[4] C. D. Johnson, "Accommodation of external disturbances in linear regulator and servomechanism problems," *IEEE Trans. Automat. Contr. (Special Issue on Linear–Quadratic–Gaussian Problem)*, vol. AC-16, pp. 635–644, Dec. 1971.
[5] S. P. Bhattacharyya, "Disturbance rejection in linear systems," *Int. J. Syst. Sci.*, vol. 5, no. 7, pp. 633–637, 1974.
[6] J. B. Pearson and C. Y. Ding, "Compensator design for linear multivariable systems," *IEEE Trans. Automat. Contr.*, vol. AC-14, pp. 130–134, Apr. 1969.

On the Stochastic Stability of Linear Systems Containing Colored Multiplicative Noise

ALAN S. WILLSKY, STEVEN I. MARCUS, AND DAVID N. MARTIN

Abstract—**The stability of linear systems containing colored multiplicative or state-dependent noise processes is considered. We present a technique for obtaining necessary and sufficient conditions for the pth moment stability of linear systems satisfying certain Lie-algebraic conditions. An example is given to illustrate the technique.**

I. Introduction

A great deal of attention has been given to the analysis of linear systems containing state-dependent or multiplicative noise processes [1]–[13]. Specifically, the question of the stochastic stability of such systems has been studied in some detail. A number of results have been obtained in the white noise case [5]–[9], but far less is known in the colored noise case. Recently, the introduction of concepts from the theory of Lie groups and Lie algebras has led to the development of several new techniques for the study of state-dependent noise systems [7]–[13]. Brockett [7], [8] has used Lie-theoretic methods in the study of white noise processes on spheres, and Willems [9] has used Lie theory concepts to derive extremely detailed stability results for a specific class of white noise systems. In the case of colored multiplicative noise, Willsky and Marcus [10], Willems [12], Martin [11], and Blankenship [13] have used Lie-theoretic methods to obtain stochastic stability results for specific classes of systems. In this note, we describe a technique for the analysis of a class of colored state-dependent noise systems that satisfy a particular Lie-theoretic condition. We give an example to illustrate the method.

II. Linear Systems with Colored Multiplicative Noise

We are interested in systems of the form

$$\dot{x}(t)=\left(A_0+\sum_{i=1}^{n}A_i\xi_i(t)\right)x(t) \tag{1}$$

where the A_i are known $n\times n$ matrices and $\xi'=(\xi_1,\cdots,\xi_n)$ is a zero-mean Gaussian random process with

$$E[\xi(t)\xi'(s)]=R(t,s). \tag{2}$$

We assume that ξ is independent of the initial condition $x(0)$.

We now define the "pth power" of (1). Following [8], recall that the number of linearly independent homogeneous polynomials of degree p in n variables [i.e., $f(cx_1,\cdots,cx_n)=c^pf(x_1,\cdots,x_n)$] is given by

$$N(n,p)=\binom{n+p-1}{p}=\frac{(n+p-1)!}{(n-1)!p!}. \tag{3}$$

Manuscript received April 25, 1975. This work was performed at the M.I.T. Electronic Systems Laboratory and was supported in part by the NSF under Grant GK-42090 and in part by NASA under Grant NGL-22-009-124. The work of S. I. Marcus was supported by an NSF Fellowship.
A. S. Willsky is with the Department of Electrical Engineering and Computer Science, Massachusetts Institute of Technology, Cambridge, Mass. 02139.
S. I. Marcus is with the Department of Electrical Engineering, University of Texas, Austin, Tex. 78712.
D. N. Martin is with the Raytheon Company, Bedford, Mass.

We choose a basis for this $N(n,p)$-dimensional space of homogeneous polynomials in $(x_1,\cdots,x_n)$ consisting of the elements

$$\sqrt{\binom{p}{p_1}\binom{p-p_1}{p_2}\cdots\binom{p-p_1-\cdots-p_{n-1}}{p_n}}\;x_1^{p_1}x_2^{p_2}\cdots x_n^{p_n},$$

$$\sum_{i=1}^{n}p_i=p,\qquad p_i\geq 0. \tag{4}$$

If we denote the vector consisting of these basic elements (ordered lexicographically) by $x^{[p]}$, then

$$\|x\|^p=\|x^{[p]}\| \tag{5}$$

where $\|x\|=\sqrt{x'x}$. It is clear that if x satisfies the linear differential equation

$$\dot{x}(t)=Ax(t), \tag{6}$$

then $x^{[p]}$ satisfies a linear differential equation

$$\dot{x}^{[p]}(t)=A_{[p]}x^{[p]}(t). \tag{7}$$

We regard this as the definition of $A_{[p]}$. We note here that $A_{[p]}$, which is closely related to Kronecker sum matrices [8], can be computed from A in a relatively straightforward manner (see [13] for a discussion of an algorithm for its computation). Two important properties of $A_{[p]}$ are linearity

$$(\alpha A+\beta B)_{[p]}=\alpha A_{[p]}+\beta B_{[p]} \tag{8}$$

and the fact that the eigenvalues of $A_{[p]}$ are all possible sums of p (not necessarily distinct) eigenvalues of A.

