Linear Estimation of Boundary Value Stochastic Processes—Part I: The Role and Construction of Complementary Models

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***Abstract*—This paper presents a substantial extension of the method of complementary models for minimum variance linear estimation introduced by Weinert and Desai in their important paper [1]. Specifically, the method of complementary models is extended to solve estimation problems for both discrete and continuous parameter linear boundary value stochastic processes in one and higher dimensions. A major contribution of this paper is an application of Green's identity in deriving a differential operator representation of the estimator. To clarify the development and to illustrate the range of applications of our approach, two brief examples are provided: one is a 1-D discrete two-point boundary value process and the other is a 2-D process governed by Poisson's equation on the unit disk.**

I. Introduction

In this paper we present an extension of the method of complementary models for minimum variance linear estimation introduced by Weinert and Desai in their important paper [1]. Weinert and Desai showed that the fixed interval smoothing problem for causal one-dimensional[1] processes described by linear state equations driven by white noise could be solved by introducing the so-called complementary process. The complementary process has the property that it is orthogonal to the observations and that, when combined with the observations, contains information equivalent to the initial conditions, driving noise and measurement noise, i.e., all of the underlying variables which determine the system state and observations. Here we build upon this general concept of complementation to solve estimation problems for both discrete and continuous parameter boundary value stochastic processes in one and higher dimensions. This class of processes is a generalization of the 1-D boundary value process introduced by Krener in [14] and includes processes governed by ordinary and partial linear differential equations and ordinary and partial linear difference equations. By employing operator descriptions for these processes we are able to unify the development of the estimators for this wide variety of processes within a single framework.

The major contribution of this paper is a differential operator representation for the estimator which is applicable to all of the types of linear boundary value problems mentioned above. A key step in our derivation is the use of Green's identity in the construction of a differential representation for the complementary process. To help clarify our presentation we carry along an example, a 2-D process governed by Poisson's equation with a white noise driving function. Finally, to illustrate the versatility of our solution, we briefly describe a 1-D discrete two-point boundary value process and derive the equations defining its estimator.

The emphasis in this paper is on the development of the differential representation for the estimator. In Part II of this paper [4], we consider a 1-D continuous parameter boundary value stochastic process. Taking the specialization of the result presented in the present paper to this 1-D setting, we address in Part II the issue of efficient implementation of the boundary-value equations which define the estimator. As both Part II and [5] illustrate, the development of efficient procedures for implementing the linear boundary value representation for the estimator provided in this paper and the computation of its estimation error variance are interesting and challenging problems themselves.

Our approach to deriving the differential form of the smoother for general processes begins with an operator representation of the complementary process for the estimation problem of interest in this paper. With this representation and Green's identity in hand, we are then in a position to derive internal differential realizations for both the complementary process and the optimal estimator. To reach this point, however, we require some machinery and notation. These are provided in Section II in which we briefly review the fundamental concept of complementation, define notation, and state our general problem. In addition, in Section II we present the operator form of the complementary process for our general problem. As is pointed out in [1] for causal 1-D models, the complementary process is closely related to the adjoint of the system which describes the process to be estimated. As one might expect, the same is true more generally, and the operator form displayed in Section II demonstrates this quite clearly and simply.

A general form for the internal differential realization for the complementary process is derived in Section III. Given this realization, we formulate an internal differential realization for the estimator. Using this recipe for the representation of the

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[1]The terminology one-dimensional (1-D), two-dimensional (2-D), or multidimensional process is used here to indicate that the dimension of the independent variable for the process is one, two, or multidimensional.

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estimator, in Section IV we present differential realizations for the estimators for both 1-D and 2-D examples. Finally, some observations and concluding remarks are offered in Section V.

II. Linear Estimation and Complementary Stochastic Processes

A. The Method of Complementary Processes

In this section we establish notation which will be used in the statement and solution of our estimation problem for second-order stochastic boundary value processes. Let $L_2(dP)$ denote the Hilbert space of finite variance random variables (on some given probability space). Let I denote an index set. A *second-order process over* I is a set of elements $\phi(\alpha)$ in $L_2(dP)$ indexed by $\alpha \in I$. The closed linear span in $L_2(dP)$ of ϕ (as α ranges over I) will be denoted by $Sp(\phi)$. The space of second-order processes over I will be denoted by $L_2(I; dP)$. Linear mappings between two such spaces will be called *second-order operators*.

Define an *underlying* second-order process over a specified index set I_ζ

$$\zeta \in L_2(I_\zeta; dP) \equiv \boldsymbol{S}_\zeta. \tag{2.1}$$

The process to be estimated X and the observations Y are defined via second-order linear operators acting on the underlying process ζ

$$X = M_x\zeta; \qquad M_x: S_\zeta \to L_2(I_x; dP) \equiv \boldsymbol{S}_x \tag{2.2a}$$

and

$$Y = M_y\zeta; \qquad M_y: S_\zeta \to L_2(I_y; dP) \equiv \boldsymbol{S}_y \tag{2.2b}$$

where M_x and M_y are known. For example, for the class of problems considered here, ζ includes the driving noise, observation noise, and uncertain boundary conditions; the mapping M_x is the input/output representation of the solution of a linear boundary value problem in one or several dimensions; and the mapping M_y is of the form $[H \ \vdots \ I]$, where the identity operator operating on the underlying process ζ produces an additive noise component in the observations.

The *complementary process* Z is also defined via a second-order mapping of ζ as

$$Z = M_z\zeta; \qquad M_z: S_\zeta \to L_2(I_z; dP) \equiv \boldsymbol{S}_z \tag{2.2c}$$

where M_z must be chosen (if possible) so that the following conditions are satisfied.

Orthogonality:

$$E[Y(\alpha)Z(\beta)] = 0 \qquad \text{for all } \alpha \in I_y, \quad \beta \in I_z. \tag{2.3}$$

Complementation: The relation between ζ and $\{Z, Y\}$

$$\begin{bmatrix} Z \\ Y \end{bmatrix} = \begin{bmatrix} M_z \\ M_y \end{bmatrix} \quad \zeta \triangleq M\zeta \tag{2.4}$$

is invertible.

