*Analysis of Bilinear Noise Models in Circuits and Devices**

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ABSTRACT: *There are a number of applications in which linear noise models are inappropriate. In this paper, the use of bilinear noise models in circuits and devices is considered. Several physical problems are studied in this framework. These include circuits involving varying parameters (such as variable resistance circuits constructed using field-effect transistors), the effect of switching jitter on sampled data system performance and communication systems involving voltage-controlled oscillators and phase-lock loops. In addition, several types of analytical techniques for stochastic bilinear systems are considered. Specifically, the moment equations of Brockett for bilinear systems driven by white noise are discussed, and closed-form expressions for certain bilinear systems (those that evolve on abelian or solvable Lie groups) driven by white or colored noise are derived. In addition, an approximate statistical technique involving the use of harmonic expansions is described.*

I. Introduction

Although linear models are extremely useful in many applications, there are large classes of problems of practical importance for which such models and the associated analytical techniques are inappropriate. It is therefore of interest to determine classes of nonlinear models that are of practical importance and that lend themselves to detailed analysis. The class of bilinear systems is such a class. A number of authors (**1**–**17**) have investigated the properties of deterministic and stochastic bilinear systems and have developed analytical tools that are almost as powerful as the corresponding tools for linear systems. The mathematical techniques used in these studies have varied, but the tools of Lie thoery and harmonic analysis have been particularly useful (**1**–**5**, **8**–**16**).

In this paper, we investigate the use of stochastic bilinear models in the study of certain networks and devices. The treatment is by no means a

* This research was performed at the M.I.T. Electronic Systems Laboratory with support provided by NASA under Grant NGL–22–009–124 and NSF under Grant GK–42090. S. I. Marcus was also supported in part by an NSF Fellowship.

thorough investigation of this subject; it is rather our intention to indicate the value of stochastic bilinear models. To this end, we present a variety of physical systems that can be modeled by noisy bilinear systems, and we then discuss several techniques for analyzing such models. In addition, several simple examples are included to indicate how these tools can be applied.

II. Random Bilinear Systems

The deterministic bilinear model studied in (**1–5**) is

$$\dot{x}(t) = \left[A_0 + \sum_{i=1}^{N} u_i(t) A_i\right] x(t), \tag{1}$$

where the A_i are given $n \times n$ matrices, the u_i are scalar inputs and x is either an n-vector or an $n \times n$ matrix. As discussed in [**1**], we note that Eq. (1) is general enough to include the case of additive control—i.e. by state augmentation we can reduce an equation of the form

$$\dot{x}(t) = \left[B_0 + \sum_{i=1}^{N} u_i(t) B_i\right] x(t) + Cu(t) \tag{2}$$

to an equation of the form (1). Also, we note that if we apply bilinear feedback to the system (1)—i.e. if we take

$$u_i(t) = v_i(t)\, c_i' x(t) + w_i(t), \tag{3}$$

where x is a vector, the v_i and w_i are controls and c_i' is the transpose of c_i, our system equation becomes

$$\dot{x}(t) = \left\{A_0 + \sum_{i=1}^{N} [v_i(t)\, c_i' x(t) + w_i(t)] A_i\right\} x(t). \tag{4}$$

This equation involves products of state variables, and by including several feedback paths around our original bilinear system we can obtain arbitrary polynomials in the state variables. Such models will be important later when we consider several systems that contain product-type nonlinearities or involve feedback around bilinear systems.

The system of primary interest to us is that given by (1) in which we allow the u_i to be stochastic processes. In considering such an equation, one must be careful to use the appropriate stochastic calculus. That is, while the rules of Stratonovich calculus are identical to the usual rules of calculus, the physically more appealing Ito calculus requires certain "correction terms" [see (**21**) for details]. For instance, if $u(t)$ is a vector zero mean white noise process with

$$E[u(t)\, u(s)] = R(t)\, \delta(t-s) \tag{5}$$

the Ito stochastic differential analog of Eq. (1) is (**8**, **12**)

$$\mathrm{d}x(t) = \left\{\left[A_0+\frac{1}{2}\sum_{i,j=1}^{N} R_{ij}(t)\,A_i A_j\right]\mathrm{d}t+\sum_{i=1}^{N} A_i\,\mathrm{d}v_i(t)\right\}x(t), \tag{6}$$

where v is the integral of u [i.e. v is a Brownian motion].

Note that there are no correction terms to Eq. (1) if u is "smoother" than white noise. For instance, this is the case if u is generated by a finite dimensional linear diffusion process

$$\mathrm{d}\xi(t) = F(t)\,\xi(t)\,\mathrm{d}t+G(t)\,\mathrm{d}w(t)+\alpha(t)\,\mathrm{d}t, \tag{7}$$

$$u(t) = H\xi(t), \tag{8}$$

where α, F, G and H are known and w is a standard Brownian motion process ($E[\mathrm{d}w(t)\,\mathrm{d}w'(t)] = I\mathrm{d}t$). Also, if we augment the state x with the variable ξ [x by itself is *not* a Markov process, but the pair (x, ξ) is], our new state equation involves products of state variables. Finally, we note that, as in (**1**), one can consider the matrix versions of Eqs. (1) and (6)—i.e. replace the n-vector $x(t)$ with the $n\times n$ matrix $X(t)$, with initial condition $X(0) = I$. We then have

$$x(t) = X(t)\,x(0) \tag{9}$$

and thus, assuming that $x(0)$ is independent of the noise, by studying the matrix versions of (1) and (6) we can obtain statistical results for the vector version with arbitrary initial conditions.