Using this notation, we obtain the "pth power" of (1),

$$\dot{x}^{[p]}(t)=\left(A_{0_{[p]}}+\sum_{i=1}^{n}A_{i_{[p]}}\xi_i(t)\right)x^{[p]}(t), \tag{9}$$

which is of the same form as (1).

III. Stochastic Stability

In this section, we describe a technique for studying the question of stochastic stability for a class of systems of the form given by (1). The next three definitions specify the type of stability of interest to us, and a Lie-algebraic condition which will be used in defining the class of systems of interest.

Definition 1: A vector random process x is *pth-order asymptotically stable* if

$$\lim_{t\to\infty}E[x^{[p]}(t)]=0. \tag{10}$$

The process is *pth-order stable* if $E[x^{[p]}(t)]$ remains bounded $\forall t$. The system (1) is pth-order (asymptotically) stable if the solution x is pth-order (asymptotically) stable for all initial conditions $x(0)$ independent of ξ and such that $E[x^{[p]}(0)]<\infty$.

Definition 2: A *Lie algebra* $\mathfrak{L}$ of $n\times n$ matrices is a subspace of $n\times n$ matrices such that

$$[A,B]\triangleq AB-BA\in\mathfrak{L},\qquad \text{for all } A,B\in\mathfrak{L}. \tag{11}$$

We use the notation $\{B_1,\cdots,B_r\}_A$ to denote the Lie algebra generated by $B_1,\cdots,B_r$, i.e., the smallest Lie algebra containing $B_1,\cdots,B_r$.

Definition 3: We associate with any Lie algebra $\mathfrak{L}$ its *derived series*

$$\mathfrak{L}^{(0)}=\mathfrak{L}$$
$$\mathfrak{L}^{(n+1)}=[\mathfrak{L}^{(n)},\mathfrak{L}^{(n)}]=\{[A,B]\,|\,A,B\in\mathfrak{L}^{(n)}\},\qquad n\geq 0. \tag{12}$$

$\mathfrak{L}$ is *Abelian* if $\mathfrak{L}^{(1)}=\{0\}$, and is solvable if $\mathfrak{L}^{(n)}=\{0\}$ for some n.

Theorem 1: A matrix Lie algebra $\mathfrak{L}$ is solvable if and only if there exists a nonsingular matrix P (possible complex-valued) such that PAP^{-1} is upper triangular for all $A\in\mathfrak{L}$.

Proof: See [14].

Consider the system (1). Let

$$\mathfrak{L}=\{A_0,A_1,\cdots,A_n\}_A. \tag{13}$$

Willems [9] studied stochastic stability for (1) when the ξ_i are white and the Lie algebra $\mathfrak{L}$ is solvable. In addition, in [12] he considered the colored noise case when $\mathfrak{L}$ is assumed to be Abelian. We now describe a method for obtaining necessary and sufficient conditions for stochastic stability for the colored noise case when $\mathfrak{L}$ is solvable.

Write the solution to (1) in the form

$$x(t)=\Phi_\xi(t,0)x(0) \tag{14}$$

where $\Phi_\xi(t,0)$ is the transition matrix for (1), thought of as an explicit function of the process ξ. If $\mathfrak{L}$ is solvable, we see that we can obtain a closed-form expression for Φ_ξ. This can be seen as follows: from Theorem 1, find the matrix P such that each $B_i=PA_iP^{-1}$ is upper triangular. Then we can solve the equation

$$\dot{\Psi}_\xi(t,0)=\left[B_0+\sum_{i=1}^{n}B_i\xi_i(t)\right]\Psi_\xi(t,0),\qquad \Psi_\xi(0,0)=I \tag{15}$$

by straightforward calculations. Then

$$\Phi_\xi(t,0)=P^{-1}\Psi(t,0)P \tag{16}$$

and Φ_ξ involves nothing more complicated than exponentials of integrals of components of ξ, polynomials in ξ, and various combinations, products, and integrals of such quantities. Since ξ is Gaussian and independent of $x(0)$, we can evaluate the expectations of such quantities in closed form, and from (14) we see that we can obtain a closed-form expression for $E[x(t)]$, and hence can determine first-order stability conditions. This procedure is illustrated in the next section. We refer the reader to [15] for an alternate, but computationally equivalent method for calculating Φ_ξ, involving the construction of a particular basis for $\mathfrak{L}$.

