The Stochastic Analysis of Dynamic Systems Moving Through Random Fields

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Abstract—In this paper we consider dynamic systems that move along specified trajectories across random fields, where the field acts as a driving force to the dynamic system. For a specific class of random fields we develop equations for the evolution of the covariance of the state of the dynamic system, and in the special case in which the trajectory is a straight line path followed by a 180° turn (i.e., an "over-and-back" trajectory) we develop a Markovian model that involves a change in the dimension of the state after the turn. For this case we also briefly discuss the estimation problem using recently developed results on "real-time smoothing."

I. Introduction

The problem we consider in this paper is depicted in Fig. 1. We have a vehicle that traverses a specified trajectory $(\eta_1(t), \eta_2(t))$ on a planar surface. Aboard this object is a dynamic system which is affected by a random field $f(\eta_1, \eta_2)$. We would like to determine the statistical characteristics of the state $x(t)$ of the system from the specified trajectory and the statistical description of the random field. Problems of this type arise in the analysis of terrestrial inertial navigation systems [1], [2], [6], [7] where f represents the errors in our knowledge of the variations in gravity and x consists of navigation errors. Since the inertial system's accelerometers measure actual acceleration plus gravity, an estimate of gravity, from a gravity map of some sort, must be subtracted from the accelerometer outputs so that the vehicle's position and velocity can be determined. Thus map errors drive the dynamics of the navigation system. In this application a problem of great practical importance is the determination of accuracy requirements in mapping the gravity field in order to achieve specified error requirements for the navigation system. To assess the effect of errors in a map one specifies typical trajectories over portions of the surface of the earth and a statistical model of the errors in a given map and then evaluates the second-order statistics of the navigation errors. This is precisely the type of problem addressed in this paper.

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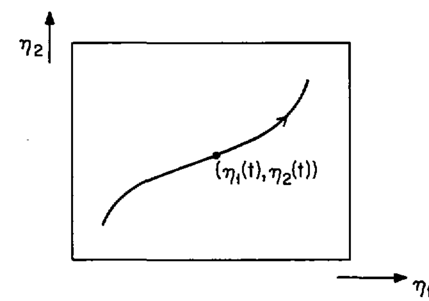


Fig. 1.

Because of the presence of turns in a given trajectory, existing methods for performing these second moment calculations involve nonrecursive calculations, i.e., numerical evaluation of integrals. In this paper we present a recursive approach to evaluating these second-order statistics in a far more efficient manner. Specifically, in the next section we develop sets of trajectory-dependent differential equations for the evolution of the covariance of $x(t)$ for a particular class of random fields. For the special case of a straight line trajectory that reverses on itself, we develop in Section III a novel Markovian representation for the process $x(t)$ and discuss the use of this representation, together with recent results on the real-time updating of smoothed estimates, to solve an estimation problem.

II. Covariance Analysis for Motion Through a Two-Dimensional Random Field

Let $f(\eta_1, \eta_2)$ be a two-dimensional (2-D) stationary random field which, for simplicity, we assume to be zero mean. The correlation matrix for this field is

$$E[f(t,s)f'(0,0)] = R(t,s). \tag{2.1}$$

It is easily seen from (2.1) that

$$R(t,s) = R'(-t,-s). \tag{2.2}$$

Let $(\eta_1(t), \eta_2(t))$ be a specified trajectory through the plane and consider a dynamic system driven by the field along the trajectory

$$\dot{x}(t) = Ax(t) + f(\eta_1(t), \eta_2(t)) + w(t) \tag{2.3}$$

where $w(t)$ is a zero-mean white process with

$$E[w(t)w'(\tau)] = Q\delta(t-\tau). \tag{2.4}$$

We assume that the initial condition $x(0)$ is zero mean and that $x(0)$, w, and f are mutually independent. We would like to determine the evolution of

$$P(t)=E[x(t)x'(t)]. \tag{2.5}$$

We will put further restrictions on the field f that, as we will see, lead to $P(t)$ being specified by a finite set of matrix differential equations. Specifically, we will assume that the covariance R is separable:

$$R(t,s)=R_1(t)R_2(s) \tag{2.6}$$

where we assume that R_1 and R_2 are square, that

$$R_1(t)=R_1'(-t), \qquad R_2(s)=R_2'(-s), \tag{2.7}$$

and that

$$R_i(t)=H_i e^{F_i t} G_i, \qquad t \geq 0, \qquad i=1,2. \tag{2.8}$$

From (2.1), (2.6), and (2.7) we can also deduce that R_1 and R_2 commute for any values of their arguments. This model is a continuous-time analog of the model examined by Attasi [3]. In particular, the 2-dimensional spectrum of f is separable and rational.

As a first step in obtaining the desired equations for $P(t)$, define

$$Q(t,s)=R(\eta_1(t)-\eta(s),\eta_2(t)-\eta_2(s)). \tag{2.9}$$

Then, writing

$$x(t)=e^{At}x(0)+\int_0^t e^{A(t-\tau)}[f(\eta_1(\tau),\eta_2(\tau))+w(\tau)]\,d\tau \tag{2.10}$$

we can obtain an expression for $P(t)$ from (2.5). By differentiating this expression we obtain the basic equations

$$\dot{P}(t)=AP(t)+P(t)A'+L(t)+L'(t)+Q \tag{2.11}$$

$$L(t)=\int_0^t Q(t,\tau)e^{A'(t-\tau)}\,d\tau. \tag{2.12}$$

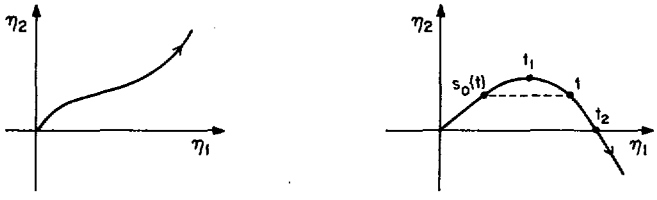
The problem then becomes one of determining a set of differential equations for $L(t)$. This calculation depends upon the nature of the trajectory. We will examine two types of trajectories in this section. For simplicity, we will assume throughout that $\eta_1(0)=\eta_2(0)=0$.