III. Linear Networks with Random Parameters

Let $\alpha(t)$ be the vector of stochastic parameters of a linear network, and suppose that the network is described by the state equations

$$\dot{x}(t) = A[\alpha(t)]\,x(t)+B[\alpha(t)]\,u(t), \tag{10}$$

$$y(t) = C[\alpha(t)]\,x(t)+D[\alpha(t)]\,u(t). \tag{11}$$

As indicated in Section II, we assume that $\alpha(t)$ is a vector stochastic process; $\alpha(t)$ may be either a vector white noise [as in Eq. (5)] or a correlated process generated by a linear diffusion process [as in Eqs. (7) and (8)]. If the system matrices in Eqs. (10) and (11) depend linearly on α (and α is correlated), we have the noisy *bilinear* equations

$$\dot{x}(t) = \left[A_0+\sum_{i=1}^{m}\alpha_i(t)\,A_i\right]x(t)+\left[B_0+\sum_{i=1}^{m}\alpha_i(t)\,B_i\right]u(t), \tag{12}$$

$$y(t) = \left[C_0+\sum_{i=1}^{m}\alpha_i(t)\,C_i\right]x(t)+\left[D_0+\sum_{i=1}^{m}\alpha_i(t)\,D_i\right]u(t). \tag{13}$$

If α is a vector of independent white noises, we must model the network with the bilinear Ito equations [see Eqs. (5) and (6)]

$$\mathrm{d}x(t) = \left[A_0\,\mathrm{d}t + \sum_{i=1}^{m} \tfrac{1}{2}A_i^2\,\mathrm{d}t + \sum_{i=1}^{m} A_i\,\mathrm{d}\beta_i(t)\right]x(t) + \left[B_0\,\mathrm{d}t + \sum_{i=1}^{m} B_i\,\mathrm{d}\beta_i(t)\right]u(t), \tag{14}$$

$$\mathrm{d}\tilde{y}(t) = \left[C_0\,\mathrm{d}t + \sum_{i=1}^{m} \tfrac{1}{2}C_i A_i\,\mathrm{d}t + \sum_{i=1}^{m} C_i\,\mathrm{d}\beta_i(t)\right]x(t) + \left[D_0\,\mathrm{d}t + \sum_{i=1}^{m} D_i\,\mathrm{d}\beta_i(t)\right]u(t) \tag{15}$$

(where $\tilde{y}$ is formally the integral of y).

An example of a class of electrical networks that give rise to noisy bilinear equations such as Eqs. (12)–(15) are networks containing junction field-effect transistors (JFET's) used as voltage-controlled resistors (VCR's) (**23, 24**). The field-effect tetrode (**25**), thin-film transitor (TFT) (**23**) and MOSFET may also be used as voltage-controlled resistors, and models similar to the one which we will develop are applicable to these devices.

Consider the JFET volt–ampere characteristics in the region before pinch-off, where the drain source voltage V_{DS} is small. These characteristics are a family of straight lines through the origin with slope equal to the drain-source conductance g_d, which is, to a close approximation, a linear function of the gate-source voltage V_{gs}. In fact, we have

$$g_d = K(V_{GS} - V_p), \tag{16}$$

where K is a function of the geometry of the channel, and the pinch-off voltage V_p is the negative voltage that removes all free charge from the JFET channel. For our purpose here, it is sufficient to consider a simplified model of the VCR which neglects noise and high frequency capacitive effects (**24**), as in Fig. 1.

By incorporating VCR's, we now construct an example of a noisy bilinear network, illustrated in Fig. 2. Assume that we have three identical VCR's with noisy gate-source voltages $V_1(t)$, $V_2(t)$, $V_3(t)$. These noisy voltages are

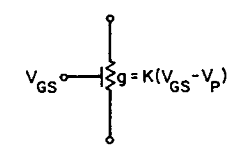

FIG. 1. A voltage-controlled resistor.

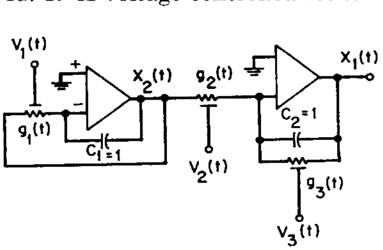

FIG. 2. A noisy bilinear network involving VCR's

assumed to be correlated noises generated by a second network containing shot and thermal noise sources, which can be modeled as white, Gaussian stochastic processes (**22**). Neglecting all other noise sources in both networks and assuming that the operational amplifiers in Fig. 2 have infinite gain, infinite input impedance and infinite output admittance, the network is described by the state equations

$$\dot{x}_1(t) = -K[V_3(t)-V_p]\,x_1(t)-K[V_2(t)-V_p]\,x_2(t), \tag{17}$$

$$\dot{x}_2(t) = -K[V_1(t)-V_p]\,x_2(t) \tag{18}$$

or

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \left\{\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}KV_p - KV_1(t)\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix} - KV_2(t)\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} - KV_3(t)\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\right\}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}$$

$$\triangleq \left[A_0 + \sum_{i=1}^{3} V_i(t)\,A_i\right]x(t). \tag{19}$$

This noisy bilinear equation evolves on a *solvable* Lie group (**26**). We analyze this case in detail in Section VII.

IV. Switching Jitter in Sampled Data Systems

Consider the system depicted in Fig. 3. We have a linear system with transfer function $G(s)$, which we assume has the time-invariant realization

$$\dot{x}(t) = Ax(t)+bv(t), \tag{20}$$

$$y(t) = c'x(t). \tag{21}$$

Assuming perfect operation of the sample and hold, we have

$$v(t) = u(kT), \quad kT \leqslant t < (k+1)\,T, \tag{22}$$

$$u(t) = r(t)-Ky(t). \tag{23}$$

Therefore, we have

$$x[(k+1)\,T] = \exp{(AT)}\,x(kT) + \left\{\int_0^T \exp{[A(T-t)]}\,\mathrm{d}t\right\} bu(kT). \tag{24}$$

One source of noise in sampled data systems is "switching jitter"—i.e. the physical devices used as switches have inherent irregularities that lead

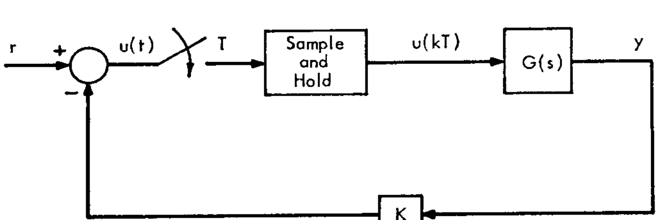

FIG. 3. A sampled data feedback system.

to randomness in switching times (**27**). Therefore, suppose there is a change δ_k in the kth switch time. Let x_k be the state at the kth switch time. Then, to first order in δ_k, Eq. (24) becomes

$$x_{k+1} = \exp(AT)(I+A\delta_k)x_k + \left\{\int_0^T \exp[A(T-t)]\,\mathrm{d}t + \exp(AT)\,\delta_k\right\} bu_k. \quad (25)$$

For simplicity, assume that $r \equiv 0$, then $u = -Kc'x$, so Eq. (25) is a discrete time stochastic bilinear system, where δ_k plays the role of the random input.