Finally, we note that the above analysis can be directly extended to the determination of necessary and sufficient conditions for pth-order asymptotic stability. This is clear, since $x^{[p]}$ consists of the same types of functionals of ξ. This can also be seen from the evolution equation (9) for $x^{[p]}$.

Let

$$\mathfrak{L}_{[p]}=\{A_{0_{[p]}},A_{1_{[p]}},A_{2_{[p]}},\cdots,A_{n_{[p]}}\}_A. \tag{17}$$

Then one can show [8] that $\mathfrak{L}_{[p]}$ is solvable if $\mathfrak{L}$ is, and therefore we can use the preceding analysis to determine first-order stability conditions for (9) (i.e., pth-order conditions for the original system).

IV. An Example

Consider the system

$$\dot{x}(t)=[A_0+A_1\xi_1(t)+A_2\xi_2(t)]x(t) \tag{18}$$

$$A_0=\alpha\begin{pmatrix}2 & -1\\ 1 & 0\end{pmatrix},\quad A_1=\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix},\quad A_2=\begin{pmatrix}1 & -1\\ 0 & 0\end{pmatrix}. \tag{19}$$

Suppose $\xi'=(\xi_1,\xi_2)$ is a zero-mean Gaussian random process with covariance given by (2). In this case, we can check that $\mathfrak{L}$ is solvable and that

$$P=\begin{bmatrix}0 & 1\\ 1 & -1\end{bmatrix} \tag{20}$$

upper triangularizes the system. That is, letting $y=Px$, we have

$$\dot{y}(t)=\begin{bmatrix}\alpha+\xi_1(t) & \alpha+\xi_1(t)\\ 0 & \alpha+\xi_2(t)\end{bmatrix}y(t) \tag{21}$$

and

$$y(t)=\Phi(t,0)y(0)=\begin{bmatrix}\eta_1(t,0) & \int_0^t\eta_1(t,s)(\alpha+\xi_1(s))\eta_2(s,0)\,ds\\ 0 & \eta_2(t,0)\end{bmatrix}y(0) \tag{22}$$

where

$$\eta_i(t_1,t_2)=\exp\left[\alpha(t_2-t_1)+\int_{t_1}^{t_2}\xi_i(s)\,ds\right],\qquad i=1,2. \tag{23}$$

Using the properties of the Gaussian distribution, we compute

$$E[\Phi_{11}(t,0)]=E[\eta_1(t,0)]=\exp\left[\alpha t+\frac{1}{2}\int_0^t\int_0^t R_{11}(s,\tau)\,ds\,d\tau\right] \tag{24}$$

$$E[\Phi_{22}(t,0)]=E[\eta_2(t,0)]=\exp\left[\alpha t+\frac{1}{2}\int_0^t\int_0^t R_{22}(s,\tau)\,ds\,d\tau\right] \tag{25}$$

$$E[\Phi_{12}(t,0)]=\int_0^t\left[\alpha+\int_s^t R_{11}(s,\tau)\,d\tau+\int_0^s R_{12}(s,\tau)\,d\tau\right]\cdot\exp\left[\alpha t+\frac{1}{2}Q(t,s)\right]ds \tag{26}$$

where

$$Q(t,s)=\int_s^t\int_s^t R_{11}(\tau,\sigma)\,d\sigma\,d\tau+2\int_s^t\left[\int_0^s R_{12}(\tau,\sigma)\,d\sigma\right]d\tau+\int_0^s\int_0^s R_{22}(\tau,\sigma)\,d\sigma\,d\tau. \tag{27}$$

Given a specific covariance matrix, we can then study the first-order stability properties of (18) and (19). For example, if

$$R_{11}(t,s)=R_{22}(t,s)=\frac{1}{2}\left[\sigma_1^2e^{-k_1|t-s|}+\sigma_2^2e^{-k_2|t-s|}\right] \tag{28}$$

$$R_{12}(t,s)=\frac{1}{2}\left[\sigma_1^2e^{-k_1|t-s|}-\sigma_2^2e^{-k_2|t-s|}\right] \tag{29}$$

with k_1, $k_2>0$, we can use (24)–(27) to show that a necessary and sufficient condition for first-order asymptotic stability is

$$\alpha<-\frac{1}{2}\left(\frac{\sigma_1^2}{k_1}+\frac{\sigma_2^2}{k_2}\right). \tag{30}$$

The system is first-order stable if equality holds in (30).