Assume that M_z can be found so that these conditions are satisfied. Partitioning the inverse of the augmented system (2.4) as

$$M^{-1} = N = [N_y \ \vdots \ N_z] \tag{2.5}$$

we can write $\zeta = \zeta_y + \zeta_z$ with

$$\zeta_y = N_y Y, \qquad \zeta_z = N_z Z. \tag{2.6}$$

It is not difficult then to see that, thanks to (2.3), the linear minimum variance estimate of ζ given y is ζ_y, that the minimum variance estimate of X given y is

$$\hat{X} = M_x\zeta_y = M_x N_y Y, \tag{2.7}$$

and that the estimation error $\tilde{X}$ is simply the linear minimum variance estimate of X given Z, which can be expressed in terms of ζ, whose probability law is known since

$$\tilde{X} = M_x\zeta_z = M_x N_z M_z \zeta. \tag{2.8}$$

The simple notation used to express the linear minimum variance estimate of X belies the complexity of the effort which may be required in 1) determining the form of the operator M_z, 2) augmenting and inverting to obtain M^{-1}, and 3) implementing the solution. Unfortunately, working with these I/O representations leads to neither a convenient nor an easily computed solution to the second step listed above. However, as in the 1-D causal problems of Weinert and Desai [1] and Levy *et al.* [8], we will find that this second step is quite easily accomplished by considering estimation problems for which the state and observations are specified in terms of an internal differential realization. The key step then is obtaining a differential realization of the complementary process. As we point out in Section II-C, the operator representation for the complementary process is specified in terms of H^*, the Hilbert adjoint of the mapping from the underlying variables to the noise-free observations.

A critical development in our research has been the recognition that Green's identity for differential operators is the key to formulating an internal realization for the Hilbert adjoint map H^* in terms of the operators involved in the internal description of the observations. Given these internal realizations, we are able to perform the augmentation and inversion yielding an internal differential realization for the estimator. We feel that this representation for the estimator is an important one. In particular, if one directly applies the projection theorem to problems of the type which we consider here, the results are generally in the form of integral equations (e.g., Wiener–Hopf integral equations) which must be factored in some way in order to produce a realization for the estimator. In contrast, our solution, obtained via the method of complementary models, directly yields a differential realization of the estimator.

Much as in the case of causal processes described by finite-dimensional state equations, these realizations provide an excellent starting point for the construction of efficient algorithms for implementing the optimal estimator. In Part II [4] we present a detailed development of a two-filter implementation of this estimator for a noncausal one-dimensional two-point boundary value stochastic process.

B. The Problem Statement

1) Differential Operators and Green's Identity: Our stochastic differential equations are defined in terms of differential operators acting on Hilbert spaces of square-integrable functions as follows. Let Ω_N be a bounded convex region in $\boldsymbol{R}^N$ with smooth boundary [10]. The space of $n \times 1$ vector functions which are square-integrable on Ω_N is represented by $L_2^n(\Omega_N)$. Let L be a formal[2] differential operator mapping into $L_2^n(\Omega_N)$ and defined on $D(L)$, the subspace of sufficiently differentiable elements of $L_2^n(\Omega_N)$.

Green's identity for L is obtained from integration by parts of the N-fold integral specified by the inner product on the left-hand side of (2.9). The result is Green's identity

[2] The term *formal differential operator* is used to denote operators which simply represent differentiation of a function. We will reserve the term *differential operator* to denote the combined action of a formal differential operator along with an appropriate boundary condition representing a well-posed boundary value problem.

$$\langle Lx, \lambda \rangle_{L_2^n(\Omega_N)} = \langle x, L^\dagger \lambda \rangle_{L_2^n(\Omega_N)} + \langle x_b, E\lambda_b \rangle_{H_b} \quad (2.9)$$

where $L^\dagger$ is referred to as the formal adjoint differential operator [2], x_b and λ_b are elements of a Hilbert space H_b of processes defined on $\partial\Omega_N$, and E is a mapping from H_b into itself; E: $H_b \to H_b$. In particular, these processes are defined through the action of an operator Δ_b: $L_2^n(\Omega_N) \to H_b$, so that

$$x_b = \Delta_b x \quad \text{and} \quad \lambda_b = \Delta_b \lambda. \quad (2.10)$$

The nature of H_b, Δ_b, and E all depend upon L and Ω_N. For Green's identity for ordinary differential operators, see [3] and [5, ch. 3]; for elliptic, hyperbolic, and parabolic second-order partial differential operators, see [2] and [5, ch. 7]. In this paper, we will restrict our discussions to operators L and regions Ω_N that admit a Green's identity.

The boundary condition associated with L is defined by a mapping V

$$V: H_b \to R(V) \quad (2.11)$$

where the nature of the range space $R(V)$ is determined by the following well-posedness condition. We will say that the pair (L, V) leads to a well-posed boundary value problem if the differential operator Λ formed by augmenting the formal differential operator L and boundary mapping V

$$\Lambda = \begin{bmatrix} L \\ V \end{bmatrix} \quad (2.12a)$$

has a unique continuous left inverse $\Lambda^\#$. We denote the components of the left inverse by

$$\Lambda^\# = \left[G_u \; \vdots \; G_v \right] \quad (2.12b)$$

where

$$G_u: L_2^n(\Omega_N) \to D(L) \quad \text{and} \quad G_v: L_2^{n_v}(\partial\Omega_N) \to D(L). \quad (2.12c)$$

The value of the vector dimension n_v in (2.12c), which is required for a well-posed problem, depends on the type and order of the operator L and the dimensions N and n. In this case, the equation

$$\Lambda x = \begin{bmatrix} u \\ v \end{bmatrix} \quad (2.13a)$$

with u and v in the domains of G_u and G_v, respectively, has a unique solution which can be written as

$$x = G_u u + G_v v. \quad (2.13b)$$

It will be assumed that all problems considered here are well posed.

A description nearly identical to that given above holds for a class of discrete processes defined by linear boundary-value partial difference equations. In this case L is a partial difference operator and Ω_N is a multidimensional discrete-valued index set. It is shown in [5] that the estimation problem statement and solution presented in this paper apply as well for this class of discrete processes.