One important problem related to sampled data systems is the problem of stability (**28**). For instance, the equations we are considering are general enough to allow us to study pulse–amplitude modulation systems (PAM) and pulse–width modulation (PWM) systems in the quasi-linear (unsaturated) range (**29–31**) and perhaps such switched electrical networks as d.c. to d.c. converters (**32**). In all of these systems the question of stability about some operating point is of great importance. A great deal of effort (**28–32**) has been devoted to the study of deterministic aspects of the stability of these systems, but relatively little is known about effects of stochastic phenomena, such as switching jitter, on system stability [see (**30**) for examples of some rather complicated and as yet unexplained simulation results for a PWM system; these phenomena may be due to system randomness].

Returning to our bilinear model including jitter effects, we note that Nelsen (**27**) found that for tunnel diodes the δ_k can be taken as an independent sequence of identically distributed Gaussian random variables. This is essentially a shot noise effect. Nelsen also found a flicker noise ($1/f$) jitter phenomenon associated with "reverse" switching [see (**27**)], but for the present treatment we will concentrate on the shot noise problem [note that our present setting can handle flicker noise effects if we use the model of Horowitz (**35**) that produced $1/f$ noise as a linear combination of the outputs of a bank of linear filters driven by independent white noises].

Some of the techniques that we will present in Sections VI–VIII can be adapted to consider discrete time bilinear models, such as Eq. (25); however, since we are essentially concerned with continuous-time models in this paper, we will derive a continuous-time analog of (25) that can be studied instead. Of course, such an analogue can be useful only in certain cases. For instance, if the switching time T is substantially smaller than any of the system time constants, the overall system may closely resemble a continuous-time system [note that this requirement—that switching frequency be much larger than system bandwidth—is one of the conditions that is often assumed when studying PWM stability in the quasi-linear range (**30**)]. Letting T become infinitesimally small and assuming that $\delta_k = \mathrm{d}w(t)$, where w is a Brownian motion process with

$$E[\mathrm{d}w^2(t)] = q(t)\,\mathrm{d}t \quad (26)$$

we obtain the stochastic differential equation

$$\mathrm{d}x(t) = \{[A-Kbc'][\mathrm{d}t+\mathrm{d}w(t)] + \tfrac{1}{2}q(t)[A-Kbc']^2\,\mathrm{d}t\}\,x(t), \quad (27)$$

where the last term inside the braces is the correction term described in Section II.

As an example, consider the case of a double integrator

$$G(s) = 1/s^2.$$

In this case, a realization of the forms (20) and (21) is obtained by taking

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad c' = [1 \quad 0]. \tag{28}$$

Then, in the unity feedback ($K = 1$) case, we have

$$A - bc' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{29}$$

$$\mathrm{d}x(t) = \begin{bmatrix} -\frac{1}{2}q(t)\,\mathrm{d}t & \mathrm{d}t + \mathrm{d}w(t) \\ -[\mathrm{d}t + \mathrm{d}w(t)] & -\frac{1}{2}q(t)\,\mathrm{d}t \end{bmatrix} x(t). \tag{30}$$

As is well known, a linear system with system matrix given by (29) defines an oscillator, and the properties of (30) have been well studied (**8–10**). In fact, as discussed in (**8–10**), (30) defines an equation evolving on the Lie group SO(2) or, equivalently, on the unit circle S^1 in the plane.

Examining (30), we see that it appears to define a *damped* harmonic oscillator; however, the damping terms $(-\frac{1}{2}q\,\mathrm{d}t)$ are simply correction terms and *do not* provide any damping [see (**8, 9**) and (**36**) for further comments on this mathematical question]. In fact [see (**8–10**) and Eq. (52)], the solution of (30) is

$$x(t) = \begin{bmatrix} \cos[t + w(t)] & \sin[t + w(t)] \\ -\sin[t + w(t)] & \cos[t + w(t)] \end{bmatrix} x(0) \tag{31}$$

and we can interpret q as the reciprocal of the "oscillator coherence time" (**37**). Thus we see that in this case switching jitter can cause phase incoherencies.

We close this section by noting that the procedure outlined here can be carried out for the case in which the circuit of interest has several switches. In this case, the continuous-time stochastic differential equations that arise involve *several* (usually independent) Brownian motion processes and will be of the form given in Eq. (6).

V. Noise Models in Communication Systems

As has been discussed in recent papers (**11, 36**), bilinear models arise quite naturally in a number of communication applications. For example, we shall see that the dynamic model of a noisy oscillator involves stochastic bilinear equations. In addition, many receiver–demodulators involve the multiplication of the received signal (usually corrupted by additive noise) by a signal at the same carrier frequency and 90° out of phase. Examples are homodyne-detectors for AM and FM signals (**29**) and phase-lock loops (PLL) for angle tracking and demodulation (**37**). Such multiplications of inputs (received signals) by internally generated signals are precisely the types of mechanisms that one finds in bilinear circuit models. For the sake of this discussion, we limit our attention to an analysis of certain angle modulation problems involving additive receiver noise. Similar analyses can also be carried out for AM problems and for communication problems involving nonadditive noise, such as the multiplicative noise model associated with Rayleigh fading channels (**37**). The reader is referred to (**11**) for results for some of these problems.