If we now consider the system (18) with system matrices given by

$$A_0=\alpha\begin{bmatrix}2 & -1\\ 1 & 0\end{bmatrix},\quad A_1=\frac{1}{\sqrt{2}}I,\quad A_3=\frac{1}{\sqrt{2}}\begin{bmatrix}-1 & 2\\ 0 & 1\end{bmatrix}, \tag{31}$$

and if we assume that the covariance of ξ is given by (28) and (29), we can show that the system is first-order asymptotically stable if and only if

$$\alpha<-\max\left(\frac{\sigma_1^2}{k_1},\frac{\sigma_2^2}{k_2}\right). \tag{32}$$

Again, the system is first-order stable if equality holds in (32).

V. Conclusions

In this note, we have studied the problem of stochastic stability for linear systems with colored multiplicative noise processes. We have described and illustrated a technique for the analysis of systems whose system matrices generate a solvable Lie algebra. Such systems certainly represent a small subclass of all linear systems containing colored multiplicative noise, and further research is needed to exploit the technique described here and to devise techniques for systems with nonsolvable Lie algebras. Some results for such systems are reported in [11] and [13].

References

[1] F. Kozin, "A survey of stability of stochastic systems," *Automatica*, vol. 5, 1969.
[2] M. B. Nevelson and R. Z. Khas'minskii, "Stability of a linear system with random disturbances of its parameters," *Appl. Math. Mech.*, vol. 30, 1966.
[3] W. Wedig, "Regions of instability for a linear system with random parametric excitation," in *Proc. Symp. on Stochastic Dynamical Syst.* (Math. Lecture Notes 294), New York: Springer-Verlag, 1972.
[4] G. L. Blankenship, "Stability of uncertain systems," Electron. Syst. Lab., Massachusetts Inst. Technol., Cambridge, Rep. ESL-R-448, June 1971.
[5] W. M. Wonham, "Random differential equations in control theory," in *Probabilistic Methods in Applied Mathematics*, vol. 2, A. T. Bharucha-Reid, Ed. New York: Academic, 1968.
[6] D. L. Kleinman, "On the stability of linear stochastic systems," *IEEE Trans. Automat. Contr.* (Corresp.), vol. AC-14, pp. 429–430, Aug. 1969.
[7] R. W. Brockett, "Lie theory and control systems on spheres," *SIAM J. Appl. Math.*, vol. 25, pp. 213–225, Sept. 1973.
[8] R. W. Brockett, "Lie algebras and Lie groups in control theory," in *Geometric Methods in Systems Theory*, D. Q. Mayne and R. W. Brockett, Ed. The Netherlands: Reidel, 1973.
[9] J. L. Willems, "Stability of higher order moments for linear stochastic systems," to be published.
[10] A. S. Willsky and S. I. Marcus, "Analysis of bilinear Noise models in circuits and devices," in *Monograph of the Colloquium on the Application of Lie Group Theory to Nonlinear Network Problems*, 1974 IEEE Int. Symp. on Circuits and Syst., San Francisco, Calif., Apr. 1974.
[11] D. N. Martin, "Stability criteria for systems with colored multiplicative noise," Ph.D. dissertation, Dep. Elec. Eng., Massachusetts Inst. Technol., Cambridge, July 1974.
[12] J. L. Willems, "Stability criteria for stochastic systems with colored multiplicative noise," to be published.
[13] G. L. Blankenship, "Lie theory and the moment stability problem in stochastic differential equations," presented at IFAC 1975, Cambridge, Mass., Aug. 1975.
[14] A. A. Sagle and R. E. Walde, *Introduction to Lie Groups and Lie Algebras*. New York: Academic, 1973.
[15] J. Wei and E. Norman, "On global representations of the solutions of linear differential equations as a product of exponentials," *Proc. Amer. Math. Soc.*, vol. 15, pp. 327–334, 1964.