Case 1: This is the simplest case in which we do not change the quadrant toward which we are heading. That is, if we choose the northeast as the direction of motion, we have the situation depicted in Fig. 2(a) where

$$\begin{aligned} &\eta_1(t)-\eta_1(s)\geq 0\\ &\eta_2(t)-\eta_2(s)\geq 0 \qquad \forall t\geq s. \end{aligned} \tag{2.13}$$

In this case, using (2.6)–(2.9) we find that (2.12) can be written as

$$\begin{aligned} L(t)&=\int_0^t H_1 e^{F_1(\eta_1(t)-\eta_1(\sigma))}G_1H_2e^{F_2(\eta_2(t)-\eta_2(\sigma))}G_2e^{A'(t-\sigma)}\,d\sigma\\ &\triangleq H_1B_1(t). \end{aligned} \tag{2.14}$$



(a) (b)

Fig. 2. Two simple trajectory types. (a) Case 1. (b) Case 2.

Differentiating $B_1(t)$, we obtain

$$\begin{aligned} \dot{B}_1(t)=\;&\dot{\eta}_1(t)F_1B_1(t)+B_1A'+G_1H_2G_2\\ &+\dot{\eta}_2(t)\int_0^t e^{F_1(\eta_1(t)-\eta_1(\sigma))}\\ &\cdot G_1H_2F_2e^{F_2(\eta_2(t)-\eta_2(\sigma))}G_2e^{A'(t-\sigma)}\,d\sigma. \end{aligned} \tag{2.15}$$

Note that the last term in (2.15) is similar to $B_1(t)$, except for the F_2 factor in the middle of the expression. This leads to the following idea. Define

$$\begin{aligned} B_j(t)=\int_0^t &e^{F_1(\eta_1(t)-\eta_1(\sigma))}G_1H_2F_2^{j-1}e^{F_2(\eta_2(t)-\eta_2(\sigma))}\\ &\cdot G_2e^{A'(t-\sigma)}\,d\sigma. \end{aligned} \tag{2.16}$$

We know that there is an integer r and coefficients $\rho_0,\cdots,\rho_{r-1}$ such that

$$F_2^r=\sum_{j=1}^{r-1}\rho_jF_2^j. \tag{2.17}$$

Then, by repeated differentiation we obtain the equations

$$L(t)=H_1B_1(t) \tag{2.18}$$

$$\begin{aligned} \dot{B}_j(t)=\;&\dot{\eta}_1(t)F_1B_j(t)+B_j(t)A'+\dot{\eta}_2(t)B_{j+1}(t)\\ &+G_1H_2F_2^{j-1}G_2 \qquad 1\leq j\leq r-1 \end{aligned} \tag{2.19}$$

$$\begin{aligned} \dot{B}_r(t)=\;&\dot{\eta}_1(t)F_1B_r(t)+B_r(t)A'+\dot{\eta}_2(t)\sum_{j=1}^{r}\rho_{j-1}B_j(t)\\ &+G_1H_2F_2^{r-1}G_2 \end{aligned} \tag{2.20}$$

$$B_j(0)=0, \qquad j=1,\ldots,r. \tag{2.21}$$

Note that we can obtain alternative analogous equations with the roles of F_1 and F_2 reversed if we use the commutativity of $R_1(t)$ and $R_2(t)$. In either case we obtain a finite set of linear matrix differential equations for L and therefore for P. Note that if the trajectory is a straight line (i.e., $\dot{\eta}_1(t)=\alpha$, $\dot{\eta}_2(t)=\beta$), as is often assumed in the literature because of an inability to handle the more general case, then these equations are time-invariant and are equivalent to a single classical covariance equation. This equation is associated with an augmented state equation consisting of the dynamics for x, together with a shaping filter for $f(\eta_1(t),\eta_2(t))$.

Case 2: In this case, illustrated in Fig. 2(b) we have a change of quadrants from northeast to southeast. Clearly the following analysis also holds for any turn from one quadrant into an adjacent one. Mathematically, if s and t are two instants of time with $s\leqslant t$, then

$$\eta_1(t)-\eta_1(s)\geqslant 0 \qquad \forall s\leqslant t \tag{2.22a}$$

$$\eta_2(t)-\eta_1(s)\begin{cases}\geqslant 0 & \text{if } t\leqslant t_1 \text{ or } s\leqslant s_0(t)\\ \leqslant 0 & \text{if } t\geqslant t_1 \text{ and } s\geqslant s_0(t)\end{cases} \tag{2.22b}$$

where $s_0(t)$ is defined in the figure. Here we assume that $\dot\eta_2(t)\neq 0$, except for isolated points in time, so that $s_0(t)$ is well-defined. If $\dot\eta_2(t)\equiv 0$ for an interval of time, a segment of the target's trajectory is a straight line, due east. This case requires special but straightforward treatment. In Fig. 2(b) t_1 is the time at which our turn takes us into another quadrant in direction, and t_2 is the time at which $s_0(t)=0$.

For $t\leqslant t_1$, the analysis of this case is identical to that for Case 1. Thus, consider $t_1\leqslant t\leqslant t_2$ and let us break up the integral expression for $L(t)$ as follows:

$$\begin{aligned}L(t)&=\int_0^{s_0(t)}H_1e^{F_1(\eta_1(t)-\eta_1(s))}G_1H_2e^{F_2(\eta_2(t)-\eta_2(s))}G_2e^{A'(t-s)}\,ds\\&\quad+\int_{s_0(t)}^{t}H_1e^{F_1(\eta_1(t)-\eta_1(s))}G_1G_2'e^{F_2'(\eta_2(s)-\eta_2(t))}\\&\quad\cdot H_2'e^{A'(t-s)}\,ds\end{aligned} \tag{2.23}$$

where we have used the fact that $R_2(\eta_2(t)-\eta_2(s))=R_2'(\eta_2(s)-\eta_2(t))$. In differentiating (2.23) we will need to calculate $\dot s_0(t)$. This can be done as follows. By definition of $s_0(t)$,

$$\eta_2(s_0(t))=\eta_2(t). \tag{2.24}$$

Therefore

$$\dot\eta_2(s_0(t))\dot s_0(t)=\dot\eta_2(t) \tag{2.25}$$

or

$$\dot s_0(t)=\frac{\dot\eta_2(t)}{\dot\eta_2(s_0(t))}. \tag{2.26}$$

Note that if $\dot\eta(s_0(t))=0$ at an isolated time instant, as it is in Fig. 3, we will have to evaluate higher derivatives. This causes no conceptual difficulty, but it complicates the development. Therefore we will assume for simplicity that there are no inflection points in the trajectory over the interval $[0,t_1)$.