2) The Problem Statement: Let u be an $m \times 1$ vector white noise on Ω_N with an invertible correlation operator Q (i.e., the correlation matrix of u is thought of as the kernel of an operator). Let v be an $n_v \times 1$ vector second-order process over $\partial\Omega_N$, uncorrelated with u and with invertible correlation operator Π_v. Then the process to be estimated is formally defined by

$$Lx = Bu \quad (2.14a)$$

with boundary condition

$$Vx_b = v. \quad (2.14b)$$

The observations are defined as follows. Let $C(t)$ be a $p \times n$ matrix continuous in $t \in \Omega_N$. Let W be an operator mapping elements of H_b into $R(W)$, a space of $n_w \times 1$ vector functions defined over the index set $\partial\Omega_N$. Let r be a $p \times 1$ vector white noise over Ω_N with invertible correlation operator R, and let r_b be a $n_w \times 1$ vector process with invertible correlation operator Π_b. It will be assumed that u, v, r, and r_b are mutually uncorrelated. The set of observations of x is given by

$$y = Cx + r \qquad \text{on } \Omega_N \quad (2.15a)$$

and

$$y_b = Wx_b + r_b \qquad \text{on } \partial\Omega_N. \quad (2.15b)$$

We will need to make some assumptions with respect to the relationship between the operators V and W. The importance of these assumptions will become apparent later in our development of Hilbert adjoint systems in Section III-A. As explained in the 1-D continuous case studied in Part II, one consequence of these assumptions is that no element of the boundary observation y_b can simply be absorbed into updating the boundary value v alone. That is, the boundary measurement contains information about the part of x_b not captured by Vx_b. In particular, we will assume that if the operator

$$\begin{bmatrix} -V \\ W \end{bmatrix} \quad (2.16a)$$

is not invertible, then there exists an operator W_c such that

$$\Gamma = \begin{bmatrix} -V \\ W_c \\ W \end{bmatrix} \quad (2.16b)$$

is invertible. (As will become clear in Section III, we have included the minus sign in defining Γ for convenience.)

Our estimation problem is to find the linear minimum variance estimate of x given the set of observations in (2.15). To transform this problem into notation similar to that used in Section II-A let the inverse of (2.13a) be denoted by

$$x = M_x \begin{bmatrix} u \\ v \end{bmatrix}. \quad (2.17)$$

This is simply the Green's function form of the solution [see (2.13b)]. Recall from (2.10) that $x_b = \Delta_b x$. If we define H as

$$H = \begin{bmatrix} C \\ W\Delta_b \end{bmatrix} M_x \quad (2.18)$$

and specify the underlying process as

$$\zeta = \begin{bmatrix} u \\ v \\ \hdashline r \\ r_b \end{bmatrix} = \begin{bmatrix} \zeta_1 \\ \hdashline \zeta_2 \end{bmatrix} \quad (2.19a)$$

then the observations (2.15) can be expressed in the desired form of signal plus noise

$$Y = \begin{bmatrix} y \\ y_b \end{bmatrix} = \left[H \; \vdots \; I \right] \zeta = H\zeta_1 + \zeta_2. \quad (2.19b)$$

(The significance of the partition of ζ into ζ_1 and ζ_2 will be made clear shortly.)

This problem formulation is illustrated in the following example. In subsequent sections, we formulate the solution to this class of problems in a differential operator form with y and y_b as the input and boundary condition, respectively, and the estimate of x as an element of the output.

Example (Poisson's Equation on the Unit Disk): In this case the dimension N of the index set is 2 and the index set itself (Ω_2) is the unit disk. Points within the disk will be represented by index variables $s, t \in \Omega_2$, and points on the unit circle $\partial\Omega_2$ will be denoted by an angle $\theta \in [0, 2\pi]$. Let u be a scalar white noise over Ω_2 with continuous covariance parameter $Q(s)$. Let v be a scalar white noise over $\partial\Omega_2$ with continuous covariance parameter $\Pi_v(\theta)$. Let $B(s)$ be a continuous function on Ω_2 and $V(\theta)$ be a nonzero continuous function on $\partial\Omega_2$. The process to be estimated is formally defined by

$$\nabla^2 x(s) = B(s)u(s) \tag{2.20a}$$

with boundary condition (in polar coordinates)

$$V(\theta)x(1,\theta) = v(\theta). \tag{2.20b}$$

For this example L is the Laplacian ∇^2 which is formally self-adjoint [2], i.e., Green's identity is

$$\langle \nabla^2 x, \lambda\rangle = \langle x, \nabla^2\lambda\rangle + \text{boundary term}. \tag{2.21}$$

The boundary term is expressed as follows [see (2.9)]. With x_n the normal derivative of x along $\partial\Omega_N$, define the function x_b as

$$x_b = \Delta_b x = \begin{bmatrix} x|_{\partial\Omega_N} \\ x_n \end{bmatrix}, \text{ or in polar coordinates}$$

$$x_b(\theta) = \begin{bmatrix} x(1,\theta) \\ x_n(1,\theta) \end{bmatrix}. \tag{2.22}$$

Thus, x_b is an element of the Hilbert space $H_b = L_2^2(\partial\Omega_2)$ with inner product

$$\langle w, z\rangle_{H_b} = (1/2\pi)\int_0^{2\pi}[w_1(\theta)z_1(\theta) + w_2(\theta)z_2(\theta)]\,d\theta. \tag{2.23a}$$

The function λ_b in (2.9) is defined in terms of λ in the same fashion as x_b in (2.22). Furthermore, the operator E in this case is simply the multiplication of elements of $L_2^2(\partial\Omega_2)$ by a 2×2 matrix which we also denote by E. Specifically,

$$E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{2.23b}$$

Let r be a scalar white noise over Ω_2 and r_b be a scalar white noise over $\partial\Omega_2$ with continuous covariance parameters $R(s)$ and $\Pi_b(\theta)$, respectively. Let $C(s)$ be a continuous function on Ω_2 and $W(\theta)$ be a nonzero continuous function on $\partial\Omega_2$. The observations are defined by

$$y(s) = C(s)x(s) + r(s) \quad \text{on } \Omega_2 \tag{2.24a}$$

and

$$y_b(\theta) = W(\theta)x_n(\rho,\theta) + r_b(\theta); \qquad \rho = 1. \tag{2.24b}$$

The estimation problem is to find the least squares estimate of x given y on Ω_2 and y_b on its boundary. We will return to this example in Section IV.

C. Operator Form for M_z and the Optimal Estimate

In this section we present expressions for the mapping M_z and for the optimal estimate. Only the continuous parameter case is addressed here; however, with a few obvious changes the same arguments can be adapted to the discrete parameter case (see [5]).

It will be convenient to partition the *underlying process* ζ into the two parts indicated in (2.19a). Here ζ_1 corresponds to the boundary value and input process, and ζ_2 represents the additive noise on the observations. The covariance parameters of the elements of ζ are assumed to be continuous and the covariance parameters and covariance matrices are all assumed invertible. The second-order statistics of ζ can be defined by way of a correlation operator. The range and domain S of this operator are identical and are defined via the following spaces

$$S_1 = L_2^n(\Omega_N) \times L_2^{n_v}(\partial\Omega_N), \tag{2.25a}$$

$$S_2 = L_2^p(\Omega_N) \times L_2^q(\partial\Omega_N) \tag{2.25b}$$

and

$$S = S_1 \times S_2. \tag{2.25c}$$

When $\partial\Omega_N$ is finite (i.e., when $N=1$), the L_2 spaces of functions over $\partial\Omega_N$ should be replaced by the Euclidean spaces R^{n_v} and R^q. The correlation operator Σ_ζ is the self-adjoint invertible mapping which we will express in partitioned form as

$$\Sigma_\zeta = \begin{bmatrix} \Sigma_{\zeta_1} & 0 \\ 0 & \Sigma_{\zeta_2} \end{bmatrix}. \tag{2.26}$$

The observations are defined via the operator M_y: $S \to S_2$, where from (2.19b)

$$M_y = [H \;\vdots\; I]. \tag{2.27}$$

The following theorem establishes an operator representation of the complementary process for this set of observations.