Partial Uniqueness: Observability and Input Identifiability

TSUNEO YOSHIKAWA AND S. P. BHATTACHARYYA

Abstract—**Necessary and sufficient conditions are derived under which a given set of linear functionals assume unique values over the solution set of a linear equation; the latter property is referred to as partial uniqueness. The usefulness of the conditions obtained for partial uniqueness is demonstrated by applying them to various problems of observability and input identifiability of linear dynamical systems.**

I. Introduction

There exists a variety of problems in linear control theory which may be reduced to the problem of determining conditions under which a given set of linear functionals assume unique values over the solution set of a linear equation. We refer to this property of the solutions as uniqueness with respect to the given set of linear functionals, or, for briefness, partial uniqueness.

The purpose of this note is to derive necessary and sufficient conditions for partial uniqueness of the solution of a linear equation, and to demonstrate their usefulness by applying them to the problem of partial observability, unknown input partial observability, and partial input identifiability of linear dynamical systems.

Manuscript received April 28, 1975. This work was supported by the National Research Council under the NRC Resident Research Associateship Program.
T. Yoshikawa is with the Automation Research Laboratory, Kyoto University, Kyoto, Japan, on leave at the Systems Dynamics Laboratory, NASA Marshall Space Flight Center, Huntsville, Ala.
S. P. Bhattacharyya is with the Department of Systems Engineering and Computation, COPPE, Federal University of Rio de Janeiro, Rio de Janeiro, Brazil, on leave at the Systems Dynamics Laboratory, NASA Marshall Space Flight Center, Huntsville, Ala.

II. Partial Uniqueness

Consider the linear equation

$$Lz = d \tag{1}$$

where L: $\mathfrak{X} \to \mathfrak{Y}$ is a linear map and $\mathfrak{X}$, $\mathfrak{Y}$ are linear vector spaces. Denote the solution set of (1) by $\mathfrak{X}_0$, i.e.,

$$\mathfrak{X}_0 = \{ z | z \in \mathfrak{X}, Lz = d \}$$

and assume that (1) is consistent, so $\mathfrak{X}_0$ is nonempty. Given the linear map P: $\mathfrak{X} \to \mathfrak{W}$ (a linear vector space), we wish to determine a necessary and sufficient condition under which for all $z_1, z_2 \in \mathfrak{X}_0$,

$$Pz_1 = Pz_2. \tag{2}$$

If P is one to one, then (2) is equivalent to the condition that (1) has a unique solution. Hence, for arbitrary P we refer to (2) as uniqueness with respect to P, or, for briefness, partial uniqueness. The proposition below solves the above problem.

Proposition: The solution of (1) is unique with respect to P if and only if

$$\text{kernel } L \subset \text{kernel } P. \tag{3}$$

Proof:

a) Sufficiency. Assume (3) is true and let $z_1, z_2 \in \mathfrak{X}_0$. Then $z_1 - z_2 \in$ kernel L, and hence $z_1 - z_2 \in$ kernel P, i.e., $Pz_1 = Pz_2$.

b) Necessity. If (3) fails, there exists $z^* \in \mathfrak{X}$ such that $Lz^* = 0$ and $Pz^* \neq 0$. Let $z_3 \in \mathfrak{X}_0$; then $z_3 + z^* \equiv z_4 \in \mathfrak{X}_0$ and $Pz_4 = Pz_3 + Pz^* \neq Pz_3$. Hence (2) fails. ■

Letting L and P denote matrix representations of the respective maps, we have the following alternative statements of the Proposition.

Corollary 1: The solution of (1) is unique with respect to P if and only if

$$\operatorname{rank} L = \operatorname{rank} \begin{bmatrix} L \\ P \end{bmatrix}. \tag{4}$$

Proof: Equation (4) is equivalent to (3). ■

Corollary 2: The solution of (1) is unique with respect to P if and only if there exists a T such that

$$TL = P. \tag{5}$$

If (5) holds, the unique value of Pz is given by

$$Pz = Td. \tag{6}$$

Proof: Equation (5) is equivalent to (4). If (5) holds, multiplying (1) from the left by T and using (5) yields (6). ■

In fact, if there exists at least one matrix T which satisfies (5), the general solution of (5) is given by

$$T = PL^{\#} + F(I - LL^{\#})$$

where $L^{\#}$ denotes a pseudoinverse of L, and F is an arbitrary matrix of appropriate dimension. The unique value of Pz is then given by

$$Pz = PL^{\#}d.$$

We remark that Corollaries 1 and 2 were stated and used in [1] for a particular form of P. The next section demonstrates some applications of the result.

III. Applications

Throughout this section, we treat the system

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k + Du_k, \qquad k = 0, 1, 2, \cdots \end{aligned} \tag{7}$$

where x_k is the n-dimensional state vector, u_k is the r-dimensional input