Let

$$B_j(t)=\int_0^{s_0(t)}e^{F_1(\eta_1(t)-\eta_1(s))}G_1H_2F_2^{j-1}e^{F_2(\eta_2(t)-\eta_2(s))}\cdot G_2e^{A'(t-s)\,ds} \qquad j=1,\cdots,r. \tag{2.27}$$

Note that if we define $s_0(t)=t$ for $t<t_1$, and B_j is precisely the quantity in (2.16) and thus the initial condition at time t_1 for B_j in (2.27) is $B_j(t_1)$ calculated from (2.19)–(2.21). If we now differentiate (2.27) and use (2.17) we find that

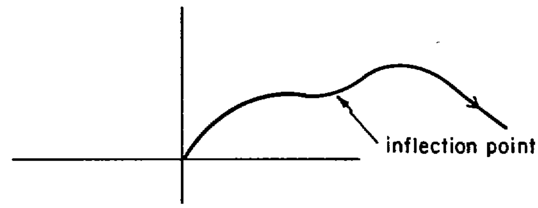


Fig. 3.

$$\begin{aligned}\dot B_j(t)&=\dot\eta_1(t)F_1B_j(t)+B_j(t)A'+\dot\eta_2(t)B_{j+1}(t)\\&\quad+\frac{\dot\eta_2(t)}{\dot\eta_2(s_0(t))}e^{F_1[\eta_1(t)-\eta_1(s_0(t))]}G_1H_2F_2^{j-1}G_2e^{A'(t-s_0(t))}\\&\qquad j=1,\cdots,r-1\end{aligned} \tag{2.28}$$

$$\begin{aligned}\dot B_r(t)&=\dot\eta_1(t)F_1B_r(t)+B_r(t)A'+\dot\eta_2(t)\sum_{j=1}^{r}\rho_{j-1}B_j(t)\\&\quad+\frac{\dot\eta_2(t)}{\dot\eta_2(s_0(t))}e^{F_1[\eta_1(t)-\eta_1(s_0(t))]}G_1H_2F_2^{r-1}G_2e^{A'(t-s_0(t))}.\end{aligned} \tag{2.29}$$

Note that

$$B_j(t_2)=0 \tag{2.30}$$

since $s_0(t_2)=0$.

Now let

$$C_j(t)=\int_{s_0(t)}^{t}e^{F_1(\eta_1(t)-\eta_1(s))}G_1G_2'(F_2')^{j-1}\cdot e^{F_2'(\eta_2(s)-\eta_2(t))}H_2'e^{A'(t-s)}\,ds. \tag{2.31}$$

Then

$$\begin{aligned}\dot C_j(t)&=\dot\eta_1(t)F_1C_j(t)+C_j(t)A'-\dot\eta_2(t)C_{j+1}(t)\\&\quad+G_1G_2(F_2')^{j-1}H_2'-\frac{\dot\eta_2(t)}{\dot\eta_2(s_0(t))}e^{F_1(\eta_1(t)-\eta_1(s_0(t)))}\\&\quad\cdot G_1G_2'(F_2')^{j-1}H_2e^{A'(t-s_0(t))}\end{aligned} \tag{2.32}$$

$$\begin{aligned}\dot C_r(t)&=\dot\eta_1(t)F_1C_r(t)+C_r(t)A'\\&\quad-\dot\eta_2(t)\sum_{j=1}^{r}\rho_{j-1}C_j(t)+G_1G_2'(F_2')^{r-1}H_2'\\&\quad-\frac{\dot\eta_2(t)}{\dot\eta_2(s_0(t))}e^{F_1(\eta_1(t)-\eta_1(s_0(t)))}\\&\quad\cdot G_1G_2'(F_2')^{r-1}H_2e^{A'(t-s_0(t))}\end{aligned} \tag{2.33}$$

$$C_j(t_1)=0, \qquad j=1,\cdots,r. \tag{2.34}$$

Then

$$L(t)=H_1[B_1(t)+C_1(t)] \qquad t_1\leqslant t\leqslant t_2. \tag{2.35}$$

Note that in the case of a piecewise linear trajectory, such as

$$\dot{\eta}_1(t)=\begin{cases}\alpha_1 & t<t_1\\ \alpha_2 & t>t_1\end{cases} \tag{2.36}$$

$$\dot{\eta}_2(t)=\begin{cases}\beta_1 & t<t_1\\ \beta_2 & t>t_1\end{cases}, \tag{2.37}$$

then we have

$$\dot{s}_0(t)=\frac{\beta_2}{\beta_1} \tag{2.38}$$

which is negative here since $\beta_1>0$, $\beta_2<0$.

We now need only piece together the situation for $t\geqslant t_2$. In this case

$$L(t)=\int_0^t H_1 e^{F_1(\eta_1(t)-\eta_1(s))}G_1G_2' e^{F_2'(\eta_2(s)-\eta_2(t))}H_2' e^{A'(t-s)}\,ds. \tag{2.39}$$

Thus in this region

$$L(t)=H_1C_1(t) \qquad t\geqslant t_2 \tag{2.40}$$

where $C_1(t_2)$ is obtained from (2.32)–(2.34), and for $t>t_2$

$$\dot{C}_j(t)=\dot{\eta}_1(t)F_1C_j(t)+C_j(t)A'-\dot{\eta}_2(t)+C_{j+1}(t) + G_1G_2'(F_2')^{j-1}H_2' \qquad j=1,\cdots,r-1 \tag{2.41}$$

$$\dot{C}_r(t)=\dot{\eta}_1(t)F_1C_r(t)+C_r(t)A'-\dot{\eta}_2(t)\sum_{j=1}^{r}\rho_{j-1}C_j(t) + G_1G_2'(F_2')^{r-1}H_2'. \tag{2.42}$$