Consider the situation depicted in Fig. 4. Let $a(t)$ denote the time function that is to be transmitted via an angle modulation system. Assuming that

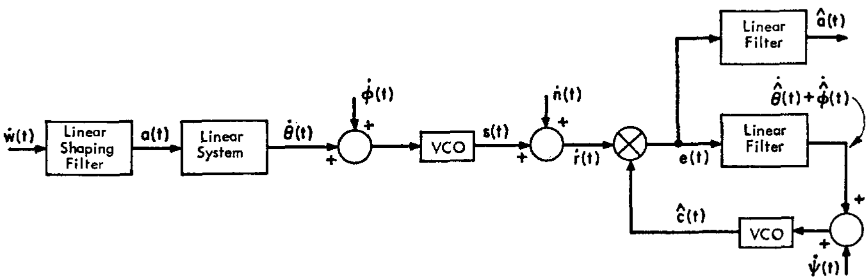

FIG. 4. Illustrating an angle modulation system, including modulator, channel and PLL demodulator.

the power spectral density of a is a proper rational function, we can think of a as being generated as the output of a "shaping filter" driven by white noise

$$\mathrm{d}\xi(t) = F\xi(t)\,\mathrm{d}t + G\,\mathrm{d}w(t), \tag{32}$$

$$a(t) = h'\xi(t), \tag{33}$$

where w is an m-dimensional standard Brownian motion process. As illustrated in Fig. 4, we wish to take the signal $a(t)$, perhaps perform some further filtering on it and then transmit the resulting signal $\theta(t)$ via a phase modulation system—i.e. we wish to transmit the signal

$$s(t) = \surd(2P)\sin[\omega_c t + \theta(t)], \tag{34}$$

where ω_c is a carrier frequency. Note that $\theta(t) = a(t)$ corresponds to simple phase modulation and $\dot{\theta}(t) = a(t)$ leads to a frequency demodulation problem. One method for generating a signal such as (34) is to use a voltage-controlled oscillator (VCO) with $\dot{\theta}(t)$ as the input. As discussed in (**37**), such a device is subject to phase drift or incoherencies which are often modeled as Brownian motion. In this case, the signal that is *actually* transmitted is

$$s(t) = \surd(2P)\sin[\omega_c t + \theta(t) + \phi(t)]. \tag{35}$$

A dynamic model of the VCO is

$$\begin{bmatrix} \mathrm{d}x_1(t) \\ \mathrm{d}x_2(t) \end{bmatrix} = \begin{bmatrix} -\tfrac{1}{2}q\,\mathrm{d}t & \omega_c\,\mathrm{d}t+\mathrm{d}\theta(t)+\mathrm{d}\phi(t) \\ -\omega_c\,\mathrm{d}t+\mathrm{d}\theta(t)+\mathrm{d}\phi(t) & -\tfrac{1}{2}q\,\mathrm{d}t \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \tag{36}$$

$$s(t) = \sqrt{(2P)}\,x_2(t), \tag{37}$$

where $q/2$ is the appropriate correction term. That is, if we write $\theta = \theta_1+\theta_2$, where θ_1 is of bounded variation and θ_2 is not (i.e. it contains Brownian motion terms), we have

$$q\,\mathrm{d}t = E[\mathrm{d}\theta_2^2(t)]+E[\mathrm{d}\phi^2(t)]. \tag{38}$$

Note that regarding $\dot{\theta}$ and $\dot{\phi}$ as noisy inputs, (36) represents a stochastic bilinear system. In fact, comparing (36) with (30), we see that (36) *also* defines an equation on SO(2).

The signal that is received and is to be demodulated [to provide an estimate of $a(t)$] is

$$\dot{r}(t) = s(t)+\dot{n}(t), \tag{39}$$

where $\dot{n}(t)$ is an additive noise process due to extraneous signals reaching the receiver and to electrical noise in the receiver circuitry. In many cases, the noise $\dot{n}(t)$ is well modeled as a white noise process [e.g. shot and thermal noise are often quite important; see (**29**)].

One often used demodulation system is the PLL depicted in Fig. 4. The PLL estimates the phase of $s(t)$ and obtains an estimate of $a(t)$ based on this phase estimate. As illustrated in Fig. 4, we generate an estimate $\dot{\hat{\theta}}+\dot{\hat{\phi}}$ of the frequency deviation (from ω_c) of s, and generate the signal

$$\hat{c}(t) = \sqrt{(2)}\cos[\omega_c t+(\hat{\theta}t)+\hat{\phi}(t)+\psi(t)] \tag{40}$$

as the output of a VCO driven by $\dot{\hat{\theta}}+\dot{\hat{\phi}}$ and by the VCO phase drift ψ. For a detailed discussion of the PLL, refer to (**37**). We note only that the VCO in the feedback loop of the PLL is described by a stochastic bilinear equation similar to (36), and the product $\dot{r}(t)\,\hat{c}(t)$ represents a bilinear feedback as in (3). Thus the overall system depicted in Fig. 4 consists of linear and bilinear components and, as discussed in Section II, we can show that the overall system equations contain only polynomial-type nonlinearities.

We note that the PLL filter in Fig. 4 is only an approximation to the optimal demodulator. As discussed in (**11**), the optimal demodulator is an infinite dimensional system consisting of an infinite bank of coupled tracking filters, each of which can be described by bilinear equations. Furthermore, finite dimensional approximations with polynomial dynamics (arising from polynomial feedback around several of the bilinear tracking filters) are derived in (**11**), and performance improvements over the optimal PLL are reported.

VI. Moment Equations for Bilinear Systems Driven by White Noise

One approach to the analysis of systems driven by white noise involves the computation of differential equations for the moments of the state $x(t)$.

