilde{x}_f(t)$ defined by (A.9), and using the backward realization for $\phi(t)$ determined by (A.14) and (A.15), we obtain for $\tilde{x}_s(\cdot)$ the following reverse-time realization:

$$-d\tilde{x}_s(t) = -\left[A + QP_f^{-1}\right]\tilde{x}_s(t)\,dt - d\tilde{w}_b(t) \tag{A.16}$$

with

$$\tilde{x}_s(T) = \tilde{x}_f(T). \tag{A.17}$$

Given (A.10) the backward realization for $\tilde{x}_s(\cdot)$ is identical to that obtained in Section II. The derivation here shows how the existence and structure of the backward realization for the smoothing errors follow from the structure of the backward model for the filtering errors alone.

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