Thus in Case 2, over the time interval $[0,t_1]$ we have one set of equations to calculate L; over the interval $[t_1,t_2]$, while the η_2 coordinate of the trajectory retraces its path over $[0,t_1]$, we have two sets of equations; and for $t>t_2$ we are back to a single equation which is essentially the equation obtained in Case 1, except that here we have a southeasterly trajectory as opposed to the northeasterly trajectory of Fig. 2(a). It should be clear that we can do this for arbitrary trajectories. Only during the "transient" of a turn do we pick up additional equations. In the Appendix we briefly describe one somewhat more complex case in which both η_1 and η_2 coordinates simultaneously retrace previous values (this does *not* mean a trajectory that retraces itself—see the Appendix, Fig. 5.). In that case, there are two additional sets of equations. From the cases considered in this section and in the Appendix it is not difficult to see that at any time we must include m additional sets of equations, where m is the total number of previous times in the trajectory that either the η_1 or η_2 coordinate of the trajectory equals the corresponding coordinate at the present time. If A is a stable matrix, then the effect on $L(t)$ [and hence $P(t)$] of a trajectory turn far in the past becomes insignificant. This can be seen in (2.28) where the driving term goes to zero exponentially as $t-s_0(t)\to\infty$ (the matrices F_1 and F_2 are stable since f is a stationary process with finite covariance). Thus in practice we need only keep track of turns within a certain number of time constants of A and correlation distances of the field (inverses of the magnitudes of the eigenvalues of F_1 and F_2).

The equations derived in this section and in the Appendix appear much more complex than the standard Lyapunov equation. However, in important special cases they simplify considerably. For example, in the case of an exponentially correlated random field,

$$R(t,s)=\sigma_f^2 e^{-|t-s|}, \tag{2.43}$$

and a Case 1 trajectory, only a single vector equation is needed in addition to (2.11), and such an equation would be needed even in the straight-line case. In any event, covariance propagation is a widely accepted technique for navigation system analysis, and the equations we have obtained permit analysis of more realistic situations than simple straight-line trajectories.

Finally, we note that since $L(t)$ is a linear functional of $Q(t,s)$, it is clear that our results extend in an obvious fashion to $R(t,s)$ of the form

$$\sum_{k=1}^{p} R_{1k}(t)R_{2k}(s)$$

where for each k, R_{1k} and R_{2k} satisfy (2.7) and (2.8). Note that any field with a separable spectrum has such a covariance, and such spectra can be used to approximate arbitrary nonseparable spectra to any desired degree of accuracy. Furthermore, we clearly can also consider random fields with nonstationary but separable correlation functions for which (2.8) is replaced by an equivalent time-varying realizability condition, which essentially guarantees that $L(t)$ can be calculated from a finite set of differential equations.

III. Markov-Type Models for Over-and-Back Trajectories

In the preceding section and in the Appendix we performed some lengthy but relatively straightforward calculations to obtain sets of differential equations for the propagation of the covariance of the state of a dynamic system moving through a random field. The primary contribution of that analysis is to provide an understanding of how the geometry of the trajectory affects the state covariance propagation. While this is quite useful, there is still a great deal left to understand about the fundamental way in which the uncertainty in the field affects the statistics of the process $x(t)$. In this section we will develop a Markov-like description for the special case of over-and-back trajectories. This not only provides us with further insight into the evolution of $x(t)$ but, together with the recent results on smoothing reported in [5], it provides us with the key to deriving efficient estimation algorithms for processes of this type.

The case that we will examine in this section involves a trajectory consisting of a straight line path followed by a reversal of direction and a return trajectory over the same path. We also will assume a constant velocity (normalized to 1) over both segments of the path, but this assumption is made only for clarity in our exposition as is our assumption that the dynamic system is time invariant. It should be clear from our analysis how our results can be modified to account for nonuniform velocity and time-varying systems. Finally, to make our discussion precise mathematically, we will use stochastic differential equations to describe the evolution of the processes of interest.

Consider the model

$$dx(t)=[Ax(t)+u(t)]\,dt+dw_1(t), \qquad 0\le t\le 2T \tag{3.1}$$

where w_1 is a Brownian motion process with

$$E[dw_1(t)\,dw_1'(t)]=S_1\,dt \tag{3.2}$$

and where

$$u(t)=\begin{cases} f(t) & 0\le t\le T \\ f(2T-t) & T\le t\le 2T \end{cases}. \tag{3.3}$$

Here f is a one-dimensional process (representing the field along the track), and we assume that f can be modeled as the output of a finite-dimensional shaping filter

$$d\xi(t)=F\xi(t)\,dt+dw_2(t) \qquad 0\le t\le T \tag{3.4}$$

$$f(t)=H\xi(t) \tag{3.5}$$

where w_2 is Brownian motion with

$$E[dw_2(t)\,dw_2'(t)]=S_2\,dt. \tag{3.6}$$

We assume that all of the processes above are zero mean and Gaussian and that $x(0)$, w_1, $\xi(0)$, and w_2 are mutually independent.

For $0\le t\le T$ we have the same situation as in Case 1 considered in the preceding section. Over this time interval, while we are going forward, the joint process

$$x_2(t)=\begin{bmatrix}\xi(t)\\ x(t)\end{bmatrix} \tag{3.7}$$

is Markovian, with the following state equation:

$$dx_2(t)=\begin{bmatrix}F & 0\\ H & A\end{bmatrix}x_2(t)\,dt+\begin{bmatrix}dw_2(t)\\ dw_1(t)\end{bmatrix} \qquad 0\le t\le T\ \text{(forward)}. \tag{3.8}$$

The meaning of the notation (forward) in (3.8) will become clear shortly. Thus the covariance $\Sigma_2(t)$ of $x_2(t)$ can be obtained from the differential equation

$$\dot{\Sigma}_2(t)=\begin{bmatrix}F & 0\\ H & A\end{bmatrix}\Sigma_2(t)+\Sigma_2(t)\begin{bmatrix}F' & H'\\ 0 & A'\end{bmatrix}+\begin{bmatrix}S_2 & 0\\ 0 & S_1\end{bmatrix}. \tag{3.9}$$

As we saw in Case 1, we needed another set of equations in addition to the standard Lyapunov equation in order to calculate the covariance of $x(t)$. Here we see that the set of equations essentially comes about by augmenting the state $x(t)$ with a shaping filter model for the field in order to obtain a process that is Markovian. Once that is done, as in (3.8), we can use standard results to write down the covariance equation (3.9).