Theorem 1 (Complementary Process): Let M_z be the mapping

$$M_z = [-I \;\vdots\; H^*]\Sigma_\zeta^{-1}; \qquad M_z\colon S \to S_1 \tag{2.28a}$$

where H^* is the adjoint of H, and I is the identity on S_1. Then the stochastic process given by the second-order mapping

$$Z = M_z\zeta \tag{2.28b}$$

is the complementary process for the observations Y, i.e., Z in (2.28b) satisfies both the orthogonality and complementation conditions as prescribed in (2.3) and (2.4).

The theorem is proved in [5] by establishing the orthogonality and complementation conditions in (2.3) and (2.4). In particular, it is shown that for the problem at hand the inverse of the augmented operator M [see (2.4)] is

$$M^{-1} = \left[\begin{array}{c|c} -\left(\Sigma_{\zeta_1}^{-1} + H^*\Sigma_{\zeta_2}^{-1}H\right)^{-1} & \Sigma_{\zeta_1}H^*\left(\Sigma_{\zeta_2} + H\Sigma_{\zeta_1}H^*\right)^{-1} \\ \hline H\left(\Sigma_{\zeta_1}^{-1} + H^*\Sigma_{\zeta_2}^{-1}H\right)^{-1} & \Sigma_{\zeta_2}\left(\Sigma_{\zeta_2} + H\Sigma_{\zeta_1}H^*\right)^{-1} \end{array}\right]. \tag{2.29}$$

Equations (2.5), (2.7), (2.8), and (2.29) provide us with explicit operator representations for the optimal estimate $\hat{x}$ as a functional of the observations Y and for the estimation error as a function of the underlying process ζ. A direct implementation of this operator form of the estimator requires a realization of the inverse of the operator $(\Sigma_{\zeta_2} + H\Sigma_{\zeta_1}H^*)$. As an alternative, in Section III we obtain a realization for the estimator without explicitly performing this inversion.

III. A Differential Operator Representation for the Estimator

A. Introduction

In this section we derive a differential operator representation for the estimator. The key to its derivation is the formulation of a differential operator representation for the complementary process whose I/O map is given in (2.28a). It is in the formulation of this differential representation for the complementary process that the Green's identity introduced in Section II-B plays an important role in that it allows us to determine a differential realization of the operator H^*. With differential representations for both the process to be estimated and the corresponding complementary process, we will find that the augmentation and inversion steps (cf. Section II-A) required in the formulation of the estimator become trivial.

B. The Hilbert Adjoint System

Theorem 1 provides us with a representation of the complementary process. Specifically, using (2.19a), (2.26), and (2.28), we obtain an expression for Z as an output signal plus noise

$$Z = H^*\Sigma_{\zeta_2}^{-1}\zeta_2 - \Sigma_{\zeta_1}^{-1}\zeta_1. \tag{3.1}$$

Our objective in this section is to formulate an internal realization for the input–output map H^*. The internal process in this realization is defined by a differential operator whose input process and boundary condition are the inputs to H^*.

To determine an internal differential realization for H^*, we temporarily leave the stochastic setting. That is, throughout the rest of this subsection all processes should be considered as elements of Hilbert spaces of deterministic functions rather than stochastic processes.

The internal realization for the input–output map H is given by (2.14), together with a noise-free version of the output equation (2.15), i.e.,

$$\Phi = \begin{bmatrix} \phi \\ \phi_b \end{bmatrix} = \begin{bmatrix} Cx \\ Wx_b \end{bmatrix} = H\begin{bmatrix} u \\ v \end{bmatrix}. \tag{3.2}$$

It will be convenient to define the spaces containing u and v as D_u and D_v, respectively, so that the domain of H can be written as $D(H) = D_u \times D_v$. Similarly, define the range spaces containing the output elements ϕ and ϕ_b as R_ϕ and R_{ϕ_b} so that the range of H is $R(H) = R_\phi \times R_{\phi_b}$.

The adjoint of H is defined to be that operator H^* which maps from the range of H into the domain of H and for which the inner product identity

$$\langle H\xi, \eta\rangle_{R(H)} = \langle \xi, H^*\eta\rangle_{D(H)} \tag{3.3}$$

is satisfied for arbitrary ξ and η in $D(H)$ and $R(H)$, respectively [7].

The first step in determining an internal realization for H^* is to rewrite (3.3) in a more convenient form. Since the input u in (2.14) enters only through the action of B, we can decompose H as

$$H = \tilde{H}\begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix}. \tag{3.4a}$$

If we denote the range of B by R_B, then $\tilde{H}$: $(R_B \times D_v) \to R(H)$. Given this decomposition of H, its adjoint H^* can be decomposed as

$$H^* = \begin{bmatrix} B^* & 0 \\ 0 & I \end{bmatrix}\tilde{H}^*. \tag{3.4b}$$

The next step is to partition the processes ζ, $H\zeta$, η, and $H^*\eta$ in a fashion compatible with the corresponding cross-product spaces of which they are elements

$$\zeta = \begin{bmatrix} u \\ v \end{bmatrix}, \qquad \eta = \begin{bmatrix} u_\lambda \\ v_\lambda \end{bmatrix} \tag{3.5}$$

where $u \in D_u$, $v \in D_v$, $u_\lambda \in R_\phi$, $v_\lambda \in R_{\phi_b}$. The partition of $H\zeta$ is given in (3.2), while

$$\tilde{H}^*\eta = \begin{bmatrix} \lambda \\ \psi_b \end{bmatrix}; \qquad \lambda \in R_B, \quad \psi_b \in D_v. \tag{3.6}$$

Substituting (3.2), (3.5), and (3.6) into (3.3), using (2.14) to express Bu and v in terms of x and x_b, and performing some straightforward manipulations, we find that (3.3) reduces to

$$\langle x, C^*u_\lambda\rangle + \langle x_b, W^*v_\lambda\rangle = \langle Lx, \lambda\rangle + \langle x_b, V^*\psi_b\rangle. \tag{3.7}$$

Up to this point we have simply combined some new notation along with that for the internal representation for H to reexpress the inner product identity (3.3). The next step is more substantial and is a key one in the development of the internal realization for H^*. In particular, we employ Green's identity from (2.9) to replace $\langle Lx, \lambda\rangle$ in (3.7). Then (3.3) can be written in terms of the formal adjoint differential (difference) operator $L^\dagger$

$$\left\langle x, \left[C^*u_\lambda - L^\dagger\lambda\right]\right\rangle = \left\langle x_b, \left[E\lambda_b + V^*\psi_b - W^*v_\lambda\right]\right\rangle. \tag{3.8}$$

Although the Hilbert adjoint H^* is a unique map, there exists a family of equivalent internal differential realizations. Using the notation introduced above, we will verify one internal realization for H^* with input η and output $\Psi = \{\psi, \psi_b\}$ by showing that it satisfies (3.8).