Our approach follows that of Brockett (**3, 4**). We assume that the n-vector $x(t)$ satisfies the Ito equation

$$\mathrm{d}x(t) = \left(A_0 + \sum_{i=1}^{m} \tfrac{1}{2}A_i^2\right) x(t)\,\mathrm{d}t + \sum_{i=1}^{m} A_i x(t)\,\mathrm{d}\beta_i(t), \tag{41}$$

where the β_i are independent Brownian motion processes with unit variance. Recall that the number of linearly independent homogeneous polynomials of degree p in n variables [i.e. $f(cx_1, \ldots, cx_n) = c^p f(x_1, \ldots, x_n)$] is given by

$$N(n,p) = \begin{pmatrix} n+p-1 \\ p \end{pmatrix} = \frac{(n+p-1)!}{p!\,(n-1)!}. \tag{42}$$

We choose a basis for this $N(n,p)$-dimensional space of homogeneous polynomials in $(x_1, \ldots, x_n)$ consisting of the elements

$$\sqrt{\left[\begin{pmatrix} p \\ p_1 \end{pmatrix}\begin{pmatrix} p-p_1 \\ p_2 \end{pmatrix}\cdots\begin{pmatrix} p-p_1-\cdots-p_{n-1} \\ p_n \end{pmatrix}\right]}$$

$$\times\, x_1^{p_1} x_2^{p_2} \ldots x_n^{p_n}; \quad \sum_{i=1}^{n} p_i = p; \quad p_i \geqslant 0. \tag{43}$$

If we denote the vector consisting of these basis elements (ordered lexicographically) by $x^{[p]}$, then

$$\| x \|^p = \| x^{[p]} \|, \tag{44}$$

where $\| x \| = \sqrt{(x'x)}$. It is clear that if x satisfies the linear differential equation

$$\dot{x}(t) = Ax(t) \tag{45}$$

then $x^{[p]}$ satisfies a linear differential equation

$$\dot{x}^{[p]}(t) = A_{[p]} x^{[p]}(t). \tag{46}$$

The matrix $A_{[p]}$ can be easily computed from A, and in fact is a linear function of A [so that $(\alpha A + B)_{[p]} = \alpha A_{[p]} + B_{[p]}$]. For an interpretation of $A_{[p]}$ as a linear operator on symmetric tensors of degree p, see (**4, 38**). We note only that the eigenvalues of $A_{[p]}$ are all possible sums of p eigenvalues of A.

Brockett has shown that if x satisfies (41), then $x^{[p]}$ satisfies

$$\mathrm{d}x^{[p]}(t) = \left[A_{0_{[p]}} + \frac{1}{2}\sum_{i=1}^{m} (A_{i_{[p]}})^2\right] x^{[p]}(t)\,\mathrm{d}t + \sum_{i=1}^{m} A_{i_{[p]}} x^{[p]}(t)\,\mathrm{d}\beta_i(t). \tag{47}$$

Taking expected values, we get the pth moment equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\{E[x^{[p]}(t)]\} = \left[A_{0_{[p]}} + \frac{1}{2}\sum_{i=1}^{m} (A_{i_{[p]}})^2\right] E[x^{[p]}(t)] \triangleq D_p E[x^{[p]}(t)]. \tag{48}$$

Note that the pth moment equation is linear and is uncoupled from the other moments. Thus, we can state the following stability definition and theorem (**15**):

Definition 1

A vector random process x is *pth-order asymptotically stable* if

$$\lim_{t\to\infty} E[x^{[p]}(t)] = 0. \tag{49}$$

The process is pth *order stable* if $E[x^{[p]}(t)]$ remains bounded for all t. A system described by a stochastic differential equation is pth-order (asymptotically) stable if the solution x is pth-order (asymptotically) stable for all initial conditions $x(0)$ independent of the noises driving the equations and such that $E[x^{[p]}(0)] < \infty$.

Theorem I

System (41) is pth-order asymptotically stable if and only if D_p has all its eigenvalues in the left half-plane [$\mathrm{Re}(\lambda) < 0$]. The system is pth-order stable if D_p has its eigenvalues in $\mathrm{Re}(\lambda) \leqslant 0$, and if λ is an eigenvalue with $\mathrm{Re}(\lambda) = 0$, then λ is a simple zero of the minimal polynomial of D_p (**39**).

For examples illustrating this result, see (**3**), (**14**) and (**15**).

VII. The Colored Noise Case—Closed Form Expressions

We now wish to consider the moment analysis of the stochastic equation

$$(\dot{X}(t) = \left[A_0 + \sum_{i=1}^{m} \alpha_i(t) A_i\right] X(t), \quad X(0) = I, \tag{50}$$

where α is a correlated Gaussian vector stochastic process and X is an $n \times n$ transition matrix (**40**) [note that (50) can be interpreted as an ordinary differential equation].

In this case the procedure of the previous section does not apply, as X is no longer a Markov process. If we augment the state with the noise variables to obtain a Markov process (as described in Section II), our dynamical equations involve products of state variables. In this case, the moment evolution equations become coupled and closed-form expressions cannot be found in general.

There is a special subclass of equations of the type (50) for which we can obtain exact closed-form expressions for the moments of X (in terms of the statistics of α). This class can be easily described with the aid of several Lie theoretic concepts.

Definition 2

A matrix Lie algebra $\mathscr{L}$ is a subspace of the vector space $R^{n\times n}$ of $n \times n$ matrices such that if $A, B \in \mathscr{L}$, then their *commutator product*

$$[A, B] \triangleq AB - BA$$

is also in $\mathscr{L}$. Given any subset of $R^{n\times n}$ we define $\{S\}_A$ to be the smallest Lie algebra which contains S.

We refer the reader to (**1–5** and **8–16**) for detailed discussions of the significance of Lie-theoretic concepts in the study of the properties of bilinear systems. We note only that the structure of the Lie algebra $\{A_0, A_1, \ldots, A_m\}_A$ is intimately related to the dynamical characteristics of Eqs. (1), (6) and (50). In the rest of this section we will explore the consequences of restricting attention to a particular subclass of Lie algebras.