The interesting part of this analysis occurs over the time interval $T\le t\le 2T$, since here we are reversing over the same sample path of ξ. Again our goal is to augment $x(t)$ with something in order to obtain a Markov model over this time interval. In order to do this we clearly must consider a model for ξ that runs in reverse. Using the results in [4] and [9] we can write a reverse time model for the augmented process $x_2(t)$ as follows:

$$-dx_2(t) = -\left\{\begin{bmatrix}F & 0\\ H & A\end{bmatrix}+\begin{bmatrix}S_2 & 0\\ 0 & S_1\end{bmatrix}\Sigma_2^{-1}(t)\right\}x_2(t)+\begin{bmatrix}d\tilde{w}_2(t)\\ d\tilde{w}_1(t)\end{bmatrix} \qquad 0\le t\le T\ \text{(backward)}. \tag{3.10}$$

The notation "backward" is used to emphasize the fact that (3.10) is a model for the process $x_2(t)$ that evolves backward in time from T to t. Here $(\tilde{w}_2'(t),\tilde{w}_1'(t))$ is a Brownian motion process whose increments backward in time are independent of the *future* values of $x_2(t)$, much as the forward increments of $(w_2'(t),w_1'(t))$ in (3.8) are independent of past values of $x_2(t)$. Note also that the increments of the $\tilde{w}_i$ processes have the same covariances as those of the increments of the w_i. Finally, it is worth pointing out one interesting aspect of the model (3.10). Specifically, note that the forward time model (3.8) has a block-triangular structure, which is a direct result of the fact that the ξ-process drives the x-process, but not vice versa. However, the reverse-time model (3.10) does *not* have this block-triangular structure, since Σ_2^{-1} is not block diagonal. Again, this has a simple explanation: since ξ drives the x dynamics forward in time, the present value of ξ is certainly not independent of the future of x.

If we now let

$$x_3(t)=x_2(2T-t), \qquad \eta_i(t)=\tilde{w}_i(2T-t) \tag{3.11}$$

we obtain a model *forward* in time over the time interval $T\le t\le 2T$ as follows:

$$dx_3(t)=-\left\{\begin{bmatrix}F & 0\\ H & A\end{bmatrix}+\begin{bmatrix}S_2 & 0\\ 0 & S_1\end{bmatrix}\Sigma_2^{-1}(t)\right\}x_3(t)\,dt +\begin{bmatrix}d\eta_2(t)\\ d\eta_1(t)\end{bmatrix} \qquad T\le t\le 2T\ \text{(forward)}. \tag{3.12}$$

The initial condition for this process is $x_3(T)=x_2(T)$, with covariance $\Sigma_2(t)$.

Consider now the following augmented process over the time interval $T\le t\le 2T$:

$$x_4(t)=\begin{bmatrix} x_3(t) \\ x(t)\end{bmatrix}=\begin{bmatrix}\xi(2T-t)\\ x(2T-t)\\ x(t)\end{bmatrix}. \tag{3.13}$$

Then, using (3.1), (3.3), (3.5), and (3.11) we obtain a *Markovian* representation for the behavior of this augmented state

$$dx_4(t)=\left[\begin{array}{c|c} -\left\{\begin{pmatrix} F & 0\\ H & A\end{pmatrix}+\begin{pmatrix} S_2 & 0\\ 0 & S_1\end{pmatrix}\Sigma_2^{-1}(t)\right\} & 0\\ (0,H) & A\end{array}\right]x_4(t)\,dt$$

$$+\begin{pmatrix} d\eta_2(t)\\ d\eta_1(t)\\ dw_1(t)\end{pmatrix}\qquad T\leqslant t\leqslant 2T\text{ (forward)} \tag{3.14}$$

where

$$E\left\{\begin{bmatrix} d\eta_2(t)\\ d\eta_1(t)\\ dw_1(t)\end{bmatrix}[d\eta_2'(t), d\eta_1'(t), dw_1'(t)]\right\}$$

$$=\begin{bmatrix} S_2 & 0 & 0\\ 0 & S_1 & 0\\ 0 & 0 & S_1\end{bmatrix}dt. \tag{3.15}$$

Basically (3.14) describes a method, starting at $t=T$, for *simultaneously* generating the future ($t>T$) of $x(t)$ and its past ($t<T$). In this fashion we can take into account the fact that the trajectory has reversed its direction.

We can use (3.14) as the basis for determining the covariance for $x(t)$. Specifically, define $N(t)$ as

$$N(t)=\begin{bmatrix} -\left\{\begin{bmatrix} F & 0\\ H & A\end{bmatrix}+\begin{bmatrix} S_2 & 0\\ 0 & S_1\end{bmatrix}\Sigma_2^{-1}(t)\right\} & 0\\ (0,H) & A\end{bmatrix}. \tag{3.16}$$

Then, letting $\Sigma_4(t)$ denote the covariance of $x_4(t)$, we obtain

$$\dot{\Sigma}_4(t)=N(t)\Sigma_4(t)+\Sigma_4(t)N'(t)$$

$$+\begin{bmatrix} S_2 & 0 & 0\\ 0 & S_1 & 0\\ 0 & 0 & S_1\end{bmatrix}. \tag{3.17}$$

To obtain the initial condition for this equation, note that

$$x_4(T)=\begin{bmatrix}\xi(T)\\ x(T)\\ x(T)\end{bmatrix}. \tag{3.18}$$

Thus, if we write

$$\Sigma_2(T)=\begin{pmatrix} (\Sigma_2(T))_{11} & (\Sigma_2(T))_{12}\\ (\Sigma_2(T))_{12}' & (\Sigma_2(T))_{22}\end{pmatrix} \tag{3.19}$$

then

$$\Sigma_4(T)=\begin{pmatrix} (\Sigma_2(T))_{11} & (\Sigma_2(T))_{12} & (\Sigma_2(T))_{12}\\ (\Sigma_2(T))_{12}' & (\Sigma_2(T))_{22} & (\Sigma_2(T))_{22}\\ (\Sigma_2(T))_{12}' & (\Sigma_2(T))_{22} & (\Sigma_2(T))_{22}\end{pmatrix}. \tag{3.20}$$

Thus we see that, as in Case 2, a reversal of motion leads to an additional equation. Also, we can regard the over-and-back example as a degenerate form of the case examined in the Appendix (referring to the notation in the Appendix, in the over-and-back case $s_1(t)=s_2(t)$ for all t and $t_x=t_y$). Thus the straightforward analysis of the Appendix will lead to equivalent equations in this case.