Let W_c be one of the family of operators which complements V and W in that Γ in (2.16b) is invertible. Define the partitioned operator

$$\begin{bmatrix} W_\lambda \\ V_{\lambda c} \\ V_\lambda \end{bmatrix} \equiv (\Gamma^*)^{-1}E. \tag{3.9}$$

This leads to an expression for E that will be useful later

$$E = -V^*W_\lambda + W_c^*V_{\lambda c} + W^*V_\lambda. \tag{3.10}$$

The following theorem establishes an internal differential realization for H^*. This realization represents a generalization of the realization found in [14] of the adjoint system for 1-D two-point boundary value processes.

Theorem 2 (Hilbert Adjoint System): An internal differential realization for the input–output map

$$\Psi = \begin{bmatrix} \psi \\ \psi_b \end{bmatrix} = H^*\begin{bmatrix} u_\lambda \\ v_\lambda \end{bmatrix}$$

is given by an internal process λ satisfying

$$L^\dagger\lambda = C^*u_\lambda \tag{3.11a}$$

with boundary condition

$$\begin{bmatrix} V_\lambda \\ V_{\lambda c} \end{bmatrix}\lambda_b = \begin{bmatrix} v_\lambda \\ 0 \end{bmatrix} \tag{3.11b}$$

and output map

$$\Psi = \begin{bmatrix} \psi \\ \psi_b \end{bmatrix} = \begin{bmatrix} B^*\lambda \\ W_\lambda\lambda_b \end{bmatrix}. \tag{3.11c}$$

This result is proved by straightforward calculation using (3.11) to show that both sides of (3.8) are zero (see [5]). Although the differential realization is not unique due to the degrees of freedom in choosing W_c, we will show that the estimator itself is invariant with respect to the choice of W_c, as it must be.

C. Augmentation and Inversion

The internal differential realization for H^* in (3.11) defines a representation for the complementary stochastic process [see (3.1)]

$$Z=\begin{bmatrix} z \\ z_b \end{bmatrix}. \tag{3.12}$$

In this subsection we augment the internal realization for Z with that for the observations. We then invert this realization to obtain an internal differential realization for the estimator.

The differential form for the augmented system is

$$\begin{bmatrix} L & 0 \\ 0 & L^{\dagger} \end{bmatrix}\begin{bmatrix} x \\ \lambda \end{bmatrix}=\begin{bmatrix} B & 0 \\ 0 & C^*R^{-1} \end{bmatrix}\begin{bmatrix} u \\ r \end{bmatrix} \tag{3.13a}$$

with boundary condition

$$\begin{bmatrix} v \\ \hline 0 \\ \Pi_b^{-1}r_b \end{bmatrix}=\begin{bmatrix} V & \vdots & 0 \\ \hline 0 & \vdots & V_{\lambda c} \\ 0 & \vdots & V_{\lambda} \end{bmatrix}\begin{bmatrix} x_b \\ \lambda_b \end{bmatrix} \tag{3.13b}$$

and outputs

$$\begin{bmatrix} y \\ z \end{bmatrix}=\begin{bmatrix} C & 0 \\ 0 & B^* \end{bmatrix}\begin{bmatrix} x \\ \lambda \end{bmatrix}+\begin{bmatrix} 0 & I \\ -Q^{-1} & 0 \end{bmatrix}\begin{bmatrix} u \\ r \end{bmatrix} \quad \text{on } \Omega_N \tag{3.13c}$$

$$\begin{bmatrix} y_b \\ z_b \end{bmatrix}=\begin{bmatrix} W & 0 \\ 0 & W_{\lambda} \end{bmatrix}\begin{bmatrix} x_b \\ \lambda_b \end{bmatrix}+\begin{bmatrix} 0 & I \\ -\Pi_v^{-1} & 0 \end{bmatrix}\begin{bmatrix} v \\ r_b \end{bmatrix} \quad \text{on } \partial\Omega_N. \tag{3.13d}$$

The inverse system we seek is one with $\{Y, Z\}=\{y, y_b, z, z_b\}$ as input and $\zeta=\{u, v, r, r_b\}$ as output. To this end, following the approach taken by Levy *et al.* for the 1-D causal case in [9], we first solve for the elements of ζ by inverting the output equations (3.13c) and (3.13d). Substituting the resulting expressions into the dynamics and boundary conditions in (3.13a) and (3.13b) yields an internal differential realization of the inverse system with dynamics

$$\begin{bmatrix} L & \vdots & -BQB^* \\ \hline C^*R^{-1}C & \vdots & L^{\dagger} \end{bmatrix}\begin{bmatrix} x \\ \lambda \end{bmatrix}=\begin{bmatrix} -BQ & \vdots & 0 \\ \hline 0 & \vdots & C^*R^{-1} \end{bmatrix}\begin{bmatrix} z \\ y \end{bmatrix} \tag{3.14}$$

and with boundary condition

$$\begin{bmatrix} z_b \\ 0 \\ \Pi_b^{-1}y_b \end{bmatrix}=\begin{bmatrix} -\Pi_v^{-1}V & \vdots & W_{\lambda} \\ \hline 0 & \vdots & V_{\lambda c} \\ \hline \Pi_b^{-1}W & \vdots & V_{\lambda} \end{bmatrix}\begin{bmatrix} x_b \\ \lambda_b \end{bmatrix}. \tag{3.15}$$

This boundary condition can be simplified so that its dependence on W_c, V_{λ}, and $V_{\lambda c}$ is eliminated. Recalling the relation between these operators and E in Green's identity from (3.10), it can be shown that operating on the left of (3.15) by $[-V^* \;\vdots\; W_c^* \;\vdots\; W^*]$ gives the boundary condition as

$$\left[W^*\Pi_b^{-1}y_b - V^*z_b\right]=\left[W^*\Pi_b^{-1}W+V^*\Pi_v^{-1}V \;\vdots\; E\right]\begin{bmatrix} x_b \\ \lambda_b \end{bmatrix}. \tag{3.16}$$