Definition 3

A matrix Lie algebra $\mathscr{L}$ is *solvable* if the derived series of subalgebras

$$\left.\begin{aligned} &\mathscr{L}^{(0)} = \mathscr{L}, \\ &\mathscr{L}^{(n+1)} = [\mathscr{L}^{(n)}, \mathscr{L}^{(n)}] \triangleq \{[A,B] \mid A, B \in \mathscr{L}^{(n)}\}, \quad n \geqslant 0, \end{aligned}\right\} \tag{51}$$

terminates in $\{0\}$. $\mathscr{L}$ is *abelian* if $\mathscr{L}^{(1)} = \{0\}$. We note that $\mathscr{L}$ being solvable is equivalent to the existence of a complex nonsingular matrix P such that PBP^{-1} is upper triangular (zero below diagonal) for all $B \in \mathscr{L}$ (**26**).

One can show (**14–17**) that closed form expressions for the moments of X in (50) can be obtained if $\{A_0, \ldots, A_m\}_A$ is solvable. We will not describe the details of this procedure and refer the reader to the references. Instead, we will illustrate the analysis by means of examples. The simplest example occurs when $\{A_0, \ldots, A_m\}_A$ is abelian, in which case the solution of (50) may be expressed as

$$X(t) = \exp\left[A_0 + \sum_{i=1}^{m} A_i \int_0^t \alpha_i(\tau)\,\mathrm{d}\tau\right]. \tag{52}$$

Thus the statistics of X are completely determined by those of the integral of the noise process α. This result is also applicable to the case when α is white, and in particular to the examples of Sections IV and V which evolve on the abelian Lie group SO(2) [see (**8–10**) for further details].

Consider now the network of Fig. 2 and Eq. (19), which evolves on the solvable Lie group of 2×2 upper triangular matrices. The equation for the transition matrix is

$$\dot{X}(t) = \begin{bmatrix} K[V_p - V_3(t)] & K[V_p - V_2(t)] \\ 0 & K[V_p - V_1(t)] \end{bmatrix} X(t), \quad X(0) = I, \tag{53}$$

where $V_i(t)$ are assumed to be zero-mean jointly Gaussian correlated stochastic processes with known covariance. By a simple change of notation, (53) becomes

$$\dot{X}(t) = \begin{bmatrix} -w_3(t) & -w_2(t) \\ 0 & -w_1(t) \end{bmatrix} X(t), \quad X(0) = I, \tag{54}$$

where $w(t) = [w_1(t), w_2(t), w_3(t)]'$ has a mean value of $-KV_p(1,1,1)'$ and known covariance $P(t,\tau) = E[\![\{w(t) - E[w(t)]\}\{w(\tau) - E[w(\tau)]\}']\!]$. Since

$$X_{21}(t) = 0$$

for all t, (54) implies

$$\dot{X}_{11}(t) = -w_3(t)X_{11}(t), \quad X_{11}(0) = 1, \tag{55}$$

$$\dot{X}_{12}(t) = -w_3(t)X_{12}(t) - w_2(t)X_{22}(t), \quad X_{12}(0) = 0, \tag{56}$$

$$\dot{X}_{22}(t) = -w_1(t)X_{22}(t), \quad X_{22}(0) = 1. \tag{57}$$

The solutions to (55)–(57) are

$$X_{11}(t) = \exp\left[-\int_0^t w_3(\sigma)\,d\sigma\right], \tag{58}$$

$$X_{12}(t) = -\int_0^t \exp\left[-\int_\tau^t w_3(\sigma)\,d\sigma - \int_0^\tau w_1(\sigma)\,d\sigma\right] w_2(\tau)\,d\tau, \tag{59}$$

$$X_{22}(t) = \exp\left[-\int_0^t w_1(\sigma)\,d\sigma\right]. \tag{60}$$

In order to evaluate the expected value of the exponential of a random variable, we make use of the characteristic function (**21**). The characteristic function of a Gaussian random vector with mean m and covariance P is

$$M_x(u) = E[\exp(iu'x)] = \exp(iu'm - \tfrac{1}{2}u'Pu). \tag{61}$$

Since the exponents in (58)–(60) are Gaussian random variables, we evaluate the characteristic function of the exponents at $u = -i$ to obtain

$$E[X_{11}(t)] = \exp\left[KV_p t + \frac{1}{2}\int_0^t\int_0^t P_{33}(\sigma_1, \sigma_2)\,d\sigma_2\,d\sigma_1\right], \tag{62}$$

$$E[X_{22}(t)] = \exp\left[KV_p t + \frac{1}{2}\int_0^t\int_0^t P_{11}(\sigma_1, \sigma_2)\,d\sigma_2\,d\sigma_1\right]. \tag{63}$$

The closed-form expression for $E[X_{12}(t)]$ is somewhat more complicated, but it is based upon the same principle. First we define the Gaussian random vector $y(\tau) = [z(\tau), w_2(\tau)]'$ where

$$z(\tau) = -\int_\tau^t w_3(\sigma)\,d\sigma - \int_0^\tau w_1(\sigma)\,d\sigma. \tag{64}$$

Then if we denote the mean and covariance of $y(\tau)$ by $m(\tau)$ and $Q(\tau)$, respectively, some simple calculations show that

$$m(t) = [KV_p t, \, -KV_p], \tag{65}$$

$$Q_{11}(\tau) = 2\int_\tau^t \left[\int_0^\tau P_{13}(\sigma_2, \sigma_1)\,d\sigma_2\right] d\sigma_1 + \int_\tau^t\int_\tau^t P_{33}(\sigma_2, \sigma_1)\,d\sigma_2\,d\sigma_1 + \int_0^\tau\int_0^\tau P_{11}(\sigma_2, \sigma_1)\,d\sigma_2\,d\sigma_1, \tag{66}$$

$$Q_{12}(\tau) = Q_{21}(\tau) = \int_\tau^t P_{23}(\tau, \sigma)\,d\sigma + \int_0^\tau P_{12}(\sigma, \tau)\,d\sigma, \tag{67}$$

$$Q_{22}(\tau) = P_{22}(\tau, \tau). \tag{68}$$

If $e_1 = [1, 0]$ and $e_2 = [0, 1]$, we have

$$E\{w_2(\tau)\exp[z(\tau)]\} = e_2 E\{y(\tau)\exp[e_1 y(\tau)]\} = -i(\mathrm{d}/\mathrm{d}u_2)[M_y(u_1,u_2)]|_{(-i,0)}$$
$$= [KV_p + Q_{12}(\tau)]\exp[KV_p t + \tfrac{1}{2}Q_{11}(\tau)]. \tag{69}$$