Note that based on the understanding gained in this and in the preceding section, we can see what will happen for more general over-and-back trajectories. For example, as illustrated in Fig. 4(a), consider the case in which we continue the process for $t\geqslant 2T$ without any further change in course. It is not difficult to show that for $t\geqslant 2T$ we can once again obtain a Markovian representation for the joint process

$$x_2(t)=\begin{bmatrix}\xi(t)\\ x(t)\end{bmatrix}$$

where the initial covariance $\Sigma_2(2T)$ for this process at time $2T$ is obtained from the solution to (3.17)

$$\Sigma_2(2T)=\begin{bmatrix} (\Sigma_4(2T))_{11} & (\Sigma_4(2T))_{13}\\ (\Sigma_4(2T))_{13}' & (\Sigma_4(2T))_{33}\end{bmatrix}. \tag{3.21}$$

In this case the time period $[T,2T]$ represents a transient due to the turn, whose effect will become negligible if A is stable. In fact, assuming that there are no further turns, x will achieve the same steady-state covariance in this case as it would from a trajectory that moves to the left for $t>0$ without *any* turns. Similarly, if we consider a second course reversal as in Fig. 4(b), we must obtain a reverse time model for x_4, reverse time once again to obtain an equation for $x_5(t)=x_4(4T-t)$ and augment this with $x(t)$ to obtain a Markovian model over the time period $2T\leqslant t<3T$. Thus in this case we obtain an additional equation for the covariance evolution.

Finally, let us comment briefly on the problem of estimating the process described by (3.1)–(3.6) given measurements. Specifically, suppose we assume that the random field has been mapped by a previous survey

$$y_1(t)=C_1\xi(t)+v_1(t),\qquad 0\leqslant t\leqslant T \tag{3.22}$$

where $E[v_1(t)v_1'(\tau)]=R_1\delta(t-\tau)$, so that we have the smoothed estimates

$$\hat{\xi}_s(t)=E[\xi(t)\,|\,y_1(\tau), 0\leqslant\tau\leqslant T]. \tag{3.23}$$

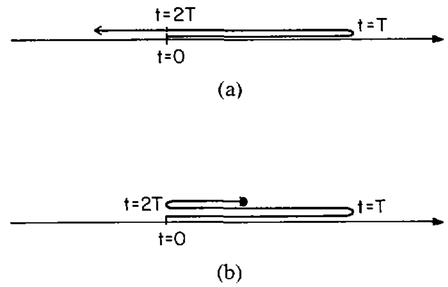


Fig. 4. Two over-and-back trajectories.

Consider now a set of real-time measurements

$$z(t)=C_2x(t)+v_2(t), \qquad 0\leqslant t\leqslant 2T \tag{3.24}$$

where $E[v_2(t)v_2'(t)]=R_2\delta(t-\tau)$ and v_1 and v_2 are independent. We wish to use the previously mapped information (3.23), together with the new data (3.24), to estimate $x(t)$.

This estimation problem can be solved completely using the results in [5]. Specifically, among the problems considered in [5] are the following: let $x'(t)=[x_1'(t), x_2'(t)]$ be a Markov process described by a linear stochastic system, and assume that $x_1(t)$ is a Markov process by itself. Suppose that we obtain linear observations y_1 of $x_1(t)$ over the time interval $[0,T]$ and process these to compute the smoothed estimate history $\hat{x}_{1s}(t)$ over this time interval. Furthermore, suppose that we obtain a second set of linear measurements y_2 of $x(t)$. The *real-time smoothing problem* is that of combining the estimate history $\hat{x}_{1s}(\tau)$, $0\leqslant\tau\leqslant T$ and the observations $y_2(\tau)$, $0\leqslant\tau\leqslant t$ to compute the optimal estimate of $x(t)$ given the full first set of observations, $y_1(\tau)$, $0\leqslant\tau\leqslant T$ and the causally obtained second pass observations up to time t, $y_2(\tau)$, $0\leqslant\tau\leqslant t$. The *smoothing update problem* is that of combining $\hat{x}_{1s}(\tau)$, $0\leqslant\tau\leqslant T$ and $y_2(\tau)$, $0\leqslant\tau\leqslant T$ to compute the optimal estimate of $x(t)$ given both full sets of observations, $y_1(\tau)$, $y_2(\tau)$, $0\leqslant\tau\leqslant T$. Efficient algorithms for solving both of these problems are presented in [5].

To see that our present problem can be solved in terms of the solutions to the problems just described, it is useful to consider the two intervals $[0,T]$ and $[T,2T]$ separately. Thus, let

$$\begin{aligned} y_2(t)&=z(t), \qquad t\in[0,T],\\ y_3(t)&=z(t), \qquad t\in[T,2T]. \end{aligned} \tag{3.25}$$

Over the interval $[0,T]$ we wish to compute

$$\hat{x}_2(t|T,t)=E[x_2(t)|y_1(\tau),0\leqslant\tau\leqslant T, y_2(\tau), 0\leqslant\tau\leqslant t]. \tag{3.26}$$

Noting from (3.7) and (3.8) that ξ is a subprocess of x_2, we see that the calculation of (3.26) in terms of $\hat{\xi}_{1s}$ and y_2 is a real-time smoothing problem.