The estimator is the solution of (3.14) and (3.16) projected onto $Sp(Y)$, i.e., the solution with $Z=\{z, z_b\}=0$

$$\begin{bmatrix} L & \vdots & -BQB^* \\ \hline C^*R^{-1}C & \vdots & L^{\dagger} \end{bmatrix}\begin{bmatrix} \hat{x} \\ \hat{\lambda} \end{bmatrix}=\begin{bmatrix} 0 \\ \hline C^*R^{-1}y \end{bmatrix} \tag{3.17a}$$

$$W^*\Pi_b^{-1}y_b=\left[W^*\Pi_b^{-1}W+V^*\Pi_v^{-1}V \;\vdots\; E\right]\begin{bmatrix} \hat{x}_b \\ \hat{\lambda}_b \end{bmatrix}. \tag{3.17b}$$

The estimates of the elements of the underlying process ζ, if desired, can be computed from the output equations (3.13c) and (3.13b) evaluated at the solution of (3.17) and with z and z_b equal to zero. Note that since L and $L^{\dagger}$ are of the same order, the order of the estimator is twice that of L. Also note the important fact that in addition to the original problem statement, we only need to know E and $L^{\dagger}$ from Green's identity in (2.9) to completely define the differential realization for the estimator. That is, it is not necessary to actually determine the complete internal differential realization for the complementary process.

D. The Estimation Error

The estimation error $\tilde{x}=x-\hat{x}$ is obtained as the solution of (3.14) and (3.16) projected onto $Sp(Z)$ rather than $Sp(Y)$. Here we formulate a differential realization of the estimation error which is driven by ζ. The second-order statistics of the estimation error can be computed from those of ζ using this relation.

Consider the boundary condition (3.16) projected onto $Sp(Z)$

$$-V^*z_b=\left[W^*\Pi_b^{-1}W+V^*\Pi_v^{-1}V \;\vdots\; E\right]\begin{bmatrix} \tilde{x}_b \\ \tilde{\lambda}_b \end{bmatrix}. \tag{3.18}$$

Substituting for z_b from (3.13d), using the basic definition

$$\tilde{\lambda}_b=\lambda_b-\hat{\lambda}_b \tag{3.19}$$

and employing the relations in (3.10) and (3.13b), the boundary condition (3.18) can be rewritten as

$$\left[V^*\Pi_v^{-1}v - W^*\Pi_b^{-1}r_b\right]=\left[W^*\Pi_b^{-1}W+V^*\Pi_v^{-1}V \;\vdots\; E\right]\begin{bmatrix} \tilde{x}_b \\ -\hat{\lambda}_b \end{bmatrix}. \tag{3.20}$$

We have chosen $-\hat{\lambda}_b$ instead of $\tilde{\lambda}_b$ to highlight the similarity between the structure of the boundary condition for the estimation error in (3.20) and that of the estimator in (3.17).

The projection of (3.14) onto $Sp(Z)$ gives the error dynamics as

$$\begin{bmatrix} L & \vdots & -BQB^* \\ \hline C^*R^{-1}C & \vdots & L^{\dagger} \end{bmatrix}\begin{bmatrix} \tilde{x} \\ \tilde{\lambda} \end{bmatrix}=\begin{bmatrix} -BQz \\ \hline 0 \end{bmatrix}. \tag{3.21}$$

Using the fact that $\tilde{\lambda}=\lambda-\hat{\lambda}$, eliminating z using (3.13c), and noting from (3.13a) that the dynamics of λ are given by

$$L^{\dagger}\lambda=C^*R^{-1}r \tag{3.22}$$

(3.21) can be rewritten as

$$\begin{bmatrix} L & \vdots & -BQB^* \\ \hline C^*R^{-1}C & \vdots & L^{\dagger} \end{bmatrix}\begin{bmatrix} \tilde{x} \\ -\hat{\lambda} \end{bmatrix}=\begin{bmatrix} Bu \\ \hline -C^*R^{-1}r \end{bmatrix}. \tag{3.23}$$

We remark that it can be readily deduced from (3.13c) that the estimate of u is $\hat{u}=QB^*\hat{\lambda}$. Thus, the first row of (3.23) simply states that $L\tilde{x}=B\tilde{u}$, where $\tilde{u}=u-\hat{u}$.

Thus, (3.20) and (3.23) completely define the estimation error in terms of $\zeta=\{u, v, r, r_b\}$ whose probability law is known. In addition, the dynamics and boundary conditions of the estimation error have been shown to be similar to those of the estimator. One should be able to take advantage of these similarities when computing the estimate and its error covariance. For example, see the discussion of the implementation of the estimator and the computation of the error covariance for the 1-D noncausal process in Part II.

IV. The Estimator for Two Examples

A. Introduction

The ease with which one can apply (3.17) to obtain an internal differential representation for the estimator of a noncausal stochastic process is demonstrated in this section. We show that the estimator for the process governed by Poisson's equation introduced earlier takes the form of a fourth-order biharmonic equation. To illustrate the versatility of our solution, we also consider a substantially different process, namely a 1-D discrete boundary value process. It is shown that a special case of the solution we obtain from (3.17) for this discrete process is a well-known form of the solution for the fixed-interval smoother for 1-D discrete *causal* processes [9].

B. 2-D Continuous Case: Poisson's Equation

The problem statement is given by (2.20)–(2.24). Substituting the Laplacian ∇^2 for $L^\dagger$ in the estimator solution (3.17), we obtain the estimator dynamics as

$$\left[\begin{array}{c:c} \nabla^2 & -B^2(s)Q(s) \\ \hdashline C^2(s)R^{-1}(s) & \nabla^2 \end{array}\right]\begin{bmatrix}\hat{x}(s)\\ \hat{\lambda}(s)\end{bmatrix} = \left[\begin{array}{c} 0 \\ \hdashline C(s)R^{-1}(s)\end{array}\right] y(s). \quad (4.1)$$