Thus $E[X_{12}(t)]$ satisfies

$$E[X_{12}(t)] = \int_0^t [KV_p + Q_{12}(\tau)]\exp[KV_p t + \tfrac{1}{2}Q_{11}(\tau)]\,\mathrm{d}\tau, \tag{70}$$

where Q_{12} and Q_{22} are given in (66) and (67). The solution $x(t)$ of (19) is then $X(t)x(0)$ and, if $x(0)$ is independent of the noise process,

$$E[x(t)] = E[X(t)]\,E[x(0)]. \tag{71}$$

The analysis of this section (and its extension to arbitrary solvable Lie algebras) can be used to study the stochastic stability of systems containing multiplicative colored noise—i.e. once we have obtained closed-form moment equations, we can study these expressions to determine stability conditions as a function of system parameters (**15–17**). For instance, in the above example suppose we assume that V_1, V_2 and V_3 are independent with $E[V_i(t)] = 0$, $i = 1, 2, 3$, and

$$E[V_i(t)\,V_i(t+\tau)] = \sigma_1^2\exp(-a_i|\tau|), \quad a_i > 0, \quad i = 1, 2, 3. \tag{72}$$

In this case the system is first-order asymptotically stable if and only if

$$V_p < -\max(K\sigma_1^2/a_1, \quad K\sigma_3^2/a_3). \tag{73}$$

In the next section we discuss a method of approximate analysis for systems for which these exact techniques are not applicable. We also refer the reader to (**17**) for a sufficient condition for the stochastic stability of a particular system of the form (50) in which $\{A_0, \ldots, A_m\}$ is not solvable. The method of analysis involves the "bounding" of the original system by a second system of the type (50) in which the relevant Lie algebra *is* solvable. The generalization of this technique to other than this one example remains an open problem.

VIII. Approximate Analysis via Harmonic Expansions

If the stochastic bilinear system of interest does not have the solvable Lie algebra structure described in the preceding section, or if we are considering a system with polynomial nonlinearities (e.g. one obtained from a feedback or series interconnection of bilinear systems), we cannot in general obtain closed-form expressions for the moments of the state of the system. One basic reason for this is that the moment equations in the colored noise or polynomial nonlinearity case are "coupled forward" (i.e. the evolution of the nth moments depend on higher moments). In the last section we avoided this problem by obtaining integral expressions for the moments. However, in the general (nonsolvable) case, this trick does not work, and we are forced to seek approximate methods for moment calculation.

One such method that can be easily applied in the case of polynomial nonlinearities is the *cumulants* method (**41**, **14**). In this method we approximate higher order moments as a function of lower ones by truncating the Taylor series expansion of the logarithm of the characteristic function of the state. We will not discuss this method here and refer the reader to (**41**, **42** and **14**) for descriptions of the approach and comments on its relationship to other methods such as the so-called second-order or Gaussian approximations (**21**).

In this section we illustrate another approach which can be applied to problems in which the state is restricted to a compact subset of Euclidean space. This approach requires the use of harmonic expansions of probability distributions, and we will describe the basis for the method via an example. We note that Fourier series methods have been utilized by several authors (**10**, **11**, **18**–**20**) to study phase tracking and demodulation problems (such as those in Section V), and our example involves the utilization of techniques similar to those used in these references. At the end of this section we will describe an extension of the concepts devised here and in (**10**, **11** and **18**–**20**) to larger classes of problems.

Consider the communications problem of Fig. 4 and Section V. An important problem is the determination of the statistics of the PLL phase error. For example, if the received signal is of the form

$$\mathrm{d}r(t) = \sin[\omega_c t + w(t)]\,\mathrm{d}t + \mathrm{d}N(t), \tag{74}$$

where w and N are independent Brownian motions, and if the PLL is first order (the linear filter feeding back to the VCO is a constant gain), then the baseband model for the error in the estimate of the phase of the received signal is of the form (37)

$$\mathrm{d}e(t) = -\alpha \sin e(t)\,\mathrm{d}t + \beta\,\mathrm{d}v(t), \tag{75}$$

where $E(\mathrm{d}v) = 0$, $E(\mathrm{d}v^2) = \mathrm{d}t$, and α and β are known constants ($\alpha > 0$).

Fokker–Planck techniques have been used to obtain a closed-form expression for the steady-state density of e (**37**), but no such exact description exists for the transient behavior of the statistics of the phase error. As such information would be extremely useful in studying the phase acquisition characteristics of the loop (**37**), it is worthwhile considering approximate analyses of the transient behavior. If we expand the density for $e(t)$ in a Fourier series

$$p(e,t) = \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} c_n(t) \exp(ine), \tag{76}$$

$$c_n^*(t) = E\{\exp[ine(t)]\} = c_{-n}, \tag{77}$$

we see that in order to determine p, we must determine the coefficients c_n, which provide a useful set of "moments" of e.

A straightforward calculation (**21**) yields the evolution equation

$$\dot{c}_n(t) = -\tfrac{1}{2}\alpha n[c_{n+1}(t) - c_{n-1}(t)] - \tfrac{1}{2}n^2\beta^2 c_n(t). \tag{78}$$

Note that these equations are linear and coupled forward and thus must be truncated or approximated in order to be solved. Several approximation methods have been suggested for problems such as these. One possibility (**20**) is to assume

$$c_n = 0, \quad \forall n \geqslant N, \tag{79}$$

but this approximation is invalid if our phase error is small. For example, if we know the phase exactly, then all of the c_n are of the same order. Thus in high signal to noise ratio situations, the approximation (79) may not yield an accurate picture of loop behavior during acquisition. However, if we assume that the initial phase error is uniformly distributed on $[0, 2\pi]$, then $c_n(0) = 0$ for all $n \neq 0$, and the approximation (79) may yield useful information about the early stages of acquisition.