Similarly, consider the estimation problem over the interval $[T,2T]$. In this case, the quantity we wish to calculate is

$$\begin{aligned} \hat{x}_4(t|T,T,t)&=E[x_4(t)|y_1(\tau),0\leqslant\tau\leqslant T, y_2(\tau),\\ &\quad\times 0\leqslant\tau\leqslant T, y_3(\tau), T\leqslant\tau\leqslant t]\\ &=E[x_4(t)|y_1(\tau),0\leqslant\tau\leqslant T, z(\tau), 0\leqslant\tau\leqslant t]. \end{aligned} \tag{3.27}$$

A procedure for calculating this quantity is as follows: we first calculate the smoothed estimate of $x_2(\tau)$ given $y_1(\tau)$, $y_2(\tau)$ $0\leqslant\tau\leqslant T$. Doing this in terms of $\hat{\xi}_{1s}$ and y_2 is a smoothing update problem. Then, since $x_3(t)=x_2(2T-t)$ *and* since $x_3(t)$ is a subprocess of $x_4(t)$, we have the real-time smoothing problem of calculating (3.27) in terms of the smoothed estimate history for x_3 given $y_1(\tau)$, $y_2(\tau)$, $0\leqslant\tau\leqslant T$ and the real-time data $y_3(\tau)$, $T\leqslant\tau\leqslant t$. We refer the reader to [8] for the detailed application of the results of [5] along the lines outlined above.

IV. Conclusions

In this paper we have examined the effect of a random field on a linear dynamic system moving through the field. We have developed a methodology for calculating the covariance of the state of the dynamic system along any trajectory. The evolution of this covariance is clearly dependent upon the nature of the trajectory, and our results indicate explicitly how this dependence is reflected in the differential equations that must be solved to determine the covariance.

In the case of one-dimensional motion we have gone several steps further in our understanding and analysis of over-and-back trajectories. Specifically, with the use of the technique for constructing backward Markovian models we have developed Markov models over each separate unidirectional segment of the trajectory. The dimension of these models decreases when the trajectory goes beyond the region covered in previous segments and increases when there is a turn. Using this model and results on real-time smoothing we then were able to describe the solution to an over-and-back estimation problem based on results in [5].

Several directions for further work suggest themselves. The first is the extension of the analysis of Section II to dynamic systems moving through three-dimensional random fields. For stationary random fields, a generalization of the analysis developed in this paper should be straightforward. However, the interesting case, motivated by missile-borne, rather than terrestrial navigation systems, is to nonstationary random fields and to spherical coordinates —the earth's gravity field is not invariant with respect to altitude! Present covariance analysis programs handle this problem in approximate and not completely satisfactory ways.

A second research direction is the detailed investigation of the estimation problem discussed at the end of Section III. While we have described the solution to this problem we have not exploited its structure as fully as is possible, either in terms of obtaining efficient on-line solutions or of gaining insight. For example, it is clear that the measure-

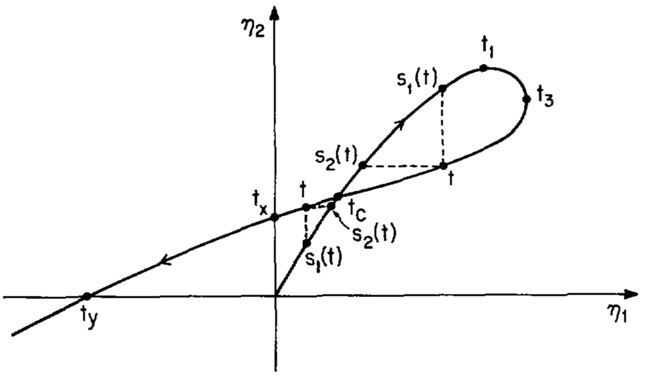


Fig. 5. A trajectory involving a sharp turn.

ments of the state x of the dynamic system provide information about the field ξ. How is this information incorporated into the solution of the real-time smoothing problem? This is potentially important in problems in which we wish to use the dynamic system to estimate the random field. Gravity mapping using inertial instruments is a potential application.

A more difficult extension of our work is the development of Markov-type models and the corresponding estimation algorithms for more general 2-D trajectories. This will involve a significant extension of the notion of a backward Markov process.

Finally, an important generalization of the problems considered in this paper are to systems moving along trajectories which are random themselves. Specifically, referring to our general model, suppose that $(\eta_1(t), \eta_2(t))$ are in fact components of x. This is in fact an important model in some applications. While our results do not address this problem, they may be of value in the case in which the trajectory is only slightly disturbed from some nominal. In that situation our analysis might form the basis for a perturbation analysis of the random trajectory problem.

Appendix

We consider one final case of the problem of Section II. In contrast to Case 2 in which the turn takes us into a neighboring quadrant, we now consider a trajectory, illustrated in Fig. 5, that has a sharp angle and takes us into the opposite quadrant—i.e., northeast (NE) to southwest (SW). This example, together with those in the text, should make clear the approach that can be used in analyzing general trajectories. Additional examples of this type of calculation are given in [8]. In the figure t_1 is the time when the direction of the trajectory changes from NE to SE, and t_3 is the time we change from SE to SW. Also t_x denotes the time at which the trajectory crosses the η_2 axis in the southwesterly direction. This corresponds to the time at which the η_1 coordinate of the trajectory has evolved from 0 to its maximum value $\eta_1(t_3)$ and has decreased back to zero. The time t_y is defined in an analogous fashion. Also t_c denotes the time at which the trajectory crosses over itself. Finally, we have indicated the definitions of $s_1(t)$ and $s_2(t)$ for two values of t, one less than t_c and one greater than t_c.

To determine the structure of the required covariance equations, let us break the trajectory into its natural component pieces, namely $0 \leqslant t \leqslant t_1$, $t_1 \leqslant t \leqslant t_3$, $t_3 \leqslant t \leqslant t_c$, $t_c \leqslant t \leqslant t_x$, $t_x \leqslant t \leqslant t_y$, and $t_y \leqslant t$. Over the first two segments we have a situation identical to Case 2 considered in the text. That is, for $0 \leqslant t \leqslant t_1$, $L(t) = H_1 B_1(t)$ [see (2.18)–(2.21)], while for $t_1 \leqslant t \leqslant t_3$, $L(t) = H_1[B_1(t) + C_1(t)]$ [(2.28)–(2.35)]. While the situation is somewhat more complex over the next few time intervals, the procedure we use is essentially the same: break the integral expression for $L(t)$ in (2.12) into pieces, each of which can be calculated by sets of differential equations.