From (2.20b) and (2.24b) the boundary condition and boundary observation can be expressed by functions on $[0,2\pi]$ as

$$(Vx_b)(\theta) = \left[V(\theta) \ \vdots\ 0\right] x_b(\theta) \quad (4.2a)$$

and

$$(Wx_b)(\theta) = \left[0 \ \vdots\ W(\theta)\right] x_b(\theta) \quad (4.2b)$$

where we recall that $x_b'(\theta) = [x(1,\theta), x_n(1,\theta)]$. Using this expression and substituting for E from (2.23b), it can be shown that the boundary condition for (4.1) is (in polar coordinates evaluated at $\rho = 1$)

$$\left[\begin{array}{c} 0 \\ \hdashline W(\theta)\Pi_b^{-1}(\theta)y_b(\theta)\end{array}\right] = \left[\begin{array}{c:c} V^2(\theta)\Pi_v^{-1}(\theta) & 0 \\ \hdashline 0 & 1\end{array}\right]\begin{bmatrix}\hat{x}(\rho,\theta)\\ \hat{\lambda}(\rho,\theta)\end{bmatrix} + \left[\begin{array}{c:c} 0 & -1 \\ \hdashline W^2(\theta)\Pi_b^{-1}(\theta) & 0\end{array}\right]\begin{bmatrix}\hat{x}_n(\rho,\theta)\\ \hat{\lambda}_n(\rho,\theta)\end{bmatrix}. \quad (4.3)$$

When $B^2(s)Q(s) > 0$ for all s, we can solve for $\hat{\lambda}$ in (4.1) as

$$\hat{\lambda}(s) = \left[B^2(s)Q(s)\right]^{-1}\nabla^2\hat{x}(s). \quad (4.4)$$

Substituting (4.4) back into (4.1), we find that the estimator dynamics are given by the biharmonic equation

$$\left\{\nabla^2\left[B^2(s)Q(s)\right]^{-1} + C^2(s)R^{-1}(s)\right\}\nabla^2\hat{x}(s) = C(s)R^{-1}(s)y(s). \quad (4.5)$$

With $\partial/\partial n$ denoting the normal derivative and substituting from (4.4), the boundary condition in (4.3) can be rewritten as

$$0 = \Pi_v^{-1}(\theta)\hat{x}(\rho,\theta) - (\partial/\partial n) \cdot\left\{\left[B^2(\rho,\theta)Q(\rho,\theta)\right]^{-1}\nabla^2\hat{x}(\rho,\theta)\right\} \quad (4.6a)$$

and

$$W(\theta)y_b(\theta) = W(\theta)^2\hat{x}_n(\rho,\theta) + \Pi_b(\theta)\left[B^2(\rho,\theta)Q(\rho,\theta)\right]^{-1}\nabla^2\hat{x}(\rho,\theta) \quad (4.6b)$$

evaluated at $\rho = 1$. We have not investigated analytical or numerical solutions for this biharmonic equation. In practice, these equations might be solved by an application of one of the many available numerical techniques such as finite difference approximations [13].

C. 1-D Discrete Case: Two-Point Boundary Value Process

This example is the discrete version of a 1-D boundary value estimation problem originally posed by Krener [14]. For this example, $N = 1$, Ω_1 is the set of integers $[0, K-1]$, and $\partial\Omega_1$ is the set $\{0, K\}$. Let u be an $m \times 1$ vector white noise over Ω_1 with nonsingular covariance matrix Q_k, $k \in \Omega_1$. Let v be an $n \times 1$ random vector with nonsingular covariance matrix Π_v. Let B_k be an $n \times m$ matrix and A_k be a $n \times n$ matrix both on Ω_1, and let V be a full rank $n \times 2n$ matrix with $n \times n$ partitions $[V^0 \ \vdots\ V^K]$. The process to be estimated is defined by the difference equation

$$x_{k+1} = A_k x_k + B_k u_k \quad (4.7a)$$

with a two-point boundary condition

$$v = V^0 x_0 + V^K x_K. \quad (4.7b)$$

If we let D denote the 1-D delay

$$(Dx)_k = x_{k-1} \quad (4.8)$$

then L is the 1-D difference operator

$$L = (D^{-1}I - A); \qquad (Lx)_k = x_{k+1} - A_k x_k. \quad (4.9a)$$

Note that the range and domain of L are properly specified by

$$L: 1_2^n[0,K] \to 1_2^n[0,K-1] \quad (4.9b)$$

where $[0,k] = \Omega_1 U \partial\Omega_1$. This illustrates an important point. That is, due to sequencing issues for discrete dynamics, it will in general be the case for discrete problems that $\partial\Omega_N$ is neither disjoint from nor a subset of Ω_N.

The Green's identity for this example is

$$\langle Lx, \lambda\rangle_{1_2^n[0,K-1]} = \langle x, L^\dagger\lambda\rangle_{1_2^n[0,K-1]} + \langle x_b, E\lambda_b\rangle_{R^{2n}} \quad (4.10)$$

where the formal adjoint difference operator is

$$L^\dagger = (I - A'D^{-1}); \qquad (L^\dagger x)_k = x_k - A_k' x_{k+1}, \quad (4.11)$$

the boundary process is

$$x_b = \Delta_b x = \begin{bmatrix} x_0 \\ x_K \end{bmatrix}, \quad (4.12a)$$

and E is a $2n \times 2n$ matrix partitioned into $n \times n$ blocks as

$$E = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix}. \quad (4.12b)$$

With V the $n \times 2n$ matrix defined earlier, the two-point boundary condition (4.7b) is given by the product

$$v = Vx_b. \quad (4.13)$$

Here $v \in R^n$, i.e., $n_v = n$.

To define the observations, let r be a $p \times 1$ white noise over Ω_1 whose covariance matrix R_k is nonsingular on Ω_1, and let r_b be a $q \times 1$ random vector with nonsingular covariance matrix Π_b. Let C_k be a $p \times n$ matrix on Ω_1 and let W be a full rank $q \times 2n$ matrix with $q \leqslant n$, with the rows of W linearly independent of the rows of V and with $q \times n$ partitions: $[W^0 \ \vdots\ W^K]$. Then the observations are given by

$$y_k = C_k x_k + r_k \qquad \text{on } \Omega_1 \tag{4.14a}$$

along with the random vector

$$y_b = W^0 x_0 + W^K x_K + r_b. \tag{4.14b}$$

The input processes u and v and the observation noises r and r_b are all assumed to be mutually uncorrelated.