Another class of approximation methods involves the use of an "assumed density" [see (**11**)]. In this approach one assumes that the density $p(e,t)$ has a particular form that allows one to express higher order coefficients in terms of lower order terms. For example, an important density in the study of random phase problems is the *folded normal density*

$$p(\theta, t) = \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} \exp\left(-\tfrac{1}{2}n^2\gamma\right) \exp\left[in(\theta - \eta)\right]. \tag{80}$$

This density has many of the properties of the Gaussian density in R^n [see (**9–11**, **13**, **43** and **44**)], and the real variables η and γ can be given the interpretation of the mode and a measure of the spread of the density, respectively. For such a density, we have

$$c_n = |c_1|^{n(n-1)} c^n. \tag{81}$$

If we use this approximation in (78), we can obtain a closed set of equations for $c_1, \ldots, c_{n-1}$. For example, if we take $n = 2$, we obtain the equation

$$\dot{c}_1 = \tfrac{1}{2}\alpha[|c_1|^2 c_1^2 - 1] - \tfrac{1}{2}\beta^2 c_1. \tag{82}$$

We note that this approximation will also yield useful information about the early stages of acquisition $[c_1(0) = 0 = > c_n(0) = 0$ from (81)], and it may perform better than the approximation (79) in providing a good picture of the overall acquisition behavior of the loop. However, its behavior as we approach steady state is not completely accurate. The reason for this comes from an examination of the analytical expression and the shape of the steady-state density $p(e, \infty)$ (**37**),[pp. 57–59]

$$p(e, \infty) = \frac{\exp\left[(2\alpha/\beta^2)\cos e\right]}{2\pi I_0(2\alpha/\beta^2)}, \tag{83}$$

where I_0 is the modified Bessel function of the first kind of order 0 (**45**). The Fourier coefficients of (83) satisfy the relations

$$c_{n-1}^{(\infty)} - c_{n+1}^{(\infty)} = \frac{n\beta^2}{\alpha} c_n(\infty) \tag{84}$$

which can be solved (**45**)

$$c_n(\infty) = \frac{I_n(2\alpha/\beta^2)}{I_0(2\alpha/\beta^2)}. \tag{85}$$

Thus the steady-state density is not a folded normal, and thus the approximation (81) is incorrect in the limit, and the resulting approximate analysis will lead to an incorrect steady state (if it reaches a steady state at all). However, the density (83) has a shape which is quite similar to the bell-shaped folded normal, and thus the approximate analysis obtained by use of the folded normal approximation may prove to be quite useful.

A final approximation method is suggested by the exact steady state density itself. Choose N such that

$$\sum_{|k|\geqslant N} |c_k(\infty)|$$

is sufficiently small [this can be done since the density is continuous (**46**)]. We then solve (78) for $|n| < N$ using the steady-state value for c_N in the $\dot{c}_{N-1}$ equation. Noting that the resulting set of linear equations are stable, we see that the solution of these equations *will* approach the correct steady-state values for $c_1, \ldots, c_{N-1}$. As we are essentially starting out with $c_N \neq 0$, we may not obtain as accurate a picture of the initial part of the transient as if we used one of the other two approximations which have $c_N(0) = 0$ [note that for N large $c_N(\infty)$ is very small, however]. Thus, we may wish to use one (of the first two approximations initially and then switch to this latter method in order to obtain a better picture of the overall transient behavior.

The type of analysis described for this example can be extended to higher order phase-lock loops and more complex modulation than the Brownian phase modulation in (74). However, these higher order problems require more than straightforward Fourier analysis of phase error (e.g. we may have to take into account the presence of an unknown frequency offset). An indication of how one might extend our analysis to these more complex cases is given in (**11**) and (**20**).

Finally, we note that the methods of analysis illustrated in this section can be extended to a much larger class of bilinear and nonlinear stochastic problems in which the tools of harmonic analysis on compact groups and homogeneous spaces can be used (**16**). For example, if we are studying rotation in three dimensions, one might consider an equation of the form (**13**)

$$\dot{x}(t) = \left[\sum_{i=1}^{3} A_i\, \xi_i(t)\right] x(t), \tag{86}$$

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \tag{87}$$

where the ξ_i represent angular velocities (containing stochastic components) and x is either the position of a particle over a sphere (if x is a 3-vector) or the orientation of a rigid body with respect to inertial space (if x is a 3×3 orthogonal matrix). Refer to (**13**) and (**16**) for more on this particular problem and note here only that much as Fourier series methods proved useful in the one-dimensional rotation (i.e. phase process) case, the use of spherical harmonic analysis is most useful in the three-dimensional case. In fact, one can obtain analogs of the PLL in three dimensions (**13**, **47**).

We also note that there is a strong relationship between the $x^{[p]}$ moment equations in the white noise (Section VI) and the corresponding harmonic expansion of the density for x (assuming that the range of x is compact). In fact, in the white noise case the harmonic coefficients do *not* couple forward and one can in principle evaluate the lower order coefficients exactly. Refer to (**3**, **4** and **16**) for more on this question.

IX. Conclusions

In this paper, we have described several techniques that can be used to study the properties of certain random nonlinear models—the class of bilinear stochastic systems. We have also given several examples to indicate how such models might arise in the examination of nonlinear circuits and devices. We have seen that several concepts from the theory of Lie groups and algebras and from the theory of harmonic analysis are extremely useful in the study of such systems.

We also note that many of the techniques described in this paper for the analysis of stochastic systems can also be used in the synthesis of nonlinear estimation systems for bilinear stochastic models. These results are discussed in (**9–13**), (**16**) and (**18–20**). Furthermore, with the development of new results (**48**, **49**) relating nonlinear systems to equivalent bilinear systems, it appears that many of the results described in this paper and in the references may be extended to far larger classes of nonlinear stochastic systems.

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