To proceed, note that from the graphical definitions of $s_1(t)$ and $s_2(t)$ in Fig. 5 we have that for $t_1 \leqslant t \leqslant t_x$

$$\begin{aligned} &\eta_1(t) - \eta_1(s) \geqslant 0 && \text{for } 0 \leqslant s \leqslant s_1(t) \\ &\eta_1(t) - \eta_1(s) \leqslant 0 && \text{for } s_1(t) \leqslant s \leqslant t \end{aligned} \tag{A.1}$$

and

$$\begin{aligned} &\eta_2(t) - \eta_2(s) \geqslant 0 && \text{for } 0 \leqslant s \leqslant s_2(t) \\ &\eta_2(t) - \eta_2(s) \leqslant 0 && \text{for } s_2(t) \leqslant s \leqslant t. \end{aligned} \tag{A.2}$$

Furthermore, for $t_3 \leqslant t \leqslant t_c$, $s_2(t) \leqslant s_1(t)$, while for $t_c \leqslant t \leqslant t_x$, $s_1(t) \leqslant s_2(t)$. From these observations and the properties of the two-dimensional correlation function, it is straightforward to show that for $t_3 \leqslant t \leqslant t_c$

$$\begin{aligned} L(t) &= \int_0^{s_2(t)} H_1 e^{F_1(\eta_1(t)-\eta_1(s))} G_1 H_2 e^{F_2(\eta_2(t)-\eta_2(s))} G_2 e^{A'(t-s)}\, ds \\ &+ \int_{s_2(t)}^{s_1(t)} H_1 e^{F_1(\eta_1(t)-\eta_1(s))} G_1 G_2' e^{F_2'(\eta_2(s)-\eta_2(t))} H_2' e^{A'(t-s)}\, ds \\ &+ \int_{s_1(t)}^{t} G_1' e^{F_1'(\eta_1(s)-\eta_1(t))} H_1' G_2' e^{F_2'(\eta_2(s)-\eta_2(t))} H_2' e^{A'(t-s)}\, ds, \end{aligned} \tag{A.3}$$

while for $t_c \leqslant t \leqslant t_x$,

$$\begin{aligned} L(t) &= \int_0^{s_1(t)} H_1 e^{F_1(\eta_1(t)-\eta_1(s))} G_1 H_2 e^{F_2(\eta_2(t)-\eta_2(s))} G_2 e^{A'(t-s)}\, ds \\ &+ \int_{s_1(t)}^{s_2(t)} G_1' e^{F_1'(\eta_1(s)-\eta_1(t))} H_1' H_2 e^{F_2(\eta_2(t)-\eta_2(s))} G_2 e^{A'(t-s)}\, ds \\ &+ \int_{s_2(t)}^{t} G_1' e^{F_1'(\eta_1(s)-\eta_1(t))} H_1' G_2' e^{F_2'(\eta_2(s)-\eta_2(t))} H_2' e^{A'(t-s)}\, ds. \end{aligned} \tag{A.4}$$

Continuing over the next interval $t_x \leqslant t \leqslant t_y$, we see that $\eta_1(t) - \eta_1(s) \leqslant 0$ for all $s \leqslant t$. Consequently,

$$\begin{aligned} L(t) = &\int_0^{s_2(t)} G_1' e^{F_1'(\eta_1(s)-\eta_1(t))} H_1' H_2 e^{F_2(\eta_2(t)-\eta_2(s))} G_2 e^{A'(t-s)}\, ds \\ &+ \int_{s_2(t)}^{t} G_1' e^{F_1'(\eta_1(s)-\eta_1(t))} H_1' G_2' e^{F_2'(\eta_2(s)-\eta_2(t))} \\ &\cdot H_2' e^{A'(t-s)}\, ds. \end{aligned} \tag{A.5}$$

Finally, for $t \geqslant t_y$, $\eta_1(t) - \eta_1(s) \leqslant 0$ and $\eta_2(t) - \eta_2(s) \leqslant 0$

for all $s \leqslant t$, and thus

$$L(t)=\int_0^t G_1' e^{F_1'(\eta_1(s)-\eta_1(t))} H_1' G_2' e^{F_2'(\eta_2(s)-\eta_2(t))} H_2' e^{A'(t-s)}\, ds. \tag{A.6}$$

From these equations it is a straightforward exercise to derive a set of differential equations for each of the terms in each expression for $L(t)$ much as we did in Case 2 (see [8] for details). Thus, as pointed out in the text, we add one set of equations at time t_1 and a second at time t_3. There is a change in these equations at t_c, but no change in their number. However, one set of equations *is* dropped at t_x and the other is deleted at t_y. Piecing these equations together from interval to interval is also no problem. In particular, for the example we have just described

$$L(t)=\begin{cases} H_1 B_1(t) & 0 \leqslant t \leqslant t_1 \\ H_1[B_1(t)+C_1(t)] & t_1 \leqslant t \leqslant t_3 \\ H_1[B_1(t)+C_1(t)]+G_1' D_1(t) & t_3 \leqslant t \leqslant t_c \\ H_1 B_1(t)+G_1'[E_1(t)+D_1(t)] & t_c \leqslant t \leqslant t_x \\ G_1'[E_1(t)+D_1(t)] & t_x \leqslant t \leqslant t_y \\ G_1' D_1(t) & t \geqslant t_y \end{cases} \tag{A.7}$$

where the letters (B, C, D, E) are consistently identified in an obvious fashion with corresponding terms in the integral expression for $L(t)$ over the specified interval. From an examination of the integral expressions (2.14), (2.23), (A.3), (A.4), (A.5), and (A.6) it is not difficult to see that these matrices are continuous, so that we simply piece together the differential equations for each set of matrices over consecutive intervals. Furthermore, terms are removed from the representation of $L(t)$ only when they are precisely equal to zero (e.g., an examination of the integral equations verifies that $B_1(t_x)=C_1(t_c)=E_1(t_y)=0$) and are initiated with zero initial conditions.

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