Again, to obtain the estimator we simply substitute from the problem statement and from Green's identity for $L^\dagger$ and E into (3.17). This gives the estimator dynamics as

$$\begin{bmatrix} \hat{x}_{k+1} \\ \hat{\lambda}_k \end{bmatrix} = \left[\begin{array}{c:c} A_k & B_k Q_k B_k' \\ \hdashline -C_k' R_k^{-1} C_k & A_k' \end{array}\right] \begin{bmatrix} \hat{x}_k \\ \hat{\lambda}_{k+1} \end{bmatrix} + \left[\begin{array}{c} 0 \\ \hdashline C_k' R_k^{-1} \end{array}\right] y_k \tag{4.15a}$$

and boundary condition as

$$\begin{aligned} \begin{bmatrix} W^{0'} \\ W^{K'} \end{bmatrix} \Pi_b^{-1} y_b &= \left[\begin{array}{c:c} V^{0'}\Pi_v^{-1}V^0 + W^{0'}\Pi_b^{-1}W^0 & -I \\ \hdashline V^{K'}\Pi_v^{-1}V^0 + W^{K'}\Pi_b^{-1}W^0 & 0 \end{array}\right] \begin{bmatrix} \hat{x}_0 \\ \hat{\lambda}_0 \end{bmatrix} \\ &= \left[\begin{array}{c:c} V^{0'}\Pi_v^{-1}V^K + W^{0'}\Pi_b^{-1}W^K & 0 \\ \hdashline V^{K'}\Pi_v^{-1}V^K + W^{K'}\Pi_b^{-1}W^K & I \end{array}\right] \begin{bmatrix} \hat{x}_K \\ \hat{\lambda}_K \end{bmatrix}. \end{aligned} \tag{4.15b}$$

If we consider the special case of *no* boundary observation y_b (i.e., $W^0 = W^K = 0$) and an *initial* condition for x (i.e., $V^0 = I, V^K = 0$), then the boundary condition in (4.15b) becomes

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} I & -\Pi_v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_0 \\ \hat{\lambda}_0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \hat{x}_K \\ \hat{\lambda}_K \end{bmatrix}. \tag{4.16}$$

This boundary condition along with the dynamics in (4.15a) is recognized as the well-known solution for the fixed-interval smoother for causal discrete 1-D stochastic processes [9]. It is shown in [5] that the solution for the estimator with the general form of the boundary condition (4.15b) can be implemented via a two-filter form similar to two-filter forms of the solution for the smoother for causal processes. See Part II for the continuous-time counterpart of this result.

V. Conclusions

Through an extension of the method of complementary models [1], we have developed a procedure for writing the estimator for both discrete and continuous parameter linear boundary value stochastic processes in a differential operator form. The two major steps in the development of the estimator have been 1) the formulation of an input–output operator representation for the complementary process in Section II and 2) the use of Green's identity in Section III in the derivation of an internal differential realization for this input–output map. We emphasize that at no point in our derivations have we required a Markovian representation for the process to be estimated. The variety of problems for which our estimator solution is applicable has been illustrated through two examples: a 1-D discrete parameter process; and a 2-D continuous parameter process.

The major advantage in having a differential realization for the estimator is that this form of representation provides an excellent starting point for the development of methods for implementing the estimator. This is in contrast to estimators derived by a direct application of the projection theorem, which usually leads to integral equations (e.g. Wiener–Hopf) requiring factorization in order to obtain an implementation. Furthermore, we have also derived an internal differential realization for the estimation errors in a form which is nearly identical to that for the estimator.

In Part II of this paper [4] we apply the estimator solution formulated in this paper to a continuous 1-D two-point boundary value stochastic process and develop a stable, recursive implementation for the resulting differential form of the estimator. In addition, by following the same procedures as used to obtain the recursive estimator implementation, we develop recursions for the computation of the smoothing error covariance. Investigations of the implementation of estimators for discrete 1-D and 2-D and continuous 2-D processes can be found in [5].

In addition to questions of implementation, there are also interesting unanswered questions which relate to the boundary conditions for multidimensional problems. For example, recall from (3.17b) that the boundary condition for our estimator is defined in terms of the operator adjoints V^* and W^* and the inverses of the correlation operators Π_v and Π_b. In our 2-D example we have tacitly avoided any complications which might arise in determining these adjoints and inverses by choosing v and r_b as white noise and by choosing V and W as a simple scaling of the process on the boundary [see (4.2a), (4.2b)]. It would be of interest to investigate the estimator for this 2-D example when the boundary value v is a 1-D periodic stochastic process on the unit circle. Another issue concerning the estimator boundary condition, namely the case when Π_v is singular, is addressed in [5]. In particular, it is shown that in this case the estimator boundary condition is somewhat more complex than that in (3.17b) but that it does allow us to determine estimator for a variety of problems including those involving 1-D periodic processes.

In summary, we feel that this paper presents an extremely useful and broadly applicable method for deriving optimal linear estimators for noncausal processes in one and several dimensions. Given this valuable tool, one is then in a position to focus one's attention on the problem of implementing the optimal estimator in an efficient fashion. As mentioned previously, this is precisely what is done in Part II for the case of 1-D continuous parameter processes and in [5] for 2-D processes. In the more general, multidimensional case many open questions remain, but the results in this paper bring us significantly closer to answering them.

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Bernard C. Levy (S'74–S'78–M'78), for a photograph and biography, see p. 220 of the March 1984 issue of this TRANSACTIONS.

Linear Estimation of Boundary Value Stochastic Processes—Part II: 1-D Smoothing Problems

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Abstract—**This paper addresses the fixed-interval smoothing problem for linear two-point boundary value stochastic processes of the type introduced by Krener [5]. As these models are not Markovian, Kalman filtering and associated smoothing algorithms are not applicable. The smoothing problem for this class of noncausal processes is solved here by an application of the estimator solution which is developed in Part I of this paper [3] via the method of complementary models. For an nth-order model, this approach yields the smoother as a $2n$th-order two-point boundary value problem. It is shown that this smoother can be realized in a stable two-filter form which is remarkably similar to two-filter smoothers for causal processes. In addition, expressions for the smoothing error and smoothing error covariance are developed. These equations are employed to perform a covariance analysis of estimating the temperature and heat flow in a cooling fin.**

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I. INTRODUCTION

BOTH linear filtering and linear smoothing for one-dimensional (1-D), nonstationary, causal processes have been extensively studied. Many of the classical solutions to these problems are discussed in the review paper by Kailath [1]. The derivations of these solutions have relied heavily on the Markovian nature of the models for these 1-D processes [2]. However, inasmuch as stochastic processes in higher dimensions (random fields) are typically noncausal, and consequently are not Markovian in the usual sense, their estimators cannot be derived through a direct extension of these 1-D derivations. Thus, linear estimation problems for noncausal processes require new approaches. One such new approach has been developed in Part I of this paper [3] where we have extended Weinert and Desai's [4] method of complementary models. This extension allows us to write solutions to estimation problems for a broad class of noncausal processes in one and higher dimensions. In this paper we build upon this solution procedure in order to perform a detailed investigation of the smoothing problem for 1-D noncausal processes.

The processes that we consider are governed by the linear noncausal 1-D dynamic models introduced by Krener in [5]. In his study of these models, he has developed results on controllability, observability, and minimality and has solved a determinis-

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